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Emergent Weyl spinors in multi-fermion systems

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Abstract

In Ref. [1] Hořava suggested, that the multi-fermion many-body system with topologically stable Fermi surfaces may effectively be described (in a vicinity of the Fermi surface) by the theory with coarse-grained fermions. The number of the components of these coarse-grained fermions is reduced compared to the original system. Here we consider the $3+1$ D system and concentrate on the particular case when the Fermi surface has co-dimension $p=3$, i.e. it represents the Fermi point in momentum space. First we demonstrate explicitly that in agreement with Hořava conjecture, in the vicinity of the Fermi point the original system is reduced to the model with two-component Weyl spinors. Next, we generalize the construction of Hořava to the situation, when the original $3+1$ D theory contains multi-component Majorana spinors. In this case the system is also reduced to the model of the two-component Weyl fermions in the vicinity of the topologically stable Fermi point. Those fermions experience the emergent gauge field and the gravitational field given by the emergent vierbein. Both these fields (the emergent gauge field and the emergent gravitational field) originate from certain collective excitations of the original system. We speculate, that the given construction may be relevant for the high energy physics in the paradigm, in which the Lorentz symmetry as well as the gravitational and gauge fields are the emergent phenomena, i.e. they appear dynamically in the low energy approximation of the underlined high energy theory.

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1. Introduction

As in particle physics, the condensed matter systems are described by the multi-component fermionic fields. In addition to spin they may have Bogoliubov spin, layer index in the multilayered $2+1$ systems, etc. In crystals the band indices are added, and the spinor acquires infinite number of components. In the low energy corner the effective number of degrees of freedom is essentially reduced. The gapped (massive) degrees of freedom are frozen out and only gapless states survive. The gaplessness is the fragile property, since it can be violated by interaction between fermions. However, there exist fermionic systems, in which the gaplessness (masslessness) is robust to interaction. These are the topological materials, where stability of nodes in the energy spectrum with respect to deformations is protected by the conservation of topological invariants of different types [1].

Examples of topologically protected zeroes in fermionic spectrum are: Fermi surface in metals [2]; Fermi points in $3+1$ D Weyl superfluid ^{3}He–A [2] and in $3+1$ D Weyl semimetals [3–7]; Dirac points in graphene [8,9]; fermionic edge modes on the surface and interfaces of the fully gapped topological insulators [10–12] and superfluids [13,14].

The Fermi or Weyl points represent the exceptional (conical, diabolic) points of level crossing, which avoid the level repulsion [15]. Topological invariants for points at which the branches of spectrum merge were introduced by Novikov [16]. In our case crossing points occur in momentum space [17,18].

The spectrum near the point nodes typically acquires the relativistic form, which is the consequence of the Atyah–Bott–Shapiro construction applied to the nodes with unit value of topological invariant [1]. This results in emergence of effective gauge and gravitational field as collective Bose modes [19,2,20,21]. This means, that the fermionic excitations reside in curved space–time. The geometry of this space–time is given by the vierbein formed by certain collective excitations of the microscopic system.

The higher values of topological invariant give rise to exotic Weyl or Dirac fermions, with nonlinear touching points of positive and negative energy branches. The $2+1$ D example of such system is given by the multilayer graphene with the ABC stacking [22]. The nonlinear Dirac spectrum results in the effective gravitational and gauge field theories, which obey anisotropic scaling of Hořava type [23–26], see [27–30]. The multilayer graphene also demonstrates the reduction of the degrees of freedom at low energy. The original tight-binding model may be described by the field theory with the multi-component fermionic field, which carries the spin, pseudospin, and layer indices. Due to the specific interaction between the fermions that belong to different layers, in the emergent low energy theory the layer index drops out. The final effective theory operates with the two-component spinors existing in the vicinity of each of the two Fermi points. These spinors also carry the flavor index that corresponds to the real spin.

The general theory that describes reduction of the fermion components and the emergent gravity experienced by the reduced fermions is not developed so far in sufficient details. The main progress in this direction has been made by Hořava [1], who considered the general case of $d+1$ dimensional condensed matter system with $d{-}p$ dimensional Fermi surface ($d{-}p$ dimensional manifold of zeroes in the d dimensional momentum space). The classification of the fully gapped topological materials [31,32] can be obtained from Hořava classification by dimensional reduction (see examples in [2]).

We are interested in the particular case of the $3+1$ D systems with Fermi-point, i.e. with the node of co-dimension $p=d=3$. In this particular case it follows from the statement of [1], that in the vicinity of the Fermi-point the system is effectively described by the two-component fermion field Ψ. The action of this two-component field is given by

$$\mathcal{S} = \int d\mu\,(\mathbf{p},\omega)\bar{\Psi}(\mathbf{p},\omega) D\Psi(\mathbf{p},\omega), \tag{1}$$

where μ is the integration measure over momentum and frequency $\mathbf{p}$, ω. It was claimed in [1], that operator D contains the construction of Atyah–Bott–Shapiro that enters the expression for the topological invariant corresponding to the nontrivial $\pi_3(GL(n,C)) = \mathbf{K}(R^3)$, where n is the original number of the fermion components:

$$D = e^{\mu}_{a}\,\sigma^{a}\bigl(p_\mu - p^{(0)}_{\mu}\bigr) + \cdots \tag{2}$$

Here p_μ is a 4-momentum; e^{μ}_{a} is an emergent vierbein; $p^{(0)}_{\mu}$ is the position of the Fermi point, whose space–time variation gives rise to the effective dynamical $U(1)$ gauge field B_μ; and dots mean the subdominant terms, which include the emergent spin connection C_μ.

The emergence of this Eq. (2) has been advocated by Froggatt and Nielsen in their random dynamics theory, where the infinite number of degrees of freedom is reduced to 2×2 subspace of Hermitian matrices (see page 147 in the book [19]). In superfluid ^{3}He–A this Eq. (2) has been explicitly obtained by expansion of the Bogoliubov–de Gennes Hamiltonian near the Weyl point [20]; for the expansion near Dirac point in $2+1$ D graphene see Refs. [33,34,30]. In both cases the complicated atomic structure of liquid and electronic structure in crystals are reduced to the description in terms of the effective two-component spinors, and this supports the conjecture of Froggatt and Nielsen and the Hořava approach.

The emergence of Weyl spinor has important consequences both in the condensed matter physics and in the high energy physics. This is because the Weyl fermions represent the building blocks of the Standard Model of particle physics (SM). Emergence of Weyl fermions in condensed matter together with Lorentz invariance, effective gravity and gauge fields and the topological stability of emergent phenomena suggest that SM and Einstein theory of gravitational field (GR) may have the status of effective theories. The chiral elementary particles (quarks and leptons), gauge and Higgs bosons, and the dynamical vierbein field may naturally emerge in the low-energy corner of the quantum vacuum, provided the vacuum has topologically protected Weyl points.

When considering the possible emergence of SM and GR, one should resolve between the symmetries which emerge in the low energy corner (Lorentz invariance, gauge symmetry, etc.) and the underlying symmetry of the microscopic system – the quantum vacuum. The discrete and continuous global symmetries of the underlying microscopic systems influence the topological classification producing the additional classes of system, which are protected by the combined action of symmetry and topology [2,31,32,35,36]. They also determine the effective symmetries emerging at low energy, such as $SU(2)$ gauge symmetry in ^{3}He–A, which follows from the discrete Z_2 symmetry of the underlying high-energy theory [2].

Especially we are interested in the case, when the original multi-fermion system consists of real fermions, i.e. it is the system of the underlying Majorana fermions of general type not obeying Lorentz invariance. This case may be related both to emergent gravity and to the foundations of quantum mechanics. The equations of ordinary quantum mechanics are described in terms of complex numbers. These are the Weyl equation; the Dirac equation obtained after electroweak symmetry breaking, when particles acquire Dirac masses; and finally the Schrödinger equation obtained for energies below the mass parameters. As is known, Schrödinger strongly resisted to introduce $i = \sqrt{-1}$ into his wave equations (see Yang [37]). The imaginary unit $i = \sqrt{-1}$ is the product of human mind, which is mathematically convenient. However, all the physical quantities are real, which implies that the imaginary unit should not enter any physical equation.

This suggests that the underlying microscopic physics is described solely in terms of the real numbers, while the complexification occurs on the way from microscopic to macroscopic physics, i.e. complexification of quantum mechanics (and of the quantum field theory) is the emergent phenomenon that appears at low energies. To see that we start with underlying microscopic system described in terms of the real-valued multi-component spinor, whose evolution is governed by the differential equation with real coefficients. We find that if the vacuum is topologically nontrivial, the low energy phenomena will be described by the emergent Weyl quantum mechanics, which is expressed in terms of the emergent complex numbers.

The quantum dynamics of the corresponding field system is described by the integral over the n-component Grassmann variables ψ that does not contain imaginary unity. In the low energy approximation the multi-component Majorana fermions are reduced to the two-component Weyl fermions, which description is given in terms of the complex-valued two component wave function. The functional integral of e^{iS} is over the two sets of 2-component Grassmann variables Ψ and $\bar{\Psi}$, where S is the action for the emergent Weyl fermions Ψ (and the conjugated fermions $\bar{\Psi}$) in the presence of the emergent vierbein e_a^μ and emergent gauge field.

It is worth mentioning that in most of the cases the main symmetry of the gravitational theory (invariance under the diffeomorphisms) does not arise. For emergence of the diffeomorphism invariance the Lorentz violation scale must be much higher than the Planck scale. If this hierarchy of scale is not obeyed, in addition to Eq. (1) the effective action contains the terms that do not depend on $\Psi, \bar{\Psi}$ but depend on e_k^j, C_μ and B_μ directly. These terms are, in general case, not invariant under the diffeomorphisms. That's why in the majority of cases we may speak of the gravity only as the geometry experienced by fermionic quasiparticles. The fluctuations of the fields e_k^j, B_μ, and C_μ themselves are not governed by the diffeomorphism-invariant theory.

We shall demonstrate, that under certain reasonable assumptions the emergent spin connection C_μ^{ab} in the considered systems is absent. This means, that we deal with the emergent teleparallel gravity, i.e. the theory of the varying Weitzenbock geometry.[1]

On the high-energy side the application of the given pattern may be related to the unification of interactions in the paradigm, in which at the extremely high energies the Lorentz-invariance as well as the general covariance are lost. In this paradigm Lorentz symmetry, the two-component Weyl fermions that belong to its spinor representation, the gravitational and gauge fields appear at low energies as certain collective excitations of the microscopic theory.

The paper is organized as follows. In Section 2 we describe the original construction of Hořava [1] and give its proof for the particular case of the $3+1$ D system, in which Fermi-surface is reduced to Fermi-point. In Section 3 we generalize the construction of Section 2 to the case, when the original system contains multi-component Majorana fermions. In Section 4 we end with the conclusions.

[1] The Riemann–Cartan space is defined by the translational connection (the vierbein) and the Lorentz group connection. There are two important particular cases. Space is called Riemannian if the translational curvature (torsion) vanishes. If the Lorentz group curvature vanishes, it is called Weitzenbock space.

2. Emergent Weyl spinors in the system with multi-component fermions (Hořava construction)

2.1. The reduction of the original multi-fermion model to the model with minimal number of spinor components

Following [1] we consider the condensed matter model with n-component spinors ψ. The partition function has the form:

$$Z = \int D\psi\, D\bar{\psi} \exp\left(i \int dt \sum_{\mathbf{x}} \bar{\psi}_{\mathbf{x}}(t)(i\partial_t - \hat{H})\psi_{\mathbf{x}}(t)\right). \tag{3}$$

Here the Hamiltonian H is the Hermitian matrix function of momentum $\hat{\mathbf{p}} = -i\nabla$. We introduced here the symbol of the summation over the points of coordinate space. This symbol is to be understood as the integral over d^3x for continuous coordinate space. First, we consider the particular case, when there is no interaction between the fermions and the coefficients in the expansion of H in powers of $\mathbf{p}$ do not depend on coordinates. We know, that there is the "repulsion" between the energy levels in ordinary quantum mechanics. Similar situation takes place for the spectrum of $\hat{H}$. The eigenvalues of $\hat{H}$ are the real-valued functions of $\mathbf{p}$.

Several branches of spectrum for the Hermitian operator $\hat{H}$ repel each other, i.e. any small perturbation pushes apart the two crossed branches. That's why only the minimal number of branches of its spectrum may cross each other. This minimal number is fixed by the topology of momentum space that is the space of parameters $\mathbf{p}$.

Let us consider the position $\mathbf{p}^{(0)}$ of the crossing of n_{reduced} branches of $\hat{H}$. There exists the Hermitian matrix Ω such that the matrix $\tilde{H}(\mathbf{p}) = \Omega^+ \hat{H} \Omega$ is diagonal. In this matrix the first $n_{\mathrm{reduced}} \times n_{\mathrm{reduced}}$ block $\hat{H}_{\mathrm{reduced}}$ corresponds to the crossed branches (i.e. all eigenvalues of $\hat{H}_{\mathrm{reduced}}(\mathbf{p})$ coincide at $\mathbf{p} = \mathbf{p}^{(0)}$). The remaining block of matrix $\hat{H}_{\mathrm{massive}}$ corresponds to the "massive" branches. The functional integral can be represented as the product of the functional integral over "massive" modes and the integral over n_{reduced} reduced fermion components

$$\Psi(\mathbf{x}) = \Pi\Omega\psi(\mathbf{x}), \qquad \bar{\Psi}(\mathbf{x}) = \bar{\psi}(\mathbf{x})\Omega^+\Pi^+ \tag{4}$$

Here Π is the projector to space spanned on the first n_{reduced} components. Let us denote the remaining components of ψ by

$$\Theta(\mathbf{x}) = (1-\Pi)\Omega\psi(\mathbf{x}), \qquad \bar{\Theta}(\mathbf{x}) = \bar{\psi}(\mathbf{x})\Omega^+\left(1-\Pi^+\right) \tag{5}$$

Let us denote the only eigenvalue of $\hat{H}_{\mathrm{reduced}}(\mathbf{p}^{(0)})$ by E_0. The transformation $\psi_{\mathbf{x}} \to e^{-iE_0t}\psi_{\mathbf{x}}$, $H(\mathbf{p}) \to H(\mathbf{p}) - E_0$ leaves the expression in exponent of Eq. (3) unchanged. That's why we can always consider the matrix H_{reduced} equal to zero at the position of the branches crossing $\mathbf{p}^{(0)}$. We are left with the following expression for the partition function:

$$Z = \int D\Psi\, D\bar{\Psi}\, D\Theta\, D\bar{\Theta} \exp\left(i \int dt \sum_{\mathbf{x}} \left[\bar{\Psi}_{\mathbf{x}}(t)(i\partial_t - \hat{H}_{\mathrm{reduced}})\Psi_{\mathbf{x}}(t) + \bar{\Theta}_{\mathbf{x}}(t)(i\partial_t - \hat{H}_{\mathrm{massive}})\Theta_{\mathbf{x}}(t)\right]\right) \tag{6}$$

where $H_{\mathrm{massive}} = (1-\Pi)\hat{H}(1-\Pi^+)$.

Spectrum of operator $\hat{H}_{\mathrm{reduced}}$ has exceptional properties around vanishing eigenvalues. The corresponding eigenfunctions do not depend on time. The key point is that at low energy the integral over $\Psi(\mathbf{x})$ dominates. The other components Θ contribute the physical quantities with the fast oscillating factors, and, therefore, may be neglected in the description of the long-wavelength dynamics. As a result at the low energies we may deal with the theory that has the following partition function:

$$Z = \int D\Psi\, D\bar{\Psi} \exp\biggl(i \int dt \sum_{\mathbf{x}} \bar{\Psi}_{\mathbf{x}}(t)(i\partial_t - \hat{H}_{\mathrm{reduced}})\Psi_{\mathbf{x}}(t)\biggr) \tag{7}$$

Here we consider the situation, when Fermi energy coincides with the value of energy at the branches crossing. It was suggested by Froggatt and Nielsen in their random dynamics theory, that this case may be distinguished due to the specific decrease of particle density as follows from the Hubble expansion [19].

2.2. Momentum space topology, and the two-component spinors

This consideration allows to prove the Hořava conjecture presented in [1]. According to this conjecture any condensed matter theory with fermions and with the topologically protected Fermi-points may be reduced at low energies to the theory described by the two-component Weyl spinors. The remaining part of the proof is the consideration of momentum space topology. It protects the zeros of $\hat{H}_{\mathrm{reduced}}$ (i.e. it is robust to deformations) only when there is the corresponding nontrivial invariant in momentum space. The minimal number of fermion components that admits nontrivial topology is two. This reduces the partition function to

$$Z = \int D\Psi\, D\bar{\Psi} \exp\biggl(i \int dt \sum_{\mathbf{x}} \bar{\Psi}_{\mathbf{x}}(t)\bigl(i\partial_t - m^L_k(\hat{\mathbf{p}})\hat{\sigma}^k - m(\hat{\mathbf{p}})\bigr)\Psi_{\mathbf{x}}(t)\biggr) \tag{8}$$

where functions m^L_k, m are real-valued.

Let us, in addition, impose the CP symmetry generated by $\mathcal{CP} = -i\sigma^2$ and followed by the change $\mathbf{x} \to -\mathbf{x}$. Its action on the spinors is:

$$\mathcal{CP}\Psi(\mathbf{x}) = -i\sigma^2\bar{\Psi}^T(-\mathbf{x}) \tag{9}$$

It prohibits the term with $m(\mathbf{p})$. Thus operator $\hat{H}$ can be represented as

$$\hat{H} = \sum_{k=1,2,3} m^L_k(\mathbf{p})\hat{\sigma}^k \tag{10}$$

The topologically nontrivial situation arises when $m^L(\mathbf{p})$ has the hedgehog singularity. The hedgehog point zero is described by the topological invariant

$$N = \frac{e_{ijk}}{8\pi} \int_{\sigma} dS^i\, \hat{m}^L \cdot \left(\frac{\partial \hat{m}^L}{\partial p_j} \times \frac{\partial \hat{m}^L}{\partial p_k}\right), \quad \hat{m}^L = \frac{m^L}{|m^L|} \tag{11}$$

where σ is the S^2 surface around the point.

For the topological invariant $N = 1$ in Eq. (11) the expansion near the hedgehog point at $p^{(0)}_j$ in $3D$ $\mathbf{p}$-space gives

$$m^L_i(\mathbf{p}) = f^j_i\bigl(p_j - p^{(0)}_j\bigr). \tag{12}$$

Here by f_i^j we denote the coefficients in the expansion. It will be seen below, that these constants are related to the emergent vierbein. As a result, Eq. (8) has the form:

$$Z = \int D\Psi\, D\bar{\Psi} \exp\left(i \int dt \sum_{\mathbf{x}} \bar{\Psi}_{\mathbf{x}}(t)\big(i\partial_t - f_k^j\big(\hat{\mathbf{p}}_j - p_j^{(0)}\big)\hat{\sigma}^k\big)\Psi_{\mathbf{x}}(t)\right) \quad (13)$$

Remark 2.1. In the absence of the mentioned above CP symmetry we have, in addition, the function $m(\mathbf{p})$ that is to be expanded around $\mathbf{p}^{(0)}$: $m(\mathbf{p}) \approx f_0^j(p_j - p_j^{(0)})$, $i, j = 1, 2, 3$. The new quantities f_0^j are introduced here. So, in general case we arrive at the expression for the partition function of Eq. (13), in which the sum is over $k = 0, 1, 2, 3$, and $j = 1, 2, 3$ while $\sigma^0 = 1$. The situation here becomes much more complicated, than in the presence of the CP symmetry. Namely, when $|f_0^j[f^{-1}]_j^a| \geqslant 1$, we have the more powerful zeros of the Hamiltonian (better to say, of its determinant). For $|f_0^j[f^{-1}]_j^a| > 1$ there is the conical Fermi-surface of co-dimension $p = 1$ given by the equation

$$f_k^j\big(p_j - p_j^{(0)}\big) = 0, \quad j = 1, 2, 3;\ k = 0, 1, 2, 3 \quad (14)$$

There exists the choice of coordinates, such that on this Fermi surface the energy of one of the two branches of spectrum of H is equal to zero. The energy corresponding to the second branch vanishes at $\mathbf{p}^{(0)}$ only, where the two branches intersect each other. However, in this situation the first branch dominates the dynamics, and we already do not deal with the Fermi-point scenario of the effective low energy theory. That's why the CP-invariance is important because it protects the system from the appearance of the Fermi surface in the vicinity of the branches crossing. It is worth mentioning, that in the marginal case $|f_0^j[f^{-1}]_j^a| = 1$ we deal with the line of zeros of the Hamiltonian (Fermi-surface of co-dimension $p = 2$). We do not consider here the other marginal cases, such as that, in which $\det f_a^j = 0$.

In the following we shall imply, that there is the additional symmetry (like the mentioned above CP symmetry) that protects the system from the appearance of the more powerful zeros in the spectrum of the Hamiltonian (i.e. Fermi surfaces and Fermi lines). The CP symmetry may be approximate instead of exact, i.e. it may be violated by small perturbations and the interactions. The approximate CP symmetry is enough to provide the inequality $|f_0^j[f^{-1}]_j^a| < 1$ that restricts the appearance of the Fermi-surfaces of co-dimension $p = 1$ and $p = 2$. In this case we may apply Lorentz transformation (boost) that brings the system to the reference frame, in which $f_0^j = 0$ for $j = 1, 2, 3$. In the following the value of $\mathbf{p}^{(0)}$ may be interpreted as the external vector potential. The interpretation of quantity f_k^j in terms of the emergent gravitational field will be given in the next subsection.

2.3. Taking into account interaction between the fermions

Next, we should consider the situation, when the coefficients of expansion of H in powers of $\mathbf{p}$, depend on coordinates and fluctuate. The original partition function for the fermions with the interaction between them can be written as follows:

$$Z = \int D\psi\, D\bar{\psi}\, D\Phi \exp\left(iR[\Phi] + i \int dt \sum_{\mathbf{x}} \bar{\psi}_{\mathbf{x}}(t)\big(i\partial_t - \hat{H}(\Phi)\big)\psi_{\mathbf{x}}(t)\right) \quad (15)$$

Here the new fields that provide the interaction between the fermions are denoted by Φ. R is some function of these fields. Now operator $\hat{H}$ also depends on these fields. In mean field approximation, when the values of Φ are set to their "mean" values we come back to the consideration of the previous subsections. However, at the end of the consideration the fluctuations of the fields Φ are to be taken into account via the fluctuations of the field f^a_k and the Fermi-point position $\mathbf{p}^{(0)}$.

Let us consider for the simplicity the low energy effective theory with only one emergent Weyl fermion. The interaction between the particles appears when the fluctuations of $p^{(0)}_k$ and f^j_k are taken into account. We assume, that these fluctuations are long-wave, so that the corresponding variables should be considered as if they would not depend on coordinates. Nevertheless, in the presence of the varied field Φ the time reversal symmetry is broken. As a result the partition function of the theory receives the form

$$Z = \int D\Psi\, D\bar{\Psi}\, D\Phi \exp\bigl(i R[\Phi]\bigr) \times \exp\biggl(i \int dt \sum_{\mathbf{x}} \bar{\Psi}_{\mathbf{x}}(t)\bigl(i\partial_t - m^L_{\Phi,k}(\hat{\mathbf{p}})\hat{\sigma}^k - m_\Phi(\hat{\mathbf{p}})\bigr)\Psi_{\mathbf{x}}(t)\biggr) \tag{16}$$

Here

$$m^L_{\Phi,i}(\mathbf{p}) \approx e\, e^j_i (p_j - B_j), \qquad m_\Phi(\mathbf{p}) \approx B_0 + e\, e^j_0 (p_j - B_j), \quad i,j = 1,2,3 \tag{17}$$

The appearance of the field B_0 reflects, that in the presence of interaction the value of energy at the position of the crossing of several branches of spectrum may differ from zero. We represented the quantity f^j_i of Eq. (12) (that depends now on the coordinates) as $f^j_i = e\, e^j_i$, where the fluctuating long-wave fields $e[\Phi]$, $B[\Phi]$ depend on the primary fields Φ. This representation for f^j_i is chosen in this way in order to interpret the field e^j_i as the vierbein. We require $e^0_a = 0$ for $a = 1,2,3$, and $e \times e^0_0 = 1$. Here $e^{-1} = e^0_0 \times \det_{3\times 3} e^i_a = e^0_0$ is equal to the determinant of the vierbein e^i_a. In the mean field approximation, Φ is set to its mean value Φ_0, while $B_0[\Phi_0] = 0$, and $e\, e^k_a[\Phi_0] = f^k_a$, where variable f was introduced in Section 2.2. It is implied (see Remark 2.1), that the approximate CP symmetry is present, that may be slightly violated by the interactions. This means, that the values of e^j_0 are suppressed compared to the values of e^j_k for $k = 1,2,3$. This allows to keep the Fermi point in the presence of interactions.

As a result, the partition function of the model may be rewritten as:

$$Z = \int D\Psi\, D\bar{\Psi}\, D e^i_k D B_k e^{iS[e^j_a, B_j, \bar{\Psi}, \Psi]} \tag{18}$$

with

$$S = S_0[e, B] + \frac{1}{2}\biggl(\int dt\, e \sum_{\mathbf{x}} \bar{\Psi}_{\mathbf{x}}(t) e^j_a \hat{\sigma}^a \hat{D}_j \Psi_{\mathbf{x}}(t) + (\text{h.c.})\biggr), \tag{19}$$

where the sum is over $a, j = 0,1,2,3$ while $\sigma^0 \equiv 1$, and $\hat{D}$ is the covariant derivative that includes the $U(1)$ gauge field B. $S_0[e, B]$ is the part of the effective action that depends on e and B only.

Remark 2.2. It is worth mentioning, that to write the expressions for the functional integral Eq. (18) and the expression for the action Eq. (19) is not enough to define the field system. Besides, we are to impose boundary conditions on the fields. Typically, the anti-periodic in time boundary conditions are imposed on the spinor fields in quantum field theory. These boundary conditions correspond to the choice of vacuum, in which all states with negative energy are occupied. This is important to point out the reference frame, in which these anti-periodic boundary conditions in time are applied. Here and below we always imply, that these boundary conditions are imposed in the synchronous reference frame, i.e. in the one, in which the mean values $\langle e^j_0 \rangle$ vanish for $j = 1, 2, 3$.

Remark 2.3. Eq. (19) is reduced to Eq. (16) with m, m^L given by Eq. (17) if the particular gauge (of the emergent $SO(3,1)$) is fixed. In this gauge $e^0_j = 0$ for $j = 1, 2, 3$. Besides, we rescale time in such a way, that $e\, e^0_0 = 1$. This means, that the term S_0 contains the corresponding gauge fixing term. Even modulo this gauge fixing the theory given by Eq. (19) is not diffeomorphism-invariant. The fermionic term alone would become diffeomorphism-invariant if the spin connection of zero curvature is added. Then, in addition Eq. (19) is to be understood as the result of the gauge fixing corresponding to vanishing spin connection. In some cases $S_0[e, B]$ may be neglected, and only the second term of Eq. (19) contributes the dynamics. Then the fields e^μ_k and B_μ may be identified with the true gravitational field (vierbein) and the true gauge field correspondingly (modulo mentioned above gauge fixing). Their effective action is obtained as a result of the integration over the fermions. It is worth mentioning, that in most of the known condensed matter systems with Fermi-points (say, in ^{3}He–A) we cannot neglect the term $S_0[e, B]$. That's why the given opportunity in the condensed matter theory remains hypothetical.

Recall, that we have considered the long-wavelength fluctuations of the emergent fields B and e. That is we neglected the derivatives of these fields. In the fermion part of the action in Eq. (19) there are no dimensional parameters. The only modification of this action that is analytical in B, e and their derivatives and that does not contain the dimensional parameters is if the covariant derivative D receives the contribution proportional to the derivative of e. That's why, even for the non-homogeneous variations of e and B in low energy approximation we are left with effective action of the form of Eq. (19) if the value of the emergent electromagnetic field is much larger than the order of magnitude of quantity $|\nabla e^k_a|$. Such a situation takes place, for example for the consideration of the emergent gravity in graphene [30].

Let us formalize the consideration of the given section as the following theorem.

Theorem 2.1. *The multi-fermion system without interaction between the particles in the vicinity of the Fermi-point (Fermi surface of co-dimension $p = 3$) is reduced to the model that is described by the two-component Weyl fermions described by partition function Eq. (16). In addition, we require, that the (approximate) CP symmetry is present. This symmetry prohibits the appearance of the Fermi surfaces of co-dimension $p = 1$ and $p = 2$ and results in the suppression of the values of $m(\mathbf{p})$ compared to the values of $m^L(\mathbf{p})$. The nontrivial momentum space topology with the topological invariant of Eq. (11) equal to unity provides that the effective low energy theory has the partition function of Eq. (13) with some constants f^j_k that depend on the underlying microscopic theory.*

When the interaction between the original fermions in this system is taken into account (while momentum space topology remains the same as in the non-interacting theory), the partition function of the low energy effective theory receives the form of Eq. (18) with the effective action Eq. (19). This is the partition function of Weyl fermion in the presence of the emergent vierbein e

and the emergent $U(1)$ gauge field B. Both these fields represent certain collective excitations of the microscopic theory. (It is assumed, that the value of the emergent electromagnetic field is much larger than the order of magnitude of quantity $|\nabla e^k_a|$.)

Remark 2.4. One can see, that in the considered long wave approximation the emergent spin connection C_μ does not arise. That's why we deal with the emergent teleparallel gravity described by the vierbein e^i_j only.

The given theorem represents the main statement given without proof in [1] in a more detailed and elaborated form (for the particular case of $3 + 1$ D system with Fermi-surface reduced to the Fermi-point). We considered only one Fermi point. This case also corresponds to the situation, when there exist several Fermi points, but the corresponding collective excitations do not correlate with each other. The situation, when the correlation is present is more involved. We make a remark on it at the end of Section 3.

3. Emergent Weyl spinors in the system of multi-component Majorana fermions

In this section we consider the generalization of the problem considered in the previous section to the case, when the original system contains multi-component Majorana fermions.

3.1. Path integral for Majorana fermions

On the language of functional integral the evolution in time of the field system is given by the correlations of various combinations of the given fields. The lagrangian density for n-component Majorana fermions ψ can be written in the form:

$$L_{\mathrm{Majorana}} = \psi^T_{\mathbf{x}}(t)(i\partial_t + i\hat{A})\psi_{\mathbf{x}}(t), \tag{20}$$

where $\hat{A}$ is the arbitrary operator that may be highly non-local. First, we consider the situation, when there is no interaction between the original Majorana fermions. This means, that operator $\hat{A}$ does not depend on the other fields. As a result the partition function is represented as

$$Z = \int D\psi \exp\left(-\int dt \sum_{\mathbf{x}} \psi^T_{\mathbf{x}}(t)(\partial_t + \hat{A})\psi_{\mathbf{x}}(t)\right) \tag{21}$$

Various correlators of the field ψ are given by

$$\begin{aligned}&\langle \psi_{\mathbf{x}_1}(t_1)\psi_{\mathbf{x}_2}(t_2)\cdots\psi_{\mathbf{x}_2}(t_2)\rangle \\ &\quad = \int D\psi \exp\left(-\int dt \sum_{\mathbf{x}} \psi^T_{\mathbf{x}}(t)(\partial_t + \hat{A})\psi_{\mathbf{x}}(t)\right)\psi_{\mathbf{x}_1}(t_1)\psi_{\mathbf{x}_2}(t_2)\cdots\psi_{\mathbf{x}_2}(t_2)\end{aligned} \tag{22}$$

Here ψ is the n-component anti-commuting variable. The Majorana nature of the fermions is reflected by the absence of the conjugated set of variables $\bar{\psi}$ and the absence of the imaginary unit in the exponent. The dynamics of the system is completely described by various correlators of the type of Eq. (22). It is worth mentioning that the complex numbers do not enter the dynamics described by Eq. (22). It can be easily seen, that if A is linear in the spacial derivatives, and is represented by the product $B_a\nabla^a$, where B_a do not depend on coordinates, then B_a should be symmetric. (For the anti-symmetric B_a expression $\sum_{\mathbf{x}} \psi^T_{\mathbf{x}} B_a \nabla^a \psi_{\mathbf{x}}$ vanishes.) We feel this instructive to give the representation of the partition function of Eq. (21) in terms of the analogues of the energy levels.

We consider the functional integral over real fermions basing on the analogy with the integral over complex fermions (see [38]). We start from the partition function of Eq. (21). In lattice discretization the differential operator $\hat{A}$ is represented as the skew-symmetric $Nn \times Nn$ matrix, where N is the total number of the lattice points while n is the number of the components of the spinor ψ. As a result there exists the orthogonal $Nn \times Nn$ transformation Ω that brings matrix $\hat{A}$ to the block-diagonal form with the 2×2 blocks of the form

$$E_k\hat{\beta} = E_k \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{23}$$

with some real values E_k. We represent ψ as $\psi(x,t) = \sum_n c_{a,n}(t)\Psi_{a,n}(x)$, where $a = 1, 2$, and $\hat{A}$ has the above block-diagonal form in the basis of $\Psi_{a,n}$. These vectors are normalized to unity ($\int d^3x\, \Psi^T_{an}\Psi_{an} = 1$). Further, we represent

$$Z = \int dc \exp\Bigl(-\sum_{\eta,n} T c^T_{-\eta,n}[-i\eta + E_n\hat{\beta}]c_{\eta,n}\Bigr), \tag{24}$$

where the system is considered with the anti-periodic in time boundary conditions: $\psi(t+T,x) = -\psi(t,x)$. We use the decomposition

$$c_n(t) = \sum_{\eta=\frac{\pi}{T}(2k+1),\, k\in Z} e^{-i\eta t} c_{\eta,n}. \tag{25}$$

Integrating out the Grassmann variables c_n we come to:

$$Z = \prod_{\eta>0}\prod_n \bigl((\eta + E_n)(-\eta + E_n)T^2\bigr) = \prod_\eta \prod_n \bigl((\eta + E_n)T\bigr) = \prod_n \cos\frac{TE^\varphi_n}{2}. \tag{26}$$

The values E_n depend on the parameters of the Hamiltonian, with the index n enumerating these values. Eq. (26) is derived as follows. Recall that in (25) the summation is over $\eta = \frac{\pi}{T}(2k+1)$. The product over k can be calculated as in [38]:

$$\prod_{k\in Z}\Bigl(1 + \frac{E_nT}{\pi(2k+1)}\Bigr) = \cos\frac{E_nT}{2}. \tag{27}$$

Formally the partition function may be rewritten as

$$Z = \mathrm{Det}^{1/2}[\partial_t + \hat{A}] = \prod_n \cos\frac{E_nT}{2} \tag{28}$$

The explanation that the square root of the determinant appears is that operator $[\partial_t + \hat{A}]$ itself being discretized becomes the skew-symmetric matrix. Via the orthogonal transformations it may be made block-diagonal with the elementary 2×2 blocks. In the latter form the functional integral is obviously equal to the square root of the determinant because for the 2-component spinor η

$$\int d\eta \exp\left[\eta^T \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \eta\right] = a = \mathrm{Det}^{1/2}\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \tag{29}$$

We get (see also [38]):

$$Z = \sum_{\{K_n\}=0,1} \exp\left(\frac{iT}{2}\sum_n E_n - iT\sum_n K_n E_n\right) \tag{30}$$

Following [38], we interpret Eq. (30) as follows. K_n represents the number of occupied states with the energy E_n. These numbers may be 0 or 1. The term $\sum_n E_n$ vanishes if values E_n come in pairs with the opposite signs (this occurs when the time reversal symmetry takes place). We can rewrite the last expression in the form, when the integer numbers represent the numbers of occupied states of positive energy and the holes in the sea of occupied negative energy states:

$$Z(T) = \sum_{\{K_n\}=0,1} \exp\left(\frac{iT}{2}\sum_n |E_n| - iT\sum_n K_n |E_n|\right) \tag{31}$$

After the Wick rotation we arrive at

$$Z(-i/\mathcal{T}) = \sum_{\{K_n\}=0,1} \exp\left(\frac{1}{2\mathcal{T}}\sum_n |E_n| - \frac{1}{\mathcal{T}}\sum_n K_n |E_n|\right), \tag{32}$$

where $\mathcal{T}$ is temperature. This shows, that in equilibrium the configuration dominates with the vanishing numbers K_n. This corresponds to the situation, when all states with negative energy are occupied. This form of vacuum is intimately related with the anti-periodic in time boundary conditions imposed on ψ. The other boundary conditions would lead to the other prescription for the occupied states in vacuum.

The values E_n are given by the solution of the system of equations

$$\hat{A}\zeta_1 = E\zeta_2$$
$$\hat{A}\zeta_2 = -E\zeta_1 \tag{33}$$

for the pair ζ^1, ζ^2 of the real-valued n-component wave functions. Alternatively, we may solve equation

$$0 = [\hat{A} + \partial_t]\xi \tag{34}$$

Here the complex-valued n-component wave function $\xi = \zeta_1 + i\zeta_2$ has the particular dependence on time $\xi(x,t) = \tilde{\xi}(x)e^{-iE_n t}$. However, Eq. (34) does not contain imaginary unity. Therefore, we may consider its real-valued solutions. These solutions may be interpreted as the time-dependent real-valued spinor wave functions of Majorana fermions. It is worth mentioning, that there are no such real valued wave functions that would correspond to definite energy.

3.2. Repulsion of fermion branches → the reduced number of fermion species at low energy

The notion of energy in the theory described by operator $\hat{A}$ may be based on the definition of the values E_n given above. Besides, we may introduce the notion of energy scale $\mathcal{E}$ as the typical factor in the dependence of various dimensionless physical quantities q on time: $q \approx f(\mathcal{E}t)$, where f is a certain dimensionless function of dimensionless argument such that its derivatives are of the order of unity. With this definition of energy it can be shown, that at low energies only the minimal number of fermion components effectively contributes the dynamics. Below we make this statement explicit and present the sketch of its proof.

As it is explained in Section 3.1, operator $\hat{A}$ in lattice discretization is given by the skew-symmetric $Nn \times Nn$ matrix, where N is the total number of the lattice points while n is the number of the components of the spinor ψ. As a result there exists the orthogonal $Nn \times Nn$ lattice transformation $\hat{\Omega}$ that brings matrix $\hat{A}$ to the block-diagonal form with the 2×2 blocks of the form $E_k \hat{\beta} = E_k \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ with some real values E_k. In the continuum language matrix Ω becomes the operator that acts as an $n \times n$ matrix, whose components are the operators acting on the coordinates. There are several branches of the values of E_k. Each branch is parametrized by the $3D$ continuum parameters. Several branches of spectrum of E_k repel each other because they are the eigenvalues of the Hermitian operator. This repulsion means, that any small perturbation pushes apart the two crossed branches. That's why only the minimal number of branches of its spectrum may cross each other. This minimal number is fixed by the topology of momentum space (see below, Section 3.3.4).

As it was mentioned, there exists the orthogonal operator $\hat{\Omega}$ (it conserves the norm $\int d^3x\, \chi_x^T\, \chi_x$) such that the operator

$$A^{\text{block diagonal}} = \hat{\Omega}^T A \hat{\Omega} \tag{35}$$

is given by the block-diagonal matrix with the elementary 2×2 blocks:

$$A^{\text{block diagonal}} = \begin{pmatrix} \hat{\beta} E_1(\mathcal{P}) & 0 & \cdots & 0 \\ 0 & \hat{\beta} E_2(\mathcal{P}) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \hat{\beta} E_n(\mathcal{P}) \end{pmatrix} \tag{36}$$

Here we denote by $\mathcal{P}$ the three-dimensional vector that parametrizes the branches of spectrum and the basis vector functions that correspond to the given form of $\hat{A}$. The first n_{reduced} values E_k coincide at $\mathcal{P} = \mathbf{p}^{(0)}$. This value is denoted by $E_0 = E_1(\mathbf{p}^{(0)}) = E_2(\mathbf{p}^{(0)}) = \cdots$. The first $2n_{\text{reduced}} \times 2n_{\text{reduced}}$ block $A^{\text{block diagonal}}_{\text{reduced}}$ corresponds to the crossed branches. The remaining block of matrix $A^{\text{block diagonal}}_{\text{massive}}$ corresponds to the "massive" branches. The functional integral can be represented as the product of the functional integral over "massive" modes and the integral over $2n_{\text{reduced}}$ reduced fermion components

$$\Psi(\mathcal{P}, t) = e^{E_0 \hat{\beta} t} \Pi\, \psi(\mathcal{P}, t) \tag{37}$$

Here by $\hat{\beta}$ we denote $2n_{\text{reduced}} \times 2n_{\text{reduced}}$ matrix $\beta \otimes 1$, while Π is the projector to space spanned on the first $2n_{\text{reduced}}$ components. Let us denote the remaining components of ψ by

$$\Theta(\mathcal{P}, t) = (1 - \Pi)\, \Omega \psi(\mathcal{P}, t) \tag{38}$$

We arrive at

$$Z = \int D\Psi\, D\Theta \exp\biggl(-\int dt \sum_{\mathcal{P}} \bigl[\Psi^T_{\mathcal{P}}(t) e^{E_0 \hat{\beta} t} \bigl(\partial_t + \hat{A}^{\text{block diagonal}}_{\text{reduced}}(\mathcal{P})\bigr) e^{-E_0 \hat{\beta} t} \Psi_{\mathcal{P}}(t) \\ + \Theta^T_{\mathcal{P}} \bigl(\partial_t + \hat{A}^{\text{block diagonal}}_{\text{massive}}(\mathcal{P})\bigr) \Theta_{\mathcal{P}}\bigr]\biggr), \tag{39}$$

where $A^{\text{block diagonal}}_{\text{reduced}}(\mathcal{P}) = \Pi A(\mathcal{P}) \Pi^T$ while $A^{\text{block diagonal}}_{\text{massive}}(\mathcal{P}) = (1 - \Pi) A(\mathcal{P}) (1 - \Pi^T)$.

The exponent in Eq. (39) contains the following term that corresponds to the contribution of the fermion fields defined in a vicinity of $\mathcal{P} = \mathbf{p}^{(0)}$:

$$\mathcal{A}_{p^{(0)}} = \int dt \sum_{\mathcal{P},\, k=1,\ldots,n_{\text{reduced}}} \Psi^T_{k,\mathcal{P}}(t)\big(\partial_t + \beta\big[E_k(\mathcal{P}) - E_k\big(\mathbf{p}^{(0)}\big)\big]\big)\Psi_{k,\mathcal{P}}(t). \tag{40}$$

We have the analogue of the $2n_{\text{reduced}} \times 2n_{\text{reduced}}$ Hamiltonian $H(\mathcal{P}) = [E_k(\mathcal{P}) - E_k(\mathbf{p}^{(0)})]$ that vanishes at $\mathcal{P} = \mathbf{p}^{(0)}$. Following Section 3.1 we come to the conclusion, that in the expression for the partition function Eq. (31) the small values of energies dominate (when the negative energy states are occupied), and these energies correspond to the reduced fermions Ψ. It is important, that in order to deal with vacuum, in which negative energy states for Eq. (40) are occupied we need to impose the antiperiodic boundary conditions in time on Ψ (not on the original fermion field ψ). The other components Θ contribute the physical quantities with the fast oscillating factors because they are "massive", i.e. do not give rise to the values of E_n from the vicinity of zero. Therefore, these degrees of freedom may be neglected in the description of the long-wavelength dynamics.

Any basis of the wave functions is related via an orthogonal operator $\tilde{\Omega}$ to the basis of the wave functions, in which $\hat{A}_{\text{reduced}}$ has the form of the block-diagonal matrix (Eq. (36)). We require, that $\tilde{\Omega}$ commutes with $\hat{\beta}$ for the transformation to the basis associated with the observed low energy coordinates. This observed coordinate space may differ from the primary one, so that $\tilde{\Omega}$ is not equal to $\hat{\Omega}$ of Eq. (35). This new coordinate space in not the primary notion, but the secondary one. $[\tilde{\Omega}, \beta] = 0$ is the requirement, imposed on the representation of the theory, that allows to recover the usual Weyl spinors and the conventional quantum mechanics with complex-valued wave functions (see the next subsection). We denote the new coordinates by $\mathbf{Z}$ to distinguish them from the original coordinates $\mathbf{x}$, in which the partition function of Eq. (21) is written. In this new basis $\hat{A}_{\text{reduced}}$ is given by the differential operator. It is expressed as a series in powers of derivatives with real-valued $2n_{\text{reduced}} \times 2n_{\text{reduced}}$ matrices as coefficients. From $[\tilde{\Omega}, \beta] = 0$ it follows, that in this basis $[\hat{A}_{\text{reduced}}, \hat{\beta}] = 0$.

3.3. *The reduced* 4-*component spinors*

3.3.1. *Analytical dependence of* A_{reduced} *on* $\mathcal{P}$

In Section 3.2 it was argued that the number of fermion components at low energies should be even. The minimal even number that admits nontrivial momentum space topology (see below) is 4. That's why we consider the effective low energy four-component spinors. This corresponds to the crossing of the two branches of the energy.

The two values E_k coincide at $\mathcal{P} = \mathbf{p}^{(0)}$. The corresponding value of $E_{1,2}$ is denoted by $E_0 = E_1(\mathcal{P}^{(0)}) = E_2(\mathcal{P}^{(0)})$. The first 4×4 block $A^{\text{block diagonal}}_{\text{reduced}}$ of Eq. (36) corresponds to the crossed branches. The remaining block of matrix $A^{\text{block diagonal}}_{\text{massive}}$ corresponds to the "massive" branches. The Fermi point appears at $\mathbf{p}^{(0)}$ if chemical potential is equal to E_0. Then the four reduced components dominate the functional integral while the remaining "massive" components decouple and do not influence the dynamics. The form $A^{\text{block diagonal}}_{\text{reduced}} = \operatorname{diag}(E_1(\mathcal{P})\hat{\beta}, E_2(\mathcal{P})\hat{\beta})$ of the reduced matrix is exceptional. It is related by the 4×4 orthogonal transformation Ω' that commutes with $\hat{\beta}$ with the 4×4 matrix $A_{\text{reduced}}(\mathcal{P})$ of a more general form. In this form $A_{\text{reduced}}(\mathcal{P})$ also commutes with $\hat{\beta}$. In general case the dependence of $A_{\text{reduced}}(\mathcal{P})$ on $\mathcal{P}$ is analytical. This is typical for the functions that are encountered in physics. The non-analytical functions represent the set of vanishing measure in space of functions. However, this is not so for the exceptional block-diagonal form $A_{\text{reduced}}^{\text{block diagonal}}$ in case of non-trivial topology that protects the levels crossing.

Example. Let us illustrate this by the example, in which

$$A_{\text{reduced}}(\mathcal{P}) = \hat{\beta}\mathcal{P}_a \Sigma^a \tag{41}$$

Here the three real-valued 4×4 Σ-matrices form the basis of the $su(2)$ algebra and have the representation in terms of the three complex Pauli matrices:

$$\Sigma^1 = \sigma^1 \otimes 1, \qquad \Sigma^2 = i_{\text{eff}}\Sigma^1\Sigma^3, \qquad \Sigma^3 = \sigma^3 \otimes 1 \tag{42}$$

There exists the orthogonal matrix Ω' that brings A to the block-diagonal form:

$$A_{\text{reduced}}^{\text{block diagonal}}(\mathcal{P}) = \sigma^3 \otimes i\tau^2 \sqrt{\sum_a \mathcal{P}_a\mathcal{P}_a} \tag{43}$$

One can see, that in the form of Eq. (41) the matrix A_{reduced} is analytical at $\mathcal{P}=0$ while in the block-diagonal representation it is not.

In the following, speaking of the low energy dynamics, we shall always imply, that $\hat{A}_{\text{reduced}}$ is discussed, and shall omit the superscript "reduced". We shall refer to space of parameters $\mathcal{P}$ as to generalized momentum space. The zeros of $\hat{A}$ in this space should be topologically protected; i.e. they must be robust to deformations.

3.3.2. Introduction of new coordinate space

Let us identify the quantities $\mathcal{P}$ with the eigenvalues of operator $\hat{\mathcal{P}} = -\hat{\beta}\frac{\partial}{\partial \mathbf{Z}}$. Here by $\mathbf{Z}$ we denote the new coordinates. They do not coincide with the original coordinates $\mathbf{x}$. This means, that the fields local in coordinates $\mathbf{x}$ are not local in coordinates $\mathbf{Z}$ and vice versa.

$1+1$ *D example* We illustrate the appearance of the new coordinates $\mathbf{Z}$ by the following simple example. Let us consider the two-component Majorana spinors in $1+1$ dimensions with original non-local operator $\hat{A}$ given by

$$\hat{A} = \exp(-\hat{G}\alpha)\begin{pmatrix} \partial_{\mathbf{x}} & 0 \\ 0 & \partial_{\mathbf{x}} \end{pmatrix}\exp(\hat{G}\alpha), \tag{44}$$

where α is parameter while the integral operator $\hat{G}$ is given by

$$[\hat{G}\phi](\mathbf{x}) = \int d\mathbf{y}\, f(\mathbf{x}-\mathbf{y})\,\hat{\sigma}^1\,\phi(\mathbf{y}) \tag{45}$$

with some odd function $f(\mathbf{x})$. This operator is well-defined for the functions ϕ that tend to zero at infinity sufficiently fast.

Our aim is to find the two representations:

(1) Generalized momentum space, where $\hat{A} = E(\mathcal{P})\hat{\beta}$ for a certain function $E(\mathcal{P})$ of generalized momenta $\mathcal{P}$.
(2) New space with coordinates $\mathbf{Z}$, related to momentum space via identification $\mathcal{P} = -\hat{\beta}\partial_{\mathbf{Z}}$.

This aim is achieved via the following operator

$$\hat{\Omega} = \exp(-\hat{G}\alpha) \tag{46}$$

It is orthogonal and brings $\hat{A}$ to the form corresponding to the new coordinates $\mathbf{Z}$:

$$\hat{\Omega}^T \hat{A} \hat{\Omega} = \begin{pmatrix} \partial_{\mathbf{Z}} & 0 \\ 0 & \partial_{\mathbf{Z}} \end{pmatrix} = \hat{\beta}\hat{\mathcal{P}} \tag{47}$$

This defines the new coordinates $\mathbf{Z}$, in which operator $\hat{A}$ is proportional to $\hat{\beta}$. Space of coordinates $\mathbf{Z}$ differs from space of coordinates $\mathbf{x}$ just like conventional momentum space differs from the conventional coordinate space: the functions local in one space are not local in another one and vice versa. In generalized momentum space operator $\hat{A}$ receives the form $\hat{A} = E(\mathcal{P})\hat{\beta}$ with $E(\mathcal{P}) = \mathcal{P}$.

3.3.3. How the fermion number conservation reduces the general form of $\hat{A}$ for $3+1$ D Majorana fermions

It was argued, that for the low energy effective fermion fields in new coordinate space operator $\hat{A}$ has the form of the series in powers of the derivatives with the 4×4 real valued constant matrices as coefficients. Moreover, the reduced operator $\hat{A}$ commutes with $\beta = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. The latter condition may be identified with the fermion number conservation, that is rather restrictive. Below we describe the general form of the 4×4 operator $\hat{A}$ that may be expanded in powers of derivatives with real-valued constant matrices as coefficients. It may always be considered as skew-symmetric ($\sum_{\mathbf{x}} \chi_1^T \hat{A} \chi_2 = -\sum_{\mathbf{x}} \chi_2^T \hat{A} \chi_1$ for real-valued spinors $\chi_{1,2}$, i.e. $\hat{A}^T = -\hat{A}$) because the combination $\sum_{\mathbf{x}} \psi^T \hat{B} \psi$ vanishes for any symmetric operator $\hat{B}$ and Grassmann valued fields ψ. We shall demonstrate how the fermion number conservation reduces the general form of such skew-symmetric operator. Let us introduce the two commuting momentum operators:

$$\hat{\mathcal{P}}_\beta = -\hat{\beta}\nabla, \qquad \hat{\mathcal{P}}_\alpha = -\hat{\alpha}\nabla \tag{48}$$

where

$$\hat{\beta} = -1 \otimes \hat{\tau}_3\hat{\tau}_1 = -1 \otimes i\tau_2, \qquad \hat{\alpha} = -\hat{\sigma}_3\sigma_1 \otimes 1 = -i\sigma_2 \otimes 1 \tag{49}$$

The two commuting operators $\hat{\mathcal{P}}_\beta$ and $\hat{\mathcal{P}}_\alpha$ have common real-valued eigenvectors corresponding to their real-valued eigenvalues. Matrix A can be represented as the analytical function

$$\hat{A} = \mathcal{F}\big(\hat{\mathcal{P}}_\beta, \hat{\mathcal{P}}_\alpha, \hat{L}^k, \hat{S}^k\big), \tag{50}$$

where

$$\begin{aligned} \hat{L}^k &= (\hat{\sigma}_1 \otimes \hat{\beta}, -\hat{\alpha} \otimes 1, \hat{\sigma}_3 \otimes \hat{\beta}), \\ \hat{S}^k &= (\hat{\alpha} \otimes \hat{\tau}_1, -\hat{1} \otimes \beta, \hat{\alpha} \otimes \hat{\tau}_3). \end{aligned} \tag{51}$$

More specifically, it can be represented as

$$\begin{aligned} \hat{A} = {} & \sum_{k=1,2,3} m_k^L(\mathcal{P}_\beta)\hat{L}^k + \sum_{k=1,2,3} m_k^S(\mathcal{P}_\alpha)\hat{S}^k \\ & + m_1^I(\mathcal{P}_\beta)\hat{I}^1 - m_2^I(\mathcal{P}_\alpha)\hat{I}^1 + m_3^I(\mathcal{P}_\beta)\hat{I}^3 - m_4^I(\mathcal{P}_\alpha)\hat{I}^3 + m^o(\mathcal{P}_\beta)\hat{\beta}. \end{aligned} \tag{52}$$

Here

$$\hat{I}^1 = \hat{\sigma}_1 \otimes \hat{\tau}_3, \qquad \hat{I}^3 = \hat{\sigma}_3 \otimes \tau_3 \tag{53}$$

while $m_k^L(\mathcal{P})$, $m_k^S(\mathcal{P})$, $m_k^I(\mathcal{P})$, $m^o(\mathcal{P})$ are real-valued functions of the momenta $\mathcal{P}$. Functions $m_k^I(\mathcal{P})$, $m_k^o(\mathcal{P})$ are odd; $\hat{L}^k$ and $\hat{S}^k$ are the generators of the two $SO(3)$ groups; $\hat{\beta}$ and $\hat{\alpha}$ are real antisymmetric matrices that commute with all $\hat{L}^k$ (or $\hat{S}^k$) correspondingly; $\hat{I}^k$ are the matrices that commute with $\hat{\alpha} \otimes \hat{\beta}$ but do not commute with either of $\hat{\alpha}$ and $\hat{\beta}$. (Notice, that $\hat{\beta}\hat{L}^2 = \hat{\alpha}\hat{S}^2 = -\alpha \otimes \beta$. That's why odd part of the function m_2^S may be set equal to zero.)

According to our condition operator $\hat{A}$ commutes with matrix $\hat{\beta} = 1 \otimes (-i\tau_2)$. The coordinates of new emergent coordinate space are denoted by $\mathbf{Z}$. Matrix $\hat{\beta}$ anticommutes with $\hat{S}_k$, $k = 1, 3$ and $\hat{I}_k$, $k = 1, 2, 3, 4$. Yet another way to look at this symmetry is to require, that the momentum defined as $\hat{\mathcal{P}}_\beta = -\hat{\beta}\frac{\partial}{\partial \mathbf{Z}}$ is conserved, i.e. commutes with $\hat{A}$. This requirement reduces the partition function to

$$Z = \int D\Psi \exp\Big(-\int dt \sum_{\mathbf{Z}} \Psi^T_{\mathbf{Z}}(t)\big(\partial_t + i_{\text{eff}} m_k^L(\hat{\mathcal{P}}_\beta)\hat{\Sigma}^k + i_{\text{eff}} m(\hat{\mathcal{P}}_\beta)\big)\Psi_{\mathbf{Z}}(t)\Big) \tag{54}$$

where $m(\mathcal{P}_\beta) = m^o(\mathcal{P}_\beta) - m_2^S(\mathcal{P}_\alpha)$. We introduced the effective 4×4 imaginary unit

$$i_{\text{eff}} = \hat{\beta}. \tag{55}$$

Thus operator $\hat{A}$ can be represented as the analytical function of $\mathcal{P}_\beta$ and $\hat{L}^k$ only: $\hat{A} = \mathcal{F}(\hat{\mathcal{P}}_\beta, \hat{L}^k)$. Here we have introduced (see Eq. (42)) the 4×4 matrices forming the quaternion units Σ_k, that can be represented in terms of the 2×2 complex Pauli matrices. Matrix $1 \otimes \tau_3$ becomes the operator of complex conjugation.

3.3.4. CP-symmetry and topology of zeroes

First, let us impose the CP symmetry generated by $\mathcal{CP} = -i\sigma^2\tau^3 = \hat{S}^3$ and followed by the change $\mathbf{Z} \to -\mathbf{Z}$. Its action on the spinors is:

$$\mathcal{CP}\psi(\mathbf{Z}) = -i\sigma^2\tau^3\psi(-\mathbf{Z}) \tag{56}$$

It prohibits the term with $m(\mathcal{P})$. Thus operator $\hat{A}$ can be represented as

$$\hat{A} = \mathcal{F}\big(\hat{\mathcal{P}}_\beta, \hat{L}^k\big) = \sum_{k=1,2,3} m_k^L(\mathcal{P}_\beta)\hat{L}^k \tag{57}$$

The topologically nontrivial situation arises when $m^L(\mathcal{P})$ has the hedgehog singularity. The hedgehog point zero is described by the topological invariant

$$N = \frac{e_{ijk}}{8\pi}\int_\sigma dS^i\,\hat{m}^L \cdot \left(\frac{\partial \hat{m}^L}{\partial p_j} \times \frac{\partial \hat{m}^L}{\partial p_k}\right), \quad \hat{m}^L = \frac{m^L}{|m^L|} \tag{58}$$

where σ is the S^2 surface around the point. For the topological invariant $N = 1$ in Eq. (58) the expansion near the hedgehog point at $P_j^{(0)}$ in $3D$ $\mathcal{P}$-space gives

$$m_i^L(\mathcal{P}) = f_i^j\big(\mathcal{P}_j - P_j^{(0)}\big). \tag{59}$$

As a result, Eq. (60) has the form:

$$Z = \int D\Psi \exp\Big(-\int dt \sum_{\mathbf{Z}} \Psi^T_{\mathbf{Z}}(t)\big(\partial_t + i_{\text{eff}} f_k^j\big(\hat{\mathcal{P}}_j - P_j^{(0)}\big)\hat{\Sigma}^k\big)\Psi_{\mathbf{Z}}(t)\Big) \tag{60}$$

Operator $\hat{A}$ can be written in the basis of the eigenvectors of $\mathcal{P}$. As a result, we arrive at the 4×4 matrix function of real variable $\mathcal{P}$: $A(\mathcal{P}) = i_{\rm eff} m^L_k(\mathcal{P})\hat{\Sigma}^k$. The matrix $A(\mathcal{P})$ near the node has the form

$$A(\mathcal{P}) = i_{\rm eff}\Sigma^i f^j_i\big(\mathcal{P}_j - \mathcal{P}^{(0)}_j\big). \tag{61}$$

In the presence of the CP symmetry the topological invariant responsible for the singularity can be written analytically if one considers the extended matrices $A(P_\mu) \equiv A(\mathcal{P}, P_4) = P_4 + A(\mathcal{P})$. As a result, for generator of π_3 we have (compare with the generator of $\pi_3(R_n)$ for $n > 3$ on page 133 of Ref. [39]):

$$N = \frac{e_{\alpha\beta\mu\nu}}{48\pi^2}\,\mathbf{Tr}\int_\sigma dS^\alpha\, A^{-1}\partial_{p_\beta}A\,A^{-1}\partial_{p_\mu}A\,A^{-1}\partial_{p_\nu}A. \tag{62}$$

Here σ is the S^3 spherical surface around the node in 4D p_μ-space.

As in Section 2.2 in the absence of CP symmetry we should introduce the new variables f^j_0 and imply summation over $k = 0, 1, 2, 3$ in Eq. (60), where Σ^0 is identified with unity matrix. In this case Eq. (62) does not represent the topological invariant. According to Remark 2.1 we require, that the CP symmetry may be violated only slightly. The explicit meaning of the word “slightly” is given in Remark 2.1. This provides, that the more powerful manifold of zeros – the Fermi surface – does not appear in the vicinity of the Fermi point.

Remark 3.1. Up to the CPT transformation (that is the overall inversion $t \to -t$, $\mathbf{Z} \to -\mathbf{Z}$) the mentioned above CP symmetry coincides with the time inversion transformation $\mathcal{T}$. (This will become clear below, when we represent these emergent four-component Majorana spinors in the form of the two-component left-handed Weyl spinors.) For the original multi-component Majorana fermions the CPT transformation understood as $t \to -t$, $\mathbf{Z} \to -\mathbf{Z}$ may already not be the symmetry. The time reversal transformation T for the original multi-component fermions may be defined as the composition of $t \to -t$ and a certain transformation of the multi-component spinor $\psi \to \mathcal{T}\psi$, such that $\mathcal{T}^2 = -1$ and its action on the reduced 4-component fermions is given by $\mathcal{T} = \mathcal{CP} = -i\sigma^2\tau^3$. The CP transformation of the original multi-component spinors may be defined as CPT × T. The CP transformation of the low energy emergent fermions may originate, for example, from CP or T symmetry of the original multi-fermion system.

3.3.5. Propagator, Hamiltonian and Schrödinger equation

It follows from the functional integral representation, that one can introduce the propagator (the Green’s function) and the Hamiltonian. In the presence of the CP-symmetry (when $f^j_0 = 0$) we have:

$$G^{-1} = i_{\rm eff}A(P_\mu) \equiv -H_{\mathcal{P}} + i_{\rm eff}P_4. \tag{63}$$

This means that the Green’s function here is determined on the imaginary axis, i.e. it is the Euclidean Green function. In terms of the Green’s function the topological invariant N in Eq. (62) has the following form:

$$N = \frac{e_{\alpha\beta\mu\nu}}{48\pi^2}\,\mathbf{Tr}\int_\sigma dS^\alpha\, G\partial_{p_\beta}G^{-1}G\partial_{p_\mu}G^{-1}G\partial_{p_\nu}G^{-1}, \tag{64}$$

where σ is S^3 surface around the Fermi point in (P_1, P_2, P_3, P_4) space.

If the Hamiltonian belongs to topological class $N = 1$ or $N = -1$, it can be adiabatically deformed to the Weyl Hamiltonian for the right-handed and left-handed fermions respectively:

$$H = N\left(\Sigma^1 P_x + \Sigma^2 P_y + \Sigma^3 P_z\right), \quad N = \pm 1, \tag{65}$$

where the emergent Pauli matrices Σ^i describe the emergent relativistic spin.

The matrix i_{eff}, which commutes with the Hamiltonian, corresponds to the imaginary unit in the time dependent Schrödinger equation. The latter is obtained, when p_4 is substituted by the operator of time translation, $p_4 \to \partial_t$:

$$i_{\mathrm{eff}} \partial_t \chi = H\chi. \tag{66}$$

The whole wave dynamics may be formulated in terms of real functions only. The Hamiltonian is expressed through the momentum operator $\hat{\mathcal{P}}_\beta$. Its eigenvalues are parametrized by the eigenvalues $\mathcal{P}$ of momentum, the projection $n = \pm 1$ of emergent spin $\hat{\Sigma}$ on vector $m(\mathcal{P})$, and the eigenvalue $C = \pm 1$ of the conjugation operator $\hat{C} = 1 \otimes \tau_3$:

$$|C, n, \mathcal{P}\rangle \equiv \left[e^{\frac{1}{2} i_{\mathrm{eff}} \hat{\Sigma} \phi[m(\mathcal{P})]} \times e^{i_{\mathrm{eff}} \mathcal{P} \mathbf{x}}\right] |C\rangle \otimes |n\rangle, \tag{67}$$

where $|C\rangle = \frac{1}{2}\binom{1+C}{-1+C}$, and $|n\rangle = \frac{1}{2}\binom{1+n}{-1+n}$, while rotation around the vector ϕ by the angle equal to its absolute value transforms a unit vector directed along the third axis into the one directed along $m(\mathcal{P})$. Vectors $|C, n, \mathcal{P}\rangle$ are the eigenvectors of Hamiltonian correspondent to the eigenvalues $E = C|m(\mathcal{P})|$. Once at $t = 0$ the wave function is given by $|C, n, \mathcal{P}\rangle$, its dependence on time is given by:

$$\chi(t) = e^{-i_{\mathrm{eff}} C |m(\mathcal{P})| t} |C, n, \mathcal{P}\rangle \tag{68}$$

3.4. *Interaction between the fermions*

3.4.1. *Effective action for reduced fermions*

In this subsection we take into account the interactions between the original Majorana fermions. We consider for the simplicity the low energy effective theory with only one emergent Weyl fermion. The consideration is in general similar to that of Section 2.3. However, there is the important complication related to the Majorana nature of the original fermions. The partition function for the fermions with the interaction between them can be written in the form:

$$Z = \int D\psi \, D\Phi \exp\left(-R[\Phi] - \int dt \sum_{\mathbf{x}} \psi^T_{\mathbf{x}}(t)\left(\partial_t + \hat{A}(\Phi)\right)\psi_{\mathbf{x}}(t)\right) \tag{69}$$

Again, the new fields that provide the interaction between the fermions are denoted by Φ. R is some function of these fields. The fields Φ are assumed to be bosonic. All existing fermionic fields of the system are included into ψ. For the applications in condensed matter physics the function R is allowed to be complex-valued. However, the situation may be considered, when R is real-valued function. In this situation the functional integral Eq. (69) does not contain imaginary unity at all, which means, that the corresponding dynamics may be naturally described without using complex numbers. Matrix $\hat{A}$ also depends on Φ. When the values of Φ are set to their "mean" values $\Phi = \Phi_0$ we come back to the consideration of the system without interaction. In this system the reduced fields Ψ and massive fields Θ are defined.

The next step is to take into account the fluctuations of the fields Φ. We write again the effective action in terms of the fields Θ and Ψ. However, now the cross terms appear in the action that correspond to the transition between the two. Besides, the operator $A_{\rm reduced}$ depends on the fields Φ and does not necessarily commute with $\hat{\beta}$. Integrating out Θ we arrive at the effective action for the reduced four-component fields Ψ and Φ. This effective action in general case contains the products of more, than two components of Ψ, but those combinations are suppressed at low energies because the fields Θ are massive.

As a result we come to the partition function

$$Z = \int D\Psi \, D\Phi \exp\Bigl(-\int dt \sum_{\mathbf{Z}} \Psi^T_{\mathbf{Z}}(t)\bigl(\partial_t + A_{\rm reduced}[\Phi]\bigr)\Psi_{\mathbf{Z}}(t)\Bigr) \tag{70}$$

Now operator $\hat{A}[\Phi]$ does not necessarily commute with β. As a result $\hat{A}[\Phi]$ has the general form of Eq. (52) with functions m that depend on Φ. We assume, that these fluctuations are long-wave, so that the functions m should be considered as if they would not depend on coordinates. Besides, we define the new two component spinors starting from the four-component spinor $\Psi = (\Psi^1, \Psi^2, \Psi^3, \Psi^4)^T$. Those two-component spinors are given by

$$\Upsilon(\mathbf{x}) = \begin{pmatrix} \Psi^1(\mathbf{x}) + i\Psi^2(\mathbf{x}) \\ \Psi^3(\mathbf{x}) + i\Psi^4(\mathbf{x}) \end{pmatrix}, \qquad \bar{\Upsilon}(\mathbf{x}) = \begin{pmatrix} \Psi^1(\mathbf{x}) - i\Psi^2(\mathbf{x}) \\ \Psi^3(\mathbf{x}) - i\Psi^4(\mathbf{x}) \end{pmatrix}^T \tag{71}$$

In terms of these new spinors the partition function receives the form:

$$Z = \int D\Upsilon \, D\bar{\Upsilon} \, D\Phi \, e^{-R[\Phi]+iS[\Phi,\bar{\Upsilon},\Upsilon]} \tag{72}$$

with

$$S = \frac{1}{2}\Bigl(\int dt \sum_{\mathbf{Z}} \bar{\Upsilon}_{\mathbf{Z}}(t)\bigl(i\partial_t - m^L_{\Phi,k}(\hat{\mathbf{p}})\hat{\sigma}^k - m_\Phi(\hat{\mathbf{p}})\bigr)\Upsilon_{\mathbf{Z}}(t) + (\text{h.c.})\Bigr) + S_{\Upsilon\Upsilon}, \tag{73}$$

where $m^L_{\Phi,k}$ and m_Φ are some real functions of momenta $\mathbf{p} = -i\nabla$ while the term $S_{\Upsilon\Upsilon}$ contains various combinations of $\Upsilon^A\Upsilon^B$ and $\bar{\Upsilon}_C\bar{\Upsilon}_D$:

$$S_{\Upsilon\Upsilon} = -\frac{1}{2}\Bigl(\int dt \sum_{\mathbf{Z}} \Upsilon_{\mathbf{Z}}(t)\bigl(u^L_{\Phi,k}(\hat{\mathbf{p}})\hat{\sigma}^k + u_\Phi(\hat{\mathbf{p}})\bigr)\Upsilon_{\mathbf{Z}}(t) + (\text{h.c.})\Bigr) \tag{74}$$

with some complex-valued functions u^L_k, u. Recall, that without the fluctuations of Φ (when we set Φ equal to its average Φ_0) the term $S_{\Upsilon\Upsilon}$ does not appear.

3.4.2. Fermion number conservation. Emergent gauge field and emergent vierbein

As in Section 2.3 we expand functions $m^L_{\Phi,i}$ and $m_{\Phi,i}$ around the Fermi-point and take into account that the parameters of the expansion fluctuate:

$$m^L_{\Phi,i}(\mathcal{P}) \approx e\, e^j_i(\mathcal{P}_j - B_j), \qquad m_\Phi(\mathcal{P}) \approx e\, B_0 + e\, e^j_0(\mathcal{P}_j - B_j), \quad i,j = 1,2,3 \tag{75}$$

We represented here $f^j_i = e\, e^j_i$. The fluctuating long-wave fields $e[\Phi], B[\Phi]$ depend on the primary fields Φ. This representation for f^j_i is chosen in this way in order to interpret the field e^j_i as the vierbein. This means, that we require

$$e^0_a = 0, \quad \text{for } a = 1,2,3; \qquad e \times e^0_0 = 1; \qquad e^{-1} = e^0_0 \times \det{}_{3\times3}\, e^i_a = e^0_0 = \det{}_{4\times4} e^i_a \tag{76}$$

If Φ is set to its mean value Φ_0, we need

$$B_0[\Phi_0] = 0, \qquad e\, e_a^k[\Phi_0] = f_a^k, \tag{77}$$

where variable f was introduced in Section 3.3.4.

Besides, we expand complex-valued functions u_k^L, u around the Fermi point:

$$u^L_{\Phi,i}(\mathcal{P}) \approx e\, W_i + e\, q_i^j(\mathcal{P}_j - B_j), \qquad u_\Phi(\mathcal{P}) \approx e\, W_0 + e\, q_0^j(\mathcal{P}_j - B_j), \quad i, j = 1, 2, 3 \tag{78}$$

with complex-valued W_i, q_i^j. We set $q_a^0 = 0$, for $a = 0, 1, 2, 3$. As a result the partition function of the low energy effective theory receives the form

$$Z = \int D\Upsilon\, D\bar{\Upsilon}\, De\, DB\, Dq\, DW\, e^{iS[e,B,q,W] + iS[e,B,q,W,\bar{\Upsilon},\Upsilon]} \tag{79}$$

with the action given by

$$S = \frac{1}{2}\int dt\, e \sum_{\mathbf{Z}} \big(\bar{\Upsilon}_{\mathbf{Z}}(t) e_a^j \hat{\sigma}^a i \hat{D}_j \Upsilon_{\mathbf{Z}}(t) + i\epsilon_{AB} \Upsilon^A_{\mathbf{Z}}(t) \Upsilon^B_{\mathbf{Z}}(t) W_2$$
$$+ \Upsilon_{\mathbf{Z}}(t) q_1^j \hat{\sigma}^1 i \hat{D}_j \Upsilon_{\mathbf{Z}}(t) + \Upsilon_{\mathbf{Z}}(t) q_3^j \hat{\sigma}^3 i \hat{D}_j \Upsilon_{\mathbf{Z}}(t) + \Upsilon_{\mathbf{Z}}(t) q_0^j i \hat{D}_j \Upsilon_{\mathbf{Z}}(t) + (\text{h.c.})\big), \tag{80}$$

where the sum is over $a, j = 0, 1, 2, 3$ while $\sigma^0 \equiv 1$, and $\hat{D}$ is the covariant derivative that includes the $U(1)$ gauge field B. $S_0[e, B, q, W]$ is the part of the effective action that depends on bosonic fields only. The second term of Eq. (80) contains the combination of Weyl spinors entering the Majorana mass terms. The other fermion number breaking terms do not have the interpretation within the model of Weyl spinors in the presence of the gravitational field.

This interpretation does appears when we imply, that there exists the mechanism that suppresses those fluctuations of the fields Φ of Eq. (69) that break the fermion number conservation for the reduced Weyl fermions Υ, $\bar{\Upsilon}$ (i.e. forbids the appearance of terms proportional to $\Upsilon_A \Upsilon_B$ and $\bar{\Upsilon}^A \bar{\Upsilon}^B$). This formulation of the fermion number conservation is equivalent to the requirement, that the operator $\hat{A}$ acting on the four-component spinors Ψ commutes with $\hat{\beta}$. Then, similar, to Section 2.3 we come to the following theorem:

Theorem 3.1. *The system of multi-component Majorana fermions without interaction between the particles in the vicinity of the Fermi-point is reduced to the model that is described by the two-component Weyl fermions with partition function Eq. (70). In addition, we require, that the (approximate) CP symmetry is present. This symmetry prohibits the appearance of the Fermi surface and results in the suppression of the values of $m(\mathcal{P})$ compared to the values of $m^L(\mathcal{P})$. The nontrivial momentum space topology with the topological invariant of Eq. (58) equal to unity provides that the effective low energy theory has the partition function of Eq. (60) with some quantities f_i^j that depend on the underlined microscopic theory.*

We assume, that in the presence of interactions the fermion number of the coarse-grained fermions remains conserved while momentum space topology is the same as in the non-interacting theory. The fluctuations of the original bosonic field Φ are supposed to be long-wave. Then there exists new coordinate space (we denote the new coordinates by **Z***), in which the partition function of the low energy effective theory receives the form*

$$Z = \int D\Upsilon\, D\bar{\Upsilon}\, De\, DB\, e^{iS[e,B] + iS[e,B,\bar{\Upsilon},\Upsilon]} \tag{81}$$

with the action given by

$$S = \frac{1}{2}\int dt\, e \sum_{\mathbf{Z}} \big(\bar{\Upsilon}_{\mathbf{Z}}(t) e^j_a \hat{\sigma}^a i \hat{D}_j \Upsilon_{\mathbf{Z}}(t) + (\text{h.c.})\big), \tag{82}$$

$S_0[e, B]$ is the part of the effective action that depends on the fields e and B only. Both these fields represent certain collective excitations of the microscopic theory. (It is assumed, that the value of the emergent electromagnetic field is much larger than the order of magnitude of quantity $|\nabla e^k_a|$.) As well as in the previous section (Remark 2.2) we impose the antiperiodic boundary conditions in time on the spinor fields in the synchronous reference frame, where $\langle e^j_0\rangle = 0$ for $j = 1, 2, 3$.

Remark 3.2. If the functional $R[\Phi]$ of Eq. (69) is real-valued, then the appearance of the term $iS[e, B]$ in exponent of Eq. (81) (with real-valued $S[e, B]$) requires some comments. Let us explain how this may occur in principle by the consideration of the following example. We start from Eq. (81), and rewrite it as:

$$Z = \int De^i_k\, DB_k\, e^{iS_0[e,B]} \mathcal{Z}\big[e^i_k, B_k\big], \tag{83}$$

where

$$\mathcal{Z}\big[e^i_k, B_k\big] = \int D\Upsilon\, D\bar{\Upsilon}\, e^{iS[e^j_a, B_j, \bar{\Psi}, \Psi]} \tag{84}$$

If there exists the transformation of fields e^j_a, B_j, $\bar{\Upsilon}$, Υ such that $\mathcal{Z}$ remains invariant while S_0 changes the sign, then we have:

$$Z = \int De^i_k\, DB_k \cos\big(S_0[e, B]\big)\mathcal{Z}\big[e^i_k, B_k\big] = \int D\Phi\, e^{-R[\Phi]}\mathcal{Z}\big[e^i_k[\Phi], B_k[\Phi]\big] \tag{85}$$

with real-valued R.

Remark 3.3. Unlike the original Hořava construction of Section 2 the action of Eq. (82) is written in coordinates $\mathbf{Z}$ that differ from the original coordinates $\mathbf{x}$ of Eq. (69). The fermions in the two coordinates are related by an operator $e^{E_0 \hat{\beta} t} \tilde{\Omega} \Pi \Omega^T$:

$$\Psi_{\mathbf{Z}} = e^{E_0 \hat{\beta} t} \tilde{\Omega} \Pi \Omega^T \psi_{\mathbf{x}} \tag{86}$$

Here Ω brings operator $\hat{A}$ of Eq. (69) to the block-diagonal form of Eq. (36). The corresponding coordinates are denoted by $\mathcal{P}$ and may be identified with the coordinates of "momentum space". Π projects to the reduced four dimensional subspace of the n-component spinor space. Operator $\tilde{\Omega}$ commutes with $\hat{\beta}$ and relates spinors in "momentum space" with spinors defined in the new coordinates $\mathbf{Z}$. E_0 is the value of "energy" at the position of the branches crossing.

Remark 3.4. The considered above pattern of the emergent gravity and emergent $U(1)$ gauge field corresponds to the approximation, when the fields living at various Fermi points are not correlated. The case, when such a correlation appears complicates the pattern considerably. This may result in the appearance of non-Abelian gauge fields [2] and the generalization of the vierbein to the form, when the field f^k_l becomes matrix in flavor space. (Flavors enumerate Fermi points and the corresponding Weyl spinors.) This remark is related also to the case of Section 2.

It is worth mentioning, that the action in the form of Eq. (82) corresponds to the left-handed Weyl fermions in the presence of the emergent vierbein e^j_a. For our purposes it is enough to consider only the emergent left-handed fermions as the right handed ones are related to them by charge conjugation. The situation, when the two emergent left-handed fermions Υ_1, Υ_2 appear may be considered as the appearance of one Dirac four-component spinor. Its left-handed component is Υ_1 while the right-handed component is defined as $\epsilon_{AB}\bar{\Upsilon}^B_2$. When these two spinors are not correlated we have two different vierbeins and two different $U(1)$ gauge fields. If, for a certain reason, the two vierbeins coincide, then the two different $U(1)$ gauge fields may be represented as one common vector $U(1)$ gauge field coupled to the Dirac fermion in a usual way and the second common axial $U(1)$ gauge field that may alternatively be considered as an axial component of torsion originated from spin connection.

4. Conclusions

In this paper we discuss the many-body systems with multi-component fermions. First of all, we consider in some details the particular case of the Hořava construction presented in [1], when the Fermi surface of $3+1$ D model is reduced to the Fermi point. We prove Theorem 2.1. It contains the original statement of Hořava given in [1] without proof. Namely, in the vicinity of the topologically protected Fermi-point with topological invariant $N=1$ the emergent two-component Weyl spinors appear. In the case, when the fields living in the vicinities of different Fermi points do not correlate with each other, we may consider each low energy Weyl spinor separately. Then, the emergent gravity given by the emergent vierbein appears that is experienced by the Weyl fermions as the geometry of space, in which the fermionic quasi-particles propagate. Besides, the emergent $U(1)$ gauge field appears.

If the fields that belong to the vicinities of different Fermi points correlate with each other, instead of the $U(1)$ gauge field the non-Abelian gauge field may appear [2]. Besides, in this case the vierbein is to be replaced by matrix in flavor space (flavor index enumerates Fermi points and emergent Weyl spinors). The consideration of this complication is out of the scope of the present paper. The particular problem, which requires further investigation, is: what global discrete or continuous symmetry of the underlying microscopic theory (including the flavor symmetry) may reproduce the emergent gauge symmetries of SM or GUT?

Then we consider the generalization of the problem discussed in Section 2 to the case, when the original system contains multi-component Majorana fermions. This case has been considered in Section 3. Theorem 3.1 is proved, that is similar to Theorem 2.1. Again, in the vicinity of the separate Fermi point the emergent two-component Weyl spinor interacting with emergent vierbein and emergent $U(1)$ gauge field appears. The important difference from the case of Section 2 is that the Weyl spinors emerge in space of generalized coordinates $\mathbf{Z}$ that are different from the original coordinates $\mathbf{x}$. Besides, in order to arrive at the model of emergent Weyl fermions we suppose, that the interactions do not break the fermion number conservation for the emergent Weyl fermions. Remarkably, we do not need this requirement, when the interaction between the fermions may be neglected. This suggests that there can be a special discrete symmetry in the underlying microscopic theory, which forbids the violation of the fermion number conservation in the lowest order terms. The higher order terms may reflect the Majorana origin of the chiral Weyl particles, manifested in particular in possibility of neutrinoless double beta decay.

The considered general constructions may have applications both in the condensed matter physics and in the high energy physics. There may exist various condensed matter systems with multi-component fermions (both usual ones and Majorana fermions) and with the Fermi-points.

In particular, certain Weyl semi-metals may belong to this class of systems. General properties considered above predict, that the effective description of such systems may be given in terms of the Weyl spinors interacting with emergent gravity and emergent gauge field.

In the high energy theory the applications may be related to the paradigm, in which Lorentz symmetry, the fermions that belong to its spinor representations, the gravitational and gauge fields appear in the low energy effective description of the underlined theory that works at extremely high energies. In the scenario, in which this theory contains multi-component Majorana fermions, the observed coordinate space corresponds to the generalized coordinates **Z**, so that the coordinate space is the emergent phenomenon, which follows from the matrix structure in momentum space (see also [40]). Besides, the corresponding construction may be related to the foundations of quantum mechanics. The original Majorana fermion is described by the n-component real-valued wave function. The differential equation that describes its evolution has the real-valued coefficients. The emergent low energy Weyl spinor, in turn, is described by the complex-valued wave function. Thus, in this pattern the complexification of quantum mechanics is the emergent low energy phenomenon.

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