Integral Operator-Based Characteristic Mode Theory for Conducting, Material, and Lossy Structures

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Abstract—Integral operator-based theory of characteristic modes (CMs) for conducting, material, and lossy structures is reviewed. CMs are defined as solutions of a generalized eigenvalue equation (GEE). These GEEs are presented for various surface and volume integral operators and material structures. Interpretation of the characteristic eigenvalues in terms of electromagnetic power is studied based on the Mie expansion and integral operator formalism. Orthogonality and diagonalizing properties of the CMs are summarized. Challenges related to dielectric-magnetic bodies, lossy structures and spurious modes are discussed, as well as differences between the surface and volume operator approaches.

I. INTRODUCTION AND BACKGROUND

CHARACTERISTIC modes (CMs) are a specific set of eigensolutions (eigenvalues, -vectors, and -fields) that enable characterization of the fundamental electromagnetic properties of arbitrarily shaped structures relying solely on their geometrical and material properties. Even though CMs have originally been developed for scattering problems [1], [2], they have been proven to be extremely useful in antenna design [3], [4]. Diverse applications of the CMs are introduced in two recent special issues [5], [6], and in [7]–[10].

Theoretical foundations of the present-day (integral-operator-based) theory of CMs (TCM) relies heavily on the pioneering work of Harrington and his co-workers in the 70’s, particularly on the seminal papers [11], [12]. Originally, the CMs were introduced as modal solutions for conductors with specific symmetry. Garbacz generalized that theory for arbitrarily shaped conducting objects and defined CMs as modal functions forming an orthogonal set on the surface of a sphere in the far-field region [1], [2]. These functions can be applied to expand the field scattered or radiated from a conducting object under any excitation or incident field [13].

Harrington and Mautz approached the same problem as Garbacz with an integral (impedance) operator relating the current on the surface of a conducting body to the tangential electric field [11]. They formulated a generalized eigenvalue equation (GEE), and defined CMs as solutions of that GEE giving a weighted orthogonal set of currents, weighted with respect to the radiated power. The main difference between these two approaches is that, in the former the CMs are radiated eigenfields, and in the latter they are eigencurrents. Nevertheless, these modal solutions are shown to have many common features [11], as elaborated recently in [14].

In [12], a volume integral operator (VIO) based TCM is presented for material (dielectric, magnetic and dielectric-magnetic) bodies. For lossless dielectric (magnetic) bodies, the obtained modes were argued to share many properties of the perfect electric conductors (PEC) of [11]. For dielectric-magnetic bodies, after symmetrizing the VIOs, a GEE was formulated analogously to the dielectric ones. In that case challenges related to the interpretation of the imaginary part of the complex power balance were pointed out [12]. For lossy structures, both material and metal, two alternative CM formulations with different weight operators and solutions have been proposed [12], [15].

In [16], following the success of [11] and [12], Chang and Harrington (CH) formulated CM equations for dielectric bodies using the surface integral operators (SIOs). This approach, where the GEE is expressed in terms of the Hermitian components of the symmetrized PMCHWT (Poggio-Miller-Chang-Harrington-Wu-Tsai) [17] operator, has been found to lead to modal solutions without clear underlying physics [18] that do not exist in the VIO-based one [19]. The origin of these extra solutions, known as "spurious modes", and their removal, has stimulated numerous studies [20]–[26]. In [27] it is shown that the CH formulation produces two sets of modes, the wanted ones and the modes of a reverse problem.

In this paper formulation of the GEE using SIOs and VIOs is presented for conducting and material structures, both lossless and lossy. By selecting the weight operator of the GEE so that the eigensolutions are linked to the radiated power, spurious modes are avoided and, in the lossless case the modes have orthogonal far fields. In the lossy case, however, the modes do not form a radiated-power orthogonal set of functions. It is demonstrated that the SIO- and VIO-based approaches are conceptually different and can not always be treated similarly. New insights into the spurious modes of the CH surface formulation [16] for lossy bodies are presented, too.

Based on the Mie expansion, a novel interpretation of the eigenvalues in terms of electromagnetic power is given. This interpretation, which is independent on the material of the sphere, is shown to be valid also for non-spherical objects. For arbitrarily shaped objects alternative interpretations are obtained with the integral operator formalism. The far-field orthogonality of the CMs is shown to depend solely on the physical problem, while diagonalizing the field-integral operator depends on the integral operators used to express the GEE. A novel VIO formulation based on the Poynting theorem is presented for dielectric-magnetic bodies. The solutions of
this formulation disagree with the conventional CMs.

II. DEFINITION OF CHARACTERISTIC MODES

Let us start by considering a time-harmonic electromagnetic scattering problem with time factor \( \exp(-i\omega t) \). Let \( F^{\text{inc}} = (E^{\text{inc}}, H^{\text{inc}}) \) denote an incident field, \( F^{\text{sca}} = (E^{\text{sca}}, H^{\text{sca}}) \) a scattered field, and \( D \) a homogeneous and bounded object with surface \( S \) with constant, possible complex, parameters \( \varepsilon_1, \mu_1 \). The background is assumed to be homogeneous with real parameters \( \varepsilon_0, \mu_0 \). By \( D_0 \) we denote the region outside \( D \) and \( D_1 \) is the interior of \( D \).

A. Transmission Operator Formulation

Garbacz [1], [2] approached the problem with generalized modal expansions of the incident, scattered, and interior fields

\[
F^{\text{inc/scat}} = \sum_n \alpha_n^{\text{inc/scat}} F^{\text{inc/out/int}}_n,
\]

where \( F^{\text{inc/out/int}}_n \) are incoming (incident), outgoing (scattered), and interior (transmitted) modal functions, respectively. On the surface \( S \) these functions are tied together via boundary and interface conditions. This gives a transmission operator \( T \) [14]

\[
T[F^{\text{inc}}] = F^{\text{sca}},
\]

and CMs are solutions of an ordinary eigenvalue equation [14]

\[
T[F_n] = \nu_n F_n,
\]

Thus, in this formulation, CMs \( F_n, n = 1, 2, \ldots \) are radiated eigenfields defined in the far field region and \( \nu_n \) is the corresponding eigenvalue. An incident field mode \( F^{\text{inc}}_n = (E^{\text{inc}}_n, H^{\text{inc}}_n) \) and a scattered field mode \( F^{\text{sca}}_n = (E^{\text{sca}}_n, H^{\text{sca}}_n) \), are related to each other as [11], [28]

\[
E^{\text{inc}}_n = \frac{1}{2\nu_n} (E^{\text{sca}}_n + (E^{\text{sca}}_n)^*),
\]

\[
H^{\text{inc}}_n = \frac{1}{2i\nu_n} (H^{\text{sca}}_n - (H^{\text{sca}}_n)^*),
\]

where * denotes complex conjugate.

B. Integral Operator Formulation

Harrington et al. [11], [12] formulated the scattering problem as an integral equation

\[
\mathcal{L}[F] = F^{\text{inc}},
\]

where \( \mathcal{L} \) is a linear integral operator and \( F \) is an unknown function (a current). For a given \( \mathcal{L} \), they defined CMs as solutions of a GEE [11], [12]

\[
\mathcal{L}[F_n] = (1 - i\lambda_n)\mathcal{W}[F_n],
\]

where \( \mathcal{W} \) is a weight operator, \( \lambda_n \) is an eigenvalue with \( \nu_n = -1/(1 - i\lambda_n) \) [11], [14], and \( F_n \) is the corresponding eigenvector (eigencurrent). The choice of the weight operator is crucial since it defines the properties of the solutions of (7). Both in [11] for PEC structures, and in [12] for lossless material bodies, the weight operator was chosen so that it relates the eigensolutions to the radiated power. In this paper this choice is called the radiated power formulation.

1The minus sign in (7) is due to the time convention \( \exp(-i\omega t) \).

C. Modes for Lossless Structures

Let us continue by repeating how Harrington and Mautz summarized the definition and properties of the CMs in [29]:

*These modes are basically solutions to a weighted eigenvalue equation involving the impedance operator \( \mathcal{Z} \), which relates the surface current on a conductor to the tangential component of the incident electric field on the conductor. The modal currents are real (or equiphase), orthogonal over the conducting surface with weight operator \( \mathcal{R}(\mathcal{Z}) \), and the modal radiation fields are Hermitian orthogonal over the radiation sphere. When used in a modal solution, they give a radiation field which converges in a least-squares sense on the radiation sphere.*

Obviously this definition is valid for lossless PEC structures only since it mentions currents on conducting surfaces. It also assumes that the integral operators used to express the GEE have special (symmetry) properties.

In [12] using the VIOs CMs were defined for material bodies. For lossless dielectric bodies the formulation of [12] gives real eigenvalues and equiphase eigenvectors with similar orthogonality and convergence properties as in [11]. Also for dielectric-magnetic bodies, the modes of [12] were argued to have similar properties as in the dielectric bodies, while interpretation of the imaginary part of the complex power balance was not clear [12], [23].

D. Modes for Lossy Structures

For lossy structures the situation is more complicated. Garbacz did not consider losses at all and Harrington et al. introduced two alternative formulations [12], [15]. In the first one, the weight operator is chosen as the radiation operator, leading to complex eigensolutions, and argued to give orthogonal far fields. This formulation agrees with the radiated power formulation mentioned above. In the second option the real part of the impedance operator was used as a weight operator. This formulation, here called accepted (radiated plus dissipated) power CM formulation, leads to real (equiphase) eigensolutions, diagonalize the field-integral operator, but does not provide orthogonal far fields.

For lossy objects, the eigenvalues are either complex or real, depending whether the radiated or accepted power formulation is used [12], [15], [25], [30–32]. The drawback of the accepted power formulation is that the dissipated power can not be distinguished from the radiated one, while with the radiated power formulation this is possible [25], [26], [30].

E. Generalized Modes of Radiated Power Formulation

To generalize classical CM formulations of [11] and [12], the weight operator \( \mathcal{W} \) is defined so that, for any integral operator \( \mathcal{L} \), it links the eigensolutions to the radiated power. For certain integral operators \( \mathcal{L} \), operator \( \mathcal{W} \) agrees with the radiated power operator \( \mathcal{R}^\text{rad} \), i.e., \( \langle F^*, \mathcal{R}^\text{rad}[F] \rangle \) gives the radiated power of \( F \). CMs (eigenvalues \( \lambda_n \), eigenvectors \( F_n \) and the corresponding eigenfields \( (E_n, H_n) \)) are solutions of

\[
i (\mathcal{L} - \mathcal{W}) [F_n] = \lambda_n \mathcal{W}[F_n].
\]

The weight operator \( \mathcal{W} \) depends on the operator \( \mathcal{L} \) used to express the field problem in (6), as shown in the next section.
III. FORMULATION OF THE GEE

The original CM formulations for PEC [11] and lossless dielectric [12] structures apply impedance-type operators, providing mappings from the electric current to the electric field. Other common features in these formulations is that the weight operator is chosen as the radiation operator and the GEE can be expressed with real-symmetric operators. Consequently, the eigenvalues are real and the eigenvectors are equiphase. Next we discuss formulation of the GEE for various integral equations and material structures based on the radiated power formulation.

A. Conducting Structures

Consider first a PEC object with a closed surface S. For the electric field integral equation (EFIE) we have \( L = \eta_0 \gamma_t T_0 \) and \( W = \eta_0 \text{Re}(\gamma_t T_0) \) (see Appendix A) giving\(^2\) [11]

\[
-\mathcal{X}^E[J_n] = \lambda_n \mathcal{R}^E[J_n].
\]  

(9)

Here \( \mathcal{R}^E \) and \( \mathcal{X}^E \) are the real and imaginary parts of the PEC-EFIO \( \eta_0 \gamma_t T_0 \), and \( J_n \) is the electric surface current. For the magnetic field integral equation (MFIE) the corresponding operators are \( L = \gamma_s K^{-} \) and \( W = i\text{Im}(\gamma_s K) \), with \( \gamma_s F = n \times F \)\(_S\). In this case the GEE is of the form [33]

\[
\mathcal{R}^M[J_n] = \lambda_n \mathcal{X}^M[J_n],
\]  

(10)

with the real \( \mathcal{R}^M \) and imaginary \( \mathcal{X}^M \) parts of the PEC-MFIO. In (10) the weight operator does not agree with the radiated power operator \( \mathcal{R}^E \), while in (9) this is the case. Despite of that, (10) gives the same eigenvalues and eigenvectors as (9), excluding internal resonance frequencies, [25], [28].

In the case of PEC objects, GEEs have also been developed for other equations, e.g., the combined field integral equation (CFIE), and the potential [34] and source [35] ones. For the CFIE, operators \( L \) and \( W \) are obtained as linear combinations of the corresponding operators of the EFIE and MFIE [28]. The potential and source formulations are based on different equations, but the same principle of defining \( W \) as the radiated power-related operator applies. In [36] the low frequency stabilization of the PEC-EFIO CM formulation is discussed.

Impedance boundary condition (IBC), \( \gamma_t E = Z_s \eta_0 \gamma_t H \), where \( Z_s \) is a normalized surface impedance, provides an approximation for non-perfectly conducting surfaces. For open IBC surfaces the integral equation formulation is based on the volume equivalence principle (VEP) [37]. Since the object is open, infinitesimally thin, the VIO reduces to a SIO with a current tangential to the surface. The GEE is expressed with operators \( L = \eta_0 (\gamma_t T_0 - Z_s I) \) [37], and \( W = \eta_0 \text{Re}(\gamma_t T_0) \) [15].

For closed impedance surfaces the situation is different since the formulation is based on the SIO-approach [37], [38]. Alternative formulations based on the IBC-EFIE are presented in [31], [39], and CM formulations for the IBC-MFIE and IBC-CFIE are given in [39]. These formulations can be seen as generalizations of the corresponding EFIE, MFIE and CFIE formulations for PEC structures. In [40] an alternative SIO formulation for the IBC surfaces, including both the electric and magnetic currents, is introduced. The GEE for this so-called self-dual formulation is presented in [30].

B. Material Bodies – Volume Formulation

Consider next the VIO formulation for penetrable material bodies. For dielectric bodies the volume EFIO is \( L = \eta_0 V + I_{\varphi} \), the weight operator is given by \( W = \eta_0 \text{Re}(V) \), and the equation involves only the electric volume current \( J^V \). For lossless bodies the GEE is formally identical with (9), and in the lossy case it reads [12]

\[
(-\text{Im}(\eta_0 V) + i \varphi) J^V_n = \lambda_n \text{Re}(\eta_0 V)[J^V_n].
\]  

(11)

For magnetic bodies \( L = 1/\eta_0 V + I_{\varphi} \) and \( W = 1/\eta_0 \text{Re}(V) \).

For dielectric-magnetic bodies the situation is more complicated since the equation includes both the electric and magnetic currents and the VIO is non-symmetric. In [12], VIO (95) is symmetrized giving \( (s \) stands for a symmetric form)

\[
L^V = \begin{bmatrix}
\eta_0 V + I_{\varphi} & iU & iU \\
U & 1/\eta_0 V + I_{\varphi} & 1/\eta_0 V + I_{\varphi}
\end{bmatrix}.
\]  

(12)

In this case, the weight operator is given by

\[
W^V_n = \begin{bmatrix}
\text{Re}(\eta_0 V) & -\text{Im}(U) \\
-\text{Im}(U) & \text{Re}(1/\eta_0 V)
\end{bmatrix}.
\]  

(13)

Symmetrization, however, is not necessary and the same solutions are obtained with the asymmetric VIO, given in (95), and \( W = R^V \) (96) [41]. In [42] an alternative power-based VIO formulation is proposed for dielectric-magnetic bodies.

C. Material Bodies – Surface Formulations

As a SIO approach for material (dielectric, magnetic, dielectric-magnetic) bodies we consider the PMCHWT formulation [17]. The symmetrized PMCHWT operator is [16]

\[
L^\mu_p = \begin{bmatrix}
\eta_0 \gamma_t T_0 + \eta_1 \gamma_1 T_1 & i\gamma_t K_0 - i\gamma_1 K_1 \\
i\gamma_t K_0 + i\gamma_1 K_1 & 1/\eta_0 \gamma_t T_0 + 1/\eta_1 \gamma_1 T_1
\end{bmatrix}.
\]  

(14)

In [16] the weight operator is chosen as \( W = \text{Re}(L^\mu_p) \) (CH formulation). Obviously, this choice does not agree with the radiated power operator [25], [27]. Hence, the CH formulation does not give the same solutions as the volume one [18], [19].

In [19] it was observed that the CH formulation, in addition to giving the correct modes, gives also a set of extra non-physical spurious modes. These extra modes of the CH formulation are argued to be due to the lack of dependent relationships of the electric and magnetic currents [18], [23], [43]. By eliminating one of the currents utilizing non-radiation operators, either by applying them as projection operators [23], or via the generalized IBC [24], gives solutions that are immune from spurious modes [44]. This, however, does not generally solve the problem, since there are formulations based on this approach [3], [18], [21], [22] that still give spurious solutions [24], [44]. For example, in [3], [18], [21] one of the current is eliminated from the PMCHWT equations and the GEE is written in terms of the Hermitian components of the modified operator. These formulations still give non-physical modes, although different than the CH formulation.

\(^2\)The minus sign on the left-hand side is due to time factor \( \exp(-i\omega t) \)
As another approach, post-processing techniques have been proposed to avoid spurious modes [19], [20], [43]. These methods are based either on extracting the non-radiating modes [19], [20] or studying the correlation of the modal currents [43]. In [20] spurious solutions of the CH formulation are linked to the internal resonances of the symmetrized formulation. This reasoning is questioned [27], [43], since the symmetric PMCHWT formulation satisfies the necessary uniqueness criterion [45]. In addition, the EFIE approach [11] suffers from internal resonances [46]. Despite this, it produces correct modes, excluding internal resonance frequencies [28].

A general procedure to avoid spurious solutions is to follow the approach of [11], [12], and define the weight operator so that the solutions are linked to the radiated power [25], [27]. For the original PMCHWT formulation with operator

\[ L_P = \begin{bmatrix} \eta_0 \gamma T_0 + \eta_1 \gamma T_1 & -\gamma_1 K_0 - \gamma_1 K_1 \\ \gamma_1 K_0 + \gamma_1 K_1 & 1/\eta_0 \gamma T_0 + 1/\eta_1 \gamma T_1 \end{bmatrix} \tag{15} \]

this yields \( \mathcal{W} = \mathcal{R}_{rad}^S \) (see Appendix A).

In [44] a comprehensive cross-validation of different SIO-based CM formulations for dielectric bodies is presented. GEEs for alternative SIE formulations of material bodies are given, e.g., in [25]. In [26] CM equations are developed for a single surface current formulation and in [47] an IBC approximation for thin dielectric sheets is introduced.

**D. SIO vs. VIO for Material Bodies and Spurious Modes**

It is often argued that the VIO-based CM formulation is immune from spurious modes, while the SIO-based one may produce these unwanted solutions. To demonstrate fundamental differences between the VIO- and SIO-based approaches, consider a dielectric non-magnetic body. Using the VIO approach, with notation \( L_V = \eta_0 \mathcal{V} + \mathcal{L}_z = \mathcal{R}_{rad}^V + \mathcal{R}_{reac}^V + i \lambda_V \), the GEE can be expressed as

\[ (-\lambda_V^{reac} + i \lambda_V^{reac}) [J_n^V] = \lambda_V^V \mathcal{R}_{rad}^V [J_n^V]. \tag{16} \]

Decomposing the PMCHWT operator analogously as \( L_P = \mathcal{R}_{rad}^S + \mathcal{R}_{reac}^S + i \lambda^{reac}_S \) gives formally an identical equation

\[ (-\lambda_S^{reac} + i \lambda^{reac}_S) [F_n^S] = \lambda_S^{S} \mathcal{R}_{rad}^S [F_n^S], \tag{17} \]

with a surface current \( F_n^S = [J_n^S, M_n^S]^T \). In the lossless case, the material dependent VIO, \( \mathcal{L}_z \), is imaginary and the dissipated power operator \( \mathcal{R}_{diss}^V = \text{Re}(\mathcal{L}_z) \) vanishes. Hence, GEE (16) can be expressed with real-symmetric operators. In (17) this is not possible since all operators (which do not agree with the real and imaginary parts of \( \mathcal{L}_P \)) are complex and asymmetric. Particularly, the surface dissipated power operator

\[ \mathcal{R}_{diss}^S = \begin{bmatrix} \text{Re}(\eta_1 \gamma T_1) & -\text{Im}(\gamma_1 K_1) \\ \text{Im}(\gamma_1 K_1) & \text{Re}(1/\eta_1 \gamma T_1) \end{bmatrix} \tag{18} \]

does not vanish in the lossless case.

To further illustrate challenges related to the operator \( \mathcal{R}_{diss}^S \), we consider the accepted power formulation, i.e., the second formulation of [12] for lossy dielectric bodies. In that case the weight operator is defined as a sum of the radiated and dissipated power operators. With the VIOs, the GEEs reads [12]

\[ -\lambda^{reac}_V [J_n^V] = \lambda^V \mathcal{R}_{rad}^V [J_n^V], \tag{19} \]

while with the PMCHWT formulation, we have

\[ -\lambda^{reac}_S [F_n^S] = \lambda^S \mathcal{R}_{rad}^S [F_n^S]. \tag{20} \]

The symmetrized form of (20) is identical with the CH formulation [16]. The problem is that the dissipated power operator \( \mathcal{R}_{diss}^S \), given in (18), has exactly the same form as the radiated power operator when the background is filled with the material of the body. Since the GEE can not separate contributions of these two operators, spurious modes appear. As a remedy, we follow [23] and define a projector operator

\[ \mathcal{P} = [I, -(1/\eta_1 \gamma T_1)^{-1} \gamma_1 K_1]^T, \tag{21} \]

and apply it to the both sides of (20)

\[ -\mathcal{P}^H \lambda^{reac}_S [F_n^S] = \lambda_n \mathcal{P} [\mathcal{R}_{rad}^S + \mathcal{R}_{diss}^S] [F_n^S]. \tag{22} \]

Formulation (22) is here called Lian-Pan-Huang (LPH) formulation, although in [23] the GEE is written differently using the Hermitian components of \( \mathcal{P}^H \mathcal{L}_P \mathcal{P} \). Figure 1 displays eigenvalues as losses of a dielectric sphere are increased. For comparison the Mie-based accepted power eigenvalues

\[ \lambda_n^{acc} = \frac{|\lambda_n|}{\text{Re}(\epsilon_r)} \tag{23} \]

are shown, too. In Fig. 1 only the correct CMs of the original CH formulation are plotted. These solutions, as losses are increased, do not agree with the solutions of the other formulations. Hence, we may conclude that to avoid spurious modes in the accepted power PMCHWT-based formulation (CH-formulation) in the lossy case, removal of one of the currents is needed. For the radiated power formulation, and for the VIO formulations, this is not necessary [25], [27].

If the object is both dielectric and magnetic, also the VIO (95) is an asymmetric operator. In the lossless case symmetrized VIO (12) is real and symmetric, and the GEE can still be expressed with its real and imaginary parts [41].
These results imply that the SIO- and VIO-based CM formulations are conceptually different and cannot be considered similarly. A methodology that works for the VIO-approach, may lead to spurious modes with the SIO-based one.

E. Combined Metallic-Dielectric Structures

SIO-based formulations have also been presented for combined metallic and dielectric structures. Formulations [48], [49] are based on [21] and thus produce spurious modes, while [50], [51] and [52] are extensions of [23] and [24], and are immune from spurious modes. In [53] a combined volume-surface integral operator (VSIO) is applied in the lossless case. The weight operator is defined as the real part of the VSIO. In [54] a SIO-based formulation giving equiphase eigencur- rents for combined metallic and lossless dielectric objects is presented. In [55], [56], a mixed potential formulation with the EFIE and layered media Green’s function is used. In [57] a similar formulation is applied for PEC objects buried in layered medium. In [58] a thin sheet-based CM formulation for lossless dielectric coated conducting bodies is proposed.

F. Sub-structure Formulation

In [59] a PEC-EFIE-based approach is proposed where the modes are found on a portion of the structure only, and the rest of the structure is treated as a background. In [60], [61] this so-called sub-structure formulation is generalized for combined PEC and dielectric structures. Sub-structure formulations have also been developed with the combined VSIO [32], [53].

CMs obtained with the sub-structure formulation are not identical with the ones of the full-structure formulation [53], [59]–[61], while they share, e.g., similar orthogonality properties [62]. These modes have been found to be particularly useful in cases where the solutions only on a part of the structure are of interest [32], [53], [61].

IV. INTERPRETATION OF EIGENVALUES

In the classical works [11], [12], interpretation of the eigenvalues was not explicitly considered. This topic has become important recently in the antenna analysis [3]. A wide consensus is that for PEC structures a characteristic eigenvalue gives the ratio of reactive and radiated powers of a mode [11]. Similar interpretation is argued to hold also for lossless dielectric (non-magnetic) bodies [23], although opposite arguments have also been presented [63]. For (lossless) dielectric-magnetic bodies the situation is less obvious, since the imaginary part of the complex power balance is not directly related to the conventional definition of reactive power [12], [14], [23], [26].

To investigate interpretation of the eigenvalues, we consider a complex power balance

\[ P_n = \frac{1}{2} \langle F_n^*, L[F_n] \rangle, \]  

with an eigenvector \( F_n \), and symmetric \( L^2 \) product

\[ \langle u, v \rangle = \int_{\Omega} u \cdot v \, d\Omega. \]  

Here \( \Omega \) is either a volume \( V \) or surface \( S \). Assume that an integral operator \( L \) is decomposed into field-related operators, \( L^{\text{fld}} \), and material-related ones, \( L^{\text{mat}} \), as follows [26], [30]

\[ L = L^{\text{fld}} + L^{\text{mat}} = (R^{\text{rad}} + i \lambda^{\text{reac, fld}}) + (R^{\text{diss}} + i \lambda^{\text{reac, mat}}). \]  

Operators \( R^{\text{diss}} \) and \( \lambda^{\text{reac}} \) are linked to the dissipated and reactive power. Applying decomposition (26) in (8) gives

\[ \langle (-i \lambda^{\text{reac, fld}} + \lambda^{\text{reac, mat}} + i R^{\text{diss}}) [F_n] \rangle = \lambda_n R^{\text{rad}} [F_n]. \]  

Multiplying with \( F_n^* \) via the symmetric \( L^2 \) product gives [26]

\[ \lambda_n = \frac{\langle F_n^* (\lambda^{\text{reac, fld}} + \lambda^{\text{reac, mat}}) [F_n] \rangle + i \langle F_n^* R^{\text{diss}} [F_n] \rangle}{\langle F_n^* R^{\text{rad}} [F_n] \rangle}. \]  

This is a generic interpretation of the eigenvalues in terms of electromagnetic power (in the radiated power formulation), and holds for all operators \( L \) allowing decomposition (26).

We continue with a sphere for which the eigenvalues can be expressed with the Mie coefficients [2], [24], [64], [65].

A. Mie Expansion

For a sphere with radius \( a \) the eigenvectors agree with the vector spherical harmonic functions, shortly vector multipoles. The incident and scattered fields have expansions [66]

\[ F_{\text{inc}} = \sum_n a_n F_{\text{inc}}^n, \quad F_{\text{sca}} = -\sum_n a_n c_n F_{\text{sca}}^n, \]  

where functions \( F_{\text{inc/sca}}^n \) are incoming/outgoing vector multipoles. From (2) and (3) it follows that \( \nu_n = -c_n \). Thus, the eigenvalues can be expressed with the coefficients \( c_n \) as [65]

\[ \lambda_n = \frac{c_n - 1}{ic_n} = \frac{\text{Im}(c_n)}{|c_n|^2} + i \frac{\text{Re}(c_n) - |c_n|^2}{|c_n|^2}. \]  

The radiated and dissipated (absorbed) powers of the \( n \)th multipole with coefficient \( c_n \) are given by [66]

\[ P_{\text{rad}}^n = \frac{2\pi a^2}{x^2} \text{Re}(c_n)^2, \]  

\[ P_{\text{diss}}^n = \frac{2\pi a^2}{x^2} (2n+1) \text{Re}(c_n) - |c_n|^2, \]  

where \( x = k_0 a \) with \( k_0 = \omega \sqrt{\varepsilon_0 \mu_0} \). From (30)–(32) we see that both the real and imaginary parts of the characteristic eigenvalues are related to the radiated power. The imaginary part of an eigenvalue gives the ratio of the dissipated and radiated power. We also note that in the lossless case, \( |c_n|^2 = \text{Re}(c_n) \), and thus

\[ \lambda_n = \frac{\text{Im}(c_n)}{\text{Re}(c_n)}. \]  

For the real part of \( \lambda_n \) we define a reactive power

\[ P_{\text{reac}}^n = \frac{2\pi a^2}{x^2} (2n+1) \text{Im}(c_n). \]
The radiated and dissipative modal powers (31) and (32) can be expressed with the modal scattered and total fields [66]

\[
P_{\text{rad}}^{\text{rad}} = \frac{1}{2} \text{Re} \int_S (E_{\text{sc}} \times (H_{\text{sc}})^*) \cdot dS,
\]

(35)

\[
P_{\text{diss}}^{\text{diss}} = -\frac{1}{2} \text{Re} \int_S (E_{\text{tot}} \times (H_{\text{tot}})^*) \cdot dS,
\]

(36)

with \(E_{\text{tot}}^n = E_{\text{inc}}^n + E_{\text{sc}}^n\) and \(H_{\text{tot}}^n = H_{\text{inc}}^n + H_{\text{sc}}^n\). Accordingly, the reactive power has an expression

\[
P_{\text{react}} = \frac{1}{2} \text{Im} \int_S (E_{\text{inc}} \times (H_{\text{sc}}^*)^* - E_{\text{sc}} \times (H_{\text{inc}}^*)^*) \cdot dS.
\]

(37)

To summarize, the eigenvalues can be expressed as

\[
\lambda_n = \frac{P_{\text{react}}^{\text{rad}}}{P_{\text{rad}}^{\text{rad}}} + \frac{i P_{\text{diss}}^{\text{diss}}}{P_{\text{rad}}^{\text{rad}}}. 
\]

(38)

This result is valid for conducting and material spheres, both lossless and lossy. Expression (37) can be modified to

\[
P_{\text{react}}^{\text{rad}} = \frac{1}{2} \text{Im} \int_S (E_{\text{inc}} \times (H_{\text{sc}}^*)^*) \cdot dS + \frac{1}{2} \text{Im} \int_S (E_{\text{tot}} \times (H_{\text{tot}}^*)^*) \cdot dS
\]

\[
- \text{Im} \int_S (E_{\text{tot}}^* \times (H_{\text{sc}}^*)^*) \cdot dS.
\]

(39)

The first two terms on the right hand side give the exterior and interior reactive powers as the interior is filled with the material of the background

\[
P_{\text{react,ext}}^{\text{rad}} = -\frac{1}{2} \text{Im} \int_S E_{\text{inc}} \times (H_{\text{sc}}^*) \cdot dS.
\]

(40)

\[
= \frac{\omega}{2} \int_{D_0} (\mu_0 \| H_{\text{inc}}^n \|^2 - \varepsilon_0 \| E_{\text{inc}}^n \|^2) \, dV.
\]

and

\[
P_{\text{react,int}}^{\text{rad}} = \frac{1}{2} \text{Im} \int_S E_{\text{inc}} \times (H_{\text{inc}}^*) \cdot dS
\]

\[
= \frac{\omega}{2} \int_{D_1} (\mu_0 \| H_{\text{inc}}^n \|^2 - \varepsilon_0 \| E_{\text{inc}}^n \|^2) \, dV.
\]

(41)

The last two terms in (39) can be interpreted as a material related reactive power, \(P_{\text{react,mat}}^{\text{rad}}\), which in the case of PEC spheres vanishes due to the boundary condition \(\gamma_n E_{\text{tot}}^{\text{tot}} = 0\).

B. Conducting Surfaces

We continue with the SIO approach for arbitrarily shaped PEC surfaces. Let \(J_n = n \times H_{\text{tot}}^n\) denote a modal electric surface current on \(S\). Due to the surface equivalence principle (SEP), \(L_\text{sd}[J_n] = \eta_0 \gamma_n J_n\) gives the scattered field in \(D_0\) and the negative of the incident field in \(D_1\). Thus, (24) reads

\[
\langle F^*_n, L_\text{sd}[J_n] \rangle = -\frac{1}{2} \int_{D_0} (E_{\text{sc}} \times (H_{\text{sc}}^*)^*) \cdot dS
\]

\[
+ \frac{i \omega}{2} \int_{D_1} (\mu_0 \| H_{\text{inc}}^n \|^2 - \varepsilon_0 \| E_{\text{inc}}^n \|^2) \, dV.
\]

(42)

The real part of this equation gives the radiated power and its imaginary part gives the exterior and interior field-related reactive powers, respectively. Hence, for closed PEC surfaces

\[
\lambda_n^{\text{PEC}} = \frac{P_{\text{react,ext}}^{\text{rad}}}{P_{\text{rad}}^{\text{rad}}} + \frac{P_{\text{react,int}}^{\text{rad}}}{P_{\text{rad}}^{\text{rad}}}.
\]

(43)

For an open PEC surface, \(P_{\text{react,int}}^{\text{rad}} = 0\). Comparing (35), (40), and (41) with (42), we find that the Mie expansion and the SIO approach give identical interpretations in the case of PEC.

Fig. 2 shows eigenvalues for a PEC sphere computed using the Mie coefficients (solid lines TM modes, dashed lines TE modes), ratio of reactive (37) and radiated (35) power (circles), and as the interior reactive power is computed with integral (41) (asterisks). Colors indicate different modes.

For impedance surfaces, we have an additional material related operator, \(Z_\text{mat} = -\eta_0 Z, J\), giving a reactive power

\[
P_{\text{react,mat}}^{\text{rad}} = \frac{1}{2} \text{Im} \langle J_n^*, L_\text{mat}[J_n] \rangle = -\frac{\eta_0 \text{Im}(Z_s)}{2} \int_S \| J_n \|^2 \, dS
\]

(44)

and, in the case of a lossy surface, a dissipated power

\[
P_{\text{diss}} = -\frac{1}{2} \text{Re} \langle J_n^*, L_\text{mat}[J_n] \rangle = \frac{\eta_0 \text{Re}(Z_s)}{2} \int_S \| J_n \|^2 \, dS.
\]

(45)

Consequently, for impedance surfaces we have

\[
\lambda_n^{\text{IBC}} = \frac{P_{\text{react,ext}}^{\text{rad}} + P_{\text{react,int}}^{\text{rad}} + P_{\text{react,mat}}^{\text{rad}}}{P_{\text{rad}}^{\text{rad}}} + \frac{P_{\text{diss}}^{\text{diss}}}{P_{\text{rad}}^{\text{rad}}}.
\]

(46)

C. Dielectric Bodies

Consider next the VIO approach for dielectric bodies with \(L_\text{sd}[J_n] = \eta_j \gamma_n (\mathcal{L}_c \varepsilon_0 J_n)\), giving a reactive power

\[
\frac{1}{2} \langle J_n^* V, L_\text{sd}[J_n^* V] \rangle = \frac{1}{2} \int_S (E_{\text{sc}}^* \times (H_{\text{sc}}^*)^*) \cdot n \, dS
\]

\[
- \frac{i \omega}{2} \int_{D_1} (\mu_0 \| H_{\text{inc}}^n \|^2 - \varepsilon_0 \| E_{\text{inc}}^n \|^2) \, dV.
\]

(47)
The real part of this equation gives the radiated power and its imaginary part gives the field-related reactive power
\[
P_{n}^{\text{preac,fld}} = P_{n}^{\text{preac,ext}} + P_{n}^{\text{preac,int}},
\]
(48)
where \( P_{n}^{\text{preac,ext}} \) is defined as in (40) and
\[
P_{n}^{\text{preac,int}} = \frac{\omega}{2} \left( \int_{D_{t}} \mu_{0} \| \mathbf{H}_{n}^{\text{ca}} \|^2 \, dV - \int_{D_{t}} \varepsilon_{0} \| \mathbf{E}_{n}^{\text{ca}} \|^2 \, dV \right).
\]
(49)
The material dependent operator gives
\[
\frac{1}{2} \langle (\mathbf{J})^{V}^{\ast}, \mathbf{L}^{\text{mat}}(\mathbf{J})^{V} \rangle = -i \omega \int_{D_{t}} (\varepsilon_{1} - \varepsilon_{0}) \| \mathbf{E}_{n}^{\text{tot}} \|^2 \, dV + \frac{1}{2} \int_{D_{t}} \sigma_{\parallel} \| \mathbf{E}_{n}^{\text{tot}} \|^2 \, dV.
\]
(50)
The real part of (50) gives the dissipated power, and its imaginary part gives material-related reactive power [23]
\[
P_{n}^{\text{preac,mat}} = -\frac{\omega}{2} \int_{D_{t}} (\varepsilon_{1} - \varepsilon_{0}) \| \mathbf{E}_{n}^{\text{tot}} \|^2 \, dV.
\]
(51)
To conclude, the VIO approach for dielectric objects gives
\[
\lambda_{n}^{\text{die}} = \frac{P_{n}^{\text{preac,fld}}}{P_{n}^{\text{rad}}} + \frac{P_{n}^{\text{preac,int}}}{P_{n}^{\text{rad}}} + \frac{P_{n}^{\text{preac,mat}}}{P_{n}^{\text{rad}}}.
\]
(52)

**D. Dielectric-Magnetic Bodies**

For dielectric-magnetic bodies with a volume current \( \mathbf{F}_{n}^{\text{V}} = [\mathbf{J}_{n}^{V}, \mathbf{M}_{n}^{V}]^T \), the field- and material-related VIOs are [12]
\[
\mathbf{L}^{\text{fld}} = \begin{bmatrix} \eta_{0} \mathbf{V} & -\mathbf{U} \\ \mathbf{U} & 1/\eta_{0} \mathbf{V} \end{bmatrix} \quad \text{and} \quad \mathbf{L}^{\text{mat}} = \begin{bmatrix} \mathbf{T}_{\varepsilon} & 0 \\ 0 & \mathbf{T}_{\mu} \end{bmatrix}.
\]
(53)
Using the definition of the volume currents gives
\[
\frac{1}{2} \langle (\mathbf{F}_{n}^{\text{V}})^{\ast}, \mathbf{L}^{\text{fld}}(\mathbf{F}_{n}^{\text{V}}) \rangle = \frac{i \omega}{2} \int_{D_{t}} (\varepsilon_{1} - \varepsilon_{0})(\mathbf{E}_{n}^{\text{tot}})^{\ast} \cdot \mathbf{E}_{n}^{\text{ca}} \, dV + \frac{i \omega}{2} \int_{D_{t}} (\mu_{1} - \mu_{0})(\mathbf{H}_{n}^{\text{tot}})^{\ast} \cdot \mathbf{H}_{n}^{\text{ca}} \, dV.
\]
(54)
The real part of (54) gives the radiated power due to \( \mathbf{F}_{n}^{\text{V}} \), and its imaginary part gives the field-related reactive power. Analogously, for the material-related operator we get
\[
\frac{1}{2} \langle (\mathbf{F}_{n}^{\text{V}})^{\ast}, \mathbf{L}^{\text{mat}}(\mathbf{F}_{n}^{\text{V}}) \rangle = -\frac{i \omega}{2} \int_{D_{t}} (\varepsilon_{1} - \varepsilon_{0}) \| \mathbf{E}_{n}^{\text{tot}} \|^2 \, dV + \frac{1}{2} \int_{D_{t}} \sigma_{\parallel} \| \mathbf{E}_{n}^{\text{tot}} \|^2 \, dV
\]
\[-\frac{i \omega}{2} \int_{D_{t}} (\mu_{1} - \mu_{0}) \| \mathbf{H}_{n}^{\text{tot}} \|^2 \, dV.
\]
(55)
The real part of (55) gives the dissipated power and its imaginary part gives the material-related reactive power, equivalent to (51). Thus, with the above definitions and notations
\[
\lambda_{n}^{\text{demag}} = \frac{P_{n}^{\text{preac,fld}} + P_{n}^{\text{preac,mat}}}{P_{n}^{\text{rad}}} + i \frac{P_{n}^{\text{preac,mat}}}{P_{n}^{\text{rad}}}.
\]
(56)

In Fig. 3 real (solid lines) and imaginary (dashed lines) parts of the eigenvalues are displayed for a lossy dielectric-magnetic brick. The results of Figs. 2 and 3 demonstrate that the interpretation of the eigenvalues in terms of the reactive power (37) is valid independently on the material and shape of the object.
Clearly, the real (active) parts are equal, while the imaginary (reactive) parts differ by a sign of the last term. Following the Poynting theorem formulation, we may derive a GEE

\[
\left( -\lambda^{\text{react.2}}_V + i\mathcal{R}^{\text{div}}_V \right) [F^V_m] = \lambda_n \mathcal{R}^{\text{rad}}_V [F^m_m] \tag{61}
\]

with an operator (the second row is multiplied with minus one)

\[
\lambda^{\text{react.2}}_V = \begin{bmatrix} \text{Im}(\eta_0 V + \mathcal{I}_c) & -\text{Re}(U) \\ -\text{Re}(U) & -\text{Im}(1/\eta_0 V + \mathcal{I}_m) \end{bmatrix}. \tag{62}
\]

Figure 4 illustrates that the GEEs based on these two alternative definitions of (reactive) power lead to different eigenvalues. Particularly, the one based on the Poynting theorem shows internal resonances (value maxima) that do not appear in the TCM one. However, they both seem to predict the same eigenvalue zero crossing, the first one near \(\mu_r = 4.5\). Clearly as \(\mu_r = 1\) these formulations give identical results. The figure shows also that the PMCHWT formulation gives the same solutions as the VIO one based on (57), although some differences can be observed due to numerical approximations.

In [26] and [63] CM formulations based on the Poynting theorem are developed using the SIO approach. The numerical results of [26], [63] indicate that those formulations give different solutions than the conventional CM formulations also in the case of dielectric bodies. With the VIO-based approach considered above differences are observed only in the case of dielectric-magnetic bodies, as suggested in [12].

V. OPERATOR DIAGNOSIS

In [11] it was mentioned that the characteristic currents diagonalize the field-integral operator, or the impedance operator, as it was called. This result was shown for real and symmetric operators, and therefore a GEE is typically formulated in terms of the Hermitian components of a symmetrized operator. Next the operator diagonalization is presented for complex-symmetric and asymmetric operators. What follows is that symmetrization is useful but not necessary.

A. Self-adjoint and Normal Operators

In the classical CM formulations [11], [12] for lossless bodies the GEE can be expressed with real-symmetric operators and the real-valued eigenvectors \(F^n_n\) diagonalize \(L\), i.e.,

\[
\langle F^*_m, L [F^m_n] \rangle = 0, \text{ if } m \neq n. \tag{63}
\]

To generalize this, we consider the adjoint of a linear (integral) operator \(L, L^n\), defined via the bilinear form

\[
\langle (L^n[g]), f \rangle = \langle g, L[f] \rangle. \tag{64}
\]

An operator is self-adjoint if \(L^n = L\), and a normal operator satisfies

\[
\langle (L^n[g]), f \rangle = \langle g, L^n(L[f]) \rangle. \tag{65}
\]

The necessary condition for the eigenvectors \(F^n_n, n = 1, 2, \ldots\), to simultaneously diagonalize \(L\) and \(W\), i.e., they satisfy

\[
\langle F^*_m, L [F^n_n] \rangle = 0, \langle F^*_m, W [F^n_n] \rangle = 0, \text{ if } m \neq n, \tag{66}
\]

is that both operators are normal and commutative [67].

Clearly, a self-adjoint operator is normal. Hence, if \(L - W\) and \(W\) are self-adjoint, they are simultaneously diagonalized according to (66). This agrees with the "classical" definition of the CM diagonalization [11] with real-symmetric operators.

If \(L - W\) is complex-symmetric, operators \(L\) and \(W\) are diagonalized with respect to the symmetric product [12]

\[
\langle F^*_m, L [F^n_n] \rangle = 0, \langle F^*_m, W [F^n_n] \rangle = 0, \text{ if } m \neq n, \tag{67}
\]

defined without the complex conjugate. In a general case of asymmetric operators, the diagonalization is satisfied with respect to the original \(f^n_n\) and adjoint eigenvectors \(g^n_m\) [68]

\[
\langle g^*_m, L [F^n_n] \rangle = 0, \langle g^*_m, W [F^n_n] \rangle = 0, \text{ if } m \neq n. \tag{68}
\]

Here \(g^n_m\) is a solution of the adjoint GEE [28], [68]

\[
L^n[g^*_m] = (1 + i\lambda^*_m)W^n[g^*_m]. \tag{69}
\]

B. Modal Expansion

An important consequence of the above results is that a solution \(F\) of the field problem (6) with a given \(F^n_{\text{inc}}\), can be expressed as a linear combination of the eigenvectors \(F^n_n\)

\[
F = \sum_n d_n F^n_n, \quad d_n = \frac{\langle h^n_n, F^n_{\text{inc}} \rangle}{1 - i\lambda^n_n}. \tag{70}
\]

Here \(d_n\) are called modal expansion or weighting coefficients [11]. The eigenvectors \(h^n_n\) needed to find coefficients \(d_n\) depend on the properties of the operators. For example, for self-adjoint (Hermitian or real-symmetric) operators we have \(h^n_n = F^n_n\) [11], [12], and for complex-symmetric ones \(h^n_n = F^n_n\) [12]. In both cases it is sufficient to find the eigenvectors \(F^n_n\) of the original GEE only. This shows that symmetrizing the operator, if possible, can avoid solving the adjoint GEE. In a case of asymmetric operators we have \(h^n_n = g^*_m\) [28], [68] and the adjoint eigenvectors are needed, too.

The fields are linearly related to the currents. Hence, also the fields can be expressed in terms of the modal fields [11]

\[
E = \sum_n d_n E^n_n, \quad H = \sum_n d_n H^n_n. \tag{71}
\]
VI. ORTHOGONALITY RELATIONS

The far-field orthogonality is often argued to be one of the most fundamental properties of the CMs [11], [12]. This equals that the eigencurrents are radiated power orthogonal and the far fields diagonalize the scattering operator [11], [14].

A. Power Orthogonality

Let us assume that an operator $\mathcal{L}$ allows a decomposition

$$\mathcal{L} = R^\text{rad} + R^\text{diss} + iX^\text{reac}. \quad (72)$$

The GEE (8) with this decomposition reads

$$(-X^\text{reac} + iR^\text{diss}) [F_n] = \lambda_n R^\text{rad} [F_n]. \quad (73)$$

If the structure is lossless, $R^\text{diss}$ vanishes, operators $R^\text{rad}$ and $X^\text{reac}$ are simultaneously diagonalizable, and the eigenvectors diagonalize both the radiated and reactive power operators

$$\langle F_m^*, R^\text{rad} [F_n] \rangle = 0, \quad \langle F_m^*, X^\text{reac} [F_n] \rangle = 0, \quad \text{if} \ m \neq n. \quad (74)$$

In other words, the eigenvectors are "power orthogonal". In a special case both $R^\text{rad}$ and $X^\text{reac}$ are real-symmetric, and $R^\text{rad}$ is positive definite, as in [11], [12] (for lossless bodies).

In the lossy case the problem is that (73) involves three operators and generally it is impossible to have orthogonality with respect to all these operators, unless $-X^\text{reac} + iR^\text{diss}$ and $R^\text{rad}$ are simultaneously diagonalizable [14]. We note that even if both $X^\text{reac}$ and $R^\text{diss}$ are real-symmetric, or Hermitian, their sum, including multiplication with the imaginary unit $i$, is at most complex-symmetric. Thus, in the lossy case, excluding special symmetric objects such as a sphere, the eigenvectors are not generally power orthogonal [14], [41].

To verify this, we investigate the EFIO for an open IBC surface [15], [30]. For a lossless surface the surface impedance is reactive, $\eta_0 Z_s = iX_s$, giving the following GEE

$$(X^E - X_s I) [J_n] = -\lambda_n R^E [J_n]. \quad (75)$$

The operators on the both sides of (75) are real and symmetric. Hence, the same orthogonality properties hold as for PEC.

Assume next that the surface is lossy with a real surface resistance $\eta_0 Z_s = R_s$. The GEE is then of the form

$$(X^E + iR_s I) [J_n] = -\lambda_n R^E [J_n]. \quad (76)$$

The operator on the left hand side is complex symmetric, indicating that the orthogonality properties are satisfied only with respect to the non-Hermitian inner products. The loss of orthogonality is illustrated in Fig. 5(a) where a factor [41]

$$\delta_n = \sum_m |A_{mn}| \quad m \neq n, \quad A_{mn} = \langle J_m^*, R^E [J_n] \rangle, \quad (77)$$

is plotted versus normalized surface resistance $R_s/\eta_0$ for a metal plate. For a sufficiently large $R_s$, $\delta_n$ differs significantly from zero, indicating that the eigencurrents are not radiated power orthogonal. As $R_s$ is increased further, the results suggest that the eigencurrents are radiated power orthogonal. This is because for a large enough $R_s$, a real symmetric term $R_s I$ dominates on the left hand side of (76).

Figure 5(b) shows the real and imaginary parts of the eigenvalues versus $R_s$ in the same case as in Fig. 5(a). The eigenvalues depict a rapid change around the same $R_s$ values as the power orthogonality is lost. Outside that region the real parts are nearly constants. The imaginary parts show an exponential increase with respect to $R_s$.

B. Far-Field Orthogonality

The far-field orthogonality is equal to the radiated power orthogonality [11], [12], [41]. Hence, in the lossless case this orthogonality is satisfied with the Hermitian inner product

$$\int_{S_\infty} E_m^* (r) \cdot E_n (r) dS = 0, \quad \text{if} \ m \neq n, \quad (78)$$

where $S_\infty$ is a spherical surface in the far-field region.

In the lossy case the orthogonality relations are satisfied with symmetric products [41], [69], defined without the complex conjugate. Since the symmetric products do not agree with any physical power quantity, for lossy structures, the CMs do not form a radiated power orthogonal set of functions and their radiated fields are not Hermitian orthogonal [41], [69].
VII. DISCRETIZATION, A.K.A., MATRIX FORMULATION

Let us next consider discretization of a GEE with the method of moments (MoM). Let \( u_l, l = 1, \ldots, L \), denote a set of basis functions used to expand a modal current, and \( v_k, k = 1, \ldots, K \), a set of suitable test functions. If Galerkin’s method is used, these functions agree, but this is not necessary [68]. With MoM the GEE (7) is converted to a matrix equation

\[
L x_n = (1 - i\lambda_n)W x_n, \tag{79}
\]

where \( L \) and \( W \) are matrices due to \( \mathcal{L} \) and \( \mathcal{W} \), and vector \( x_n \) includes the coefficients of the basis function approximation of an eigenvector \( f_n \). The elements of these matrices are

\[
L_{kl} = \langle v_k, \mathcal{L}[u_l] \rangle, \quad W_{kl} = \langle v_k, \mathcal{W}[u_l] \rangle. \tag{80}
\]

For more details on numerical evaluation of matrix elements with Rao-Wilton-Glisson (RWG) functions, see e.g. [70].

In (79) vectors \( x_n \) are so called right eigenvectors. We may also consider the left eigenvectors \( z_m \), the solutions of

\[
z_{m}^{\dagger} L = (1 - i\lambda_n)z_{m}^{\dagger} W. \tag{81}
\]

Also adjoint GEE (69) can be converted to a matrix equation

\[
L^a y_m = (1 + i\lambda_m^a)W^a y_m. \tag{82}
\]

Here \( L^a \) and \( W^a \) are matrices due to the adjoint operators \( \mathcal{L}^a \) and \( \mathcal{W}^a \) [68]. Taking the Hermitian transpose of (82) gives

\[
y_m^{\dagger} (L^a)^{\dagger} = (1 - i\lambda_m^a)y_m^{\dagger} (W^a)^{\dagger}. \tag{83}
\]

Since \( y_m^{\dagger} (A x) = (A^d y_m)^{\dagger} x \), for matrices the operator adjoint agrees with the Hermitian transpose, i.e., \( A^d = A^d \). Thus, \( (L^a)^{\dagger} = L \) and \( (W^a)^{\dagger} \approx W \), and the adjoint eigenvectors \( y_m \) can be approximated with the left eigenvectors \( z_m \). To summarize, for matrices orthogonality relations (68) with the weighted inner products read

\[
y_m^{\dagger} L x_n = 0, \quad y_m^{\dagger} W x_n = 0, \quad \text{if } m \neq n. \tag{84}
\]

For real-symmetric matrices, \( L^T = L, W^T = W \), or for Hermitian ones, \( L^H = L, W^H = W \), relations (84) are [11]

\[
x_m^{\dagger} L x_n = 0, \quad x_m^{\dagger} W x_n = 0, \quad \text{if } m \neq n. \tag{85}
\]

To verify (84), we study properties of the eigenvectors obtained as solutions of the GEE written with a non-symmetric PEC-MFIE. Let \( Z^{(E)} = R^{(E)} + iX^{(E)} \) and \( Z^{(M)} = R^{(M)} + iX^{(M)} \) denote the matrices due to the Galerkin-RWG discretized EFIO and the dual-tested MFIO [71]. In the latter, the basis functions are RWG functions and the test functions are rotated Buffa–Christiansen (BC) [72] functions. Fig. 6 shows a weighted inner product

\[
y_m^{\dagger} (M x_n), \quad m, n = 1, 2, \ldots, 20, \tag{86}
\]

with different eigenvectors \( x_n, y_m \), and matrix \( M \). These results indicate that the (right) eigenvectors \( x_n \) do not diagonalize the MFIE matrix, rather that matrix is diagonalized with respect to the right \( x_n \) and left \( y_m = z_m \) eigenvectors. Nevertheless, the right eigenvectors \( x_n \) are radiated and reactive power orthogonal. Thus, the eigensolutions where the GEE is discretized using a non-Galerkin scheme, and leading to a non-symmetric matrix, satisfy the same power and farfield orthogonalities as the eigensolutions obtained with the classical approach of [11] involving real-symmetric matrices.

\[
\begin{array}{cccc}
\text{(a)} & \text{(b)} & \text{(c)} & \text{(d)} \\
\end{array}
\]

\section*{VIII. CONCLUSIONS}

Integral operator-based TCM is reviewed for conductors, material bodies, and lossy structures. The CMs are defined as solutions of a GEE. Formulation of the GEE with both the SIOs and VIOs, interpretation of the eigenvalues, and properties of the CMs are discussed. The main results include:

1) Defining the weight operator of the GEE so that it links the eigensolutions to the radiated power (called here as a radiated power formulation) avoids spurious modes [25]. In the lossless case, this gives radiated-power orthogonal set of CMs with Hermitian orthogonal far fields.

2) VIO- and SIO-based CM formulations are conceptually different and can not be treated similarly. For example, with the VIO approach, after symmetrization, the GEE can be expressed with real symmetric operators for lossless dielectric-magnetic bodies, while for the SIO one, this is not possible even for non-magnetic bodies.

3) The CH surface formulation [16] produces spurious modes since the dissipated power operator is identical with the radiated power operator of the reverse problem and the GEE can not distinguish them [27]. The contributions of these operators can be separated by proper projection operators [23].

4) Real parts of the eigenvalues give the ratio of reactive and radiated power, while their imaginary parts give the ratio of dissipated and radiated power [25], [30], [31]. For dielectric-magnetic bodies the reactive power does not agree with its usual definition [12], [23], [26]. Reformulating the GEE so that the reactive power agrees with its conventional definition leads to different eigensolutions than obtained with conventional TCM.

5) Orthogonality of the characteristic far fields depends

\[
\begin{array}{cccc}
\text{Fig. 6: Weighted inner product (86) for a rectangular PEC brick with dimensions 146 mm \times 71 mm \times 1 mm:} & \text{(a) MFIE matrix } Z^{(M)} \text{ with right eigenvectors,} & \text{(b) MFIE matrix } Z^{(M)} \text{ with left and right eigenvectors,} & \text{(c) radiated power matrix } R^{(E)} \text{ with right eigenvectors,} & \text{(d) reactive power matrix } X^{(E)} \text{ with right eigenvectors. The axes indicate the mode index.}
\end{array}
\]
solenoidal, and the electric and magnetic surface currents, \( \eta J \) and \( J \), respectively. The radiated power operator due to (89) reads

\[
\mathcal{T}_j[f](r) = \frac{i}{k_j} \nabla \int_S G_j(r, r') \nabla_s \cdot f(r') dS',
\]

\( + ik_j \int_S G_j(r, r') f(r') dS', \) \hspace{1cm} (87)

\[
K_j^\pm[f](r) = \nabla \times \int_S G_j(r, r') f(r') dS'(-1)^j \frac{1}{2} \eta n(r) \times f(r).
\]

\hspace{1cm} (88)

Here \( k_j \) and \( G_j \) are the wavenumber and Green’s function in domain \( D_j \), \( j = \{0, 1\} \), \( \nabla_s \) is the surface divergence, and \( n \) is the normal vector of \( S \) pointing into \( D_j \). Let \( E_{inc}, H_{inc}^\tau \) denote the incident fields with sources in \( D_j \). Using the SEP, the tangential field-integral equations on \( S \) are [37]

\[
\begin{bmatrix}
\eta \gamma \gamma_j \mathcal{T}_j \\
\gamma_j \mathcal{K}_j^\pm \\
1/\eta \gamma_j \mathcal{K}_j^\pm
\end{bmatrix} = \begin{bmatrix}
J \\
M
\end{bmatrix} = - \begin{bmatrix}
\gamma_j E_{inc} \\
\gamma_j H_{inc}^\tau
\end{bmatrix},
\]

\hspace{1cm} (89)

Here \( J = n \times H_{tot} \) and \( M = -n \times E_{tot} \) are the equivalent electric and magnetic surface currents, \( \eta_0 = \sqrt{\mu_0/\varepsilon_0} \), and \( \gamma_j F = -n \times (n \times F)|_S \) denotes tangential component of \( F \). The radiated power operator due to (89) reads

\[
\mathcal{R}_S^{rad} = \mathcal{R}_V^{rad} = \begin{bmatrix}
\text{Re}(\eta_0 \gamma_j \mathcal{T}_j) \\
\text{Im}(\eta_0 \gamma_j \mathcal{K}_j^\pm) \\
\text{Re}(1/\eta_0 \gamma_j \mathcal{K}_j^\pm)
\end{bmatrix}.
\]

\hspace{1cm} (90)

**APPENDIX A: SURFACE INTEGRAL OPERATORS**

In this paper we use the following SIOs

\[
\mathcal{T}_j[f](r) = \frac{i}{k_j} \nabla \int_S G_j(r, r') \nabla_s \cdot f(r') dS',
\]

\( + ik_j \int_S G_j(r, r') f(r') dS', \) \hspace{1cm} (87)

\[
K_j^\pm[f](r) = \nabla \times \int_S G_j(r, r') f(r') dS'(-1)^j \frac{1}{2} \eta n(r) \times f(r).
\]

\hspace{1cm} (88)

Here \( k_j \) and \( G_j \) are the wavenumber and Green’s function in domain \( D_j \), \( j = \{0, 1\} \), \( \nabla_s \) is the surface divergence, and \( n \) is the normal vector of \( S \) pointing into \( D_j \). Let \( E_{inc}, H_{inc}^\tau \) denote the incident fields with sources in \( D_j \). Using the SEP, the tangential field-integral equations on \( S \) are [37]

\[
\begin{bmatrix}
\eta \gamma \gamma_j \mathcal{T}_j \\
\gamma_j \mathcal{K}_j^\pm \\
1/\eta \gamma_j \mathcal{K}_j^\pm
\end{bmatrix} = \begin{bmatrix}
J \\
M
\end{bmatrix} = - \begin{bmatrix}
\gamma_j E_{inc} \\
\gamma_j H_{inc}^\tau
\end{bmatrix},
\]

\hspace{1cm} (89)

Here \( J = n \times H_{tot} \) and \( M = -n \times E_{tot} \) are the equivalent electric and magnetic surface currents, \( \eta_0 = \sqrt{\mu_0/\varepsilon_0} \), and \( \gamma_j F = -n \times (n \times F)|_S \) denotes tangential component of \( F \). The radiated power operator due to (89) reads

\[
\mathcal{R}_S^{rad} = \mathcal{R}_V^{rad} = \begin{bmatrix}
\text{Re}(\eta_0 \gamma_j \mathcal{T}_j) \\
\text{Im}(\eta_0 \gamma_j \mathcal{K}_j^\pm) \\
\text{Re}(1/\eta_0 \gamma_j \mathcal{K}_j^\pm)
\end{bmatrix}.
\]

\hspace{1cm} (90)

**APPENDIX B: VOLUME INTEGRAL OPERATORS**

The VIOs, with volume \( V \), used in this paper are given by

\[
\mathcal{V}[f](r) = \frac{i}{k_0} (\nabla \cdot + k_0^2) \int_V G_0(r, r') f(r') dv',
\]

\hspace{1cm} (91)

\[
\mathcal{U}[f](r) = \nabla \times \int_V G_0(r, r') f(r') dv'.
\]

\hspace{1cm} (92)

Here \( G_0 \) is the free space Green’s function. In addition,

\[
\mathcal{I}_\varepsilon = \frac{1}{i \omega (\varepsilon_1 - \varepsilon_0)} \mathcal{I} \quad \text{and} \quad \mathcal{I}_\mu = \frac{1}{i \omega (\mu_1 - \mu_0)} \mathcal{I}.
\]

\hspace{1cm} (93)

By defining the volume currents as follows

\[
\mathcal{J}^V = -i \omega (\varepsilon_1 - \varepsilon_0) E_{tot}, \quad \mathcal{M}^V = -i \omega (\mu_1 - \mu_0) H_{tot},
\]

\hspace{1cm} (94)

using the VEP, the electric and magnetic field VIEs read [12]

\[
\begin{bmatrix}
\eta_0 \mathcal{I}_\varepsilon + \mathcal{I}_\varepsilon \\
\mathcal{U}
\end{bmatrix} = \begin{bmatrix}
\mathcal{J}^V \\
\mathcal{M}^V
\end{bmatrix} = \begin{bmatrix}
\mathcal{E}_{inc} \\
\mathcal{H}_{inc}^\tau
\end{bmatrix}.
\]

\hspace{1cm} (95)

The radiated power operator for (95) is given by

\[
\mathcal{R}_V^{rad} = \begin{bmatrix}
\text{Re}(\eta_0 \mathcal{I}_\varepsilon) \\
-\text{Im}(\mathcal{U}) \\
\text{Re}(1/\eta_0 \mathcal{I}_\varepsilon)
\end{bmatrix}.
\]

\hspace{1cm} (96)


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