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# Vortex-bound solitons in topological superfluid <sup>3</sup>He

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#### Abstract

The different superfluid phases of <sup>3</sup>He are described by p-wave order parameters that include anisotropy axes both in the orbital and spin spaces. The anisotropy axes characterize the broken symmetries in these macroscopically coherent quantum many-body systems. The systems' free energy has several degenerate minima for certain orientations of the anisotropy axes. As a result, spatial variation of the order parameter between two such regions, settled in different energy minima, forms a topological soliton. Such solitons can terminate in the bulk liquid, where the termination line forms a vortex with trapped circulation of mass and spin superfluid currents. Here we discuss possible soliton-vortex structures based on the symmetry and topology arguments and focus on the three structures observed in experiments: solitons bounded by spin-mass vortices in the B phase, solitons bounded by half-quantum vortices (HQVs) in the polar and polar-distorted A phases, and the composite defect formed by a half-quantum vortex, soliton and the Kibble-Lazarides-Shafi wall in the polar-distorted B phase. The observations are based on nuclear magnetic resonance (NMR) techniques and are of three types: first, solitons can form a potential well for trapped spin waves, observed as an extra peak in the NMR spectrum at shifted frequency; second, they can increase the relaxation rate of the NMR spin precession; lastly, the soliton can present the boundary conditions for the anisotropy axes in bulk, modifying the bulk NMR signal. Owing to solitons' prominent NMR signatures and the ability to manipulate their structure with external magnetic field, solitons have become an important tool for probing and controlling the structure and dynamics of superfluid <sup>3</sup>He, in particular HQVs with core-bound Majorana modes.

Keywords: half-quantum vortex, composite topological defect, spontaneous symmetry breaking

(Some figures may appear in colour only in the online journal)

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#### 1. Introduction

Solitary waves, or solitons, often refer to localized wave packets that maintain their shape as they propagate through a medium or interact with other solitons. In the context of this paper, the word 'soliton' refers to continuous variation of the superfluid order parameter between two regions settled in distinct yet degenerate energy minima. Within the soliton the free energy is not minimized—instead, it is stabilized by the topology of the underlying superfluid. Topological solitons are protected by the same mechanism as linear topological defects, quantized vortices, on which solitons can terminate in bulk liquid. Such composite objects are in the focus of this work.

We discuss vortex-bound solitons in the *p*-wave superfluid <sup>3</sup>He [1], where complex spontaneous symmetry breaking (SSB) patterns allow for the existence of a plethora of different topological objects [2]. The topological protection of composite objects is described by relative homotopy groups [3], which reflect on the relation between residual symmetries in two ordered phases. These phases may appear in a sequence of SSB phase transitions, similar to some scenarios of the evolution of the early Universe. Alternatively, they may refer to the same ordered state viewed at two distinct length scales set by different orientational energies. Both scenarios are realized in superfluid <sup>3</sup>He and discussed in this paper: Several superfluid phases are observed in bulk and more are engineered with nanostructured confinement. In these phases condensation, magnetic anisotropy, and spin-orbit interaction energies provide the hierarchy of relevant length scales ranging from tens of nanometers to a fraction of a millimeter. Diversity of superfluid <sup>3</sup>He even allows for the existence of an object where a quantized vortex both terminates a planar topological soliton and simultaneously serves as a starting line of another planar object, the Kibble-Lazarides-Shafi (KLS) wall-we call such composite defect a nexus.

Vortex-bound solitons provide an experimental handle to probing topological defects, as the characteristic size of a soliton is often a few orders of magnitude larger than that of the accompanying linear defects. The nuclear magnetic resonance (NMR) signature of solitons may be used to identify such defects, including half-quantum vortices (HQVs) in the polar and polar-distorted superfluid phases [4]. Interesting on their own right, HQVs can also provide experimental insight into grand unified theories (GUTs) when taken through successive phase transitions [5], or host Majorana zero modes in the chiral superfluid or superconducting phases [6]. Experimental systems with known and sufficiently complex order parameters for testing the predictions of GUTs are extremely rare and therefore valuable, while Majorana zero modes are highly sought after for their promise in topological quantum computing scenarios [7]. The superfluid phases of  ${}^{3}$ He provide experimental access to such scenarios; to date the polar-distorted A (PdA) phase of  ${}^{3}$ He is the only experimental platform known to host HQVs in a chiral superfluid. Therefore, understanding vortex-bound solitons is important for many research directions.

This work aims to provide the necessary theoretical background for understanding vortex-bound solitons in the superfluid phases of <sup>3</sup>He, as well as review the relevant experimental results. In section 2 we discuss the symmetries that govern the order parameters of the normal fluid, as well as superfluid phases encountered in bulk or confined samples. We further discuss how these symmetries are linked to topological defects, and how they are affected by the presence of orienting energy terms. In section 3 we introduce relative homotopy groups and their connection to topological and composite defects, and solitons. The rest of the paper focuses on experimental aspects: section 4 lays out the theoretical background for understanding the NMR measurements and solitons' effect on the observed NMR spectrum, while the experimental observations of different types of vortex-bound solitons in various superfluid phases of <sup>3</sup>He are discussed in section 5. Finally, we provide concluding remarks in section 6.

#### 2. Residual symmetries in <sup>3</sup>He

#### 2.1. Superfluid phases in the bulk liquid

At temperatures well below the Fermi temperature  $T_{\rm F} \sim 1$  K but above the superfluid transition temperature  $T_{\rm c} \sim 1$  mK, <sup>3</sup>He behaves as Fermi liquid, as described by Landau [8]. Properties of the Fermi liquid can be modeled through a weakly interacting gas of excitations in the vicinity of the Fermi surface. According to the Bardeen–Cooper–Schrieffer theory [9], the presence of any attractive interaction, in the case of <sup>3</sup>He arising from Van der Waals interaction accompanied by spin-fluctuation exchange mechanism [1], there exists a temperature below which the fermions tend to form a coherent macroscopic condensate via Cooper pairs. The spin-exchange mechanism favors *p*-wave pairing with angular momentum l = 1. Simultaneously, the antisymmetry of the fermionic wave function requires s = 1, where *s* is the spin quantum number for the Cooper pair.

The formation of Cooper pairs becomes energetically favorable below the critical temperature  $T_c$ , and the liquid undergoes a phase transition to the superfluid state. Above the superfluid transition bulk <sup>3</sup>He is described by the symmetry group [10]

$$G = SO(3)_{\mathbf{L}} \times SO(3)_{\mathbf{S}} \times U(1)_{\phi} \times T \times C \times P, \qquad (1)$$

which includes continuous symmetries, three-dimensional rotations of coordinates  $SO(3)_L$ , rotations of the spin space  $SO(3)_S$ , and the global phase transformation group  $U(1)_{\phi}$ , as well as discrete symmetries, *T* is the time-reversal symmetry, *C* is the particle-hole conversion symmetry, and *P* is the space parity symmetry. The transitions from normal fluid to superfluid phases as well as transitions between different superfluid phases are accompanied by spontaneous breaking of continuous and/or discrete symmetries in *G*. In bulk <sup>3</sup>He three superfluid phases have been realized [1], figure 1; the fully-gapped superfluid B phase characterized by broken relative



**Figure 1.** Superfluid phase diagram of bulk <sup>3</sup>He. At 1 bar pressure <sup>3</sup>He liquefies at 3.2 K. It remains liquid all the way to the absolute zero temperature for pressures P < 34 bar. In bulk fluid, three different superfluid phases are observed below the critical temperature—the time-reversal symmetric B phase, the chiral equal spin-pairing A phase, and at high magnetic fields the A<sub>1</sub> phase which only allows Cooper pairs with both spins oriented along the magnetic field.

spin-orbit symmetry, the chiral  $p_x + ip_y$  superfluid A phase, and finally, the spin-polarized A<sub>1</sub> phase close to  $T_c$  in high magnetic fields. In the rest of the section 2.1, we discuss the symmetry-breaking patterns in the B and A phases, while the infomation on  $A_1$  phase is included for completeness in appendix A.

2.1.1. *B phase.* In the absence of magnetic field, the superfluid B phase has the total angular momentum  $\mathbf{j} = \mathbf{l} + \mathbf{s} = 0$ . The requirement  $\mathbf{j} = 0$  allows three spin configurations for Cooper pairs, i.e.  $s_z = \{-1, 0, 1\}$  and, respectively,  $l_z = \{1, 0, -1\}$ . The B phase is characterized by broken spin-orbit symmetry, i.e. the relative orientation of the spin and orbital vectors becomes locked. It is reflected in the order parameter of the B phase [1]

$$A_{\mu j}^{\rm B} = \Delta_{\rm B} e^{i\phi} \mathbf{R}_{\mu j}(\hat{\mathbf{n}}, \theta), \qquad (2)$$

where  $\Delta_{\rm B}$  is the pressure- and temperature-dependent superfluid gap characterizing the energy required to break a Cooper pair,  $\phi$  is the superfluid phase, and matrix  $\mathbf{R}(\hat{\mathbf{n}}, \theta)$  describes the rotation of spins with index  $\mu$  relative to the orbital coordinates with index j around the vector  $\hat{\mathbf{n}}$  by angle  $\theta$ . In equilibrium the spin vector  $\hat{\mathbf{s}}$  can be obtained from the orbital vector  $\hat{\mathbf{l}}$  by rotating  $\hat{\mathbf{l}}$  around axis  $\hat{\mathbf{n}}$  by the 'Leggett angle'  $\theta_{\rm L} = \arcsin(-1/4) \approx 104^{\circ}$ , minimizing the spin-orbit interaction [11].

Taking into account the broken relative symmetry between the spin and orbital parts of the order parameter and the broken phase symmetry, the remaining symmetry in the B phase is [1, 10]

$$H_{\rm B} = SO(3)_{\rm J} \times T \times C \times PU_{\pi},\tag{3}$$

where  $SO(3)_J$  denotes the joint three-dimensional rotation in the orbital and spin spaces, and  $PU_{\pi}$  denotes the joint discrete symmetry of the parity P and  $\pi$  phase rotation  $U_{\pi}$ . 2.1.2. A phase. The A phase is an equal-spin-pairing state, where the Cooper pairs consist of spins with the same sign and thus  $s_z = \{-1, 1\}$ . The order parameter of the A phase can be written as

$$A^{\rm A}_{\mu j} = \Delta_{\rm A} e^{i\phi} \hat{\mathbf{d}}_{\mu} (\hat{\mathbf{m}}_j + i\hat{\mathbf{n}}_j), \tag{4}$$

where the vectors  $\hat{\mathbf{m}}$  and  $\hat{\mathbf{n}}$  (different from the  $\hat{\mathbf{n}}$  vector in the B phase) form an orthogonal triad with the Cooper pair orbital angular momentum axis  $\hat{\mathbf{l}} = \hat{\mathbf{m}} \times \hat{\mathbf{n}}$ ,  $\hat{\mathbf{d}}$  is the spin anisotropy vector along which the total spin of a Cooper pair vanishes, and  $\Delta_A$  is the maximum superfluid gap in the A phase. In the A phase, the gauge symmetry  $U(1)_{\phi}$  is broken and any change of the order parameter phase  $\phi \rightarrow \phi + \Delta \phi$  may be compensated by rotating the orbital component of the order parameter about  $\hat{\mathbf{l}}$  by an angle  $-\Delta\phi$ , leaving the order parameter invariant under that particular combination of transformations. This combination corresponds to a remaining relative  $U(1)_{\phi+\mathbf{L}}$  symmetry. Additionally, rotation of the spin space about vector  $\hat{\mathbf{d}}$  leaves the order parameter unchanged, corresponding to a remaining  $SO(2)_{S}$  symmetry. The order parameter is also symmetric under the simultaneous rotation about perpendicular axis and  $\pi$  phase change, i.e. under transformation  $(\hat{\mathbf{d}}, \phi) \rightarrow (-\hat{\mathbf{d}}, \phi + \pi)$ , corresponding to a discrete symmetry described by the cyclic group  $\mathbb{Z}_2$  (denoted  $\mathbb{Z}_{2(\phi+\mathbf{S})}$ ) of order 2. In the A phase, the time-reversal symmetry is partially broken due to the non-zero imaginary part in the orbital space, reducing it to the combined discrete symmetry  $\mathbb{Z}_{2(T+L)}$ corresponding to simultaneous time reversal and  $\pi$ -rotation of the orbital space about  $\hat{\mathbf{m}}$ . The remaining symmetries are described by the group [1]

$$H_{\rm A} = U(1)_{\phi+\mathbf{L}} \times \tilde{D}_{\infty S} \times \mathbb{Z}_{2(T+\mathbf{L})} \times C \times PU_{\pi}, \qquad (5)$$

where we have combined the symmetries  $\tilde{D}_{\infty \mathbf{S}} = SO(2)_{\mathbf{S}} \rtimes \mathbb{Z}_{2(\phi+\mathbf{S})}$ .

#### 2.2. <sup>3</sup>He under confinement by nafen

The presence of nanostructured confinement, i.e. thin slabs [12, 13] or various aerogels [14–18], modifies the superfluid phase diagram. Anisotropic confinement may also alter the symmetry group of the normal phase of <sup>3</sup>He by explicitly breaking the three-dimensional rotational symmetry in the coordinate space. Here we consider the effect of commercially available nematically ordered material called nafen [16], which breaks the three-dimensional continuous rotational symmetry  $SO(3)_{\rm L}$  in equation (1). The total symmetry group of the normal phase is reduced to [10]

$$G' = D_{\infty \mathbf{L}} \times SO(3)_{\mathbf{S}} \times U(1)_{\phi} \times T \times C \times P, \qquad (6)$$

where  $D_{\infty \mathbf{L}}$  contains rotations about the nafen anisotropy axis  $\hat{\mathbf{z}}$  and  $\pi$  rotations about perpendicular axes. This symmetry may also be written as a product of rotations of the space around the anisotropy axis and reflection with respect to the perpendicular plane, i.e.  $D_{\infty \mathbf{L}} \stackrel{c}{=} SO(2)_{\mathbf{L}} \rtimes \mathbb{Z}_{2\mathbf{L}}$ .



**Figure 2.** Schematic phase diagram of superfluid <sup>3</sup>He under confinement by nafen. Nanostructured confinement by uniaxial cylinders modifies the symmetries of <sup>3</sup>He in the normal fluid and below the superfluid transition temperature. The superfluid transition with the highest critical temperature under these conditions occurs to the polar phase [16]. The phase diagram shown in this figure sketches the measured phase diagram in a sample of nematically ordered aerogel called nafen-90, which consists of nearly uniaxial strands of  $Al_2O_3$  with density 90 mg cm<sup>-3</sup>. In addition to the polar phase, PdA and PdB phases are encountered at lower temperatures. The PdA phase can be supercooled, as illustrated by the patterned area. The phase diagram in the low-pressure and low-temperature region (white) is not well known and there is a possibility that the direct transition from the polar to the PdB phase is realized in this area.

The resulting phase diagram in nafen with 90 mg  $cm^{-3}$ density [16] is shown in figure 2. There are notable differences to the bulk phase diagram as novel superfluid phasesthe polar, PdA, and polar-distorted B (PdB) phases-are observed. In all cases the superfluid gap becomes anisotropic due to the effect of the confinement. Schematic illustrations of the superfluid gap in these phases are shown in figure 3. It is interesting that despite strong scattering by the confining strands, the critical temperature is suppressed only by a few percent compared to the bulk liquid. The robustness of the polar phase which appears immediately below transion is provided by extension of the Anderson theorem. This theorem has been derived in the case of *p*-wave pairing for the specular scattering from parallel strands [19]. This robustness of the polar phase is also demonstrated by the characteristic  $T^3$  lowtemperature dependence of the gap resulting from the presence of the nodal line [20]. In realistic samples scattering is never completely specular, nor are the strands perfectly aligned, and a small suppression of  $T_c$  is thus observed.

2.2.1. Symmetry breaking patterns and fibrations of vacuum manifolds. Besides stabilizing novel superfluid phases, confinement by nafen features even more complicated symmetry breaking patterns not possible in bulk samples. Such symmetry-breaking phase transitions can be used to study cosmological and dark matter models with SSB [21–28] and, in particular, topological defects that appear in such theories. In nafen samples, the following symmetry breaking pattern can be realized:

Here solid arrows represent the directions of symmetry reduction through SSB, and the dashed line represents the first order transition. The sequence of transitions from normal  $\rightarrow$ polar  $\rightarrow$  PdA  $\rightarrow$  PdB phases is seen on the phase diagram in figure 2, while a direct transition form the polar to the PdB phase remains a possibility at low pressures and temperatures.

The SSB pattern, equation (7), suggests that, starting from the normal phase, both the PdA and PdB phases are accessible via two consecutive second-order steps. Such situation cannot be realized in bulk superfluid <sup>3</sup>He. One can calculate the vacuum manifolds *R* of the confined phases from their respective residual symmetry groups *H* by taking the quotient space of *G'* over *H*. Denoting the vacuum manifolds generated through SSB from the normal phase as

$$R^{\rm P} = G'/H_{\rm P}, R_1^{\rm PdA} = G'/H_{\rm PdA}, R_1^{\rm PdB} = G'/H_{\rm PdB},$$
 (8)

and the vacuum manifolds accessible via the polar phase as

$$R_2^{\rm PdA} = H_{\rm P}/H_{\rm PdA}, R_2^{\rm PdB} = H_{\rm P}/H_{\rm PdB},$$
 (9)

we find  $R_2 \subset R_1$ . In the rest of the subsection we omit PdA or PdB indices at *R* and *H*, since equations apply in both phases. Detailed discussion of residual symmetries in each of the confined phases is presented in the following subsections.



**Figure 3.** Schematic illustration of superfluid gaps in superfluid phases presented in figure 2. The polar phase and PdB phase gaps are symmetric under rotation about the nafen anisotropy axis, and the PdA phase gap is shown in two projections as it lacks the rotational symmetry. For each phase, the maximum gap is oriented along the anisotropy axis of nafen strands, which have characteristic diameter  $d_1 \approx 9$  nm and separation  $d_2 \sim 50$  nm. Gap asymmetry is not drawn to scale.

Applying the third isomorphism theorem  $R_1/R_2 = (G'/H)/(H_P/H)$  [29], we get

$$R_1/R_2 \cong R^{\mathrm{P}} = G'/H_{\mathrm{P}}.$$
 (10)

Equation (10) suggests that the normal phase vacuum manifold  $R_1$  consists of infinite disjoint subspaces (cosets)  $R_2^{\rho}$ , which relate to  $R_2$  though action of  $\rho \in R_1$  i.e.  $R_2^{\rho} = \rho R_2$ . We also notice that every  $R_2^{\rho}$  is isomorphically mapped to an element of the vacuum manifold of the polar phase. Mathematically speaking, this signifies that the vacuum manifold  $R_1$ covers  $R^{\rm P}$  [30]. The resulting fibration *p* between the vacuum manifolds  $R_1$  and  $R^{\rm P}$  is

$$R_2 \hookrightarrow R_1 \xrightarrow{p} R^{\mathrm{P}}, \tag{11}$$

leading to the relationship between the homotopy groups of different vacuum manifolds

$$\pi_n(R_1, R_2) \cong \pi_n(R^{\mathbf{P}}). \tag{12}$$

Consequently, for n = 1 this relationship suggests that the polar phase linear topological defects are conserved as linear topological objects in the subsequent second order phase transition. These defects, as we will see later, are HQVs and singly quantized vortices. To assess whether the resulting linear objects are composite topological defects and serve as a termination line of a planar object, one must calculate the exact sequences of  $\pi_1(R_1, R_2)$ . Similarly, one can extract information about composite defects where monopole terminates a string [31] by setting n = 2 in equation (12). The resulting topological objects are discussed in section 3.

The presence of various effects, such as liquid-surface interactions, Zeeman splitting, and spin-orbit interaction affect the underlying symmetries of the superfluid phases. Since these orientational energies have characteristic length scales (see appendix B), both residual symmetry H and vacuum manifold R of a superfluid depend on the length scale at which superfluid is considered. The maximum H is found at the smallest length scales, limited by coherence length from below and, depending on conditions, by the magnetic  $\xi_{\rm H}$  or the spin-orbit  $\xi_{\rm D}$  lengths or by the confining geometry from above. At larger scales the symmetry is reduced. For polar and PdA phases below we provide one example of such reduction relevant to the soliton bounded by HOVs found in those phases. For the PdB phase we discuss reduced symmetries in more details in section 2.2.5 as they are relevant for analysis of composite objects in section 3.

2.2.2. *Polar phase.* The order parameter of the polar phase can be written as

$$A^{\rm P}_{\mu j} = \Delta_{\rm P} e^{i\,\phi} \hat{\mathbf{d}}_{\mu} \hat{\mathbf{m}}_j, \tag{13}$$

where  $\Delta_P$  is the maximum superfluid gap in the polar phase. The order parameter is invariant with respect to rotations of the spin space about  $\hat{\mathbf{d}}$  and rotations of the orbital space about  $\hat{\mathbf{m}}$ , as well as about combined rotations about perpendicular axes and  $\pi$  phase changes, i.e. with respect to transformations  $(\hat{\mathbf{d}}, \phi) \rightarrow (-\hat{\mathbf{d}}, \phi + \pi)$  and  $(\hat{\mathbf{m}}, \phi) \rightarrow (-\hat{\mathbf{m}}, \phi + \pi)$ . Similarly to other superfluid phases, the gauge symmetry  $U(1)_{\phi}$  is broken in the transition. The maximum group of remaining symmetries in the polar phase is

$$H_{\rm P} = \tilde{D}_{\infty {\bf L}} \times \tilde{D}_{\infty {\bf S}} \times T \times C \times P U_{\pi} \,, \tag{14}$$

where  $\tilde{D}_{\infty \mathbf{L}}$  and  $\tilde{D}_{\infty \mathbf{S}}$  are the symmetries involving  $\pi$  rotations about axes transverse to  $\hat{\mathbf{m}}$  or  $\hat{\mathbf{d}}$  in the orbital and spin spaces, respectively, combined with a phase rotation by  $e^{\pi i}$ , i.e.  $\tilde{D}_{\infty \mathbf{L}} = SO(2)_{\mathbf{L}} \rtimes \mathbb{Z}_{2(\phi+\mathbf{L})}$  and  $\tilde{D}_{\infty \mathbf{S}} = SO(2)_{\mathbf{S}} \rtimes \mathbb{Z}_{2(\phi+\mathbf{S})}$ . The vacuum manifold  $R^{\mathbf{P}}$  of the polar phase is [1]

$$R_{\rm P} = G'/H_{\rm P} = (S_{\rm S}^2 \times U(1)_{\phi})/\mathbb{Z}_{2(\phi+{\rm S})}, \qquad (15)$$

where  $S_{\mathbf{S}}^2$  denotes inversion.

The spin-orbit coupling (SOC) energy in the polar phase is proportional to  $(\hat{\mathbf{d}} \cdot \hat{\mathbf{m}})^2$ , which favors  $\hat{\mathbf{d}}$  orientation in the plane perpendicular to the nafen anisotropy axis. In addition, there is a magnetic energy term proportional to  $(\mathbf{H} \cdot \hat{\mathbf{d}})^2$ , which orients  $\hat{\mathbf{d}} \perp \mathbf{H}$  at sufficiently large magnetic fields  $\mathbf{H}$  above the dipole field  $H_D \sim 3$  mT. For a tilted field  $\mathbf{H} \not\parallel \hat{\mathbf{m}}$ , the combination of the spin-orbit and magnetic energies breaks the  $SO(2)_{\mathbf{S}}$ symmetry of  $H_P$  in equation (14), resulting in the reduced residual symmetries at scales larger than  $\xi_{\mathbf{H}}$  and  $\xi_{\mathbf{D}}$ 

$$\tilde{H}_{\rm P} = SO(2)_{\rm L} \times \mathbb{Z}_{2(\phi+{\rm S})} \times T \times C \times PU_{\pi}.$$
(16)

Here we additionally took into account strong orienting effect of nafen strands on the orbital space which fix  $\hat{\mathbf{m}}$  along the strands (common direction of  $\hat{\mathbf{m}}$  can be assumed since  $(\hat{\mathbf{d}}, \hat{\mathbf{m}})$ and  $(-\hat{\mathbf{d}}, -\hat{\mathbf{m}})$  is the same state), while keeping the symmetry  $SO(2)_{\mathbf{L}}$  of orbital rotation about the strand direction unaffected by any other orientational energy.

The existence of the symmetry group  $\mathbb{Z}_{2(\phi+\mathbf{S})}$  in equation (16) signifies that there are two degenerate minimum energy states which can be obtained via transformation  $(\hat{\mathbf{d}}, \phi) \rightarrow (-\hat{\mathbf{d}}, \phi + \pi)$ . Such degenerate regions are connected by topological  $\hat{\mathbf{d}}$  solitons, which can be terminated in bulk by HQVs with  $\pi$  phase winding, discussed in detail in section 3.3.

2.2.3. PdA phase. In the 90 mg cm<sup>-3</sup> nafen sample, the PdA phase is reached on cooling via a second-order phase transition from the polar phase [16]. The order parameter of the PdA phase is

$$A_{\mu j}^{\rm PdA} = \Delta_{\rm PdA} e^{i\,\phi} \hat{\mathbf{d}}_{\mu} (\hat{\mathbf{m}}_j + ib\,\hat{\mathbf{n}}_j), \tag{17}$$

where 0 < b < 1 is a dimensionless parameter characterizing the gap distortion and  $\Delta_{PdA}(b)$  is the maximum gap in the PdA phase. For b = 0 the order parameter of the polar phase is obtained, while b = 1 recovers the order parameter of the bulk A phase. The order parameter is symmetric under rotations of the spin space about  $\hat{\mathbf{d}}$ , about the combined transformation  $\mathbb{Z}_{2(\phi+\mathbf{S})}$ , and with respect to  $\pi$  rotation about  $\hat{\mathbf{l}}$  and  $\pi$  phase change, i.e. about  $(\hat{\mathbf{m}}, \hat{\mathbf{n}}, \phi) \rightarrow (-\hat{\mathbf{m}}, -\hat{\mathbf{n}}, \phi + \pi)$ . Moreover, the  $\tilde{D}_{\infty S}$  symmetry is preserved in the polar-PdA transition due to similar spin structure of the order parameter. In the orbital space,  $\pi$  rotation about the axis  $\hat{\mathbf{l}}$ , in combination with a phase rotation  $e^{\pi i}$ , form a symmetry, i.e. the orbital symmetry is reduced to a discrete symmetry  $U_{\pi(\phi+\mathbf{L})}$ . Similar to the bulk A phase, the time-reversal symmetry is partially broken in the PdA phase and the PdA phase is invariant under the combined discrete symmetry  $\mathbb{Z}_{2(T+\mathbf{L})}$ . The maximum residual symmetry group in the PdA phase is

$$H_{\text{PdA}} = U_{\pi \, (\phi + \mathbf{L})} \times \tilde{D}_{\infty \mathbf{S}} \times \mathbb{Z}_{2(T + \mathbf{L})} \times C \times PU_{\pi} \,. \tag{18}$$

Assuming phase transition from the normal phase, the vacuum manifold of the PdA phase becomes

$$R_1^{\rm PdA} = G'/H_{\rm PdA} = SO(2)_{\rm L} \times S_{\rm S}^2 \times U(1)_{\phi} \,. \tag{19}$$

In addition, the group  $H_{PdA}$  is also the subgroup of  $H_P$ , reflecting the fact that the PdA-polar phase transition is of the second order. For the polar-PdA transition, the vacuum manifold becomes

$$R_2^{\rm PdA} = H_{\rm P}/H_{\rm PdA} = SO(2)_{\rm L}.$$
 (20)

The SOC  $\propto (\hat{\mathbf{d}} \cdot \hat{\mathbf{l}})^2$  in the PdA phase affects distribution of both spin  $\hat{\mathbf{d}}$  and orbital  $\hat{\mathbf{l}}$  anisotropy axes since, unlike the polar phase, the orbital degrees of freedom are not fully fixed by the confinement. The effect of the magnetic anisotropy is similar to that in the polar phase. The reduced symmetry at scales larger than  $\xi_{\rm H}$  and  $\xi_{\rm D}$  is

$$\tilde{H}_{PdA} = \mathbb{Z}_{2(\phi+\mathbf{S})} \times \mathbb{Z}_{2(T+\mathbf{L})} \times C \times PU_{\pi}.$$
 (21)

A similar consideration holds for the spin part of the bulk A phase, where the residual symmetries in the presence of magnetic and spin-orbit energies become

$$H_{\rm A} = U(1)_{\phi + \mathbf{L}} \times \mathbb{Z}_{2(\phi + \mathbf{S})} \times \mathbb{Z}_{2(T + \mathbf{L})} \times C \times PU_{\pi} \,.$$
(22)

In both cases of bulk A and confined PdA phases the residual symmetry groups contain the two degenerate ground states connected by the  $\mathbb{Z}_{2(\phi+S)}$  symmetry and thus support HQVs, as in the polar phase.

It is worth noting that the symmetry considerations above concern an ideal confinement without random inhomogeneity. In practical samples, fluctuations of the density of confining strands leads to formation of the Larkin-Imry-Ma (LIM) orbital glass state in the PdA phase. Originally, LIM state was discovered in the A phase confined in the isotropic aerogels [32, 33], where random anisotropy breaks long-range order of the orbital vector  $\hat{\mathbf{I}}$  and orientation of  $\hat{\mathbf{I}}$  fluctuates over all possible directions in 3D space with a characteristic length scale significantly smaller that the dipolar length  $\xi_D$ . In the PdA phase under strong anisotropic columnar confinement,  $\hat{\mathbf{I}}$ is fixed into the plane perpendicular to the confining strands. Experimental evidence [34, 35] suggests, that also in this case the long-range orientational order of  $\hat{\mathbf{I}}$  within this plane is lost, and the 2D LIM state is formed. Formation of LIM state does not eliminate the  $\mathbb{Z}_{2(\phi+S)}$  symmetry responsible for the existence of solitons considered in this work, but it does modify the NMR response.

Another way to introduce a discrete  $\mathbb{Z}_2$  symmetry is to confine superfluid <sup>3</sup>He into a narrow slab. Independently of the original superfluid phase, the boundary orients the orbital momentum of the Cooper pairs along the surface normal. Two states connected with  $\mathbb{Z}_2$  symmetry correspond to the orbital momentum oriented toward or away from the wall. The corresponding solitons or domain walls connecting two states were observed both in the B [36] and A [37–39] phases. At the conditions near the phase transition between A and B phases such walls can proliferate and create a spatially modulated order parameter [36]. In the A phase, reorientation of the orbital momentum can take place independently or in combination with  $\hat{\mathbf{d}}$ , owing to the SOC. In the latter case the soliton is relatively thick, of the order of the slab thickness, and has been directly visualized by magnetic resonance imaging [39].

2.2.4. PdB phase. Depending on the confinement, the phase transition to the PdB phase may occur directly from the normal phase [40], via a first-order transition from the PdA phase [16] (as realized in the nafen-90 sample), or, in principle, via a second-order phase transition from the polar phase. The order parameter of the PdB phase in zero magnetic field can be written as

$$A_{\mu j}^{\text{PdB}} = \Delta_{\text{PdB}} e^{i\phi} (\hat{\mathbf{d}}_{\mu} \hat{\mathbf{z}}_{j} + q_1 \hat{\mathbf{e}}_{\mu}^{1} \hat{\mathbf{x}}_{j} + q_2 \hat{\mathbf{e}}_{\mu}^{2} \hat{\mathbf{y}}_{j}), \qquad (23)$$

where  $|q_1| = |q_2| \equiv q \in (0, 1)$  describes the relative gap size in the plane perpendicular to the strands. Vectors  $\hat{\mathbf{e}}^1$  and  $\hat{\mathbf{e}}^2$ are unit vectors in the spin space and form an orthogonal triad with vector  $\hat{\mathbf{d}}$ . The maximum gap  $\Delta_{\text{PdB}}(q)$  is achieved along the nafen anisotropy axis  $\hat{\mathbf{z}}$ . For q = 0, the order parameter of the polar phase is obtained while q = 1 recovers the order parameter of the bulk B phase. The order parameter is symmetric under the joint rotation of the spin and orbital spaces about the nafen anisotropy axis  $\hat{\mathbf{z}}$ . The remaining symmetries in the PdB phase are

$$H_{\rm PdB} = SO(2)_{\rm J} \times T \times C \times PU_{\pi}, \tag{24}$$

where the subscript notation  $\mathbf{J}$  refers to the symmetry of the simultaneous rotation of orbital and spin spaces.

The vacuum manifold for the normal-PdB phase is obtained from

$$R_1^{\rm PdB} = G'/H_{\rm PdB} = SO(3)_{\rm J} \times U(1)_{\phi} \,. \tag{25}$$

The group  $H_{PdB}$  is again a subgroup of  $H_P$ , reflecting the fact that also the polar-PdB phase transition is of second order. In this case, the resulting vacuum manifold becomes

$$R_2^{\text{PdB}} = H_{\text{P}}/H_{\text{PdB}} = SO(2)_{\mathbf{J}} \times \mathbb{Z}_{2(\phi + \mathbf{S})}.$$
 (26)

We note that  $R_2^{\text{PdB}}$  is a disconnected space characterized by states  $(\mathbf{d}, \mathbf{e}^1, \mathbf{e}^2, \phi)$  and  $(-\mathbf{d}, -\mathbf{e}^1, \mathbf{e}^2, \pi + \phi)$ . These right-handed and left-handed spaces can not be made equal by any action from  $H_P$ . Moreover, calculating  $\pi_0(R_2^{PdB})$  yields  $\mathbb{Z}_{2(S+\phi)}$  describing a domain wall structure (whereas solitons are described by the  $\pi_1$  homotopy group). As we will see further, the domain wall may combine with various types of solitons to form composite topological defects.

2.2.5. Residual symmetries in the presence of orientation energies in the PdB phase. The orientational energies with different coupling strengths and hierarchy of characteristic length scales reduce the normal phase symmetry group, resulting in reduced degrees of freedom for the order parameter vacuum manifolds after SSB phase transitions. The reduced vacuum manifolds introduce a different set of topological objects than those generated through fibration, discussed in section 2.2.1. For example, in the PdB phase the orientation energies allow the topological objects described by the relative homotopy groups  $\pi_n(R_1^{\text{PdB}}, R_2^{\text{PdB}})$  to expand into mesoscopic length scales, forming a nexus [41–43]. In this section we will discuss the reduction of the vacuum manifolds at mesoscopic length scales.

In <sup>3</sup>He the relevant length scales are the magnetic length  $\xi_{\rm H}$  and the dipole length  $\xi_{\rm D}$  [1, 4], which characterize the spatial ranges in which the gradient energy dominates over the orientational energy. When the length scale of the spatial variation is larger than these characteristic length scales, the vacuum manifolds are reduced to minimize the orientation energies. See appendix B for details on calculation of values of  $\xi_{\rm H}$  and  $\xi_{\rm D}$ .

If a static magnetic field  $\mathbf{H}^{(0)}$  with fixed direction is turned on [5], the degenerate space of the PdB order parameter reduces to

$$\boldsymbol{R}_{1}^{\mathrm{H}} = \boldsymbol{S}_{\mathbf{S}}^{1} \times \boldsymbol{U}(1)_{\phi} \tag{27}$$

in the region where length scale of spatial variation is larger than  $\xi_{\rm H}$ . Here  $S_{\rm S}^{\rm l}$  is the reflection symmetry. Since the magnetic energy locks the  $\hat{\bf d}$  vector into the plane perpendicular to  ${\bf H}^{(0)}$ ,  $R_2^{\rm PdB}$  retains its form inside the region characterized by the length scale  $\xi_{\rm H}$ . It follows that  $R_2^{\rm PdB} = SO(2)_{\rm J} \times \mathbb{Z}_{2(\phi+{\bf S})}$ in the region where  $\hat{\bf d}$  is approximately constant. In figure 4, we illustrate the  $R_1^{\rm H}$  and  $\xi_{\rm H}$  in the presence of the KLS wall.

When the SOC is taken into account, the vacuum manifolds are further reduced from  $R_1^{\rm H}$  and  $R_2^{\rm PdB}$ . In general, the requirement of minimizing the SOC energy in the region with  $r > \xi_{\rm D}$ fixes the relative directions between spin and orbital vectors, resulting in the broken relative spin-orbit symmetry [1]. Thus  $R_1^{\rm H}$  reduces to

$$\tilde{R}_1^{\text{SOC}} = R_{\mathbf{S}}^{\text{SOC}} \times U(1)_\phi \tag{28}$$

in the region with  $r > \xi_D$ , where  $R_S^{SOC}$  is the reduced vacuum manifold of the spin degree of freedom. Generally speaking,  $R_S^{SOC}$  has a complicated form, which may be simplified by using the following parametrization

$$\hat{\mathbf{d}} = \hat{\mathbf{x}}\cos\theta - \hat{\mathbf{z}}\sin\theta, \ \hat{\mathbf{e}}^1 = -\hat{\mathbf{x}}\sin\theta - \hat{\mathbf{z}}\cos\theta, \ \hat{\mathbf{e}}^2 = \hat{\mathbf{y}}, \ \mathbf{H}^{(0)} = H\hat{\mathbf{y}},$$
(29)



**Figure 4.** Illustration of relevant length scales in the PdB phase. The smallest length scale in the PdB phase is provided by the coherence length  $\xi$ , which sets the size of the vortex core. The KLS wall, described by  $\pi_1(R_1^{PdB}, R_2^{PdB})$ , is larger than the the coherence length by a factor  $q^{-1}$ . In the region  $r < \xi_H$  the vacuum manifolds are described by the groups  $R_1^{PdB}$  and  $R_2^{PdB}$ , while the presence of magnetic and SOC energies extends the hierarchy of length scales to  $R_1^{H}$  and the spin vector  $\hat{\mathbf{d}}$  becomes fixed perpendicular to the static magnetic field  $\mathbf{H}^{(0)}$ , while the  $R_2^{PdB}$  remains unchanged. When the SOC energy is taken into account,  $R_1^{H}$  further reduces to  $\tilde{R}_1^{SOC}$  and  $R_2^{PdB}$  reduces to  $\tilde{R}_2$ .

where  $\theta$  is the angle between **d** and local orbital-coordinate frame [5]. In this case, we find

$$R_{\mathbf{S}}^{\text{SOC}} = \{\theta_0, \pi - \theta_0, -\theta_0, \pi + \theta_0\},\tag{30}$$

where  $\theta_0 = \arcsin(q/(1-|q|))$ . In the region where  $\hat{\mathbf{d}}$  is approximately constant, the SOC energy fixes the relative rotation of  $SO(2)_{\mathbf{J}}$ , reducing  $R_2^{\text{PdB}}$  to

$$\tilde{R}_2 = \mathbb{Z}_{2(\phi + \mathbf{S})} \tag{31}$$

in the region  $r > \xi_D$ . The regions described by groups  $R_1^H$ ,  $\tilde{R}_1^{SOC}$ , and  $\tilde{R}_2$  are illustrated in figure 4.

#### 3. Composite topological objects

The relative homotopy groups may be utilized to investigate topological objects in the presence of multiple characteristic length scales [3, 44, 45]. In addition to the PdB phase discussed in the previous section, examples of such systems include e.g. solitons terminated by HQVs in spinor Bose condensates with quadratic Zeeman energy [46, 47]. In this section, we will discuss various composite objects connected by solitons in the superfluid phases of <sup>3</sup>He.

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## 3.1. Relative homotopy groups – from polar to polar-distorted phases

In experimentally reachable magnetic field  $\mathbf{H}^{(0)}$  the magnetic healing length is much larger than the coherence length,  $\xi_{\rm H} \gg \xi$ . The symmetry group G' and the symmetry breaking pattern described by equation (7) are valid in the region  $\xi \ll r \leq \xi_{\rm H}$ . In section 2.2.1, we have seen that the polar-phase topological objects survive a second symmetry-breaking transition into either PdA or PdB phases. In this section, we will study how these objects change in the transition to the PdA and PdB by analyzing the relative homotopy groups  $\pi_n(R_1, R_2)$ . The structures of the relative homotopy groups  $\pi_n(R_1, R_2)$  can be extracted from their short exact sequences (SESs). The SESs are calculated by splitting the long exact sequences of  $\pi_n(R_1, R_2)$ . In this section we use the results calculated in appendices C.1 and C.2.

The SES of  $\pi_1(R_1^{\text{PdA}}, R_2^{\text{PdA}})$  is

$$0 \longrightarrow \mathbb{Z}_{\phi} \longrightarrow \pi_1(R_1^{\text{PdA}}, R_2^{\text{PdA}}) \xrightarrow{\partial^*} 0 \longrightarrow 0 \tag{32}$$

and SES of  $\pi_2(R_1^{\text{PdA}}, R_2^{\text{PdA}})$  is

$$0 \longrightarrow \mathbb{Z}_{\mathbf{S}} \longrightarrow \pi_2(R_1^{\mathrm{PdA}}, R_2^{\mathrm{PdA}}) \xrightarrow{\partial^*} 0 \longrightarrow 0 , \qquad (33)$$

where the vacuum manifolds  $R_1^{PdA}$  and  $R_2^{PdA}$  are given by equations (19) and (20). From the SESs of  $\pi_{1(2)}(R_1^{PdA}, R_2^{PdA})$ in equations (32) and (33), we find that the boundary homomorphisms  $\partial^*$  are trivial both for  $\pi_1(R_1^{PdA}, R_2^{PdA})$  and  $\pi_2(R_1^{PdA}, R_2^{PdA})$ . In other words, linear and point-like defects (HQVs and **d**-monopoles, respectively), which PdA phase inherits from polar phase, are simple (i.e. not composite) topological objects. Due to similar spin structure, the HQVs and **d**-monopoles are similar in both phases [41].

In contrast, the symmetry breaking pattern resulting in the PdB phase yields the SESs for  $\pi_1(R_1^{\text{PdB}}, R_2^{\text{PdB}})$  and  $\pi_2(R_1^{\text{PdB}}, R_2^{\text{PdB}})$ , which have non-trivial boundary homomorphisms. Specifically, they are

$$0 \longrightarrow \mathbb{Z}_{\phi} \longrightarrow \pi_1(R_1^{\text{PdB}}, R_2^{\text{PdB}}) \xrightarrow{\partial^*} \mathbb{Z}_{2(\phi+\mathbf{S})} \longrightarrow 0 \quad (34)$$

and

$$0 \longrightarrow 0 \longrightarrow \pi_2(R_1^{\text{PdB}}, R_2^{\text{PdB}}) \xrightarrow{\partial^*} 2\mathbb{Z}_{\mathbf{J}} \longrightarrow 0 , \qquad (35)$$

where  $R_1^{\text{PdB}}$ ,  $R_2^{\text{PdB}}$  are given by equations (25) and (26). The homotopy group  $\mathbb{Z}_{2(\phi+S)} = \pi_0(R_2^{\text{PdB}})$  gives rise to the KLS domain walls [48, 49] owing to the disconnected vacuum manifold  $R_2^{\text{PdB}}$  discussed in section 2.2.4, while the homotopy group  $2\mathbb{Z}_J \subset \pi_1(R_2^{\text{PdB}})$  describes spin vortices with an even winding number [41].

Non-trivial boundary homomorphisms  $\partial^*$  in equations (34) and (35) describe how low-dimensional objects, e.g. KLS walls and spin vortices, connect to objects with higher dimensionality. Here we have considered HQVs and **d**-monopoles, inherited from the polar phase. We refer to the resulting composite objects consisting of *D*-dimensional and (D + 1)dimensional objects, as string-walls and string-monopoles, respectively, based on the rank of their relative homotopy groups [41].

#### 3.2. Vortex-bound solitons in the PdB phase and the nexus object

We now know that the PdB phase supports several composite topological objects with different dimensions, contained in length scales smaller than  $\xi_{\rm H}$ . However, typical NMR experiments probe superfluid properties at length scales larger than  $\xi_{\rm H}$ , which poses a question: what happens to composite objects at length scales approaching  $\xi_{\rm H}$ ? This question is both theoretically interesting and relevant for understanding experimental observations, as composite objects with characteristic length scales  $\xi$  or  $\xi/|q|$  cannot be observed directly with NMR methods.

Composite defects, such as walls bounded by strings studied in [5], can be identified from the NMR signature of the related spin solitons. HQVs can therefore be considered as one dimensional (1D) nexuses connecting defects with different characteristic sizes [42, 43]. In this section, we will discuss the relative homotopy groups for the reduced vacuum manifolds analyzed in section 2.2.5. We will see how HQVs and KLS walls with characteristic length scales  $\xi$  and  $\xi/|q|$ , respectively, connect to spin solitons with much larger characteristic length scale, set by the dipole length  $\xi_D$ . In this problem the length scales are well separated,  $\xi_D \gg \xi/|q| \gg \xi$  [42]. For technical details, we refer the reader to appendix C.3.

3.2.1. Walls bounded by strings. In the region  $\xi_{\rm H} < r < \xi_{\rm D}$ , the SES of  $\pi(R_1^{\rm H}, R_2^{\rm PdB})$  is

$$0 \longrightarrow \mathbb{Z}_{\phi} \longrightarrow \pi_1(R_1^{\mathrm{H}}, R_2^{\mathrm{PdB}}) \xrightarrow{\partial^*} \mathbb{Z}_{2(\phi+\mathbf{S})} \longrightarrow 0 \quad . \tag{36}$$

Equation (36) determines how linear defects with a characteristic length scale  $\xi_{\rm H} < r < \xi_{\rm D}$  connect with a possible domain wall. Equation (36) suggests

$$\pi_1(\mathbf{R}_1^{\mathrm{H}}, \mathbf{R}_2^{\mathrm{PdB}}) \cong \tilde{\mathbb{Z}},\tag{37}$$

which is isomorphic to  $\pi_1(R_1^{\text{PdB}}, R_2^{\text{PdB}})$  within  $\xi_{\text{H}}$ . In other words the KLS wall, determined by two length scales  $\xi$ and  $\xi/q$ , extends into the region  $\xi_{\text{H}} \leq r \leq \xi_{\text{D}}$ . However, equation (36) only contains the phase factor  $\phi$ . In particular, all information about the spin degrees of freedom is lost as they are trivial elements of  $\pi_1(R_1^{\text{H}}, R_2^{\text{PdB}})$ . To restore the spin part of the KLS wall, we recall that the domain wall connects regions with ( $\mathbf{d}, \mathbf{e}^1, q, \phi$ ) and ( $-\mathbf{d}, -\mathbf{e}^1, -q, \pi + \phi$ ) [50]. However, equation (37) only catches the phase degree of freedom, i.e. phase vortices. Similarly, the spin degree of freedom can be described by the group

$$M \equiv \{n_{\mathbf{S}}/2 | n_{\mathbf{S}} \in \mathbb{Z}\},\tag{38}$$

such that  $M/\pi_1(S_{\mathbf{S}}^1) \cong \mathbb{Z}_2 = \{[0], [1/2]\}$ , where  $\pi_1(S_{\mathbf{S}}^1) \subset \pi_1(R_1^{\mathrm{H}})$  represents free spin vortices. The cosets [0] and [1/2] correspond to the absence or presence of the KLS string wall in the region  $\xi_{\mathrm{H}} < r \leq \xi_{\mathrm{D}}$ , respectively. Coset  $[0] \cong 2\mathbb{Z}$  contains all integer spin vortices, while the Coset  $[1/2] \cong \{n + 1/2 | n \in \mathbb{Z}\}$  contains all half-integer spin vortices, including the HQV. These properties are equivalent to SES

$$0 \longrightarrow \mathbb{Z}_{\mathbf{S}} \longrightarrow M \xrightarrow{\partial^*} \mathbb{Z}_{2(\phi + \mathbf{S})} \longrightarrow 0 .$$
 (39)

3.2.2. Spin solitons. For  $r > \xi_D$ ,  $R_1^H$  reduces to  $\tilde{R}_1^{SOC} = R_S^{SOC} \times U(1)_{\phi}$ . The resulting linear objects are classified by the homotopy group  $\pi_1(R_1^H, \tilde{R}_1^{SOC})$  which has the SES

$$0 \longrightarrow \mathbb{Z}_{\mathbf{S}} \longrightarrow \pi_1(R_1^{\mathrm{H}}, \tilde{R}_1^{\mathrm{SOC}}) \xrightarrow{\partial^*} \mathbb{Z}_4 \longrightarrow 0 . \quad (40)$$

It follows that

$$\pi_1(\boldsymbol{R}_1^{\mathrm{H}}, \tilde{\boldsymbol{R}}_1^{\mathrm{SOC}}) = \{\boldsymbol{n}_{\mathbf{S}}/4 | \boldsymbol{n}_{\mathbf{S}} \in \mathbb{Z}\} \cong \mathbb{Z},$$
(41)

and

$$\pi_1(R_1^{\mathrm{H}}, \tilde{R}_1^{\mathrm{SOC}}) / \mathbb{Z}_{\mathbf{S}} \cong \mathbb{Z}_4.$$
(42)

Because equation (40) is determined by  $\mathbb{Z}_{\mathbf{S}} = \pi_1(S_{\mathbf{S}}^1)$  and  $\mathbb{Z}_4 = \pi_0(R_{\mathbf{S}}^{\text{SOC}})$  (see details in appendix C.3), we have

$$\pi_1(R_1^{\rm H}, \tilde{R}_1^{\rm SOC}) = \pi_1(S_{\rm S}^{\rm I}, R_{\rm S}^{\rm SOC}).$$
(43)

Equation (43) signifies that the linear objects classified by  $\pi_1(R_1^{\rm H}, \tilde{R}_1^{\rm SOC})$  only involve the spin degree of freedom, i.e. they are spin solitons and spin vortices [41, 45]. The four cosets of  $\pi_1(S_5^{\rm L}, R_5^{\rm SOC})$  are

$$[0] = \{n_{\mathbf{S}}\}, \left[\frac{1}{4}\right] = \left\{n_{\mathbf{S}} + \frac{1}{4}\right\},$$

$$\left[\frac{2}{4}\right] = \left\{n_{\mathbf{S}} + \frac{2}{4}\right\}, \text{and} \left[\frac{3}{4}\right] = \left\{n_{\mathbf{S}} + \frac{3}{4}\right\}.$$

$$(44)$$

These cosets give the topological invariants of the four types of linear objects corresponding to spin vortices and three types of spin solitons. Figure 5 shows the representatives of spin solitons of  $\pi_1(S_S^1, R_S^{SOC})$ . Following the terminology in [5], they are the big-soliton ( $|\Delta\theta| = \pi + 2\theta_0$ ), the soliton ( $|\Delta\theta_0| = \pi - 2\theta_0$ ), the KLS-soliton ( $|\Delta\theta_0| = 2\theta_0$ ), and the  $\pi$ -soliton ( $|\Delta\theta| = \pi$ ). Spin vortices, i.e. the coset [0], are not discussed further as they are outside the scope of this manuscript.

A significant property of  $\pi_1(S_{\mathbf{S}}^{\mathbf{I}}, R_{\mathbf{S}}^{\text{SOC}})$  is that it has a subgroup  $G_{\text{sub}} \equiv \{[0], [2/4]\}$  such that  $G_{\text{sub}}/\mathbb{Z}_{\mathbf{S}} \cong \mathbb{Z}_2$ . It follows that the SES of  $G_{\text{sub}}$  is given by equation (40) as

$$0 \longrightarrow \mathbb{Z}_{\mathbf{S}} \longrightarrow G_{\mathrm{sub}} \xrightarrow{\partial^*} \mathbb{Z}_2 \longrightarrow 0.$$
 (45)

The mapping diagram of equation (45) is shown as the dashed panel in figure 6. Comparing equations (39) and (45) leads to

$$G_{\rm sub} = \pi_1(S^1_{\mathbf{S}}, \tilde{R}_2) \cong \hat{\mathbb{Z}} = M, \tag{46}$$

where  $\hat{\mathbb{Z}} \equiv \{n_{\mathbf{S}}/2 | n_{\mathbf{S}} \in \mathbb{Z}\}.$ 

Equation (46) suggests that one can continuously transform spin solitons, classified by the [2/4] coset of  $\pi_1(S_S^1, \tilde{R}_2)$ , to half spin vortices of M. In other words, the KLS wall smoothly connects to the [2/4] spin soliton via a HQV. Similarly to a 2D nexus connecting the string monopole and the vortex skyrmion [41, 43], the HQV is a 1D nexus connecting the KLS wall and the [2/4] spin soliton [42]. The composite object formed by the [2/4] spin soliton and the KLS wall is called the 1D nexus object.



**Figure 5.** Soliton structures in the PdB phase. The black fine dotted lines represent the four elements of  $R_{\rm S}^{\rm SOC}$  i.e.  $\pm \theta_0$  and  $\pi \pm \theta_0$ . The colored dashed, dotted, dash-dotted, and solid lines correspond to  $\pi$ -soliton ( $|\Delta \theta| = \pi$ ), soliton ( $|\Delta \theta| = \pi - 2\theta_0$ ), KLS-soliton ( $|\Delta \theta| = 2\theta_0$ ) and big-soliton ( $|\Delta \theta| = \pi + 2\theta_0$ ), respectively. Panel (a) sketches the spin solitons with topological invariants 1/4, 2/4 and 3/4 originating at  $\theta_0$  (orange) and  $\pi - \theta_0$  (blue), respectively, and panel (b) sketches similar spin solitons originating at  $-\theta_0$  (pink) and  $\pi + \theta_0$  (green).



**Figure 6.** LES, SES, and spin solitons in the PdB phase. The mapping diagram drawn here demonstrates that the linear objects given by  $\pi_1(R_1^{\text{H}}, \tilde{R}_1^{\text{SOC}})$  are spin solitons. This is because the mapping between  $\pi_1(\tilde{R}_1^{\text{SOC}})$  and  $\pi_1(R_1^{\text{H}})$  is a projection, the image of homomorphism  $i^* : \pi_1(\tilde{R}_1^{\text{SOC}}) \to \pi_1(R_1^{\text{H}}) = \mathbb{Z}_{\phi}$ , i.e. the topological invariant of all phase vortices. As a result, all the trivial linear objects given by  $\pi_1(R_1^{\text{H}}, \tilde{R}_1^{\text{SOC}}) \to \pi_1(R_1^{\text{H}}) = \mathbb{Z}_{\phi}$ , i.e. the topological invariant of all phase vortices. As a result, all the trivial linear objects given by  $\pi_1(R_1^{\text{H}}, \tilde{R}_1^{\text{SOC}})$  are phase vortices as  $\text{Im}[i^*] \cong \text{ker}[j^*]$ . Since  $\text{ker}[k^*] \cong \text{Im}[\partial^*] = \mathbb{Z}_4$ , there are three types of spin solitons (and one type of spin vortex). In addition, the subgroup  $G = \{[n], [n+2/4]\}$  is an extension of  $\mathbb{Z}_{\text{S}}$  by  $\pi_0(\tilde{R}_2^{\text{SOC}}) = \mathbb{Z}_2$  and therefore isomorphic to M. The corresponding SES is marked by the dashed line. The HQV acts as a 1D nexus between the spin soliton given by the coset [2/4] and the KLS wall [42].

3.2.3. The 1D nexus object. We note that since  $\pi_1(S_{\mathbf{S}}^1, R_{\mathbf{S}}^{\text{SOC}})/\mathbb{Z}_{\mathbf{S}} \cong \mathbb{Z}_4$ , we have [2/4] = [1/4] + [1/4]. It follows that the relative homotopy group  $\pi_1(S_{\mathbf{S}}^1, \tilde{R}_2)$  can be represented both as  $\{[0], [1/4] + [1/4]\}$  and as  $\{[0], [2/4]\}$ . In other words, there are two types of spin soliton configurations connecting to a KLS wall via a HQV for a given element of  $\pi_1(S_{\mathbf{S}}^1, \tilde{R}_2)$ . When the topological invariant is 2/4, the spin soliton is spatially inseparable and forms a  $\pi$ -soliton sketched in figure 7(a).

On the other hand, when the topological invariant is 1/4 + 1/4, the spin soliton is a combination of two spatially separable spin solitons, each described by the topological invariant 1/4. These two spatially separated spin solitons are the KLS soliton ( $|\Delta\theta| = 2\theta_0$ ) and the soliton ( $|\Delta\theta| = \pi - 2\theta_0$ ).

As illustrated in figure 7(b), the 1D nexus object contains two spin solitons. As we will see in section 4, the free energies and spin textures of the two cases are quite different, resulting in distinguishable dynamic spin response properties and NMR frequency shifts. The corresponding minimum energy configurations based on 2D calculations [42] for both configurations are shown in figure 8.

#### 3.3. Vortex-bound solitons in the polar phase

In the polar phase, the solitons are bound by HQVs. The existence of HQVs was predicted decades ago in <sup>3</sup>He-A [51] and observed recently in the polar [4] and polar-distorted [5] phases of <sup>3</sup>He. Previously, HQVs have been observed in



**Figure 7.** Nexus object configurations. The 1D nexus object consists of spin soliton(s), a HQV and a KLS wall. The green and cyan arrows represent the  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{e}}^1$  vectors respectively. (a) Configuration of the inseparable spin soliton. In this configuration, the topological invariant of spin soliton is 1/2, which corresponds to  $\pi$ -soliton. (b) In this configuration there are two spin solitons with topological invariant 1/4 when the group  $\pi_1(S_{\mathbf{S}}^1, \tilde{R}_2)$  is implemented in alternative way. Following the requirement of continuity of order parameter, these two spin solitons are the KLS-soliton ( $\Delta \theta = 2\theta_0$ ) and the soliton ( $\Delta \theta = \pi - 2\theta_0$ ).



**Figure 8.** Minimum energy configurations for different spin solitons. Numerically calculated equilibrium configurations of inseparable and separable spin solitons in one-half unit cell with |q| = 0.18 and  $D = 18\xi_D$ . (a) Inseparable configuration with the  $\pi$ -soliton, corresponding to the sketch in figure 7(a). (b) Separable configuration with combination of the KLS-soliton and the soliton, corresponding to the sketch in figure 7(b).

grain boundaries of *d*-wave cuprate superconductors [52], in superconductor rings [53], and in Bose condensates [54, 55]. In *p*-wave superfluids such as <sup>3</sup>He, HQVs provide access to vortex-core-bound fermion states, which have been predicted to harbor non-Abelian anyons in 2D  $p_x + ip_y$  superconductors and superfluids [6, 7].

In the polar phase the presence of magnetic field larger than the dipole field, H > 3 mT, fixes the spin anisotropy vector  $\hat{\mathbf{d}} = \hat{\mathbf{i}}\cos\theta(\mathbf{r}) + \hat{\mathbf{j}}\sin\theta(\mathbf{r})$  in equation (13) via the spinorbit interaction  $F_{so} \propto (\hat{\mathbf{d}} \cdot \hat{\mathbf{m}})^2$  to the plane perpendicular to **H**. Vectors  $\hat{\mathbf{i}}$  and  $\hat{\mathbf{j}}$  are mutually orthogonal unit vectors in the plane normal to **H**. The orbital anisotropy vector  $\hat{\mathbf{m}}$  is pinned parallel to nafen strands,  $\hat{\mathbf{m}} \parallel \hat{\mathbf{z}}$ . The combined effect of the confinement and magnetic field affects the distribution of the  $\hat{\mathbf{d}}$  vector such that  $\theta$  is governed by the Sine-Gordon equation

$$\nabla^2 \theta = \frac{1}{2\xi_{\rm D}^2} \sin^2 \mu \sin 2\theta. \tag{47}$$

Here  $\xi_D \sim 10 \,\mu\text{m}$  is the dipole length and  $\mu$  is the angle of the magnetic field with respect to  $\hat{z}$ .

The polar phase order parameter supports three different vortex types illustrated in figure 9—the singly quantized phase vortices, the spin-current vortices (SCVs), and the HQVs. Reorientation of  $\hat{\mathbf{d}}$  outside of SCV and HQV cores is governed by solitonic solutions of equation (47). In tilted  $\mathbf{H} \not\parallel \hat{\mathbf{z}}$  field the SCV terminates two  $\pi$ -solitons, while the HQV terminates just one. Additionally, solitons may be terminated at the sample boundary. The soliton width is fixed by the characteristic length scale  $\sim \xi_D / \sin \mu$ . For  $\mathbf{H} \parallel \hat{\mathbf{z}}$  or at zero magnetic field, all states with  $\hat{\mathbf{d}} \perp \hat{\mathbf{m}}$  are degenerate, and solitons are not created.



**Figure 9.** Vortex types in the polar phase. The phase of the order parameter  $\phi$  is shown as the background color. The spin anisotropy vector **d** is locked to the plane (green disks) perpendicular to the magnetic field *H* by magnetic energy. In this plane, the vector  $\hat{\mathbf{d}}$  rotates by  $\pi$  around the HQV core and by  $2\pi$  around the spin vortex core. In a tilted magnetic field ( $\mathbf{H} \not\parallel \hat{\mathbf{m}}$ ) the reorientation of the  $\hat{\mathbf{d}}$  vector is concentrated in one or two solitons (illustrated with dashed lines) with  $\pi$  winding, terminating at the HQV or at the spin vortex core, respectively. The nafen strands, oriented along  $\hat{\mathbf{m}}$ , and the vortex lines are orthogonal to the figure. The phase vortex and the HQV have hard cores (red discs) of the size of coherence length  $\xi \sim 40$  nm, while the spin vortex has a soft core (blue disc) of much larger dipolar size  $\xi_D \sim 10 \ \mu$ m.



**Figure 10.** Structure of the SMV in <sup>3</sup>He-B. (*Left*) Mass vortex traps winding of phase  $\phi$  around a hard core. (*Center*) Disclination in the orbital rotation field takes form of a soliton tail terminated by a spin vortex with a hard core. (*Right*) When cores of mass and spin vortices merge, a stable configuration of SMV in rotating container is possible.

#### 3.4. Composite spin-mass vortex with soliton tail in <sup>3</sup>He-B

The existence of the spin-mass vortex (SMV)-a composite defect in superfluid <sup>3</sup>He-B [56, 57]-may be seen from the B-phase order parameter, equation (2). The B phase order parameter supports different types topological defects sketched in figure 10. In rotation the most usual defect is a mass-current vortex, a conventional quantized vortex line. Around it the phase  $\phi$  winds by  $2\pi$ , which results in a supercurrent proportional to  $\nabla \phi$  around its singular core. Within the core the amplitude of the order parameter  $|A_{\alpha i}|$  is depleted from its equilibrium value  $\Delta_{\rm B}$ . A second defect is a disclination in the  $R_{\alpha j}$  field. It also has a singular core which is encircled by a spin current. On moving once around the core  $\hat{\mathbf{n}}$  reverses its direction twice: First by smooth rotation while the angle  $\theta$  remains at the equilibrium value  $\theta_{\rm L} \approx 104^{\circ}$ , which minimizes the spin-orbit interaction energy. Later by increasing  $\theta$  to 180°, where both directions of  $\hat{\mathbf{n}}$  are equivalent, and then decreasing back to  $\theta_{\rm L}$ . The second leg in the direction reversal does not minimize the spin-orbit interaction and hence it becomes confined in space within a planar structure, a soliton sheet, which terminates on the linear singular core or on the wall of the container. This structure becomes possible through the existence of two different energy (and length) scales: The superfluid condensation energy defines the scale of the coherence length  $\xi \sim 10-100$  nm, which is roughly the radius of the singular core. The much weaker spin-orbit interaction defines the scale of the dipolar healing length  $\xi_D \sim 10$   $\mu$ m, which is approximately the width of the soliton sheet. The detailed topological analysis of the B-phase soliton structures can be found in [58], where experimentally realized configurations correspond to (-+1) or (+-1) classes in terms of that work.

The spin vortex by itself is an unstable structure: The surface tension of the soliton leads to its annihilation and disappearance. Another composite defect—the combined SMV can be stabilized in the rotating container. It has both phase  $\phi$  winding and a  $R_{\alpha j}$  disclination trapped on the same core. If the number of mass-current vortex lines in the container is less than that in the equilibrium state in rotation, then the existing lines are confined to a cluster in the center of the container by the Magnus force from the uncompensated normalsuperfluid counterflow. Due to the trapped mass current the SMV also experiences the Magnus force which pulls it toward the vortex cluster. This force is opposed by the surface tension of the soliton and as a result its equilibrium position is slightly outside the cluster of mass vortices. The soliton tail of the SMV has a characteristic NMR absorption response which allows its identification.

Another possible stable configuration is the soliton sheet bounded by two SMVs and embedded within the vortex cluster. The size of such pair is set by competition of the surface tension of the soliton sheet, which tries to pull vortices together and repulsion of two mass vortices of the same circulation. The equilibrium size is estimated to be about  $6\xi_D$  and it is too small to identify contribution of a single pair in the NMR spectrum.

#### 4. Spin dynamics and NMR

NMR techniques are used as a non-invasive way to probe the properties of different superfluid phases of <sup>3</sup>He. NMR methods are particularly useful for extracting information about the SOC. This is because the long range coherence in superfluid states enhances the SOC energy [59, 60]. In the superfluid state, the observable NMR resonance frequency is often shifted from the Larmor value since the precessing spin experiences an additional torque from SOC. The topological solitons form potential wells for bound spin-wave states, which result in satellite peaks with characteristic frequency shifts. Since the characteristic time scale of SOC is much longer than the microscopic time scales of superfluid, i.e.  $\hbar\Delta^{-1}$ , the microscopic processes remain in equilibrium under weak magnetic perturbation. In other words, the NMR response is the hydrodynamic response of the spin densities  $\delta S_a$  and spin vectors [1, **6**1].

Under weak magnetic perturbation  $\delta H_a \equiv \delta \mathbf{H}$  along y-axis, the linear response of spin density is

$$\delta S_{+}(\mathbf{r},t) = \int \mathrm{d}\sigma' \int \mathrm{d}t' \frac{\delta S_{+}}{\delta H_{a}}(\mathbf{r},t,\mathbf{r}',t') \delta H_{a}(\mathbf{r}',t') + O(\delta H_{a}^{2}),$$
(48)

where  $\delta S_+ = [\delta S_1 + i\delta S_3]/\sqrt{2}$  is transverse spin density and a = 1, 2, 3 are spatial coordinate indices. To calculate the response function and its poles, we use method based on hydrodynamic equations [61]. In the hydrodynamic limit, the system of dynamic equations of spin densities  $S_{\alpha}$  and spin vectors are a system of Liouville equations

$$\frac{\partial S_{\alpha}}{\partial t} = \{F_{\rm hy}, S_{\alpha}\}, \ \frac{\partial V_{\alpha}^a}{\partial t} = \{F_{\rm hy}, V_{\alpha}^a\}, \tag{49}$$

where  $V_{\alpha}^{a=1} = \hat{\mathbf{d}}$ ,  $V_{\alpha}^{a=2} = \hat{\mathbf{e}}^1$  and  $V_{\alpha}^{a=3} = \hat{\mathbf{e}}^2$ , such that  $\hat{\mathbf{e}}^2 = \hat{\mathbf{d}} \times \hat{\mathbf{e}}^1$  is directed along spin polarization, denote the three spin vectors of order parameter. These equations are known as the Leggett equations [1]. The hydrodynamic free energy of a superfluid dominated by the SOC energy is

$$F_{\rm hy} = \int_{\Sigma} (f_{\rm H} + f_{\rm soc} + f_{\rm grad}) \mathrm{d}\Sigma.$$
 (50)

Equation (49) can be rewritten as

$$\frac{\partial S_{\alpha}}{\partial t} = \int_{\Sigma} d^{3}r' \frac{\delta F_{\text{hy}}}{\delta S_{\beta}}(r') \{S_{\beta}(r'), S_{\alpha}(r)\} + \int_{\Sigma} d^{3}r' \frac{\delta F_{\text{hy}}}{\delta V_{\beta}^{a}}(r') \{V_{\beta}^{a}(r'), S_{\alpha}(r)\}$$
(51)

and

$$\frac{\partial V_{\alpha}^{a}}{\partial t} = \int_{\Sigma} \mathrm{d}^{3} r' \frac{\delta F_{\mathrm{hy}}}{\delta S_{\beta}}(r') \{ S_{\beta}(r'), V_{\alpha}^{a}(r) \}, \qquad (52)$$

where  $\alpha, \beta = 1, 2, 3$  are indexes of spatial components of hydrodynamic variables. The Poisson brackets between  $S_{\alpha}$  and  $V_{\alpha}^{a}$  are [62]

$$\{S_{\alpha}(r_1), S_{\beta}(r_2)\} = \epsilon_{\alpha\beta\gamma}S_{\gamma}\delta(r_1 - r_2) \{S_{\alpha}(r_1), V^a_{\beta}(r_2)\} = \epsilon_{\alpha\beta\gamma}V^a_{\gamma}\delta(r_1 - r_2),$$
(53)

where  $r_1$  and  $r_2$  are the spatial coordinates and  $\epsilon_{\alpha\beta\gamma}$  is the Levi-Civita symbol.

#### 4.1. Satellite shifts induced by spin solitons in the PdB phase

After plugging equation (53) into equations (51) and (52), the coupled first order dynamic equations for spin densities  $S_{\alpha}$  and  $V_{\alpha}^{a}$  for the PdB phase are given as

$$\frac{\partial S_{\alpha}}{\partial t} = \gamma H_{\beta} \epsilon_{\alpha\beta\gamma} S_{\gamma} - \frac{6}{5} g_{\rm D} V_j^d V_{\gamma}^b \epsilon_{\alpha\beta\gamma} Q_{\beta j}^{bd} + (\partial_i \partial_i V_{\beta}^b) V_{\gamma}^a \epsilon_{\alpha\beta\gamma} K_{ja}^{ba},$$
(54)

$$\frac{\partial V_{\alpha}^{a}}{\partial t} = \gamma H_{\beta} \epsilon_{\alpha\beta\gamma} V_{\gamma}^{a} - \delta \gamma^{2} \chi_{\perp}^{-1} S_{\eta} V_{\eta}^{3} V_{\beta}^{3} \epsilon_{\alpha\beta\gamma} V_{\gamma}^{a} - \gamma^{2} \chi_{\perp}^{-1} S_{\beta} \epsilon_{\alpha\beta\gamma} V_{\gamma}^{a},$$
(55)

where  $\delta = (\chi_{\perp} - \chi_{\parallel})/\chi_{\parallel}$  in which  $\chi_{\perp}$  and  $\chi_{\parallel}$  are the PdB phase transverse and longitudinal magnetic susceptibility, respectively. Here

$$\begin{aligned}
K_{ij}^{ba} &= K_1 \delta_{ij} X_m^b X_m^a + K_2 X_j^a X_i^b + K_3 X_j^b X_i^a, \\
Q_{\beta j}^{bd} &= X_{\beta}^b X_j^d + X_{\beta}^d X_j^b
\end{aligned} (56)$$

with  $X_i^1 = \Delta_{\perp 1} \hat{x}_i, X_i^2 = \Delta_{\perp 2} \hat{y}_i, X_i^3 = \Delta_{\parallel} \hat{z}_i$ . Starting from the first order equations of spin densities and degenerate parameters in equations (54) and (55), we can further derive the second order spin dynamic response equations of  $\delta S_{\alpha}$  under weak magnetic drive  $\delta H_{\alpha}$  by plugging

$$H_{\alpha} = H_{\alpha}^{(0)} + \delta H_{\alpha}(t)$$

$$S_{\alpha} = S_{\alpha}^{(0)} + \delta S_{\alpha}(\mathbf{r}, t)$$

$$V_{\alpha}^{a} = V_{\alpha}^{a(0)} + \delta V_{\alpha}^{a}(\mathbf{r}, t)$$
(57)

into equations (54) and (55). Here the  $S_{\alpha}^{(0)}$  and  $V_{\alpha}^{a(0)}$  are the equilibrium spin densities and equilibrium degenerate parameters, respectively. The  $\delta S_{\alpha}(\mathbf{r},t)$  and  $\delta V_{\alpha}^{a}(\mathbf{r},t)$  are the

dynamic parts of the perturbed spin densities and degenerate parameters, the  $H_{\alpha}^{(0)}$  is the static magnetic field and  $\delta H_{\alpha}(t) = |\delta \mathbf{H}| \hat{x} e^{-i\omega t}$  is the RF drive. The derived spin dynamic response equations are

$$i\omega\delta S_{\alpha}(\omega) = \gamma\epsilon_{\alpha\beta\gamma}H^{(0)}_{\beta}\delta S_{\gamma}(\omega) + \gamma\epsilon_{\alpha\beta\gamma}S^{(0)}_{\gamma}\delta H_{\beta}(\omega) + \frac{\Xi_{\alpha\lambda}}{i\omega}\delta S_{\lambda}(\omega) + \frac{C_{\alpha\eta}}{i\omega}\delta H_{\eta}(\omega)$$
(58)

and

$$\Xi_{\alpha\lambda} = \frac{\gamma^2}{\chi_{\perp}} K^{ba}_{ij} \Lambda^{ba}_{ij\alpha\lambda} + \frac{6g_{\rm D}\gamma^2}{5\chi_{\perp}} R^{db}_{j\lambda\alpha\beta} Q^{bd}_{\beta j} + \frac{6g_{\rm D}\gamma^2}{5\chi_{\perp}} V^{d(0)}_{\zeta} V^{b(0)}_{\gamma} \epsilon_{j\lambda\zeta} \epsilon_{\alpha\beta\gamma} Q^{bd}_{\beta j} C_{\alpha\eta} = \gamma G^{ba}_{ij\alpha\eta} K^{ba}_{ij} - \frac{6g_{\rm D}\gamma}{5} R^{db}_{j\eta\alpha\beta} Q^{bd}_{\beta j} - \frac{6g_{\rm D}\gamma}{5} V^{d(0)}_{\zeta} V^{b(0)}_{\gamma} \epsilon_{\eta\gamma\zeta} \epsilon_{\alpha\beta\gamma} Q^{bd}_{\beta j},$$
(59)

where

$$\begin{split} R_{j\eta\alpha\beta}^{db} &= V_{j}^{d(0)} V_{\beta}^{b(0)} \delta_{\eta\alpha} - V_{j}^{d(0)} V_{\alpha}^{b(0)} \delta_{\eta\beta}, \\ G_{ij\alpha\gamma}^{ba} &= (\partial_{i}\partial_{j}V_{\alpha}^{b(0)}) V_{\gamma}^{a(0)} - (\partial_{i}\partial_{j}V_{\beta}^{b(0)}) \delta_{\beta\gamma}V_{\alpha}^{a(0)}, \\ \Lambda_{ij\alpha\lambda}^{ba} &= (\partial_{i}\partial_{j}V_{\beta}^{b(0)}) \delta_{\beta\lambda}V_{\alpha}^{a(0)} + (V_{\gamma}^{b(0)}V_{\gamma}^{a(0)} \delta_{\alpha\lambda} \\ &- \delta_{\gamma\lambda}V_{\alpha}^{b(0)}V_{\gamma}^{a(0)}) \partial_{i}\partial_{j} + [(\partial_{i}V_{\gamma}^{b(0)})V_{\gamma}^{a(0)} \delta_{\alpha\lambda} \\ &- (\partial_{i}V_{\alpha}^{b(0)}) V_{\gamma}^{a(0)} \delta_{\gamma\lambda}]\partial_{j} + [(\partial_{j}V_{\gamma}^{b(0)})V_{\gamma}^{a(0)} \delta_{\alpha\lambda} \\ &- (\partial_{j}V_{\alpha}^{b(0)})V_{\gamma}^{a(0)} \delta_{\gamma\lambda}]\partial_{i} - \delta_{\gamma\lambda}(\partial_{i}\partial_{j}V_{\alpha}^{b(0)})V_{\gamma}^{a(0)}. \end{split}$$
(60)

The first two terms of equation (58) correspond to the Larmor precession with frequency  $\omega_{\rm L} = \gamma H^{(0)}$ , while the last two terms, known as torque terms, describe the NMR response related to the superfluid order parameter texture.

The torque terms in equations (59) and (60) are fully determined by the equilibrium order parameter texture, i.e. the NMR frequency shift arises from the equilibrium texture of spin solitons. Taking the static magnetic field  $\mathbf{H}^{(0)} = |\mathbf{H}^{(0)}|\hat{\mathbf{y}}$  and the parametrization equation (29) into account, the poles of the dynamic response equations for the transverse spin density  $\delta S_+ = \frac{1}{\sqrt{2}} [\delta S_1(\omega) + i\delta S_3(\omega)]$  under weak magnetic drive  $\delta \mathbf{H}(t)$  become

$$(\omega^2 - \omega_{\rm L}^2)\delta S_+(\omega) = (\Xi_{11} + \Xi_{33}) + i(\Xi_{13} - \Xi_{31})\delta S_+(\omega).$$
(61)

In experiments, the observed transverse frequency shift  $\lambda$  is found from an eigen-equation using the parametrization in equation (29). For the full calculation, we refer the reader to [42, 63]. The end result of the calculations is

$$\lambda \delta S_{+}(\omega) = \xi_{\rm D}^{2} [(6q_{2}^{2} + q_{1}^{2} + 1)\partial_{y}\partial_{y} + (3q_{1}^{2} + 2q_{2}^{2} + 1)\partial_{x}\partial_{x}]\delta S_{+}(\omega) - (2\xi_{\rm D}^{2}iV - U)\delta S_{+}(\omega)$$
(62)

with

$$V = (1 + 3q_1^2 \cos 2\theta) \partial_x \theta \partial_x + (1 + q_1^2) \partial_y \theta \partial_y + \frac{1}{2\xi_D^2} [(1 + q_1)^2 \sin 2\theta - (1 + q_1)q_2 \cos \theta], U = (1 + q_1) [-(1 + q_1) \cos 2\theta - 5q_2 \sin \theta] + 1 + q_1^2 + 4q_2^2,$$
(63)

and

$$\lambda = \frac{(\omega^2 - \omega_{\rm L}^2)}{\Omega_{\rm PdB}^2}, \quad \Omega_{\rm PdB}^2 = \frac{6\gamma^2 \Delta_{\rm PdB}^2 g_{\rm D}}{5\chi_{\perp}}, \tag{64}$$

where  $\Omega_{PdB}$  is the Leggett frequency in the PdB phase. The Eigen-equation (62) was solved through Galerkin Eigen-value method with a finite element mesh [42, 63–65]. The Eigenvalues  $\lambda$  resulting from stationary configuration of spin solitons are plotted in figures 11 and 12. The relative intensity of the soliton satellite, determined as the ratio of the area of the satellite  $I_{sat}$  to the total area  $I_{tot}$  under the absorption spectrum can be calculated using found eigenfunctions  $\delta S_+$  as

$$\frac{I_{\text{sat}}}{I_{\text{tot}}} = n \frac{\left|\int \delta S_+ \, \mathrm{d}V\right|^2}{\int |\delta S_+|^2 \, \mathrm{d}V},\tag{65}$$

where  $n = D^{-2}/2$  is the density of the solitons. The numerical results for  $\lambda$  agree well with experimental observations, discussed in more detail in section 5.4, while  $I_{\text{sat}}/I_{\text{tot}}$  scales proportionally to  $\sqrt{\Omega}$ , also similar to the experiment.

#### 5. Experimental observations

In many experiments, vortex-bound solitons result in observable signatures in the NMR spectrum, playing an integral role in distinguishing various topological and composite defects in superfluid phases of <sup>3</sup>He. This section aims to provide a review on the experimental observations of such defects to date, underlining the role of solitons that led to these observations.

#### 5.1. Identification of the SMV in <sup>3</sup>He-B

Spin-mass vortices are formed in <sup>3</sup>He-B as a rare event in two different processes: When the front of the transition between A and B phases of superfluid <sup>3</sup>He sweeps through the rotating sample [56] or by the Kibble-Zurek mechanism (KZM) [57]. Identification of the SMV is based on the NMR spectrum.

Owing to spin-orbit interaction the NMR absorption in <sup>3</sup>He-B is shifted from the Larmor frequency by an amount  $\Delta \nu$  which depends on the local orientation of the anisotropy axis  $\hat{\mathbf{n}}$  with respect to the applied magnetic field *H*. In the experiment in figure 13, the field is oriented along the rotation axis  $\Omega$ , which is also the symmetry axis of the sample cylinder. At the soliton sheet  $\hat{\mathbf{n}} \perp \mathbf{H}$  (figure 10), which produces the maximum possible frequency shift  $\Delta \nu_{max}$ . Everywhere else in the rotating cylinder the angle between  $\hat{\mathbf{n}}$  and  $\mathbf{H}$  (and thus the frequency shift) is smaller: The cluster of vortex lines in the center of the container gives rise to absorption with frequency shifts close to zero. The annular region with



**Figure 11.** Intensity and frequency shift for inseparable  $\pi$ -soliton. The relative frequency shift  $\lambda$  corresponds to the lowest eigenvalue of equation (62) with the equilibrium texture of  $\pi$ -solitons in the London limit. Filled circles represent the numerical results, while the lines are guides for the eye. (a) For experimentally relevant values of  $|q| \leq 0.2$  the relative frequency shift  $\lambda \approx -1$  for all tested inter-vortex separations *D*, in agreement with the experimental data from [5] (red diamonds). (b) The relative satellite intensity remains fairly constant with *q*, but decreases with increasing inter-vortex distance. (c) The relative frequency shift increases when the inter-vortex distance is small and saturates to a value  $\lambda \approx 1$  for larger separation, regardless of *q*. (d) The relative satellite intensity scales with intervortex separation approximately as  $I_{\text{sat}}/I_{\text{tot}} \propto \sqrt{\Omega}$ , in agreement with the experimental observations.

vortex-free counterflow between normal and superfluid component around the cluster is responsible for the large absorption maximum at  $0.8 \Delta \nu_{max}$ . The height of this peak decreases with the decrease of the counterflow velocity, that is, when new vortex lines are added to the cluster or if rotation velocity decreases with given number of vortices. The spectrum drawn with the solid line was measured after the neutron irradiation and shows the absorption from the soliton sheet, centered around  $\Delta \nu_{\rm max}$ . A top view of the rotating cylinder with the vortex cluster is depicted on the top row of the figure. This illustrates how the SMV can be selectively removed by reducing  $\Omega$  to where the cluster has almost expanded to the wall (at 0.2 rad  $s^{-1}$ ) and the SMV as the outermost vortex has been pushed to the cylinder wall. After increasing  $\Omega$  back to the original 1.6 rad  $s^{-1}$  the spectrum plotted with the dashed line was recorded. The difference from the original spectrum is the absence of the soliton signal (shown by the gray area).

#### 5.2. HQVs in the polar phase

In the polar phase, the minimum energy configuration of the order parameter forms the main peak in the NMR spectrum at the frequency [66]

$$\Delta\omega_{\rm P} = \omega_{\rm P} - \omega_{\rm L} \approx \frac{\Omega_{\rm P}^2}{2\omega_{\rm L}} \cos^2 \mu.$$
 (66)

Here  $\Omega_P$  is the Leggett frequency in the polar phase, which characterizes the spin-orbit torque.

Winding of the  $\hat{\mathbf{d}}$  vector, e.g. in the form of a soliton, provides an additional potential energy term for spin waves as the spin-orbit energy is not at minimum. Excitation of standing spin waves within these potential wells leads to a satellite peak in the NMR spectrum at frequency

$$\Delta\omega_{\rm Psat} = \omega_{\rm Psat} - \omega_{\rm L} \approx \frac{\Omega_{\rm P}^2}{2\omega_{\rm L}} \left(\cos^2\mu + \lambda_{\rm P}\sin^2\mu\right), \qquad (67)$$



**Figure 12.** Intensity and frequency shift for separable solitons. The relative frequency shift  $\lambda$  corresponds to the lowest eigenvalue of equation (62) with the equilibrium texture of 1/4 + 1/4-solitons in the London limit. Filled circles represent the numerical results, while the lines are guides for the eye. (a) The relative frequency shift  $\lambda$  increases with increasing |q|, since only solitons ( $|\Delta\theta| = \pi - 2\theta_0$ ) contribute to the lowest transverse spin dynamic response mode. (b) The relative satellite intensity remains fairly constant with q, but decreases with increasing inter-vortex distance. (c) The relative frequency shift increases when the inter-vortex distance is small and saturates to a constant value determined by |q| for larger separation. (d) The relative satellite intensity scales with intervortex separation approximately as  $I_{\text{sat}}/I_{\text{tot}} \propto \sqrt{\Omega}$ .

where the parameter  $\lambda_{\rm P}(\mu)$  is specific to the type of the topological object. For example, an infinite planar  $\hat{\mathbf{d}}$  soliton gives  $\lambda_{\rm P} = -1$  for the zero mode on the soliton [1, 66, 67]. The frequency shift for the satellite in this case becomes equal, but opposite than for the main peak, i.e.  $\Delta \omega_{\text{Psat}}(\mu = \pi/2) =$  $-\Delta\omega_{\rm P}(\mu=0)$ . In reality, the finite soliton length (similar to figure 12(c)) and disorder in the nafen confinement may lead to reduced shift, and  $|\lambda_{\rm P}| < 1$  is expected [68]. In the experiments [4], a controlled amount of polarized HQVs was created by slowly cooling the sample in rotation with constant angular velocity  $\Omega$  from above  $T_c$  in zero or axial (along  $\hat{\Omega} \parallel \hat{\mathbf{m}}$ ) magnetic field. The presence of HQVs is apparent from the NMR spectrum via the related spin soliton peak, figure 14(a). The experimental results, summarized in figures 14 and 15 indeed yield  $\lambda_{\rm P}(\mu = \pi/2) = -0.93 \pm 0.07$ , in good agreement with theoretical expectations. We note that the predicted spin polarization of the HQV core [69] does not affect the signal, as the winding of the  $\hat{\mathbf{d}}$  vector (and thus the spin polarization) is always opposite for a pair of HQVs connected by the spin soliton.

In the polar phase HQVs are energetically preferable to SQVs in axial or zero magnetic field [70, 71]. Application of tilted magnetic field changes the situation via the spin-orbit interaction related to winding of the  $\hat{\mathbf{d}}$  vector within the HQV-bound spin solitons, making SQVs preferable to HQVs. In addition, SCVs can be created during the cooldown if strong time-dependent magnetic field is applied to generate a random distribution of vector  $\hat{\mathbf{d}}$  [32, 34].

HQVs can be distinguished from SCVs by their connection to rotation. In particular, the dependence of the relative satellite peak intensity  $I_{sat}$  on the angular velocity  $\Omega$ , shown in figure 16(a), is expected to follow  $\propto \sqrt{\Omega}$ . This dependence originates from the following considerations; for solitons with their width set by the dipole length  $\xi_D$ , the expected signal intensity is  $I_{sat} = (n_v/2) g_s L \xi_D$  [72]. Here  $L = b n_v^{-1/2}$  is the average soliton length set by the average inter-vortex distance,



**Figure 13.** Observation of the SMV in <sup>3</sup>He-B. The SMV can be identified by comparing two NMR spectra [57]. The first spectrum contains the signal from a SMV created by neutron irradiation, and the second spectrum was measured after selectively removing the SMV as illustrated at the top of the figure. The contribution of the SMV (gray area) is centered around  $\Delta \nu_{\text{max}}$ , distinguishable from the larger feature centered at  $0.8\Delta\nu_{\text{max}}$ , resulting from the mismatch of the normal fluid and superfluid velocities, i.e. from counterflow.



**Figure 14.** NMR spectra in the polar phase in the presence of HQVs. (a) Normalized spectra measured in transverse field  $\mu = \pi/2$  (blue thick line) shows the HQV satellite at the negative frequency shift  $\Delta \omega_{Psat}$  and the main line at zero frequency shift [4]. In the axial field  $\mu = 0$  (red thin line) only the main line at positive shift  $\Delta \omega_P$  is seen. This spectrum is not sensitive to presence of vortices as the spin-orbit interaction results in no trapping potential for spin waves. (b) Temperature dependencies of the satellite position in the transverse field  $|\Delta \omega_{Psat}(\mu = \pi/2)|$  (blue circles) and the main line position in the axial field  $\Delta \omega_P(\mu = 0)$  (red squares) closely match as expected for HQVs. The error bars show full width at half maximum of the main line as an estimate of possible systematic error.

while  $g_s \sim 1$  and  $b \sim 1$  are numerical factors. For a very low vortex density and long solitons  $L \to \infty$  one has  $g_s \to 2$ . Since the vortex density  $n_v \propto \Omega$ , one expects  $I_{\text{sat}} \propto \Omega^{1/2}$ , as indeed confirmed in figure 16(a).

We note that the observation of solitons bounded by HQVs relies on a crucial experimental detail. Namely, the tension related to the energy cost of the solitons is overcome by the pinning of HQVs by the nafen strands. In fact, the pinning force is stronger than any relevant energy scale in the system. Each HQV core with a characteristic size given by the coherence length  $\xi \sim 40$  nm, is penetrated by a few nafen strands of ~10 nm diameter, leading to reduced energy cost of the HQV core thus leading to effective pinning of HQVs in place as they are created. For example, after stopping the rotation the satellite in the NMR spectrum remained unchanged for weeks, while the Magnus force, pulling vortices toward the sample boundary, exceeds the soliton tension by a large factor  $10^3$ . It is worth pointing out that the experiments shown



**Figure 15.** Frequency shift of the HQV satellite. (a) Measured values of the dimensionless frequency shift  $\lambda_P$  as a function of the field tilt angle  $\mu$  (symbols) are compared with numerical calculations for the uniform polar phase (solid line) using theoretical value of  $\xi_D = 17 \ \mu m$  [4]. Leggett frequency  $\Omega_P$  is determined from a separate measurement at  $\mu = 0$ . Deviation from the infinitely-long  $\hat{\mathbf{d}}$  soliton value  $\lambda_P = -1$  increases towards small  $\mu$ . The disagreement between the experiment and calculations likely originates from disorder in the nafen strand orientation [68], which leads to fluctuations of the spin-orbit interaction energy within the solitons. (b) Values of  $\lambda_P$  are found from positions of the HQV satellite  $\Delta \omega_{Psat}$  (blue circles) and of the main line  $\Delta \omega_P$  (red squares). The red and blue solid lines show results of equations (66) and (67), respectively, for  $\lambda_P = -1$ . The bars show full width at half maximum of the spectral lines in both panels.



**Figure 16.** Intensity of the HQV satellite. (a) The satellite intensity  $I_{sat}$  measured in slow ( $\tau_Q \approx 1.5 \cdot 10^5$  s, magenta circles) and fast ( $\tau_Q \approx 5.5 \cdot 10^2$  s, blue triangles) zero-field cooldowns as a function of  $\Omega$ . The solid lines are theoretical fits assuming that HQV creation by the KZM is independent from rotation. The dash line shows fitted equilibrium  $I_{sat}$ , corresponding to vanishing HQV density from KZM. Applying rf drive at the resonance during the cooldown in the axial field creates SCVs (red squares), seen as extra rotation-independent satellite intensity. (b) The satellite intensity  $I_{sat}$  measured in the absence of rotation and bias fields (green diamonds) follows the KZM power law  $I_{sat} \propto \tau_0^{-1/4}$  (solid line). The fitted satellite intensities at  $\Omega = 0$  from panel a are marked with the corresponding symbols and colors.

in figure 16 were performed with a stationary cryostat, after rotating it with constant angular velocity during the superfluid transition. Due to the strong pinning of HQVs by the nafen strands [73], the relation  $I_{\text{sat}} \propto \Omega^{1/2}$  holds for pinned vortices, as long as one takes  $\Omega$  at the time of the superfluid transition.

In the rotating experiments with the magnetic field oriented transverse to the nafen strands during cooldown [4], the NMR satellite peak related to HQV-bound solitons was absent—consistent with the absence of HQVs. However, since the angular velocity is conserved in the superfluid phase transition, this implies that SQVs were created. The effect of the applied magnetic field on HQV creation was further studied in stationary (non-rotating) measurements in [74], where HQVs were created by temperature quenches via the KZM. The conclusion was that the HQV density is suppressed at the normal-polar phase transition when the soliton width, controlled by the magnetic field, becomes smaller than the Kibble-Zurek length  $l_{\text{KZ}}$ , cf figure 17 and equation (68). One way to understand this observation is by assuming that the applied magnetic field then fixes the  $\hat{\mathbf{d}}$  vector on a characteristic length scale already during the phase transition. If the size of this length scale is smaller than  $l_{\text{KZ}}$ , the  $\hat{\mathbf{d}}$  vector orientation becomes correlated at length scales exceeding  $l_{\text{KZ}}$ —in contrast to the original idea assuming a random realization of the order parameter, i.e. lack



**Figure 17.** Suppression of the HQV density created by the KZM under a symmetry-breaking bias. (a) Filled red circles, magenta triangles, and black diamonds correspond to quench rates of  $\tau_Q \approx 3.8 \times 10^2$  s,  $\tau_Q \approx 1.4 \times 10^3$  s, and  $\tau_Q \approx 7.7 \times 10^3$  s, respectively, while applying a constant H = 11 mT magnetic field [74]. The field is rotated to achieve different bias fields  $H_{\perp} = H \sin \mu$ . Open blue squares  $(\tau_Q \approx 6.0 \times 10^2 \text{ s})$  correspond to measurements with zero axial field component,  $H_{\perp} = H$ . Vortex density is constant for  $H_{\perp} < H_{\perp t}$  and suppressed for higher bias fields. The suppression starts when the characteristic length scale of the bias field  $\xi_{\text{bias}}(H_{\perp})$  becomes smaller than the relevant Kibble-Zurek length. Solid lines correspond to theoretical model (see text for details). (b) The extracted threshold bias length  $\xi_t$  as a function of  $l_{\text{KZ}}$  with the same symbols. The dashed line corresponds to  $\xi_t = l_{\text{KZ}}$ . The patterned gray diamond is the same measurement as the black diamond, but with  $l_{\text{KZ}}$  on the horizontal axis replaced with an estimation of the transition front thickness  $l_{\text{F}}$  [75]. For other measurements,  $l_{\text{F}}$  lies beyond the right border of the plot.



**Figure 18.** Detection of vorex-bound solitons using magnon BEC. (a) In continuous-wave NMR coherently precessing magnetization state is formed with the sweep of the frequency of the pumping field *f* of sufficient magnitude via Larmor frequency  $f_L$  in the upward direction [73]. It is manifested by a characteristic non-linear response. (b) When external pumping is switched off, the pumped magnons decay, but magnetization of the sample continues to precess coherently while the precession frequency *f* returns to  $f_L$  in exponential decay with time constant  $\tau$ . (c) When solitons are present in the sample, the relaxation rate  $\tau^{-1}$  increases proportionally to the volume occupied by the solitons, which is in the plot is characterized by the intensity of the soliton satellite in the NMR spectrum measured independently. The measurements were performed with the nafen-243 sample at P = 7 bar and  $T = 0.4 T_c$ , where the polar phase is the equilibrium phase.

of correlation, at such length scales. Therefore, the very same properties that define the characteristic length scales of topological solitons also play an integral role in formation of topological defects during phase transitions.

Additionally, one might ask what is the interplay of the KZM [76, 77] and rotation. The KZM is expected to create various order-parameter defects, including vortices of all possible types [78–81]. In the transition, the inter-vortex distance is set by the KZ length

$$l_{\rm KZ} = \xi_0 (\tau_{\rm Q}/\tau_0)^{1/4}, \tag{68}$$

where  $\tau_{\rm Q}^{-1} = -\frac{{\rm d}(T/T_{\rm c})}{{\rm d}t}\Big|_{T=T_{\rm c}}$  is the cooldown rate at  $T_{\rm c}$ ,  $\xi_0 = \xi(T=0)$  and the order-parameter relaxation time  $\tau_0 \sim 1$  ns.

For HQVs the inter-vortex distance sets the length of the interconnecting solitons and thus the amplitude of the satellite signal. The resulting dependence  $I_{\text{sat}} \propto n_V l_{\text{KZ}} \propto l_{\text{KZ}}^{-1} \propto \tau_Q^{-1/4}$  is indeed observed in the experiment, figure 16(b). The magnitude of the signal corresponds to the averaged soliton length of 1.4  $l_{\text{KZ}}$ , as has been estimated also in the B phase of <sup>3</sup>He [81, 82]. The shift of experimental data in figure 16(a) above the theoretical expectation indicates that the KZM is important also in cooldowns with applied rotation.

Alternative method of observation of the solitons bounded by HQVs in the polar phase utilizes coherent magnetization precession state forming Bose–Einstein condensate of magnon quasiparticles, figure 18. The relaxation rate of the magnon condensate increases proportionally to the volume occupied



**Figure 19.** HQVs in thermal cycling. (a) The plot shows the measured NMR spectra in transverse ( $\mu = \pi/2$ ) magnetic field in the presence of HQVs. HQVs were created by rotation with 2.5 rad s<sup>-1</sup> during the transition from normal phase to the polar phase. The NMR spectrum includes the response of the bulk liquid and the  $\hat{\mathbf{d}}$ -solitons, which appear as a characteristic satellite peak at lower frequency. The satellite intensity in the PdA phase remains unchanged after thermal cycling. The NMR spectrum in the PdB phase at the same temperature, measured between the two measurements in the PdA phase, is shown for [5]. (b) The <sup>3</sup>He sample is confined within a cylindrical container filled with nafen-90, which consists of nearly parallel Al<sub>2</sub>O<sub>3</sub> strands with  $d_2 \approx 8$  nm diameter, separated by  $d_1 \approx 50$  nm on average. The strands are oriented predominantly along the axis denoted  $\hat{\mathbf{z}}$ . The sample can be rotated with angular velocities up to 3 rad s<sup>-1</sup> around the same axis. The sample is surrounded by rectangular NMR pick-up coils. The static magnetic field transverse to the NMR coils can be oriented at an arbitrary angle  $\mu$  with respect to the  $\hat{\mathbf{z}}$  axis. (c) The magnetic field, oriented along the *y*-direction ( $\mu = \pi/2$ ) in this figure, locks the  $\hat{\mathbf{e}}^2$ -vector in the PdB phase order parameter, equation (23). Vectors  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{e}}^1$  are free to rotate in the *xz*-plane by angle  $\theta$ .

by the solitons. This method is especially useful with small amounts of solitons present in the sample, as accurate measurement of the satellite intensity in cw NMR might require hours of averaging while the relaxation rate can be measured with sufficient precision in seconds. At larger soliton densities, though, the relaxation rate becomes too fast to be reliably measured.

#### 5.3. HQVs in the PdA phase

The transverse resonance frequency of the bulk fluid in the PdA phase with magnetic field in the direction parallel to the strand orientation, i.e. for  $\mu = 0$ , is [16]

$$\Delta\omega_{\rm PdA} = \omega_{\rm PdA} - \omega_{\rm L} \approx \frac{\Omega_{\rm PdA}^2}{2\omega_{\rm L}},\tag{69}$$

where  $\Omega_{PdA}$  is the frequency of the longitudinal resonance in the PdA phase. The NMR line retains its shape during the second order phase transition from the polar phase but renormalizes the longitudinal resonance frequency due to appearance of the order parameter component  $b \hat{\mathbf{n}}$  in equation (17).

Since the  $\hat{\mathbf{m}}$  vector is fixed by nafen parallel to the anisotropy axis, the  $\hat{\mathbf{l}}$  vector then lies on the plane perpendicular to it, prohibiting the formation of continuous vorticity [83] like the double-quantum vortex in <sup>3</sup>He-A [84]. Some planar structures in the  $\hat{\mathbf{l}}$ -vector field, such as domain walls [39] or disclinations, remain possible but the effect of the  $\hat{\mathbf{l}}$  texture on the trapping potential for spin waves is negligible due to the large polar distortion [16] (i.e. for  $|b| \ll 1$ ).

In the presence of HQVs the excitation of standing spin waves localized on the soliton leads to a characteristic NMR satellite peak in transverse ( $\mu = \pi/2$ ) magnetic field, c.f. figure 19, with frequency shift

$$\Delta \omega_{\rm PdAsat} = \omega_{\rm PdAsat} - \omega_{\rm L} \approx \lambda_{\rm PdA} \frac{\Omega_{\rm PdA}^2}{2\omega_{\rm I}},\tag{70}$$

where  $\lambda_{PdA}$  is a dimensionless parameter dependent on the spatial profile (texture) of the order parameter across the soliton. For an infinite  $\hat{\mathbf{d}}$ -soliton, one has  $\lambda_{PdA} = -1$ , corresponding to the zero-mode of the soliton, as in the polar phase [66, 67, 72, 85]. The measurements in the supercooled PdA phase, figure 19, at temperatures close to the transition to the PdB phase give value  $\lambda_{PdA} \approx -0.9$ , which is in good agreement with theoretical predictions and the polar phase measurements with the 243 mg cm<sup>-3</sup> nafen sample. This confirms that the structure of the  $\hat{\mathbf{d}}$ -solitons connecting the HQVs is similar in polar and PdA phases and the effect of the orbital part to the trapping potential can be neglected. Furthermore, the satellite intensity shown in figure 20(b) scales with  $\sqrt{\Omega}$  as in the polar phase, indicating that the observed signal is linked to topological solitons bounded by HQVs also in the PdA phase.

#### 5.4. Walls bounded by strings in the PdB phase

As established in section 3.2, isolated HQVs cease to be protected by topology in the PdB phase as its order parameter lacks the relevant  $\mathbb{Z}_{2(\phi+S)}$  symmetry. The experimental data, figure 19, strongly suggests that HQVs survive the phase transition to the PdB phase as composite defects, walls bounded by strings (or KLS walls), see section 3.1. Let us now take a closer look at the experiments that led to this conclusion.

For a magnetic field oriented transverse to the uniaxial nafen anisotropy axis  $\hat{z}$ , the order parameter of the PdB phase



**Figure 20.** NMR spectra in the PdB phase. (a) The plot shows the measured NMR spectrum in the PdB phase at 0.38  $T_c$  for different HQV densities, controlled by the angular velocity  $\Omega$  at the time of crossing the  $T_c$  [5]. The presence of KLS walls produces characteristic features seen both as widening of the main line (located at small positive frequency shift) and as a satellite peak with a characteristic negative frequency shift. The inset shows magnified view of the satellite peak. (b) The satellite intensity in the PdA phase at  $0.60T_c$  (blue circles) and in the PdB phase multiplied by a factor of 9 (red triangles) at  $0.38T_c$  show the expected  $\sqrt{\Omega}$ -scaling. The solid black line is a linear fit to the measurements including data from both phases. The non-zero  $\Omega = 0$  intersection corresponds to vortices created by the KZM [4, 76, 77]. (c) The FWHM of the main line, determined from the spectrum in panel (a), gives FWHM  $\approx 3$  kHz for 2.5 rad s<sup>-1</sup>. FWHM for other angular velocities is recalculated from the amplitude of the main NMR line, shown in panel (a), assuming constant area.

is given by equation (23). We denote with  $\theta$  the rotation of the spin space with respect to the orbital space, with  $\sin \theta_0 = q_2(2-2q_1)^{-1}$  corresponding to the minimum energy configuration for  $\theta$ . The transverse frequency shift with uniform  $\theta = \theta_0$  (i.e. the response of the bulk) is given by [40]

$$\frac{\omega_{\perp}^2 - \omega_{\rm L}^2}{\Omega_{\rm PdB}^2} = q_1 - q_2^2.$$
(71)

In the axial field, with **H** oriented along the anisotropy axis, the homogeneous transverse frequency shift with uniform  $\theta = \theta_{0,\parallel} = \operatorname{sgn}(q_2)\pi/2$  is given by [40]

$$\frac{\omega_{\parallel}^2 - \omega_{\rm L}^2}{\Omega_{\rm PdB}^2} = 1 + \frac{5}{2}|q_2|.$$
(72)

The *q*-parameter value is determined from the frequency shifts in equations (71) and (72), following a method described in [40]. In the experimental region of interest, the distortion factor is given by

$$q = \frac{2 - 5C}{4} - \frac{1}{4}\sqrt{25C^2 - 36C + 4},$$
 (73)

where

$$C = \frac{\omega_{\perp} - \omega_{\rm L}}{\omega_{\parallel} - \omega_{\rm L}}.\tag{74}$$

The expression (73) is valid in the range  $q \in [0, (\sqrt{14} - 2)/5 \approx 0.348]$ . To measure q, we carefully prepare the state by cooling the sample through the superfluid transition temperature at zero rotation in the transverse magnetic field to avoid creation of HQVs. Then we cool the sample down to the lowest temperatures and start warming it up slowly, continuously



**Figure 21.** The measured distortion parameter q as a function of temperature. The dots represent the measured values for q. The solid red line is an estimation of q, calculated based on Ginzburg-Landau theory with strong-coupling corrections using two fitting parameters in the spirit of [40] and taking  $\beta$  parameter values from [86]. The PdB phase critical temperature is shown by dashed line for the transition to the PdA phase on warming. The jump in q at this temperature reflects the fact that the PdB-PdA transition is of the 1st order.

monitoring the NMR spectrum. We perform two temperature sweeps, first in the axial and then in the transverse field. This way we can measure the q parameter in the coexistence region of the PdA and PdB phases. The results of our measurements are shown in figure 21.

In the transverse magnetic field **H** exceeding the dipolar field  $\sim 3$  mT, the order parameter vector  $\hat{\mathbf{e}}^2$  in equation (23)



**Figure 22.** Composite defect of KLS wall and spin  $\pi$ -soliton connected by HQVs. Each HQV core terminates one soliton—reorientation of the spin part of the order parameter denoted by the angle  $\theta$ —and one KLS wall. The orientation of the  $\hat{\mathbf{d}}$ -vector is shown as cones where their color indicates the angle  $\theta$ , based on numerical calculations. The  $\pi$ -soliton (blue), responsible for the observed NMR feature, corresponds to a situation where the KLS wall (light green) is bound between a different pair of HQV cores (green) than the soliton. The order parameter is continuous as the KLS walls are accompanied by virtual jumps, where  $\phi \rightarrow \phi + \pi$ ,  $\theta \rightarrow \theta + \pi$ , and  $q_2 \rightarrow -q_2$ .

becomes locked along the field, while vectors  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{e}}^1$  are free to rotate around the axis  $\hat{\mathbf{y}}$ , directed along **H**, with the angle  $\theta$  between **d** and  $\hat{\mathbf{z}}$ . The order parameter of the PdB phase on a loop around a HQV has the following properties. The phase  $\phi$  around the HQV core changes by  $\pi$  and the angle  $\theta$  (and thus vectors  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{e}}^1$ ) winds by  $\pi$ . Consequently, there is a phase jump  $\phi \rightarrow \phi + \pi$  and related sign flips of vectors  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{e}}^1$  along a perpendicular direction. In the presence of order-parameter components with q > 0, equation (23) remains single-valued if, and only if,  $q_2$  also changes sign. We conclude that the resulting domain wall separates the degenerate states with  $q_2 = \pm q$  and together with the bounding HQVs has a structure identical to the domain wall bounded by strings-the KLS wall-proposed by Kibble, Lazarides, and Shafi in cosmological context [48, 49] and discussed in detail in section 3.2.

The KLS wall and the topological soliton have distinct defining length scales [50, 87], see figure 4. The KLS wall has a hard core of the order of  $q^{-1}\xi_0$  and the soliton has a soft core of the size of the dipole length  $\xi_D \gg q^{-1}\xi_0$ . The combination of these two objects may emerge in two different configurations, see figure 7. The observed frequency shift suggests the configuration, depicted in figure 22, where  $\pi$ -soliton and KLS wall connect at the HQV acting as nexus.

The appearance of KLS walls and the associated  $\hat{\mathbf{d}}$ -solitons leads to a characteristic frequency shift

$$\Delta \omega_{\rm PdBsat} = \omega_{\rm PdBsat} - \omega_{\rm L} \approx \lambda_{\rm PdB} \frac{\Omega_{\rm PdB}^2}{2\omega_{\rm L}}.$$
 (75)

The dimensionless parameter  $\lambda_{PdB}$  has been calculated in section 4.1. Comparison of calculations with the experiment for all possible soliton structures from figure 5 is shown in figure 23. Numerical calculations give the low-temperature values  $\lambda_{soliton} \sim -0.6$  for the soliton ( $\Delta \theta = \pi - 2\theta_0$ ) and  $\lambda_{big} \sim -1.5$  for its antisoliton, the big soliton ( $\Delta \theta = \pi + 2\theta_0$ ). The KLS-soliton ( $\Delta \theta = 2\theta_0$ ) provides too shallow potential to have a bound spin-wave state distinguishable from the main line. The last possibility, the inseparable  $\pi$ -soliton, gives  $\lambda_{\pi} \approx$ -1 for the accessible temperature range, in good agreement with the measured value,  $\lambda_{PdB} \sim -1$ . The measured values



**Figure 23.** Frequency shift of the HQV satellite as a function of temperature in the polar-distorted phases. In the PdA phase the measured values are in reasonable agreement with the theoretical prediction for a  $\hat{\mathbf{d}}$ -soliton with  $\pi$  winding, shown as the red dashed line. The corresponding values in the PdB phase for the lowest-energy  $\hat{\mathbf{d}}$ -soliton (soliton) and its antisoliton (big soliton), as well as the  $\pi$ -soliton (see text) are shown as dashed blue lines. The experimental data is taken from [5] and the dashed lines are based on 2D numerical calculations, section 4.1, using  $D = 20\xi_{\rm D}$ . The error bars denote the uncertainty in the position of the satellite peak by 1.0 and 0.5 kHz in the PdB and PdA phases, respectively. The uncertainty is taken as the full width at half maximum (FWHM) of the satellite peak in the PdB phase and as half of the FWHM due to improved signal-to-noise ratio in the PdA phase.

for  $\lambda_{PdB}$ , together with the fact that the total winding of the  $\hat{\mathbf{d}}$ -vector is also equal to  $\pi$  in the PdA and polar phases, suggest that the observed soliton structure in the PdB phase corresponds to the  $\pi$ -soliton in the presence of a KLS wall.

In addition, the KLS wall possesses a tension  $\sim \xi q^2 \Delta_{PdB}^2 N_0$ [87, 88], where  $N_0$  is the density of states. Thus the presence of KLS walls applies a force pulling the two HQVs at its ends towards each other. The fact that the number of HQVs remains unchanged through the phase transitions signifies that the KLS wall tension does not exceed the maximum pinning force in the studied nafen sample. Strong pinning of single-quantum vortices in B-like phase in silica aerogel has also been observed previously [89]. An alternative way to remove a KLS wall is to create a hole within it, bounded by a HQV [48]. However, growth of such a HQV ring is also prohibited by the strong pinning by the nafen strands. We also note that for significantly larger values of q creation of a HQV vortex-antivortex pair within the KLS wall may become energetically favorable and as a result the HQV pairs bounded by KLS walls would eventually shrink to singly-quantized vortices.

The satellite intensity in the PdB phase, cf figure 20(b), scales as  $\sqrt{\Omega}$ —as in the polar and PdA phases. Although the scaling is identical, one striking difference appears in the PdB phase—the satellite intensity normalized to the total absorption integral in the PdB phase is smaller by a factor of ~9 compared to the PdA phase. Simultaneously, the original satellite intensity in the PdA phase is restored after a thermal cycle. There is currently no explanation for this observation.

Another effect of rotation in the PdB phase transverse  $(\mu = \pi/2)$  NMR spectrum is observed at the main peak, cf figure 20(a). The full-width-at-half-maximum (FWHM), extracted from the amplitude of the main peak assuming  $w \cdot h = \text{const}$ , where w is its width and h is height, scales as  $\propto \sqrt{\Omega}$ ; figure 20(c). Increase in the FWHM may indicate that the presence of KLS walls enhances scattering of spin waves and thus results in increased dissipation. The small frequency shift of this feature may originate from spin waves weakly bound between KLS walls.

#### 6. Conclusions and outlook

In superfluid <sup>3</sup>He topological solitons are manifested as localized winding of the order parameter anisotropy vectors, connecting two energetically degenerate regions with one another. The soliton width is set by the magnetic or the SOC energy arising from deflection of the order parameter from the lowestenergy configuration within the soliton. The topological protection of the soliton structure is given by the  $\pi_1$  homotopy group, which is the same as for linear topological defects, quantized vortices. Thus solitons may terminate in bulk liquid on linear topological defects, in addition to the possibility to terminate at the sample boundary. Vortex cores typically are of coherence length size, and are too small to be directly observed with the available experimental methods. Vortexbound solitons provide an experimental tool for accessing properties of a range of exotic objects via NMR methods.

In this review article, we have shown the connection between solitons and the underlying order parameter symmetries, and how solitons are manifested in the experiments. So far experiments have demonstrated three well-identified cases of solitons terminated at the quantized vortices: Solitons bounded by HQVs in the polar and PdA phases, solitons accompanying the KLS-walls in the PdB phase, and solitons connecting spinmass vortices in the B phase of <sup>3</sup>He. The frequency shift of the associated feature in the NMR spectrum in all cases agrees with theoretical predictions. This observation, together with the scaling of the soliton peak intensity with the angular velocity of the sample container confirmed the existence of HQVs [4] and of the spin-mass vortices [56]. The soliton peak has also been used to study the KZM of topological defect formation, in particular to demonstrate variety of defect types created by the KZM [57] and to find modification of the KZM in the presence of a symmetry-breaking bias field [74], where the number of defects created in the phase transition was suppressed by applying magnetic field tilted with respect to the system symmetry axis during the superfluid transition. In the PdB phase, the NMR properties of the spin soliton were used to identify the type of soliton accompanying the KLS walls, where multiple possibilities existed [5, 42]—the experiments and numerics are in beautiful agreement. More recent numerical work on a spinor BEC [90, 91] studies the effect of the strong coupling correction on the domain wall structure and the instability of the domain wall induced by spin current.

In conclusion, vortex-bound solitons are ubiquitous in different superfluid phases of <sup>3</sup>He. They have proven important for experimental identification of composite topological defects. Some composite topological defects, such as the KLS walls bounded by strings, require a specific hierarchy of symmetry-breaking phase transitions in order to be realized. Simply proving their existence in the cosmological vacuum would be a major step forward in the study of the early times of our Universe, immediately ruling out some GUTs [48]. Additionally, it has been proposed that solitons could provide observable signatures of axion dark matter [28]. Recently, topological classifications of Yang-Mills solitons, non-Abelian sine-Gordon solitons, and skyrmions in quantum field theories and quantum chromodynamics have also been discussed in terms of composite objects [92]. HQVs, identified by the NMR signature of the vortexbound solitons, hold promise for topological, or error-tolerant, quantum computation with non-Abelian Majorana corebound states [6]. Therefore, understanding and identifying vortex-bound solitons is important for a number of research directions.

#### Data availability statement

The data cannot be made publicly available upon publication because they are not available in a format that is sufficiently accessible or reusable by other researchers. The data that support the findings of this study are available upon reasonable request from the authors.

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# Appendix A. A<sub>1</sub> phase—order parameter and remaining symmetry

In bulk fluid, the  $A_1$  phase exists only at a very narrow temperature region close to  $T_c$  in the presence of magnetic field. The  $A_1$  phase consists of Cooper pairs with both spins oriented along the direction of the magnetic field. The order parameter of the  $A_1$  phase can be written as

$$A_{\mu i}^{\mathrm{A}_{1}} = \Delta_{\mathrm{A}_{1}} e^{i\phi} (\hat{\mathbf{d}}_{\mu} + i\hat{\mathbf{e}}_{\mu}^{1}) (\hat{\mathbf{m}}_{j} + i\hat{\mathbf{n}}_{j}), \qquad (\mathrm{A}.1)$$

where  $\hat{\mathbf{d}}$  and  $\hat{\mathbf{e}}^1$  form an orthogonal triad in spin space with vector  $\hat{\mathbf{e}}^2 = \hat{\mathbf{d}} \times \hat{\mathbf{e}}^1$  and  $\Delta_{A_1}$  is the maximum gap in the A<sub>1</sub> phase. Similar to the A phase, a gauge transformation  $A_{\mu j}^{A_1} \rightarrow e^{i \Delta \phi} A_{\mu j}^{A_1}$ can be compensated by simultaneous rotation of the orbital space by  $-\Delta\phi$  about **I**, corresponding to remaining  $U(1)_{\phi+\mathbf{L}}$ symmetry. On the other hand, the spin part of the order parameter has identical structure to the orbital part, and the gauge transformation can be compensated also by rotation of the spin space around  $\hat{\mathbf{e}}^2$  by  $-\Delta\phi$ , resulting in a symmetry  $U(1)_{\phi+\mathbf{S}}$ corresponding to simultaneous rotation of the spin space and change of phase. Alternatively, one of the U(1) symmetries may be interpreted as a simultaneous rotation of the orbital and spin spaces. Additionally, the time-reversal symmetry is partially broken due to the non-zero imaginary part both in orbital and spin spaces, reducing it to the combined discrete symmetry  $\mathbb{Z}_{2(T+\mathbf{J})}$  corresponding to simultaneous time reversal,  $\pi$ -rotation of the orbital space about  $\hat{\mathbf{m}}$ , and  $\pi$  rotation of the spin space about  $\hat{\mathbf{d}}$ . Thus, the order parameter of the A<sub>1</sub> phase is invariant under [1]

$$H_{A_1} = U(1)_{\phi + \mathbf{L}} \times U(1)_{\phi + \mathbf{S}} \times \mathbb{Z}_{2(T + \mathbf{J})} \times C \times PU_{\pi}.$$
 (A.2)

#### Appendix B. Length scales in the presence of orientation energies

The magnetic healing length  $\xi_{\rm H}$  is determined by the competition of the gradient energy density

$$f_{\nabla} = \frac{1}{2} K_1 \partial_i A_{\alpha j}^{\text{PdB}} \partial_i \left( A_{\alpha j}^{\text{PdB}} \right)^* + \frac{1}{2} K_2 \partial_j A_{\alpha i}^{\text{PdB}} \partial_i \left( A_{\alpha j}^{\text{PdB}} \right)^* + \frac{1}{2} K_3 \partial_i A_{\alpha i}^{\text{PdB}} \partial_j \left( A_{\alpha j}^{\text{PdB}} \right)^*, \qquad (B.1)$$

where  $K_1 = K_2 = K_3$  [1], with the magnetic energy density, which is

$$f_{\rm H} = -\frac{1}{2}\chi_{\alpha\beta}H_{\alpha}H_{\beta} = \frac{1}{2}\gamma^2 S_a S_b(\chi^{-1})_{ab} - \gamma H_a S_a, \quad (B.2)$$

where the  $\chi_{\alpha\beta}$  is the uniaxial magnetic susceptibility tensor of the PdB phase,  $H_{\alpha}$  are magnetic field strengths with  $\alpha =$ 1,2,3,  $S_a$  are spin densities with a = 1,2,3, and  $\gamma$  is the gyromagnetic ratio of <sup>3</sup>He [1]. Using equations (B.1) and (B.2) allows writing the magnetic length as

$$\xi_{\rm H} = \sqrt{\frac{K_1 \Delta_{\rm PdB}^2}{(\chi_\perp - \chi_\parallel) H^2}},\tag{B.3}$$

where  $\chi_{\perp}$  and  $\chi_{\parallel}$  are transverse and longitude spin magnetic susceptibilities of PdB phase. Following the same idea, the dipole length  $\xi_{\rm D}$  is determined by the gradient energy density  $f_{\nabla}$  and the SOC energy density

$$f_{\rm SOC} = \frac{3}{5} g_{\rm D} \left( \left( A_{ii}^{\rm PdB} \right)^* A_{jj} + \left( A_{ij}^{\rm PdB} \right)^* A_{ji} - \frac{2}{3} \left( A_{ij}^{\rm PdB} \right)^* A_{ij} \right),$$
(B.4)

where  $g_D$  is the strength of the SOC. Then we have

$$\xi_{\rm D} = \sqrt{\frac{5K_1}{6g_{\rm D}}}.\tag{B.5}$$

# Appendix C. Relative homotopy groups and exact sequences

The homotopy groups and relative homotopy groups of vacuum manifolds  $R_1$  and  $R_2$  form a long exact sequence (LES) [44]

$$\dots \pi_k(R_2) \xrightarrow{x_*^{n-1}} \pi_k(R_1) \xrightarrow{x_*^n} \pi_k(R_1, R_2) \xrightarrow{\partial^*} \pi_{k-1}(R_2) \dots$$
(C.1)

The exact sequence of (relative) homotopy groups means that the image of any homomorphism  $x_*^{n-1}: M \to N$  in equation (C.1) (the sets of the elements of the group N into which the elements of A are mapped) is the kernel of the next homomorphism  $x_*^{n+1}: N \to W$  (the sets of the elements of N which are mapped to the zero or unit element of W), i.e.  $\operatorname{Im} x_*^n \cong \ker x_*^{n+1}$  with  $n \in \mathbb{Z}$  [44].

The relative homotopy classes of  $\pi_{k+1}(R_1, R_2)$  are mapped to the homotopy classes of  $\pi_k(R_2)$  by mapping the *k*dimensional subspace of k+1 sphere, which surrounds the defects, into  $R_2$ . This mapping between two homotopy classes with different dimensions is called boundary homomorphism  $\partial^*$  [44]. When this mapping is not trivial, i.e. Im  $\partial^* \neq 0$ , topological defects given by  $\pi_k(R_1, R_2)$  can be mapped to those given by  $\pi_{k-1}(R_2)$ , living on the k+1 sphere enclosing the original defects. In other words, boundary homomorphism describes how topological objects with different dimensionalities connect to one another—it is therefore a convenient tool for describing composite topological objects. In contrast, the commonly used homotopy group  $\pi_k(R_1)$  lacks boundary homomorphism and therefore does not provide information about such connections.

The LES, equation (C.1), has infinite number of terms. It is therefore useful to split it up to SESs [93]. For every relative homotopy group  $\pi_1(R_1, R_2)$ , the LES can be split as

$$0 \longrightarrow \operatorname{Im} x_*^n \xrightarrow{x_*^n} \pi_k(R_1, R_2) \xrightarrow{\partial^*} \operatorname{Im} \partial^* \longrightarrow 0$$
 (C.2)

by the image of  $\partial^*$  and  $x_*^n$ . In this case, the relative homotopy group  $\pi_k(R_1, R_2)$  is an extension of  $\text{Im } x_*^n$  by  $\text{Im } \partial^*$ .

#### *C.1.* SESs of $\pi_n(R_1^{\text{PdA}}, R_2^{\text{PdA}})$

Following the definition of LES in equation (C.1) and the vacuum manifolds mentioned in section 2.2, we have the LESs

$$\dots \pi_1(R_2^{\text{PdA}}) \xrightarrow{x_*^*} \pi_1(R_1^{\text{PdA}}) \xrightarrow{x_*^*} \pi_1(R_1^{\text{PdA}}, R_2^{\text{PdA}}) \xrightarrow{\sigma_1^*} \pi_0(R_2^{\text{PdA}}) \dots \text{ (C.3)}$$
  
and

$$\dots \pi_2(R_2^{\text{PdA}}) \xrightarrow{x_*^{n-1}} \pi_2(R_1^{\text{PdA}}) \xrightarrow{x_*^n} \pi_2(R_1^{\text{PdA}}, R_2^{\text{PdA}}) \xrightarrow{\partial_2^*} \pi_1(R_2^{\text{PdA}}) \dots, (C.4)$$

which in fact are

$$\mathbb{Z}_{\mathbf{L}} \xrightarrow{x_*^{n-1}} \mathbb{Z}_{\mathbf{L}} \times \mathbb{Z}_{\phi} \xrightarrow{x_*^n} \pi_1(R_1^{\mathrm{PdA}}, R_2^{\mathrm{PdA}}) \xrightarrow{\partial_1^*} 0 \dots$$
 (C.5)

and

$$\dots \xrightarrow{a_*^{n-1}} \mathbb{Z}_{\mathbf{S}} \xrightarrow{x_*^n} \pi_2(R_1^{\mathrm{PdA}}, R_2^{\mathrm{PdA}}) \xrightarrow{\partial_2^n} \mathbb{Z}_{\mathbf{L}} \dots$$
(C.6)

It is easy to see that in equation (C.5)  $\text{Im} \partial_1^* = 0$ , while  $\text{Im} x_*^{n-1} = \mathbb{Z}_L$  as the orbital vertices formed by  $\hat{\mathbf{n}}$  (or  $\hat{\mathbf{l}} = \hat{\mathbf{m}} \times \hat{\mathbf{n}}$ ) form same group  $\mathbb{Z}_L$ . It follows that ker  $x_*^n \cong \mathbb{Z}_L$  and

$$\operatorname{Im} x_*^n \cong (\mathbb{Z}_{\mathbf{L}} \times \mathbb{Z}_{\phi}) / \mathbb{Z}_{\mathbf{L}} = \mathbb{Z}_{\phi}, \qquad (C.7)$$

leading to the SES of  $\pi_1(R_1^{\text{PdA}}, R_2^{\text{PdA}})$ 

$$0 \longrightarrow \mathbb{Z}_{\phi} \longrightarrow \pi_1(R_1^{\text{PdA}}, R_2^{\text{PdA}}) \xrightarrow{\partial^*} 0 \longrightarrow 0.$$
 (C.8)

Following similar reasoning, we have  $\operatorname{Im} \partial_2^* = 0$ , where 0 is the kernel of the mapping  $x_*^{n-1} : \mathbb{Z}_{\mathbf{L}} \mapsto \mathbb{Z}_{\mathbf{L}} \times \mathbb{Z}_{\phi}$ , resulting in

$$0 \longrightarrow \mathbb{Z}_{\mathbf{S}} \longrightarrow \pi_2(R_1^{\mathrm{PdA}}, R_2^{\mathrm{PdA}}) \xrightarrow{\partial^*} 0 \longrightarrow 0.$$
 (C.9)

## C.2. SESs of $\pi_n(R_1^{\text{PdB}}, R_2^{\text{PdB}})$

We now conduct the calculations of SESs for the PdB phase. Similarly to the previous section, we start from

$$\dots \mathbb{Z}_{\mathbf{J}} \xrightarrow{x_*^{n-1}} \mathbb{Z}_{2\mathbf{J}} \times \mathbb{Z}_{\phi} \xrightarrow{x_*^n} \pi_1(R_1^{\mathrm{PdB}}, R_2^{\mathrm{PdB}}) \xrightarrow{\partial_1^n} \mathbb{Z}_{2(\phi+\mathbf{S})} \longrightarrow 0 \dots \quad (C.10)$$

and

$$\dots 0 \xrightarrow{x_*^{n-1}} 0 \xrightarrow{x_*^n} \pi_2(R_1^{\text{PdA}}, R_2^{\text{PdA}}) \xrightarrow{\partial_2^*} \mathbb{Z}_{\mathbf{J}} \xrightarrow{x_*^{n+1}} \mathbb{Z}_{\phi} \times \mathbb{Z}_{2\mathbf{J}} \dots, \quad (C.11)$$

where

$$\mathbb{Z}_{\mathbf{J}} = \pi_1(R_2^{\text{PdB}}), \mathbb{Z}_{\phi} \times \mathbb{Z}_{2\mathbf{J}} = \pi_1(R_1^{\text{PdB}}), 0 = \pi_0(R_2^{\text{PdB}}) = \pi_0(R_1^{\text{PdB}}).$$
(C.12)

Mapping  $x_*^{n-1}: \pi_1(R_2^{\text{PdB}}) \mapsto \pi_1(R_2^{\text{PdB}})$   $(x_*^{n+1}$  in equation (C.11)) plays a significant role in splitting equations (C.10) and (C.11). The  $\mathbb{Z}_J$  vertices are mapped into  $\mathbb{Z}_{2J}$  through a surjection, i.e.  $\text{Im} x_*^{n-1} = \mathbb{Z}_J$ . A natural choice of ker  $x_*^{n-1}$  is  $2\mathbb{Z}_J$ , which is a group of vortices with even winding number [41]. As a result,  $\text{Im} \partial^*$  in equations (C.10) and (C.11) are  $\mathbb{Z}_{2(\phi+S)}$  and  $2\mathbb{Z}_J$ , respectively. Since  $\text{Im} x_*^{n-1} = \mathbb{Z}_{2J} = \ker x_*^n$  in equation (C.10), we also have

$$\operatorname{Im} x_*^n = (\mathbb{Z}_{2\mathbf{J}} \times \mathbb{Z}_{\phi}) / \ker x_*^n = \mathbb{Z}_{\phi}.$$
 (C.13)

Then we get the SES

$$0 \longrightarrow \mathbb{Z}_{\phi} \longrightarrow \pi_1(R_1^{\text{PdB}}, R_2^{\text{PdB}}) \xrightarrow{\partial^*} \mathbb{Z}_{2(\phi+\mathbf{S})} \longrightarrow 0 \quad (C.14)$$
  
for  $\pi_1(R_1^{\text{PdB}}, R_2^{\text{PdB}})$  and

$$0 \longrightarrow 0 \longrightarrow \pi_2(R_1^{\text{PdB}}, R_2^{\text{PdB}}) \xrightarrow{\partial^*} 2\mathbb{Z}_{\mathbf{J}} \longrightarrow 0 \quad (C.15)$$
  
for  $\pi_2(R_1^{\text{PdB}}, R_2^{\text{PdB}}).$ 

C.3. SES of  $\pi_1(R_1^H, \tilde{R}_1^{SOC})$ 

The long exact sequence for  $\pi_1(R_1^{\rm H}, \tilde{R}_1^{\rm SOC})$  with reduced vacuum manifolds is written as

 $\pi_1(\tilde{R}_1^{SOC}) \longrightarrow \pi_1(R_1^{H}) \xrightarrow{x_1^n} \pi_1(R_1^{H}, \tilde{R}_1^{SOC}) \xrightarrow{\partial^*} \pi_0(\tilde{R}_1^{SOC}) \longrightarrow \pi_0(R_1^{H})$ , (C.16) where  $\partial^*$  is the boundary homomorphism [44, 93]. Plugging in the homotopy groups of the reduced manifold, equation (C.16), gives

$$\mathbb{Z}_{\phi} \longrightarrow \mathbb{Z}_{\mathbf{S}} \times \mathbb{Z}_{\phi} \xrightarrow{x_*^n} \pi_1(R_1^{\mathrm{H}}, \tilde{R}_1^{\mathrm{SOC}}) \xrightarrow{\partial^*} \mathbb{Z}_4 \longrightarrow 0. \quad (C.17)$$
  
It is worth noting that  $\mathrm{Im}\,\partial^* = \mathbb{Z}_4$  and  $\mathrm{Im}\,x_*^n = \mathbb{Z}_{\mathbf{S}}$ , resulting in

$$0 \longrightarrow \mathbb{Z}_{\mathbf{S}} \xrightarrow{x_*^n} \pi_1(R_1^{\mathrm{H}}, \tilde{R}_1^{\mathrm{SOC}}) \xrightarrow{\partial^*} \mathbb{Z}_4 \longrightarrow 0 . \quad (C.18)$$

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