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Loss–gain compensated anti-Hermitian magnetodielectric medium to realize Tellegen nihility effects

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Generalized duality transformations significantly modify the constitutive relations of electromagnetic media, preserving principal electromagnetic properties. Here, we contemplate transformation of Tellegen nihility as a specific type of extreme-property nonreciprocal bi-isotropic media and show that some intriguing magnetodielectric properties of that medium can be realized in a particular class of isotropic magnetodielectric media without magnetoelectric coupling. We show that the permittivity and permeability of the corresponding transformed medium have equal absolute values and opposite signs. Depending on the value of the Tellegen parameter of the original medium, the transformed magnetodielectric medium can be Hermitian, non-Hermitian, or anti-Hermitian, which simultaneously exhibits loss and gain. Focusing on the latter class of anti-Hermitian media, we theoretically and numerically demonstrate that this extraordinary medium allows propagation of electromagnetic plane waves having zero time-averaged Poynting vector, similarly to the original Tellegen nihility media. Hopefully, this work can open novel opportunities for manipulating electromagnetic fields. © 2023 Optica Publishing Group
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Introduction. One of intriguing electromagnetic media with exotic properties is the magnetodielectric medium with zero-valued permittivity and permeability. In such media, the equations for electric and magnetic fields are decoupled and static, leading to practically important features, e.g., [1–3]. Making media with vanishing permittivity and permeability offers enticing possibilities in realizing negative refraction and maximizing chirality effects [4]. Even more intriguing exotic medium is the nonreciprocal bi-isotropic medium with zero relative permittivity and permeability. This medium, called Tellegen nihility [5], is governed by the following constitutive relations in the frequency domain:

\[
\mathbf{D}(\mathbf{r}, \omega) = \sqrt{\mu_0 \varepsilon_0} \mathbf{H}(\mathbf{r}, \omega), \quad \mathbf{B}(\mathbf{r}, \omega) = \sqrt{\mu_0 \varepsilon_0} \chi \mathbf{E}(\mathbf{r}, \omega),
\]

where \(\mathbf{D}\) and \(\mathbf{B}\) are the electric and magnetic flux densities, \(\mathbf{E}\) and \(\mathbf{H}\) denote the electric and magnetic fields, and \(\chi\) represents the Tellegen parameter, and \(\mu_0\) and \(\varepsilon_0\) are the free-space permeability and permittivity, respectively. The Tellegen nihility is a very special sub-class of bi-isotropic media. General classification of these materials is discussed in Refs. [6,7]. Some examples of the studies of waves in nonreciprocal biisotropic media can be found in Refs. [8–11]. Substituting these material relations into source-free Maxwell’s equations, we see that electricity and magnetism are completely decoupled, but the fields obey dynamic equations. These simultaneously decoupled and dynamic equations certainly provide seductive and potentially useful effects. However, the key problem is the actual realizations of such a medium which appear to be extremely difficult if at all possible. Indeed, while the Tellegen effect exists in some natural materials [12], it is very weak. Some recipes on realizations of strong magnetodielectric coupling in artificial materials can be found in Refs. [13,14], but it is not known if and how this response could be combined with nihility of the electric and magnetic response.

In classical electrodynamics, field transformations, including the generalized duality transformation [15,16], are used to find solutions of new problems by applying transformations to known solutions of other problems. However, another important advantage of this approach is to uncover simple media realizing exotic field effects of very complex materials. This is possible due to the invariance of those effects under the transformation. Therefore, exploiting duality transformation gives hope to realize unusual effects of Tellegen nihility in simpler materials. It is known that a nonreciprocal bi-isotropic medium with parameters \(\varepsilon, \mu, \chi\) can be transformed into a reciprocal isotropic medium [17,18] using the generalized duality transformation. In this work, we apply this transformation to the Tellegen nihility medium \((\varepsilon = 0, \mu = 0, \chi \neq 0)\) and show that a magnetodielectric medium whose permittivity and permeability are equal in magnitude and opposite in sign provides a great opportunity for realizing the most interesting Tellegen nihility effects. Scrutinizing this transformed magnetodielectric medium, we introduce the concept of anti-Hermitian media that offer alluring possibilities for manipulation of electromagnetic waves in unprecedented ways.
Tellegen nullity. The constitutive relations of Tellegen nullity are given by Eq. (1). Using those relations, the Faraday and Maxwell–Ampère laws read
\[ \nabla \times \mathbf{E} = -j\omega \varepsilon_0 \mu_0 \chi \mathbf{E}, \quad \nabla \times \mathbf{H} = j\omega \varepsilon_0 \mu_0 \chi \mathbf{H}. \] (2)

From these two equations, unequivocally, we observe that \( \mathbf{E} \) and \( \mathbf{H} \) are eigenvectors of the curl operator, whose corresponding eigenvalues have different signs (\( \lambda_+ = -j\omega \varepsilon_0 \mu_0 \chi \) and \( \lambda_- = +j\omega \varepsilon_0 \mu_0 \chi \)). Eigenvectors of the curl operator with constant eigenvalues are called Tellegen fields [19] or, in electromagnetics, wave fields [6]. They are special cases of Beltrami vector fields (e.g., [20]) that, in general, correspond to the case when parameter \( \chi \) depends on the spatial coordinates. For a Tellegen field, since the eigenvalue is constant, and the divergence of the curl is always zero, one of the salient characteristics is the zero divergence of the eigenvector. For Tellegen nullity, this fact is fully consistent with Maxwell’s equations in source-free regions (counterintuitively, zero divergence of electric field is related to the nonexistence of magnetic charge; and zero divergence of magnetic field is associated with the known Gauss law).

Based on the Tellegen nullity field equations, we draw some important conclusions. First, the cross product of the field and the curl of the field is zero: \( \nabla \times (\nabla \times \mathbf{F}) = 0 \), where \( \mathbf{F} \) represents one of the fields. Second, in the source-free regions, both eigenvectors satisfy the Helmholtz equation that is expressed as
\[ \nabla^2 \mathbf{F} - \omega^2 \varepsilon_0 \mu_0 \chi \mathbf{F} = 0. \] (3)

This simply shows that propagating plane wave solutions at real angular frequencies can exist only if \( \chi^2 < 0 \), that is, when the Tellegen parameter is purely imaginary. Third, we readily write that \( \mathbf{F} \cdot \nabla \times (\nabla \times \mathbf{F}) = -j\omega \varepsilon_0 \mu_0 \chi \mathbf{F} \cdot \nabla \mathbf{F} \). Here, the sign “−” is in the equation for the electric field, and the sign “+” corresponds to the magnetic field equation. Concerning a uniform plane wave propagating in the \( z \) direction, we can replace \( \nabla \) by \( -jk \), where \( k \) is the wave vector, and conclude that in this case, the field is a complex lamellar vector field, i.e., \( \mathbf{F} \cdot \nabla \times (\nabla \times \mathbf{F}) = 0 \). Consequently, we find that fields of homogeneous plane waves satisfy \( \mathbf{F} \cdot \nabla \times (\nabla \times \mathbf{F}) = 0 \). This property means that the plane wave fields are circularly polarized. Interestingly, for the same propagation direction, the polarization handedness of electric and magnetic fields are opposite to each other, due to the sign difference in the corresponding curl equations. Indeed, if one of the fields is right-handed circularly polarized, the other one is left-handed and vice versa. This alluring feature can be seen by directly calculating the polarization vector \( \mathbf{p}(\mathbf{F}) = \mathbf{F} \times \mathbf{F}^* / (|\mathbf{F} \cdot \mathbf{F}^*|) \) [15] which has opposite signs for \( \mathbf{F} = \mathbf{E} \) and \( \mathbf{F} = \mathbf{H} \). The main implication of this characteristic is that if the phase velocity of electric- and magnetic-field waves is the same, then \( \mathbf{E} \cdot \mathbf{H}^* = 0 \) and \( \mathbf{E} \times \mathbf{H}^* = 0 \). Thus, the averaged Poynting vector is zero for waves with a real-valued propagation constant.

Here, it is worth noting that since the divergence of the Poynting vector also becomes zero, the medium is lossless. We can understand this fact based on the frequency-domain version of the Poynting theorem. By writing \( \nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) \) and using Maxwell’s equations, we deduce that
\[ \nabla \cdot S = k_0 \text{Im} \chi |\text{Re} \mathbf{E} \cdot \mathbf{H}^*|, \] (4)
where Im and Re represent the imaginary and real parts of the expressions inside square brackets, respectively, and \( k_0 \) is the free-space wave number. This result shows that for propagating plane waves, when the real part of \( \mathbf{E} \cdot \mathbf{H}^* \) vanishes, regardless of the value of the imaginary part of the Tellegen parameter, the medium behaves as lossless since the divergence of the Poynting vector becomes zero. Next, we will use the generalized dualization transformation to find a simpler and reciprocal material having the same property of supporting propagating plane waves without carrying power.

Generalized duality transformation. The generalized duality transformation [16–18], which is the rotation of normalized field vectors by an angle \( \theta \), is written as
\[ \left( \begin{array}{l} \mathbf{E} \\ \eta_0 \mathbf{H} \end{array} \right)_T = T(\theta) \left( \begin{array}{l} \mathbf{E} \\ \eta_0 \mathbf{H} \end{array} \right), \] (5)
where the transformation matrix \( T(\theta) \) is given by
\[ T(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \] (6)
Here, \( \eta_0 \) represents the free-space intrinsic impedance. While Maxwell’s equations are invariant with respect to this transformation, the material relations of Tellegen nullity are changed to
\[ \begin{align*}
\mathbf{D}_T(\mathbf{r}, \omega) &= \frac{\chi}{c} \sin(2\theta) \mathbf{E}_T(\mathbf{r}, \omega) + \frac{\chi}{c} \cos(2\theta) \mathbf{H}_T(\mathbf{r}, \omega), \\
\mathbf{B}_T(\mathbf{r}, \omega) &= \frac{\chi}{c} \cos(2\theta) \mathbf{E}_T(\mathbf{r}, \omega) - \eta_0 \frac{\chi}{c} \sin(2\theta) \mathbf{H}_T(\mathbf{r}, \omega),
\end{align*} \] (7)
where \( c \) denotes the speed of light \((c = 1/\sqrt{\varepsilon_0 \mu_0})\). By contemplating these relations associated with the transformed medium, we observe that it is in fact a bi-isotropic nonreciprocal medium with the Tellegen parameter \( \chi_T = (\chi/c) \cos(2\theta) \).

Importantly, the Poynting vector is an invariant of this transformation [16]. This property is actually obvious from the fact that the transformation is a rotation operation. Rotation of two vectors in their plane does not change their lengths and the angle between them, so that the cross product of the vectors does not change. Thus, we can simplify the material relations choosing a convenient angle \( \theta \), while the main property of our interest, existing of propagating waves that do not carry power, will be preserved. Here, we chose \( \theta = \pi/4 \) which makes \( \cos(2\theta) = 0 \), and, as a consequence, \( \chi_T = (\chi/c) \cos(2\theta) = 0 \). Accordingly, this particular angle transforms the Tellegen nullity material into a reciprocal isotropic magnetodielectric with the following material relations:
\[ \begin{align*}
\mathbf{D}_T(\mathbf{r}, \omega) &= \varepsilon_0 \chi \mathbf{E}_T(\mathbf{r}, \omega), \\
\mathbf{B}_T(\mathbf{r}, \omega) &= -\mu_0 \chi \mathbf{H}_T(\mathbf{r}, \omega).
\end{align*} \] (8)
Note that in this case, the relative permittivity and permeability are expressed as
\[ \varepsilon_T = \chi, \quad \mu_T = -\chi. \] (9)

At this point, first, we would like to remark that although the Poynting vector is invariant, the field polarization is not conserved. We saw that the eigenfields in Tellegen nullity are circularly polarized. However, the plane wave solutions for isotropic media subject to Eq. (8) can have arbitrary polarization, including linear. Second, it is also important to mention that “vacuum” material relations can be shown to be invariant under the above transformation. Therefore, as illustrated in Fig. 1, the interaction of electromagnetic waves with a Tellegen nullity object located in vacuum can be analyzed by studying the interaction of waves with the counterpart magnetodielectric.
Fig. 1. (a) Transformation of vacuum and Tellegen nihility under the specific angle of $\theta = \pi/4$. (b) Circularly polarized plane wave, whose fields have opposite handedness, propagating in Tellegen-nihility medium. (c) Propagation of plane wave in the transformed medium (anti-Hermitian type).

object placed in vacuum. Finally, let us note that exactly similar conversion results are achieved also by applying a special duality transformation

$$
\begin{bmatrix}
E' \\
H'
\end{bmatrix} = \hat{T}(\theta) \begin{bmatrix}
E \\
H
\end{bmatrix},
$$

(10)

whose transformation matrix is given by

$$
\hat{T}(\theta) = \frac{1}{\sqrt{1 - \sin (2\theta)}} \begin{pmatrix}
\frac{\sin \theta}{\cos \theta} & \sqrt{2} \nu_0 \sin(\theta) \\
\frac{\sin \theta}{\cos \theta} & -1
\end{pmatrix}.
$$

(11)

This matrix was introduced by Lindell, and its properties are explained in Ref. [15].

Anti-Hermitian medium. The first conclusion that can be drawn from the constitutive relations illustrated by Eq. (8) is that the intrinsic impedance of the transformed medium is purely imaginary. In other words,

$$
\eta = \sqrt{\frac{\mu_0 \mu_r}{\epsilon_0 \epsilon_r}} = \pm j \eta_0.
$$

(12)

Accordingly, for a linearly polarized plane wave, the magnetic field has 90° phase difference with the electric field, which gives rise to the zero-averaged Poynting vector. Indeed, this difference in phase causes $E_r \times H_r = \text{constant}$ purely imaginary. If $E_r = E_0 \exp(-j\beta z)a$, then $E_r \times H_r = \pm j E_0 \frac{\mu_r}{\epsilon_r} \exp(-j(\beta - \beta')z)a$. This is in agreement with the previous discussion about Tellegen-nihility media.

The second conclusion is associated with the phase constant. We know that this constant is expressed as $\beta = k_0 \sqrt{\mu_r \epsilon_r}$, where $k_0$ denotes the free-space wave number. From Eq. (9), we find that $\beta = k_0 \sqrt{-\chi^2}$. Here, we are faced with two distinct scenarios. The first one corresponds to complex-valued $\chi^2$. It is simple to conclude that the propagation constant $\beta$ is complex-valued, having nonzero real and imaginary parts. The second scenario corresponds to real-valued $\chi^2$ which is more interesting. If $\chi^2$ is positive, meaning that the Tellegen parameter $\chi$ is a real number, the phase constant is purely imaginary, and we have spatial attenuation of field solutions. In this case, the corresponding permittivity and permeability are real with different signs (for instance, a lossless plasmonic material with $\epsilon_r < 0$ and $\mu_r = 1$). However, if $\chi^2$ is negative, meaning that the Tellegen parameter is imaginary, the phase constant is real. For this exotic case, intriguingly, we have propagating eigenwaves that do not carry power. In ordinary media, this is possible only for standing waves. The case of purely imaginary $\chi$ is a special case of complex conjugate or pseudosparse media [21–23] in which the permittivity is equal to the complex conjugate of the permeability. Complex conjugate media with purely imaginary material parameters have been called purely imaginary metamaterials [24] and considered in the context of coherent perfect absorbers and lasers [24,25]. Alternatively, such media can be also called anti-Hermitian because the negative of the conjugate of permeability is equal to the permittivity (permeability) itself:

$$
\epsilon_r = -\mu_r, \quad \mu_r = -\epsilon_r.
$$

(13)

Denoting $\chi = j \chi_0$, where $\chi_0$ is a real-valued parameter, we write for this anti-Hermitian medium with loss-gain compensation

$$
\epsilon_r = j \chi_0, \quad \mu_r = -j \chi_0.
$$

(14)

Interestingly, the case of real-valued propagation constant comes from the duality transformation of a lossy or active Tellegen medium, because the imaginary part of the Tellegen parameter measures loss or gain due to the nonreciprocal magnetoelectric coupling [16]. Actually, the lossy or active character of the medium survives in the transformation, since both permittivity and permeability are imaginary numbers, but loss is balanced by gain. Note that the net absorption is zero only for propagating plane wave eigenwaves. For general fields, the material can exhibit overall loss or gain. In contrast, duality transformation of a lossless Tellegen medium results in a magnetodielectric material with real-valued permittivity and permeability, but propagating waves are not supported due to the opposite signs of the two material parameters.

To illustrate the exotic phenomenon of zero averaged Poynting vector and real phase constant, we have made frequency-domain numerical simulations by using Ansys HFSS software. The numerical results fully agree with the theoretical expectations, as shown in Fig. 2. This anti-Hermitian medium may provide possibilities to achieve intriguing effects, some of which were discussed earlier in Refs. [21–25]. For instance, it is easy to see that for normally incident plane waves illuminating a half-space filled by an anti-Hermitian medium, the reflection coefficient is

$$
\Gamma = \frac{\eta_r - \eta_0}{\eta_r + \eta_0} = -j,
$$

(15)

where the intrinsic impedance of the illuminated medium is $\eta_r$. What is important is that while the incident power is fully reflected back, it does not mean that the fields are zero or attenuate fast in the illuminated medium. The amplitudes of the electric and magnetic fields are nonzero and constant in the whole half-space, and the transmitted wave is propagating toward infinity (see Fig. 2). Since one of the material parameters is lossy, there is power absorption that allows measurements of the incident field. Importantly, the fact that the incident field has been detected and measured cannot be revealed by observing the reflected field.
Fig. 2. Numerical simulations by employing Ansys HFSS software. The left half-space is the vacuum in which the incident plane wave is propagating, and the right half-space is the anti-Hermitian medium. In simulations, we set \( c_V = -j \) and \( \mu_V = -j \). Note that the \(-j\)-directed arrows represent the magnetic field. In the vacuum, we have standing waves due to the full reflection, and in the anti-Hermitian half-space, we have propagating waves with zero averaged power.

which makes the sensor “invisible”. We note that due to the presence of active components, field solutions can be unstable, which can open up possibilities for creation of lasing devices, e.g., [24,25].

Using this simple solution for the half-space reflection problem, we can use the duality transformation and find the reflected and transmitted waves for a Tellegen nihility half-space. To do that, we assume that the incident electric field is in the form

\[
\mathbf{E}_t^{inc} = \frac{E_0}{\sqrt{2}} (a_x + a_y) \exp(-j k_0 z),
\]

(16)

The reason that we chose this incident electric field is the fact that if we apply the inverse duality transformation, such incident field corresponds to simply \( \mathbf{E}_t^{inc} = E_0 \exp(-j k_0 z) a_x \). Employing Eq. (15), the reflected and transmitted electric fields are calculated as \( \mathbf{E}_t^{ref} = -j (E_0 / \sqrt{2})(a_x + a_y) \exp(j k_0 z) \) and \( \mathbf{E}_t^{trans} = (1 - j / \sqrt{2})(a_x + a_y) \exp(j \beta z) \), respectively. Having the electric fields, it should be simple to write magnetic fields in the vacuum and anti-Hermitian medium as well. Now, we use the inverse duality transformation and obtain the corresponding fields in the vacuum-Tellegen nihility structure. As we mentioned, upon transformation, the incident electric field has only an \( x \)-component. The reflected and transmitted fields read

\[
\mathbf{E}_t^{ref} = -j E_0 \exp(j k_0 z) a_y
\]

(17)

and

\[
\mathbf{E}_t^{trans} = E_0 (a_x - j a_y) \exp(j \beta z),
\]

\[
\mathbf{H}_t^{trans} = -j E_0 (a_x + j a_y) \exp(-j \beta z).
\]

(18)

The reflected electric field is rotated completely 90° so that it is in the \( y \)-direction. This is a manifestation of the non-reciprocal magnetoelectric coupling measured by the Tellegen parameter. Furthermore, we explicitly see that the fields are circularly polarized with different handedness in the Tellegen-nihility half-space. As a check, we have directly solved the plane wave reflection problem for a vacuum-Tellegen nihility structure using Maxwell’s equations (without applying the transformation matrix method), and obtained the same results.

In summary, we have discussed a possibility to realize some complex-media effects in simpler media using the generalized duality transformation. As an example, we have considered non-reciprocal Tellegen materials with zero-valued permittivity and permeability. This exotic material is known to support propagating waves that do not carry power. We have shown that the same effects can be realized in reciprocal magnetoelectric media with anti-Hermitian permittivity and permeability and presented some interesting properties of this material.

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Data availability. Data underlying the results presented in this paper are not publicly available at this time but may be obtained from the authors upon reasonable request.

REFERENCES

2. A. Mahmoud and N. Engheta, Nat. Commun. 5, 5638 (2014).
3. I. Liberov and N. Engheta, Nat. Photonics 11, 149 (2017).