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Geometry of Random Sparse Arrays

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***Abstract*—We consider random sparse arrays whose sensors are randomly placed on a grid of fixed size. Although deterministic sparse array geometries such as nested, coprime and their many variants have been extensively studied, less is known about difference/sum sets of random arrays. In this work, we analytically characterize the size of the contiguous segment of the so-called difference coarray of random sparse arrays. The difference coarray determines fundamental performance limits of sparse arrays and is therefore an essential object of study. Moreover, a large contiguous coarray is usually desired to guarantee unambiguous identification of many signal sources. We show that i.i.d. sampling schemes can be inadequate for guaranteeing a large contiguous coarray with high probability. Instead, one needs to design alternative random sampling schemes. We propose such a scheme and verify numerically that it yields random arrays with a difference coarray whose contiguous segment scales super-linearly with the expected number of sensors.**

Keywords*—Random Arrays, Sparse Arrays, Difference Coarray, Tapering

I. Introduction

Sparse arrays have recently drawn significant research interest [1], [2]. Unlike Uniform Linear Arrays (ULA), which can resolve at most $m-1$ sources using m sensors, sparse arrays, with non-uniform sensor positions, are capable of resolving $O(m^2)$ sources using m sensors, and thereby achieving higher spatial resolution [3]–[11]. This property is due to the fact that the difference coarray of a properly designed sparse array contains a large ULA segment of length $O(m^2)$.

Prime examples of widely used sparse geometries are nested and coprime arrays, and their extensions [12]–[16]. These geometries achieve the optimal number of contiguous elements in difference coarray $O(m^2)$ in addition to having simple closed-form sensor positions. However, certain properties are difficult to verify (or may not hold) for these deterministic sampling schemes. One example is the Restricted Isometry Property (RIP) used to guarantee uniform recovery in compressive sensing using computationally efficient techniques [17]–[21]. RIP is typically established using independent random sampling schemes, due to the challenge of analyzing deterministic or correlated schemes [22]–[26]. Proving RIP for suitable deterministic array manifolds remains a challenging problem.

Motivated by the above, we study random sparse arrays. Specifically, we provide insights into the properties of such sparse arrays, whose sensors are randomly placed on a grid of fixed size. The study of random sparse arrays can be useful for understanding the properties of difference sets beyond specific deterministic designs. For example one may ask whether sparse array geometries with a large contiguous difference set are rare or not.

Randomly generated arrays have empirically been observed to have many advantages, such as benign grating lobes and reduced side-lobe levels using fewer elements [27]–[30]. Consequently, several works have analyzed the performance of algorithms utilizing random arrays for applications ranging from DoA estimation [31], [32] to collaborative distributed beamforming [33]–[35]. However, these properties of random arrays have not been extensively investigated in the coarray domain. The most relevant body of work on difference sets of randomly generated sets can actually be found in the additive combinatorics literature. Many of these works nevertheless focus on comparing the size of the difference and sum sets of random sets for various sampling distribution [36]–[39]. An exception is Harvey-Arnold *et al.* [40], who recently characterized the probability distribution of the number of missing elements in the difference set, when elements of the original set are chosen uniformly at random with a fixed probability. However, questions regarding the geometry of the difference set for variable and nonuniform probability distributions remain unaddressed.

In this work, we focus on characterizing the size of the contiguous difference coarray segment of random arrays. We ask the question "How large can the (contiguous) difference coarray of random sparse array be (in expectation or with high probability) given an expected number of sensors m?". Specifically, we consider a random sparse array model consisting of independent, but not necessarily identical Bernoulli random variables. We are ultimately interested in which choices of the Bernoulli probabilities of this model are likely to give rise to a contiguous coarray of size $O(m^2)$, similarly to deterministic sparse array designs. Note that deterministic arrays are a corner case of this random array model, where the Bernoulli random variables corresponding to the deterministic sensor positions are assigned unit probabilities (the remaining probabilities being equal to zero). However, such distributions do not give rise to a *diversity* of array geometries with a large contiguous difference coarray or other desired properties. Hence, we focus on nontrivial random distributions to understand whether arrays with a large contiguous coarray are rare or common. We first show that standard independently and identically distributed (i.i.d.) sampling schemes fail to generate arrays with large contiguous coarrays. Since properties of the difference coarray fundamentally rely on the sampling distribution (the Bernoulli probabilities) used to generate the random array, we shape this distribution to yield arrays that achieve a large contiguous difference coarray with high probability (w.h.p.). In particular,

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we establish that the average weight of each lag in a large contiguous ULA segment of the coarray is larger than zero for a carefully chosen distribution. Simulations demonstrate that our design achieves a difference coarray whose contiguous ULA segment scales super-linearly with the expected number of sensors m.

II. Problem Formulation

Given an aperture of length $N \in \mathbb{N}$, and probabilities $\{p_0, p_1, \ldots, p_{N-1}\} \in [0,1]^N$, we choose a random subset $\mathbb{S}$ of $\mathbb{I} = \{0, 1, \ldots, N-1\}$, such that each $i \in \mathbb{I}$ is independently chosen to $\mathbb{S}$ with probability p_i. We call a linear array whose sensors are located at $\{n\lambda/2, n \in \mathbb{S}\}$ a *random sparse array*, where λ is the wavelength of the incoming far-field, narrowband, and uncorrelated sources. We denote each selection by a Bernoulli random variable B_i with probability p_i, i.e. $B_i = 1$, $i \in \mathbb{S}$ and $B_i = 0$ otherwise. Note that the cardinality $|\mathbb{S}|$ of the set $\mathbb{S}$ is also a random variable. Hence, we assume the expected number of sensors

$$\mathbb{E}\big[|\mathbb{S}|\big] = \sum_{i=0}^{N-1} p_i = m \tag{1}$$

is fixed to some positive number $m \in (0, N]$.

The difference coarray is an important concept in the context of sparse arrays:

Definition 1: The difference coarray $\mathbb{D}_\mathbb{S}$ of a physical array $\mathbb{S} = \{d_1, \ldots, d_m\}$ is defined as

$$\mathbb{D}_\mathbb{S} := \{d_m - d_n ; d_m, d_n \in \mathbb{S}\}.$$

By a simple counting argument we note that $\mathbb{D}_\mathbb{S} = O(|\mathbb{S}|^2)$.

Next, we define some relevant quantities for the difference coarray. Let R denote the largest integer satisfying $\{0, 1, \ldots, R-1\} \subset \mathbb{D}_\mathbb{S}$. By the symmetry of $\mathbb{D}_\mathbb{S}$, we then also have $\{-R+1, \ldots, 0\} \subset \mathbb{D}_\mathbb{S}$. Define set $\mathbb{U}_\mathbb{S} := \{0, \pm 1, \ldots, \pm R-1\}$ and call it the *central ULA segment*. The cardinality of $\mathbb{U}_\mathbb{S}$ plays an important role in DOA estimation using sparse arrays, since algorithms such as coarray MUSIC explicitly leverage the contiguous segment of the difference coarray [1]. The contiguous coarray determines fundamental performance limits of sparse arrays, such as the number of unambiguously resolvable sources. Another important quantity is the number of repeated elements in the difference coarray, given by the so-called *weight function*, defined as follows.

Definition 2: The weight W_j of a sparse array $\mathbb{S}$ is the multiplicity of sensor pairs in $\mathbb{S}$ which is separated by the distance j:

$$W_j := |\{(n_1, n_2) \in \mathbb{S}^2 \; : \; n_1 - n_2 = j\}|, \quad |j| \leq N-1.$$

Weights W_j are non-negative random variables since $\mathbb{S}$ is a random set. It can be shown that the sequence $W_0, W_1, \ldots, W_{N-1}$ is the auto-correlation of the sequence of Bernoulli random variables $B_0, B_1, \ldots, B_{N-1}$, i.e.,

$$W_j = \sum_{i=0}^{N-1-j} B_i B_{i+j}, \quad 0 \leq j \leq N-1. \tag{2}$$

Eq. (2) shows that even if the sensor positions in $\mathbb{S}$ are selected independently at random, the corresponding weights W_j in the difference coarray $\mathbb{D}_\mathbb{S}$ are not independent. However, we can easily characterize the expected weights $\mathbb{E}[W_j]$ using the independence of the B_i's:

$$w_j \triangleq \mathbb{E}[W_j] = \begin{cases} m, & \text{if } j = 0 \\ \sum_{i=0}^{N-1-j} p_i p_{i+j}, & \text{if } 1 \leq j \leq N-1. \end{cases} \tag{3}$$

For $-(N-1) \leq j \leq -1$, we can characterize W_j and w_j by the well-known relationship $W_j = W_{-j}$ and $w_j = w_{-j}$. Finally, we define the limit of w_j as $N \to \infty$ as

$$w_j^\infty \triangleq \lim_{N \to \infty} \mathbb{E}[W_j].$$

We can use (2) and (3) to characterize the expected length of the contiguous segment of the difference coarray $\mathbb{U}_\mathbb{S}$ of a random array. First, note that $j \in \mathbb{D}_\mathbb{S}$ if and only if $W_j > 0$. Hence, the event of interest, the positive part of the central ULA segment is of length at least $L-1$, can be mathematically denoted as

$$\bigcap_{j=0}^{L-1} W_j \geq 1. \tag{4}$$

There are many well-known examples of deterministic sparse array geometries whose difference coarray $\mathbb{D}_\mathbb{S}$ contains a contiguous ULA segment of length $O(m^2)$ [1], [2], [41]. Indeed, this is the order-wise optimal scaling for an array with m physical sensors. In the next two sections, we investigate how the difference coarray of a random array compares to these deterministic designs.

III. Necessity of probability tapering as $N \to \infty$

The geometry of the random array is dependent on sampling probabilities $\{p_0, p_1, \ldots, p_{N-1}\}$. Carefully designing these probabilities is therefore needed to achieve a difference coarray with a large contiguous segment. To demonstrate this, we first show that naive *uniform i.i.d. sampling* does not provide a large contiguous difference set when the expected number of sensor m in (1) does not scale with $N \to \infty$. We then propose a *tapered sampling scheme* that yields a random array whose difference coarray has a large ULA segment.

We ultimately wish to characterize the probability of the event of interest in (4), i.e., $\Pr\big(\bigcap_{j=0}^{L-1} W_j \geq 1\big)$. If this probability tends to 1 as $N \to \infty$, then the random array $\mathbb{S}$ has a central ULA of length at least L w.h.p. In this work, we nevertheless restrict our analysis to the expected weights in (3) for the i.i.d. and tapered distributions. Since each W_j is a non-negative random variable, we have the following relationship due to Markov's inequality:

$$w_j \geq \Pr\big(\{W_j \geq 1\}\big) \geq \Pr\left(\bigcap_{j=0}^{L-1} \{W_j \geq 1\}\right) \forall j. \tag{5}$$

Eq. (5) shows that $w_j > 0$ for $0 \leq j \leq L-1$ is a *necessary* condition for the difference coarray to have a central ULA segment of length at least L with nonzero probability. Establishing sufficient conditions using (other) concentration inequalities is ongoing work.

A. Uniform i.i.d. Sampling

A simple sampling scheme is the uniform i.i.d. distribution $p_i = \frac{m}{N}$. This choice clearly satisfies the expected sensor condition in (1). However, as we now show, this distribution cannot generate arrays with a large coarray when $N \gg m$. Following (3), the expected weights are $w_0 = m$ and

$$w_j = (N-j)\frac{m^2}{N^2}, \text{ when } 1 \leq j \leq N-1. \tag{6}$$

For a fixed m, we then have

$$w_j^{\infty} = \begin{cases} m, & \text{if } j = 0 \\ 0, & \text{if } 1 \leq j \leq N-1. \end{cases}$$

Consequently, achieving $w_j^{\infty} > 0$ for $j \geq 1$ in the regime of *fixed* m requires either nonuniform probabilities p_i (tapering), or introducing correlation between the Bernoulli random variables B_i.[1]

B. Tapered Sampling Scheme

We now show that tapering schemes can asymptotically achieve a large contiguous coarray segment (of length L) in expectation for any fixed value of m. We define probabilities p_i as

$$p_i = \begin{cases} \frac{m}{2L} + (1-t)\frac{m}{2L}, & 0 \leq i \leq L-1 \\ \frac{m}{2(N-L)}t, & L \leq i \leq N-1 \end{cases} \tag{7}$$

for some integer $L \leq N-1$ and $t \in [0,1]$. Parameter t controls the fraction of probability mass placed beyond index $L-1$. Note that (7) satisfies condition (1). Our main result states that the expected weights in (7) also satisfy $w_j > 0$ and $w_j^{\infty} > 0$ for $j \leq L-1$, where L is fixed.

Theorem 1: Suppose the elements of a random array are sampled according to the tapering scheme defined in (7). Then for fixed m and fixed L we have

$$w_j^{\infty} = \begin{cases} m, & \text{if } j = 0 \\ \frac{m^2}{4}\frac{L-j}{L^2}(2-t)^2, & \text{if } 1 \leq j \leq L-1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof: Since we operate in the regime where N is arbitrarily large and $L = O(1)$, we can assume $L \leq \frac{N}{2}$ without loss of generality. Due to (3), for $j = 0$, we have $\mathbb{E}[W_j] = m$. For $j \geq 1$, we distinguish three cases. If $j \leq L-1$, then

$$\begin{aligned} \mathbb{E}[W_j] &= \sum_{i=0}^{L-j-1} p_i p_{i+j} + \sum_{i=L-j}^{L-1} p_i p_{i+j} + \sum_{i=L}^{N-1-j} p_i p_{i+j} \\ &= \frac{m^2}{4L^2}(L-j)(2-t)^2 + j\frac{m^2(2-t)t}{4L(N-L)} + \frac{m^2t^2(N-L-j)}{4(N-L)^2}. \end{aligned}$$

If $L \leq j \leq N-1-L$, then

$$\begin{aligned} \mathbb{E}[W_j] &= \sum_{i=0}^{L-1} p_i p_{i+j} + \sum_{i=L}^{N-1-j} p_i p_{i+j} \\ &= L\frac{m^2}{4L}\frac{(2-t)t}{(N-L)} + \frac{m^2t^2(N-L-j)}{4(N-L)^2}. \end{aligned}$$

If $j \geq N-L$, then

$$\mathbb{E}[W_j] = \sum_{i=0}^{N-1-j} p_i p_{i+j} = \frac{m^2}{4L}\frac{(2-t)t(N-j)}{(N-L)}.$$

Overall, we therefore have

$$\mathbb{E}[W_j] = \begin{cases} m, & \text{if } j = 0 \\ \frac{m^2}{4}\left(\frac{(L-j)(2-t)^2}{L^2} + \frac{j(2-t)t}{L(N-L)} + \frac{t^2(N-L-j)}{(N-L)^2}\right), & \text{if } 1 \leq j \leq L-1 \\ \frac{m^2}{4(N-L)}\left((2-t)t + \frac{t^2(N-L-j)}{N-L}\right), & \text{if } L \leq j \leq N-1-L \\ \frac{m^2}{4L}\frac{(2-t)t(N-j)}{(N-L)}, & \text{otherwise.} \end{cases}$$

Note that in the last case $N-j \leq L$. Hence, taking the limit of the above expressions as $N \to \infty$ completes the proof. ■

Remark 1: Theorem 1 ensures that a random array generated according to the tapering defined in (7) satisfies the necessary condition (5) for the array to have a difference coarray with a central ULA segment of length at least $O(L)$ with nonzero probability. However, condition $w_j^{\infty} > 0$ may not be sufficient for the given array to achieve this desired property.

Remark 2: Placing non-zero mass beyond index $L-1$ in Theorem 1, i.e., setting $t > 0$, does not affect the value of the expected weight w_j^{∞} of difference coarray lags as $N \to \infty$. However, for finite N, there is still a non-zero probability that sensors are placed at indices beyond $L-1$. Even though these sensors may not increase the size of the largest ULA segment in the difference coarray, they might still be useful for other purposes, such as array interpolation [42]. We plan to explore this possibility in future work.

In the next section, we empirically establish that the tapered sampling scheme in (7) can indeed generate random arrays whose difference coarray contains with nonzero probability a central ULA segment of length $O(m^a)$, where $a > 1$.

IV. Numerical Results

In the first experiment, we compare the tapered sampling scheme with i.i.d. scheme. We fix the expected number of sensors to $m = 5$ and set the desired length of the contiguous segment of the coarray to $L = 10$ and parameter $t = 0$ in (7). Fig. 1 shows the probability distributions of the random arrays generated with tapered and i.i.d. sampling schemes for $N = 21$ and $N = 101$ (zoomed in on the first 21 indices) on the top left and right, respectively. The corresponding average empirical weights of the lags in the difference coarray are plotted below. Clearly, as N increases, the average weights in difference coarray of the i.i.d. random array decay to 0, whereas the corresponding average weights are nonzero (when $j < L = 10$) for the random array generated by the tapered sampling scheme. This is in agreement with Theorem 1 and verifies our previous conclusion that tapering is required to ensure non-vanishing difference coarray weights as N increases.

In the second experiment, we study how the expected length of the largest central ULA segment of the coarray $\mathbb{E}[|\mathbb{U}_{\mathbb{S}}|]$ scales with m and L as a function of (growing) N. We focus on the tapered sampling scheme in (7) with $t = 1$. For each N, we generate 10^4 random array realizations, and average the lengths of the central ULA segment to estimate the expected length. Fig. 2 shows the relationship between N and the expected length of the central ULA segment for

[1] Alternatively, m could be allowed to scale with N in a suitable manner.

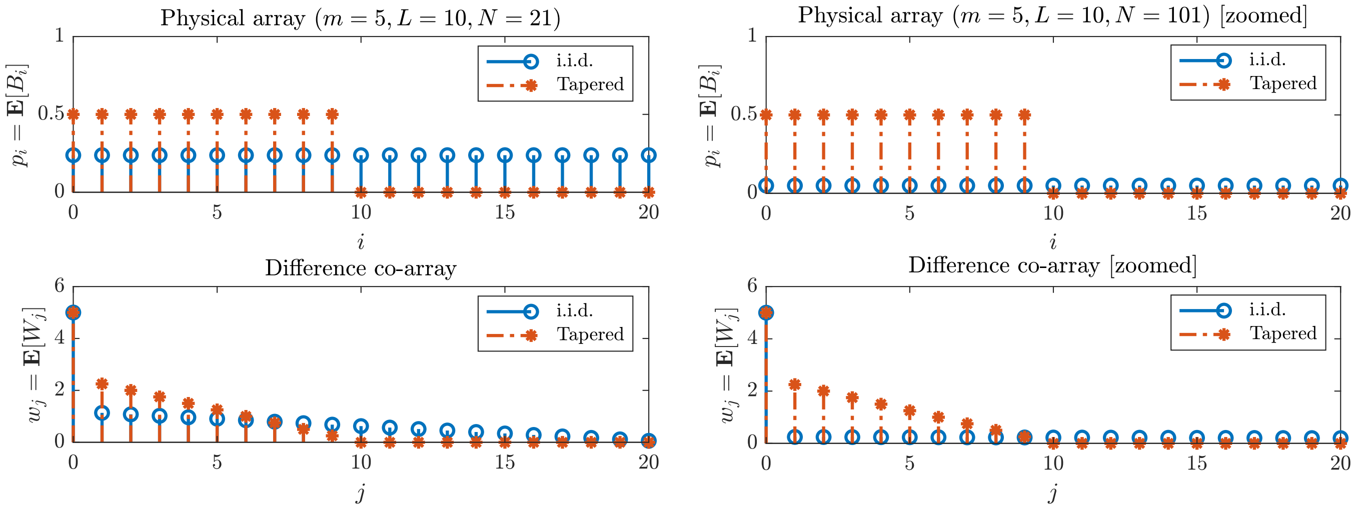

Fig. 1. Example of sensor selection probabilities of i.i.d. and tapered sampling schemes (top), along with the corresponding expected coarray weights (bottom). Tapering is needed to ensure the expected weights in the desired contiguous segment ($L = 10$) do not vanish as N increases (left: $N = 21$, right: $N = 101$).

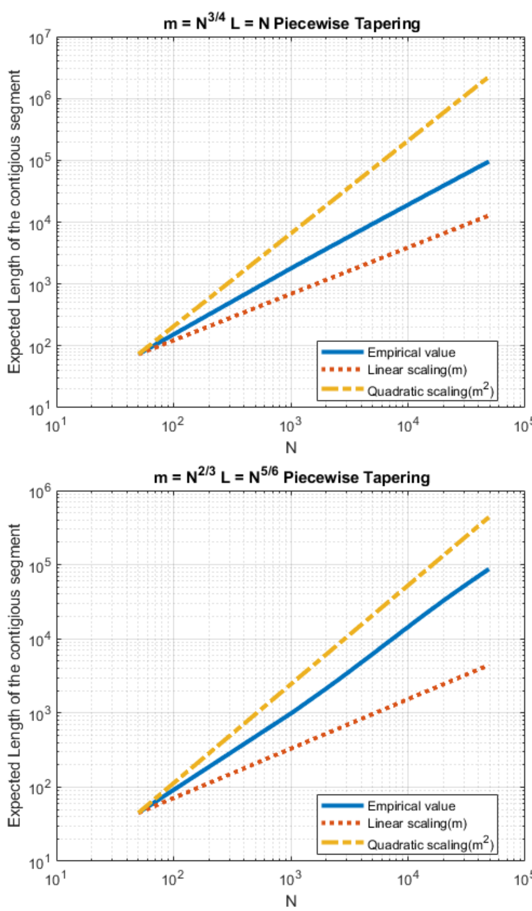

Fig. 2. Average length of the central ULA segment $|\mathbb{U}_{\mathbb{S}}|$ as a function of the aperture N for two scaling regimes of the expected number of sensors m and desired contiguous segment length L (top: $m = N^{3/4}$ and $L = m^{4/3} = N$, bottom: $m = N^{2/3}$ and $L = m^{5/4} = N^{5/6}$). The average largest ULA segment of the coarray scales super-linearly (in m) when the random array is generated using the proposed tapered sampling distribution.

m and L obeying two different scaling laws with respect to N. For comparison, we also plot the curves of naive linear scaling m and the optimal quadratic scaling m^2. We observe that the average length of the central ULA segment grows super-linearly in m, although still being below the optimal quadratic rate.

V. Conclusion

This work investigated the difference coarray of random arrays generated from independent Bernoulli random variables under the condition that the average number of sensors is fixed. We first showed that the standard i.i.d. sampling fails to generate a large contiguous coarray when the array aperture is very large compared to the expected number of sensors. We then proposed a different random sampling strategy called tapered sampling. We established that the expected weights of the difference coarray lags of this tapering scheme are strictly positive, which is necessary for achieving a large difference coarray. Furthermore, we empirically demonstrated that the size of the contiguous segment of the coarray indeed scales super-linearly with the expected number of sensors. This suggests that random arrays can achieve a contiguous difference coarray of nontrivial size by judiciously designing the probability taper. In future work, we will provide analytical bounds on the probability of the length of central ULA segment being large. We also plan to explore alternative sampling strategies for maximizing the size of the contiguous coarray segement.

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