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On Realized Gain-Optimal Feeding Coefficients of Antenna Arrays

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***Abstract*—The feeding coefficients maximizing the realized gain are often solved from an eigenvalue problem. We derive an alternative method for obtaining the optimal feed coefficients in this paper and show that it gives the same solution. The derived method obtains the coefficients from a direct equation. We compare the realized gain obtained with the optimal feeding coefficients to the case where the elements are progressively phased. The examples show that with optimal feeding coefficients the realized gain can be improved in cases where the mutual coupling is high, edge-element effect is strong, or the embedded element patterns are unequal. Also, an example demonstrates that circularly polarized realized gain can be improved in dual polarized array by feeding the elements with optimal coefficients.**

***Index Terms*—antenna arrays, beam steering, beamforming, feeding coefficients.**

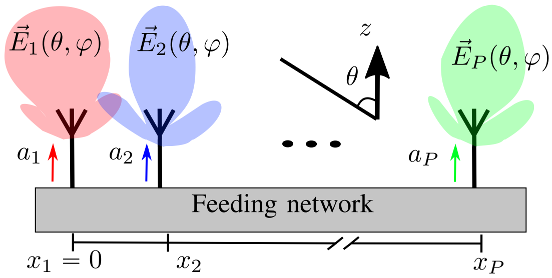

Fig. 1: Illustration of the feeding coefficients and embedded element patterns.

I. Introduction

Adaptation of higher frequencies necessitates use of electrically beam-steerable antenna arrays. The beam direction of an electronically steered antenna array can be controlled with feeding coefficients of antenna elements.

Traditionally, the array elements are fed using uniform amplitude and progressive phase shift between the elements [1], [2]. The resulting radiation pattern of the array is then a product of an embedded element pattern and an array factor. If the locations of the elements are known, the array factor can be analytically calculated. By assuming identical element patterns and negligible mutual coupling, progressive phasing focuses the radiated power to a desired direction.

However, the assumed conditions are never fully met in practical antenna arrays [3], [4]. Moreover, the exact locations of the array elements, i.e. phase centers, can be laborious to define [5]. Therefore, progressive phasing often results to a suboptimal gain pattern where more than minimum amount of power is radiated to an undesired direction or dissipated inside the device. In addition, polarization of the radiated field cannot be systematically adjusted in progressive phasing method.

The optimal feeding coefficients can be determined when the embedded element patterns are known. In typical modern antenna design process, the radiation patterns are obtained using a simulator, or by characterizing the antenna elements by measurements. Thus, computing the optimal feeding coefficients does not typically require any extra effort compared to the progressive phasing. The optimal excitations that maximize the gain to a desired direction with an appropriate polarization can be computed numerically by solving an eigenvalue problem derived from the Rayleigh quotient [6]–[9].

However, solving the eigenvalue problem might be computationally heavy in large arrays. Also, these obtained feeding coefficients may have variation in amplitude which can be cumbersome to implement in practical arrays.

In this paper, we first show that the optimal feeding coefficients can be obtained with a direct method. We prove that the resulting feeding coefficients are equal to the ones obtained from the eigenvalue problem. In addition, we show that the phasing of the optimal feeding coefficients is also optimal in the case where the amplitudes of the coefficients are fixed to be same and constant. Formally, that is an optimization problem with constant modulus constraint (CMC). The derived results are in line with the results of [10]–[13].

We demonstrate the effect of differently chosen feeding coefficients in two cases. First, we study a four-element connected slot antenna array (CSAA) where the embedded element patterns are unequal and the mutual coupling is strong. The second example considers 4x4-element circularly polarized patch antenna array (CPPA). The example shows that if a patch element has separate driving points for both polarizations, the circularly polarized realized gain can be improved by feeding the driving points separately with optimal coefficients, instead of keeping constant 90 degree phase shift between the points. In both examples, we compare progressive phasing to optimal feeding with and without CMC.

II. Determination of optimal feeding coefficients

In this section, we derive formulas for the antenna array feeding coefficients maximizing the realized gain. Of course, there might also be other goals for the array patterns, such as side-lobe level, location of zeros, and main beam width. Those quantities could be optimized for example with Dolph-Chebyshev's, Schelkunoff's or Orchard's synthesis methods [1], [2], [14] but they are left outside the scope of this paper.

We assume that the generator impedances are given. Let a_p be an incident voltage wave, called as feeding coefficient, of an antenna element p and $p = 1...P$. When the feeding coefficient is unity, $\vec{E}_p(\theta,\varphi)$ is the embedded element pattern at direction (θ,φ) radiated by the element p, as illustrated in Fig. 1.

Let us define a complex unit vector $\vec{u}_\alpha = \alpha_\theta \vec{u}_\theta + \alpha_\varphi \vec{u}_\varphi$ $(\alpha_\theta, \alpha_\varphi \in \mathbb{C})$ that describes a polarization of an electric field. An α-polarized component of the far field is $\vec{E}_p(\theta,\varphi)\cdot\vec{u}_\alpha$. For the polarization of the field radiated by an element p, we define $\vec{u}_\alpha = \vec{E}_p(\theta,\varphi)/\|\vec{E}_p(\theta,\varphi)\|$, where $\|\cdot\|$ denotes vector length.

The realized gain of an antenna array to the direction (θ,φ) with desired polarization $\vec{u}_\alpha$ can be expressed as

$$G(\theta,\varphi,\vec{u}_\alpha) = \frac{4\pi}{\eta}\frac{\left|\sum_{p=1}^{P} a_p(\theta,\varphi,\vec{u}_\alpha)\vec{E}_p(\theta,\varphi)\cdot\vec{u}_\alpha\right|^2}{\sum_{p=1}^{P}|a_p(\theta,\varphi,\vec{u}_\alpha)|^2}, \quad (1)$$

where η is the wave impedance. The goal is to find the feeding coefficients $a_p(\theta,\varphi,\vec{u}_\alpha)$ that maximize the realized gain to a desired direction with an appropriate polarization.

A. Solution derived from Cauchy-Schwarz inequality

We analyze the situation at a point frequency, to a single desired direction (θ,φ), and with one given polarization $\vec{u}_\alpha$. Let us collect the feeding coefficients $a_p(\theta,\varphi,\vec{u}_\alpha)$ and the polarized embedded element patterns $\vec{E}_p(\theta,\varphi)\cdot\vec{u}_\alpha$ to the column vectors $\boldsymbol{a} \in \mathbb{C}^P$ and $\boldsymbol{E} \in \mathbb{C}^P$, respectively. Then (1) can be formulated as

$$G = \frac{4\pi}{\eta}\frac{|\boldsymbol{E}^T\boldsymbol{a}|^2}{\|\boldsymbol{a}\|^2} = \frac{4\pi}{\eta}\frac{|\langle\boldsymbol{E}^*,\boldsymbol{a}\rangle|^2}{\|\boldsymbol{a}\|^2}, \quad (2)$$

where $(\cdot)^T$ is transpose, $(\cdot)^*$ denotes complex conjugation, and the inner product is defined as $\langle\boldsymbol{f},\boldsymbol{g}\rangle = \boldsymbol{f}^H\boldsymbol{g}$, where $(\cdot)^H$ is conjugate transpose.

According to the Cauchy-Schwarz inequality [15], the upper bound for the numerator of (2) is

$$|\langle\boldsymbol{E}^*,\boldsymbol{a}\rangle|^2 \leqslant \|\boldsymbol{E}\|^2\|\boldsymbol{a}\|^2. \quad (3)$$

The upper bound is reached if and only if $\boldsymbol{a} = c\boldsymbol{E}^*$ where $c \in \mathbb{C}$ is an arbitrary scaling factor. Thus, the maximum achievable realized gain is

$$G^{\max}(\theta,\varphi,\vec{u}_\alpha) = \frac{4\pi}{\eta}\|\boldsymbol{E}\|^2 = \frac{4\pi}{\eta}\sum_{p=1}^{P}|\vec{E}_p(\theta,\varphi)\cdot\vec{u}_\alpha|^2 \quad (4)$$

and it can be obtained by feeding the array with coefficients

$$a_p(\theta,\varphi,\vec{u}_\alpha) = c\vec{E}_p^*(\theta,\varphi)\cdot\vec{u}_\alpha^*. \quad (5)$$

B. Solution derived from Rayleigh quotient

The realized gain can also be maximized by utilizing Rayleigh quotient as is done e.g. in [6]–[9]. Then, the realized gain is formulated as

$$G = \frac{4\pi}{\eta}\frac{(\boldsymbol{E}^T\boldsymbol{a})^*\boldsymbol{E}^T\boldsymbol{a}}{\|\boldsymbol{a}\|^2} = \frac{4\pi}{\eta}\frac{\boldsymbol{a}^H\boldsymbol{E}^*\boldsymbol{E}^T\boldsymbol{a}}{\boldsymbol{a}^H\boldsymbol{a}} \quad (6)$$

which is equal to (2) and has the form of Rayleigh quotient. The gain is maximized when $\boldsymbol{a}$ is the eigenvector of matrix $\boldsymbol{E}^*\boldsymbol{E}^T$ that corresponds to the largest eigenvalue of $\boldsymbol{E}^*\boldsymbol{E}^T$, namely $\lambda^{\max}$. The maximal realized gain is obtained directly from the largest eigenvalue: $G^{\max}(\theta,\varphi,\vec{u}_\alpha) = (4\pi/\eta)\lambda^{\max}$.

C. Proof of equality of the solutions

We prove that the feeding coefficients obtained from Rayleigh quotient are equal to the ones derived using Cauchy-Schwarz inequality. In practice, we show that a vector $\boldsymbol{a} = c\boldsymbol{E}^*$ is the eigenvector of $\boldsymbol{E}^*\boldsymbol{E}^T$ that corresponds to the largest eigenvalue $\lambda^{\max}$.

First, let us show that $\boldsymbol{a} = c\boldsymbol{E}^*$ is an eigenvector of $\boldsymbol{E}^*\boldsymbol{E}^T$ by applying it to the eigenvalue equation:

$$\begin{aligned} \boldsymbol{E}^*\boldsymbol{E}^T\boldsymbol{a} &= \lambda\boldsymbol{a} \\ \Leftrightarrow \|\boldsymbol{E}\|^2\boldsymbol{E}^* &= \lambda\boldsymbol{E}^*. \end{aligned} \quad (7)$$

Obviously, $\boldsymbol{a} = c\boldsymbol{E}^*$ is the eigenvector of $\boldsymbol{E}^*\boldsymbol{E}^T$ that corresponds to the eigenvalue $\lambda = \|\boldsymbol{E}\|^2$.

Next, we show that $\lambda = \|\boldsymbol{E}\|^2$ is the maximum eigenvalue of $\boldsymbol{E}^*\boldsymbol{E}^T$. The rank of $\boldsymbol{E}^*\boldsymbol{E}^T$ is one since all it's columns (or rows) are mutually linearly dependent. A column p of $\boldsymbol{E}^*\boldsymbol{E}^T$ can be expressed as

$$[\boldsymbol{E}^*\boldsymbol{E}^T]_p = E_p\boldsymbol{E}^*, \quad (8)$$

where E_p is the pth element of vector $\boldsymbol{E}$. Thus, there exists only one linearly independent column (or row) in the matrix $\boldsymbol{E}^*\boldsymbol{E}^T$.

The rank-nullity theorem [15] states that

$$\begin{aligned} \mathrm{rank}(\boldsymbol{E}^*\boldsymbol{E}^T) + \dim(\mathrm{Ker}(\boldsymbol{E}^*\boldsymbol{E}^T)) &= \dim(\boldsymbol{E}^*\boldsymbol{E}^T) \\ &= P, \end{aligned} \quad (9)$$

which indicates that dimension of the kernel of $\boldsymbol{E}^*\boldsymbol{E}^T$ is $\dim(\mathrm{Ker}(\boldsymbol{E}^*\boldsymbol{E}^T)) = P-1$. That is, there exist $P-1$ linearly independent vectors $\boldsymbol{v}$ that satisfy $\boldsymbol{E}^*\boldsymbol{E}^T\boldsymbol{v} = \boldsymbol{0}$. This implies that $\boldsymbol{E}^*\boldsymbol{E}^T$ has $P-1$ eigenvectors that correspond to the eigenvalue $\lambda = 0$. Since the total number of eigenvalues of $\boldsymbol{E}^*\boldsymbol{E}^T$ is P, there can be only one non-zero eigenvalue. It must be then $\lambda = \|\boldsymbol{E}\|^2$.

Thus, the Cauchy-Schwarz inequality and Rayleigh quotient based methods give equal feeding coefficients. They are given in (5) and the obtained realized gain is (4) in both cases.

D. Optimal phases of constant modulus feeding coefficients

We prove that if the amplitudes of the feeding coefficients are set to be one, the phases of the coefficients given in (5) are the optimal ones. Denote the polarized field of element p to a certain direction as $E_p = \hat{E}_p e^{j\gamma_p}$, where a real number $\hat{E}_p \geqslant 0$

is the magnitude of the field, j imaginary unit, and γ_p describes the phase of the field. Note that parameters describing the direction and polarization have been omitted due to simplicity.

If the amplitudes of the feeding coefficients are unity, the coefficients are of form $a_p = e^{j\beta_p}$. Then the realized gain of the array is

$$G = \frac{4\pi}{P\eta}\left|\sum_{p=1}^{P} a_p E_p\right|^2 = \frac{4\pi}{P\eta}\left|\sum_{p=1}^{P} \hat{E}_p e^{j(\beta_p+\gamma_p)}\right|^2. \quad (10)$$

Using the triangle inequality [15], we can find upper limit for the gain:

$$G \leqslant \frac{4\pi}{P\eta}\left(\sum_{p=1}^{P} |a_p E_p|\right)^2 =: G^{\max}. \quad (11)$$

Because $\hat{E}_p \geqslant 0$, then $|a_p E_p| = |\hat{E}_p| = \hat{E}_p$. That is,

$$G^{\max} = \frac{4\pi}{P\eta}\left(\sum_{p=1}^{P} \hat{E}_p\right)^2. \quad (12)$$

By applying $\beta_p = -\gamma_p$ in (10), we obtain

$$G = \frac{4\pi}{P\eta}\left|\sum_{p=1}^{P} \hat{E}_p\right|^2 = \frac{4\pi}{P\eta}\left(\sum_{p=1}^{P} \hat{E}_p\right)^2 = G^{\max}. \quad (13)$$

Thus, the maximum achievable realized gain is obtained when the phases of the feeding coefficients are equal to the phases of the complex conjugates of the element pattern fields in the desired direction and at the desired polarization. Formally, the realized gain is maximized with constant modulus feeding coefficients when

$$a_p(\theta,\varphi,\vec{u}_\alpha) = e^{-j\gamma_p} = \frac{\vec{E}_p^*(\theta,\varphi)\cdot\vec{u}_\alpha^*}{|\vec{E}_p(\theta,\varphi)\cdot\vec{u}_\alpha|}, \quad (14)$$

where $\gamma_p \in \mathbb{R}$ is the phase of the field. Analogous result was derived in [13].

III. Comparison to Progressive Phasing

In this section, we compare three different methods for calculating the feeding coefficients:

1) Progressive phasing
2) Optimal excitations calculated from element patterns as described in (5)
3) Optimal excitations with constant modulus constraint (CMC) as given in (14)

As discussed in [2], the coefficients for progressive phasing can be calculated as

$$a_p(\theta,\varphi) = e^{-jk\sin(\theta)(x_p\cos(\varphi)+y_p\sin(\varphi))}, \quad (15)$$

where k is the wavenumber and (x_p, y_p) are the coordinates of the element, similarly as in Fig. 1.

We demonstrate the effect of the feeding method using 4-element connected slot antenna array (CSAA) and 4x4-element circularly polarized patch antenna array (CPPA). The test antennas are simulated in CST Microwave Studio and the feeding coefficients and results are calculated in Matlab using far-field data exported from CST. The input signals are modeled in CST using 50-Ω discrete S-parameter ports.

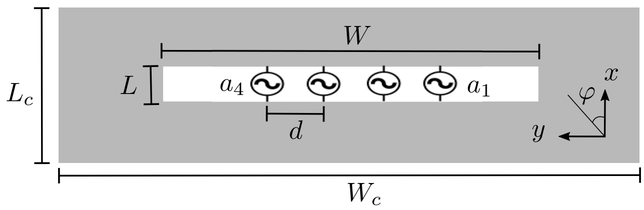

Fig. 2: The connected slot antenna array

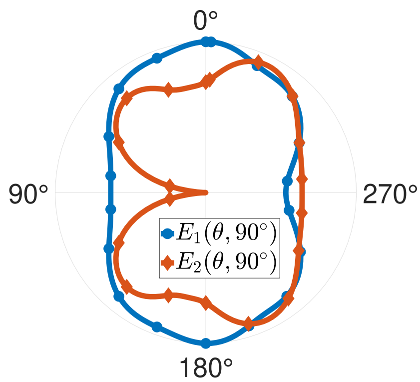

Fig. 3: CSAA's embedded element patterns on yz plane

A. *Connected slot antenna array*

The 4-element CSAA used for testing the feeding methods is similar as used in [16]. The structure is presented in Fig. 2 and the dimensions are L_c = 50 mm, W_c = 200 mm, L = 0.864 mm, W = 159 mm and d = 26.7 mm. The material is thin plane of perfect electric conductor. The excitations are placed symmetrically at the center of the slot. The antenna operates at 3 GHz.

The CSAA is chosen for testing since the coupling between ports is relatively high and the embedded element patterns are not identical. These issues are faced in practical arrays since the edge elements have different environment than the center ones, and avoiding mutual coupling is generally difficult [17]. The S-parameters that describe coupling are S_{21} = -7.4 dB and S_{32} = -3.9 dB. Fig. 3 illustrates the embedded element patterns of elements 1 and 2.

Fig. 4 shows the total scan gain to the direction $(\theta, 90°)$. The curves compare the gain when the feeding coefficients are calculated using progressive phasing (15) to a case with optimal excitations calculated using (5) and (14). Polarization is chosen to be the polarization of field radiated by excitation 2, i.e. $\vec{u}_\alpha = \vec{E}_2(\theta,\varphi)/\|\vec{E}_2(\theta,\varphi)\|$ in (5) and (14).

Even 1.0 dB improvement in realized gain has been achieved when optimal feeding coefficients are used. This happens because the antenna excitations cannot be seen as individual elements, like in traditional arrays. In this structure, each excitation uses the whole slot for generating radiation although the excitations are placed regularly like in linear arrays.

Fig. 5 illustrates the amplitudes of the optimal feeding coefficients. They differ significantly from unity. For instance, when steering the beam to the broadside direction, the

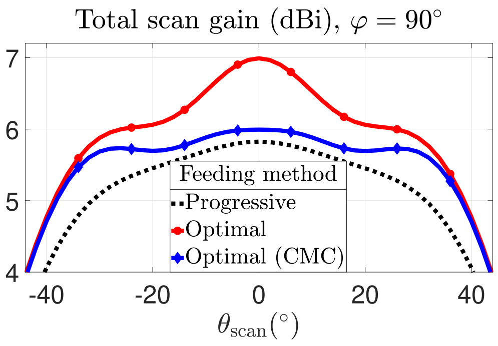

Fig. 4: Total scan gain of CSAA when the beam is steered on yz plane.

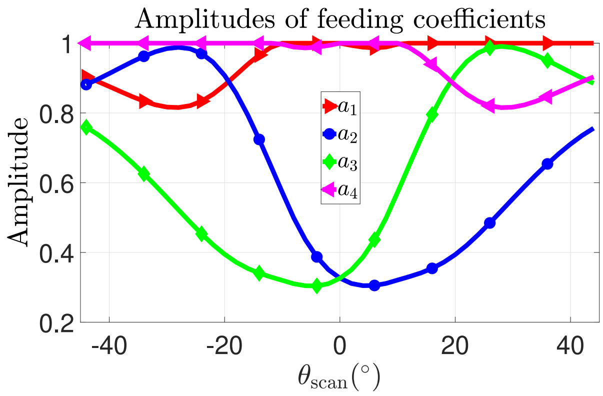

Fig. 5: Amplitudes of the optimal feeding weights of the CSAA when the beam is steered.

amplitudes are 1 for elements 1 and 4, and 0.33 for elements 2 and 3.

Fig. 6 shows the phase differences of the feeding coefficients when element 1 is fed in phase zero. The figure compares progressive phasing to optimal phasing. The phase difference is not linear in optimal case. Also, the curves are generally steeper in optimal case than in progressive phasing. It can be inferred that the phase centers of the elements are further from each other than the excitation locations could indicate.

B. Planar circularly polarized patch antenna array

The optimal feeding coefficients can be calculated for arbitrary polarization. With dual polarized planar 4x4-element patch antenna array we demonstrate how the polarization efficiency can be improved using optimal excitations.

Fig. 7b illustrates a single element of the array. The dimensions are in millimeters: $W_s = 50$, $W_p = 46.42$, and $s = 13.07$. The substrate (green) is 1.5-mm thick FR-4 with $\epsilon_r = 4.3$ and $\tan\delta = 0.025$. The black dots are the driving points of the patch. The 16 array elements are connected directly side-by-side to each other. The operation frequency is 1.5 GHz, which gives the inter-element distance of 0.3λ.

A traditional method for implementing circular polarization in a such patch element is to feed the two driving points with 90 degree phase shift. Then one patch is considered as a single

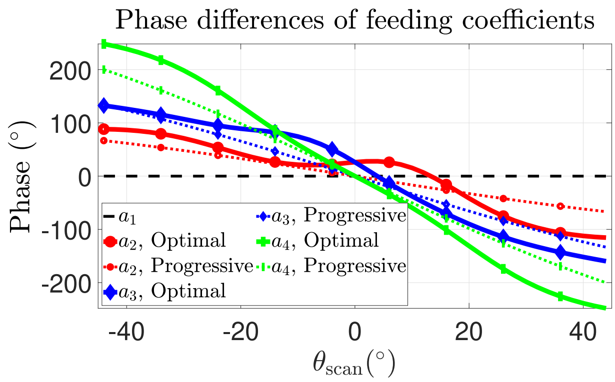

Fig. 6: Phase differences of the excitations compared to the phase of a_1 in CSAA.

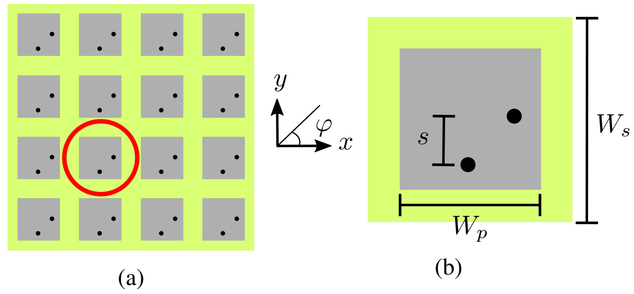

Fig. 7: a) Illustration of the circularly polarized patch antenna array. b) Closer look of a CPPA element.

element with one excitation since the 90 degree phase shift is fixed. However, by calculating the feeding weights for each excitation separately, we can improve the circularly polarized gain. A simple method for calculating feeding weights for all 32 excitations is to use formula (5) or (14). The polarization vector is then

$$\vec{u}_\alpha = \frac{1}{\sqrt{2}}(\vec{u}_\theta \pm j\vec{u}_\varphi). \tag{16}$$

The sign $\pm$ gives the handedness of the polarization.

Fig. 8 shows how much the optimal phasing inside an element differs from the conventional 90 degree phase shift. In the following example, the beam is steered on diagonal plane, where $\varphi = 45°$. The element under this analysis is circled in Fig. 7a.

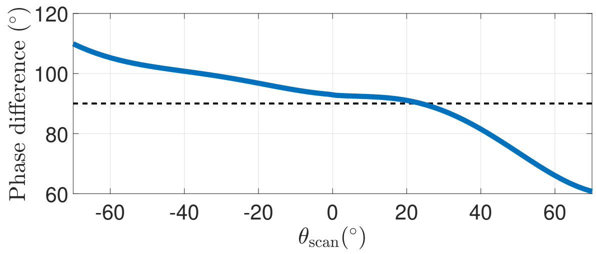

Fig. 8: Phase difference between two excitations in the CPPA patch element when the beam is steered diagonally.

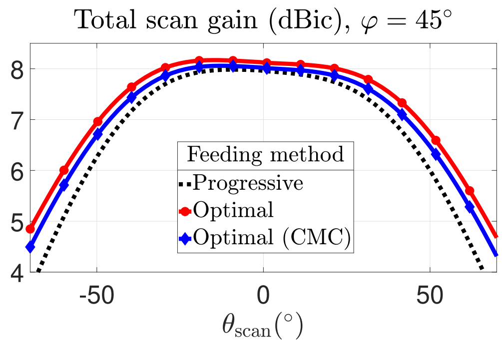

Fig. 9: Total scan gain of CPPA when the beam is steered on diagonal plane.

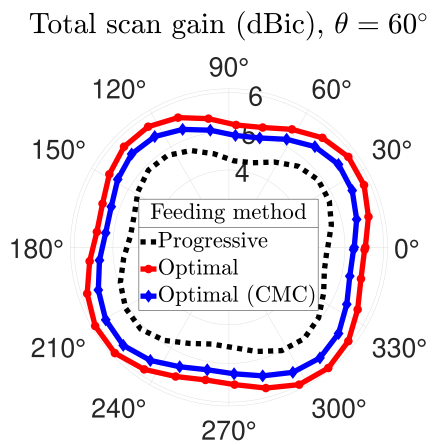

Fig. 10: Total scan gain of CPPA when the beam is steered around z axis.

The improvement in realized gain can be seen in Fig. 9 and Fig. 10. Both figures present the circularly polarized gain. In Fig. 9 the beam is steered diagonally such as in Fig. 8. We see that the benefit of using optimal excitations is more significant with large θ angles. Fig. 10 illustrates the scan gain when the beam is steered around the z axis with constant $\theta = 60°$.

IV. Conclusion

We have studied and compared three different methods for feeding an antenna array: traditional progressive phasing, optimal feedings with amplitude variation, and optimal feedings with constant amplitudes. We have shown that the optimal feeding coefficients can be calculated directly from the far-field data without solving eigenvalue problem, which increases the computational efficiency especially in case of large antenna arrays.

The examples showed that the amplitude adjustment of the feeding coefficients can improve the gain when the embedded element patterns are not identical or mutual coupling is strong. However, amplitude manipulation typically decreases the efficiency of the driving amplifier, and therefore it is not beneficial in all situations. Nevertheless, even with fixed amplitudes, the optimal phases can produce a significant improvement to realized gain compared to progressive phases.

Acknowledgment

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