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Research Article

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Supersolutions to nonautonomous Choquard equations in general domains

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Abstract: We consider the nonlocal quasilinear elliptic problem:

$$-\Delta_m u(x) = H(x)((I_\alpha * (Qf(u)))(x))^\beta g(u(x)) \quad \text{in } \Omega,$$

where Ω is a smooth domain in $\mathbb{R}^N$, $\beta \geq 0$, I_α, $0 < \alpha < N$, stands for the Riesz potential, $f, g : [0, a) \to [0, \infty)$, $0 < a \leq \infty$, are monotone nondecreasing functions with $f(s), g(s) > 0$ for $s > 0$, and $H, Q : \Omega \to \mathbb{R}$ are nonnegative measurable functions. We provide explicit quantitative pointwise estimates on positive weak supersolutions. As an application, we obtain bounds on extremal parameters of the related nonlinear eigenvalue problems in bounded domains for various nonlinearities f and g such as e^u, $(1+u)^p$, and $(1-u)^{-p}$, $p > 1$. We also discuss the Liouville-type results in unbounded domains.

Keywords: quasilinear elliptic equations, m-Laplace operator, Liouville-type theorems, eigenvalue problems

MSC 2020: 35J92, 35A23, 35B09, 35B53, 47J10

1 Introduction

This work discusses positive supersolutions to the nonlocal quasilinear elliptic problem:

$$-\Delta_m u(x) = H(x)((I_\alpha * (Qf(u)))(x))^\beta g(u(x)) \quad \text{in } \Omega, \tag{1.1}$$

with $\beta \geq 0$. Here, Ω is a domain in $\mathbb{R}^N$:

$$\Delta_m u(x) = \operatorname{div}(|\nabla u(x)|^{m-2}\nabla u(x)), \quad 1 < m < \infty,$$

is the m-Laplace operator and

$$(I_\alpha * (Qf(u)))(x) = \int_\Omega I_\alpha(x-y)Q(y)f(u(y))\mathrm{d}y,$$

where $I_\alpha : \mathbb{R}^N \to \mathbb{R}$,

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad \text{with } A_\alpha = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)\pi^{\frac{N}{2}}2^\alpha},$$

which is the Riesz potential of order $0 < \alpha < N$. We assume that

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(C) $f, g : [0, a) \to [0, \infty)$, $0 < a \leq \infty$, are the monotone nondecreasing functions with $f(s), g(s) > 0$ for $s > 0$. Moreover, we assume that $H, Q : \Omega \to \mathbb{R}$ are nonnegative measurable functions.

Our motivation for the study of (1.1) comes from the general Choquard equation:

$$-\Delta u(x) = (I_\alpha * u^p)(x)u(x)^q \quad \text{in } \Omega, \tag{1.2}$$

and some of its variants have received a lot of attention in the literature [1,4,5,8,12–17,23–26,28,30,33,34,36–41,46]. For $\alpha = p = 2$, $q = 1$, and $\Omega = \mathbb{R}^3$, Problem (1.2) was introduced in [42] and it is known as the Choquard or Choquard-Pekar equation. It arises, for example, as a model in quantum theory of a Polaron at rest [19,42], an electron trapped in its own hole, in an approximation to the Hartree-Fock theory of one-component plasma [34], and in a self-gravitating matter model (see [32,35]), where it is referred to as the Schrödinger-Newton equation.

Using nonvariational methods, Moroz and Schaftingen [37] obtained sharp conditions for the nonexistence of nonnegative supersolutions to (1.2) in an exterior domain of $\mathbb{R}^N$, $N \geq 3$. They accomplished this by using nonlocal version of the Agmon-Allegretto-Piepenbrink positivity principle and an integral version of the comparison principle for the Laplacian in exterior domains. Very recently, Ghergu et al. [23] studied the existence and nonexistence of positive supersolutions for the quasilinear elliptic problem:

$$-\operatorname{div}\mathcal{A}(x, u, \nabla u) = (I_\alpha * u^p)(x)u(x)^q \quad \text{in } \Omega,$$

for a large class of operators, which includes the m-Laplace and the m-mean curvature operators, and obtained optimal ranges of exponents p, q, and α for which positive solutions exist.

This work provides pointwise estimates on positive weak supersolutions to (1.1). We emphasize that we allow $\beta = 0$, and hence, our results apply to quasilinear equations

$$-\Delta_m u(x) = H(x)g(u(x)) \quad \text{in } \Omega.$$

Our approach is based on the maximum principle and thus applies to many nonlocal quasilinear elliptic problems. For instance, as an application of our main results (Theorems 2.3 and 2.5), we obtain the Liouville-type results for

$$-\Delta_m u(x) = H(x)\left(\int_\Omega \frac{Q(y)u(y)^p}{|x-y|^{N-\alpha}}\,dy\right)^\beta u(x)^q \quad \text{in } \Omega,$$

in an unbounded domain Ω such as $\mathbb{R}^N$, $\mathbb{R}^N\setminus\{0\}$, exterior domains or more general unbounded domains with the property

$$\sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega) = \infty \quad \text{and/or} \quad \limsup_{x\in\Omega,\ |x|\to\infty} \frac{d_\Omega(x)}{|x|^s} = \infty$$

for some $0 < s < 1$, with the weights $H(x), Q(x) = |x|^\gamma$, $e^{\gamma x_1}$ or $|x_1|^\gamma$, $\gamma > 0$.

We also consider the eigenvalue problem:

$$\begin{cases} -\Delta_m u(x) = \lambda |x|^\gamma (I_\alpha * f(u)(x))^\beta g(u(x)) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and f and g satisfy (C) and some further assumptions. The extremal parameter of Problem (1.3) is defined as:

$$\lambda^* = \sup\{\lambda > 0\colon \text{(1.3) has a positive supersolution}\}. \tag{1.4}$$

With $\beta = \gamma = 0$ and with the zero Dirichlet boundary condition, Problem (1.3) becomes

$$\begin{cases} -\Delta_m u(x) = \lambda g(u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $g : [0, a) \to \mathbb{R}$, $0 < a \leq \infty$, is an increasing smooth function such that $g(0) > 0$ and $\lim_{s \nearrow a} \frac{g(s)}{s^{m-1}} = \infty$, is interesting already in the case $m = 2$. Typical examples of nonlinearities g are e^u, $(1+u)^p$, and $(1-u)^{-p}$ for

$p > m - 1$. Under the assumptions on g, it is known that there exists an extremal parameter $0 < \lambda^* < \infty$ such that if $0 < \lambda < \lambda^*$, then Problem (1.5) admits a minimal regular solution u_λ, while if $\lambda > \lambda^*$, then it admits no regular solution. Furthermore, the family $\{u_\lambda\}$ is increasing in λ, every u_λ is stable and we may consider $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$, which is a weak solution of (1.5) with $\lambda = \lambda^*$. The solution u^* is also stable, and it is called the extremal solution (see [7,11]). Regularity properties of the extremal solution u^* and estimates for the extremal parameter λ^* of Problem (1.5) have attracted a lot of attention.

We consider the following special case of (1.3) with a singular nonlinearity

$$\begin{cases} -\Delta_m u(x) = \lambda |x|^\gamma \left(I_\alpha * \dfrac{1}{(1-u(x))^p} \right)^\beta \dfrac{1}{(1-u(x))^q} & \text{in } \Omega, \\ 0 < u(x) < 1 & \text{in } \Omega, \end{cases} \tag{1.6}$$

where $p, q > 0$, $\Omega = B_R(0) = B_R$ and prove that the extremal parameter of Problem (1.6) satisfies

$$\lambda^* \le \left(\frac{\alpha}{N\omega_N A_\alpha} \right)^\beta \mathcal{B}(\gamma + N, 1 + \alpha\beta)^{-1} \left(\frac{\alpha\beta + m + \gamma}{\beta p + q + m - 1} \right)^{m-1} R^{-(\alpha\beta+m+\gamma)}, \tag{1.7}$$

where $\omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$ is the volume of the unit ball in $\mathbb{R}^N$ and $\mathcal{B}$ stands for the beta function. Note that Problem (1.6) with $\beta = 0$ becomes

$$\begin{cases} -\Delta_m u(x) = \dfrac{\lambda |x|^\gamma}{(1-u(x))^q} & \text{in } \Omega, \\ 0 < u(x) < 1 & \text{in } \Omega. \end{cases} \tag{1.8}$$

This equation appears in the so-called MEMS (micro-electro-mechanical systems) technology. In the two-dimensional case with $q = 2$, this equation models a steady state of a simple MEMS device, consisting of a dielectric elastic membrane covered by a thin conducting film attached to $\partial\Omega$. Here, λ is proportional to the applied voltage, and the permittivity profile $|x|^\gamma$ allows for varying dielectric properties of the membrane (see [18,20,21,27,29]). Since λ^* is the critical voltage beyond which a snap-through occurs and u^* is the optimal membrane deflection, it is important for the design of MEMS devices to know how the critical voltage λ^* (called pull-in voltage) and the pull-in distance $\|u^*\|_{L^\infty}$ depend on the membrane geometry and permittivity profile [18,20]. Our main result gives an explicit upper bound (1.7) for λ^* and lower bound for $\|u_\lambda\|_\infty$ for any positive supersolution u_λ. As far as we are aware, this result is new for Problem (1.3) even for the classical case $\beta = 1$.

2 Main results

We begin with a definition of weak solution to (1.1).

Definition 2.1. Let $1 < m < \infty$, $\beta \ge 0$, $0 < \alpha < N$ and assume that $\Omega \subset \mathbb{R}^N$ is a domain. A function $u \in W^{1,m}_{\mathrm{loc}}(\Omega)$ is a positive weak supersolution to (1.1)

(i) if $u > 0$ and

$$H(x) \left(\int_\Omega \frac{Q(y) f(u(y))}{|x-y|^{N-\alpha}} \mathrm{d}y \right)^\beta g(u(x)) \in L^1_{\mathrm{loc}}(\Omega);$$

(ii) if $\beta > 0$, then u satisfies

$$\int_\Omega \frac{Q(y) f(u(y))}{1 + |y|^{N-\alpha}} \mathrm{d}y < \infty;$$

(iii) for any $\phi \in C_c^\infty(\Omega)$ with $\phi \ge 0$, we have

$$\int_\Omega |\nabla u(x)|^{m-2}\nabla u(x)\cdot\nabla\phi(x)\mathrm{d}x \ge \int_\Omega H(x)((I_\alpha*(Qf(u)))(x))^\beta g(u(x))\phi(x)\mathrm{d}x.$$

For $x \in \Omega$ and $0 < r < d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$, we denote

$$m_x(r) = \inf_{y\in B_r(x)} u(y), \quad H_x(r) = \inf_{y\in B_r(x)} H(y), \quad \text{and} \quad Q_x(r) = \inf_{y\in B_r(x)} Q(y), \tag{2.1}$$

where $\inf_A$ denotes the essential infimum on the set A and $B_r(x) = \{y \in \mathbb{R}^N : |y - x| < r\}$ is the open ball with the center x and radius $r > 0$.

Remark 2.2. We note that if $u \in W^{1,m}_{\mathrm{loc}}(\Omega)$ is a solution to (1.1), then it is a weak supersolution to the m-Laplace equation. By [31, Theorem 3.63], we conclude that u is locally essentially bounded from below and that there exists a lower semicontinuous representative of u that satisfies

$$u(x) = \operatorname{ess}\liminf_{y\to x} u(y)$$

for every $x \in \Omega$. Here,

$$\operatorname{ess}\liminf_{y\to x} u(y) = \lim_{r\to 0} \operatorname*{ess\,inf}_{B_r(x)} u.$$

In particular,

$$m_x(0) = \lim_{r\to 0} m_x(r) = u(x).$$

Weak supersolutions to (1.1) are, *a priori*, defined only up to a set of Lebesgue measure zero, but the aforementioned lower semicontinuous representative allows us to discuss pointwise defined supersolutions. When it is useful, we may replace u by its lower semicontinuous representative denoted again by u.

The following pointwise estimate is our main result.

Theorem 2.3. *Let $1 < m < \infty$, $\beta \ge 0$, $0 < \alpha < N$ and assume that $\Omega \subset \mathbb{R}^N$ is a domain. Assume that f, g, H, and Q satisfy (C) and let u be a positive supersolution to (1.1) in Ω. Then,*

$$\int_{m_x(r)}^{u(x)} (f^\beta(s)g(s))^{-\frac{1}{m-1}}\mathrm{d}s \ge C_{\alpha,\beta,N}^{\frac{1}{m-1}} \int_0^r (s^{\alpha\beta+1}H_x(s)Q_x(s)^\beta)^{\frac{1}{m-1}}\mathrm{d}s, \tag{2.2}$$

for every $x \in \Omega$ and $0 < r < d_\Omega(x)$, with

$$C_{\alpha,\beta,N} = \left(\frac{N\omega_N A_\alpha}{\alpha}\right)^\beta \int_0^1 t^{N-1}(1-t)^{\alpha\beta}\mathrm{d}t. \tag{2.3}$$

Here, $\omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}$ is the volume of the unit ball in $\mathbb{R}^N$.

Proof. Let $x \in \Omega$ with $0 < r < d_\Omega(x)$. From Definition 2.1, we obtain

$$\int_{B_r(x)} |\nabla u(z)|^{m-2}\nabla u(z)\cdot\nabla\phi(z)\mathrm{d}z \ge \int_{B_r(x)} H(z)g(u(z))((I_\alpha*(Qf(u)))(z))^\beta\phi(z)\mathrm{d}x$$

for every $\phi \in C_c^\infty(B_r(x))$ with $\phi \ge 0$. For short, we write

$$-\Delta_m u(z) \ge H(z)g(u(z))((I_\alpha*(Qf(u)))(z))^\beta \quad \text{in } B_r(x).$$

For $z \in B_r(x)$, we set $r_z = r - |x - z|$. Then, we have $|y - x| \le |y - z| + |x - z| < r_z + |x - z| = r$ for $y \in B_{r_z}(z)$, which implies that $y \in B_r(x)$. We observe that

$$\begin{aligned}H(z)g(u(z))((I_\alpha*(Qf(u)))(z))^\beta &= H(z)g(u(z))\left(\int_\Omega I_\alpha(z-y)Q(y)f(u(y))\mathrm{d}y\right)^\beta\\ &\geq H_x(r)g(m_x(r))\left(\int_{B_{r_z}(z)} I_\alpha(z-y)Q(y)f(u(y))\mathrm{d}y\right)^\beta\\ &\geq H_x(r)g(m_x(r))Q_x(r)^\beta f(m_x(r))^\beta\left(\int_{B_{r_z}(z)} \frac{A_\alpha}{|z-y|^{N-\alpha}}\mathrm{d}y\right)^\beta\end{aligned}$$

for almost every $z \in B_r(x)$. A simple calculation gives

$$\begin{aligned}\int_{B_{r_z}(z)} \frac{A_\alpha}{|z-y|^{N-\alpha}}\mathrm{d}y &= \int_0^{r_z}\int_{\partial B_s(z)} \frac{A_\alpha}{|z-y|^{N-\alpha}}\mathrm{d}\sigma\mathrm{d}s\\ &= \int_0^{r_z}\frac{A_\alpha}{s^{N-\alpha}}N\omega_N s^{N-1}\mathrm{d}s\\ &= N\omega_N A_\alpha\int_0^{r_z}s^{\alpha-1}\mathrm{d}s\\ &= \frac{N\omega_N A_\alpha}{\alpha}r_z^\alpha,\end{aligned}$$

which implies that

$$\begin{aligned}&\int_{B_r(x)} |\nabla u(z)|^{m-2}\nabla u(z)\cdot\nabla\phi(z)\mathrm{d}z\\ &\quad\geq \left(\frac{N\omega_N A_\alpha}{\alpha}\right)^\beta H_x(r)g(m_x(r))Q_x(r)^\beta f(m_x(r))^\beta\int_{B_r(x)}(r-|x-z|)^{\alpha\beta}\phi(z)\mathrm{d}z\end{aligned}$$

for every $\phi \in C_c^\infty(B_r(x))$ with $\phi \geq 0$, i.e.,

$$-\Delta_m u(z) \geq \left(\frac{N\omega_N A_\alpha}{\alpha}\right)^\beta H_x(r)g(m_x(r))Q_x(r)^\beta f(m_x(r))^\beta(r-|x-z|)^{\alpha\beta} \quad \text{in } B_r(x). \tag{2.4}$$

We consider an auxiliary function

$$\Phi(z)=\Phi(|z|)=\int_{|z|}^1\left(\int_0^s\left(\frac{t}{s}\right)^{N-1}(1-t)^{\alpha\beta}\mathrm{d}t\right)^{\frac{1}{m-1}}\mathrm{d}s,$$

which is the unique radial solution of

$$\begin{cases}-\Delta_m\Phi(z)=(1-|z|)^{\alpha\beta} & \text{in } B_1(0),\\ \Phi'(0)=\Phi(1)=0.\end{cases}$$

By a scaling and translation argument, we observe that the function

$$\Psi(z)=r^{\frac{m+\alpha\beta}{m-1}}\Phi\left(\frac{|z-x|}{r}\right)$$

is a solution to

$$\begin{cases}-\Delta_m\Psi(z)=(r-|z-x|)^{\alpha\beta} & \text{in } B_r(x),\\ \Psi=0 & \text{on } \partial B_r(x).\end{cases}$$

From (2.4), we obtain

$$
\begin{aligned}
-\Delta_m u(z) &\geq \left(\frac{N\omega_N A_\alpha}{\alpha}\right)^{\beta} H_x(r) g(m_x(r)) Q_x(r)^{\beta} f(m_x(r))^{\beta} (r - |x - z|)^{\alpha\beta} \\
&\geq -\left(\frac{N\omega_N A_\alpha}{\alpha}\right)^{\beta} H_x(r) g(m_x(r)) Q_x(r)^{\beta} f(m_x(r))^{\beta} \Delta_m \Psi(z) \quad \text{in } B_r(x).
\end{aligned}
$$

Let $v(z) = u(z) - m_x(r)$, $z \in B_r(x)$, and

$$
w(z) = A H_x(r)^{\frac{1}{m-1}} g(m_x(r))^{\frac{1}{m-1}} Q_x(r)^{\frac{\beta}{m-1}} f(m_x(r))^{\frac{\beta}{m-1}} \Psi(z), \quad z \in B_r(x),
$$

with

$$
A = \left(\frac{N\omega_N A_\alpha}{\alpha}\right)^{\frac{\beta}{m-1}}. \tag{2.5}
$$

By the facts that Δ_m is positively homogeneous of order $m - 1$ and that constants can be added to a solution, we obtain

$$
-\Delta_m v(z) \geq -\Delta_m w(z) \quad \text{in } B_r(x),
$$

i.e.,

$$
\int_{B_r(x)} |\nabla v(z)|^{m-2} \nabla v(z) \cdot \nabla \phi(z) \, dz \geq \int_{B_r(x)} |\nabla w(z)|^{m-2} \nabla w(z) \cdot \nabla \phi(z) \, dz
$$

for every $\phi \in C_c^{\infty}(B_r(x))$ with $\phi \geq 0$. Since $v \geq 0$ on $\partial B_r(x)$ and $w = 0$ on $\partial B_r(x)$, by a comparison result, see Tolksdorf [45] or [43, Corollary 3.4.2], we have $v \geq w$ in $B_r(x)$. It follows that

$$
u(z) - m_x(r) \geq A H_x(r)^{\frac{1}{m-1}} g(m_x(r))^{\frac{1}{m-1}} Q_x(r)^{\frac{\beta}{m-1}} f(m_x(r))^{\frac{\beta}{m-1}} r^{\frac{m+\alpha\beta}{m-1}} \Phi\left(\frac{|x - z|}{r}\right)
$$

in $B_r(x)$. Since Φ is decreasing, we have $\Phi\left(\frac{|x-z|}{r}\right) \geq \Phi\left(\frac{h}{r}\right)$ for $z \in B_h(x)$ with $0 < h < r$. It follows that

$$
u(z) - m_x(r) \geq A H_x(r)^{\frac{1}{m-1}} g(m_x(r))^{\frac{1}{m-1}} Q_x(r)^{\frac{\beta}{m-1}} f(m_x(r))^{\frac{\beta}{m-1}} r^{\frac{m+\alpha\beta}{m-1}} \Phi\left(\frac{h}{r}\right).
$$

By taking essential infimum on the left-hand side over $z \in B_h(x)$, we obtain

$$
m_x(h) - m_x(r) \geq A H_x(r)^{\frac{1}{m-1}} g(m_x(r))^{\frac{1}{m-1}} Q_x(r)^{\frac{\beta}{m-1}} f(m_x(r))^{\frac{\beta}{m-1}} r^{\frac{m+\alpha\beta}{m-1}} \Phi\left(\frac{h}{r}\right),
$$

and then dividing both sides by $r - h$, we arrive at

$$
\frac{m_x(h) - m_x(r)}{r - h} \geq A H_x(r)^{\frac{1}{m-1}} g(m_x(r))^{\frac{1}{m-1}} Q_x(r)^{\frac{\beta}{m-1}} f(m_x(r))^{\frac{\beta}{m-1}} r^{\frac{m+\alpha\beta}{m-1}} \frac{\Phi\left(\frac{h}{r}\right)}{r - h}.
$$

By letting $h \to r$ and using the fact that

$$
\lim_{h \to r} \frac{\Phi\left(\frac{h}{r}\right)}{r - h} = -\frac{\Phi'(1)}{r} = \frac{1}{r}\left(\int_0^1 t^{N-1}(1 - t)^{\alpha\beta} \, dt\right)^{\frac{1}{m-1}},
$$

we obtain the following ordinary differential inequality with an initial value condition:

$$
\begin{cases}
-m_x'(r) \geq A H_x(r)^{\frac{1}{m-1}} g(m_x(r))^{\frac{1}{m-1}} Q_x(r)^{\frac{\beta}{m-1}} f(m_x(r))^{\frac{\beta}{m-1}} r^{\frac{\alpha\beta+1}{m-1}} \\
\quad \cdot \left(\int_0^1 t^{N-1}(1 - t)^{\alpha\beta} \, dt\right)^{\frac{1}{m-1}} \quad \text{a.e} \quad r \in (0, d_\Omega(x)), \\
m_x(0) = u(x).
\end{cases} \tag{2.6}
$$

Dividing through by $g(m_x(r))^{\frac{1}{m-1}}f(m_x(r))^{\frac{\beta}{m-1}}$, we may rewrite (2.6) as:

$$J'(r) \geq C_{\alpha,\beta,N}^{\frac{1}{m-1}} r^{\frac{\alpha\beta+1}{m-1}} H_x(r)^{\frac{1}{m-1}} Q_x(r)^{\frac{\beta}{m-1}} \quad \text{a.e. } r \in (0, d_\Omega(x)), \tag{2.7}$$

where $J : (0, d_\Omega(x)) \to \mathbb{R}$ is defined by:

$$J(r) = \int_{m_x(r)}^{u(x)} (f^\beta(s)g(s))^{-\frac{1}{m-1}} \mathrm{d}s.$$

Since $m_x(r)$ is nonincreasing and f and g are the positive functions, the function J is nondecreasing. By the Lebesgue differentiation theorem,

$$\int_0^r J'(s)\mathrm{d}s \leq J(r) - J(0) = J(r).$$

Thus, integrating (2.7) from 0 to r yields

$$\int_{m_x(r)}^{u(x)} (f^\beta(s)g(s))^{-\frac{1}{m-1}} \mathrm{d}s \geq C_{\alpha,\beta,N}^{\frac{1}{m-1}} \int_0^r (s^{\alpha\beta+1} H_x(s) Q_x(s)^\beta)^{\frac{1}{m-1}} \mathrm{d}s,$$

which proves (2.2). □

The following result is an immediate consequence of Theorem 2.3 with $\beta = 0$.

Corollary 2.4. *Let $1 < m < \infty$ and $0 < \alpha < N$ and assume that $\Omega \subset \mathbb{R}^N$ is a domain. Assume that g and H satisfy (C) and let u be a positive supersolution to*

$$-\Delta_m u(x) = H(x)g(u(x)) \quad \text{in } \Omega.$$

Then,

$$\int_{m_x(r)}^{u(x)} g(s)^{-\frac{1}{m-1}} \mathrm{d}s \geq N^{-\frac{1}{m-1}} \int_0^r (sH_x(s))^{\frac{1}{m-1}} \mathrm{d}s,$$

for every $x \in \Omega$ and $0 < r < d_\Omega(x)$. In particular, if $H \equiv 1$, then

$$\int_{m_x(r)}^{u(x)} g(s)^{-\frac{1}{m-1}} \mathrm{d}s \geq \frac{m-1}{mN^{\frac{1}{m-1}}} r^{\frac{m}{m-1}},$$

for every $x \in \Omega$ and $0 < r < d_\Omega(x)$.

Observe that in those points $x \in \Omega$, where $H_x(x) = 0$ or $Q_x(x) = 0$, the right-hand side of (2.2) becomes zero; hence, we gain nothing. In this case, we have the following result, which can be proved by making a slight modification to the proof of Theorem 2.3. We only consider the case when $H(x) = |x - x_0|^\gamma$ for some $\gamma \geq 0$ and $x_0 \in \Omega$.

Theorem 2.5. *Let $1 < m < \infty$, $\gamma, \beta \geq 0$, $0 < \alpha < N$, Ω a domain in $\mathbb{R}^N$, and $x_0 \in \Omega$. Assume that f and g satisfy (C) and let u be a positive supersolution to*

$$-\Delta_m u(x) = |x - x_0|^\gamma ((I_\alpha * f(u))(x))^\beta g(u(x)) \quad \text{in } \Omega. \tag{2.8}$$

Then,

$$\int_{m_{x_0}(r)}^{u(x_0)} (f^\beta(s)g(s))^{-\frac{1}{m-1}} \mathrm{d}s \geq C_{\gamma,\alpha,\beta,\gamma,N}^{\frac{1}{m-1}} \frac{m-1}{\gamma+\alpha\beta+m} r^{\frac{\gamma+\alpha\beta+m}{m-1}}$$

for every $0 < r < d_\Omega(x_0)$, *with*

$$C_{\gamma,\alpha,\beta,N} = \left(\frac{N\omega_N A_\alpha}{\alpha}\right)^\beta \int_0^1 t^{N-1+\gamma}(1-t)^{\alpha\beta}\mathrm{d}t. \tag{2.9}$$

Proof. As in the proof of Theorem 2.3, by (2.8), we obtain

$$\begin{aligned}\int_{B_r(x_0)} &|\nabla u(z)|^{m-2}\nabla u(z)\cdot\nabla\phi(z)\mathrm{d}z\\ &\geq \left(\frac{N\omega_N A_\alpha}{\alpha}\right)^\beta g(m_{x_0}(r))f(m_{x_0}(r))^\beta \int_{B_r(x_0)} |z-x_0|^\gamma (r-|x_0-z|)^{\alpha\beta}\phi(z)\mathrm{d}z\end{aligned}$$

for every $\phi \in C_c^\infty(B_r(x_0))$, $0 < r < d_\Omega(x_0)$, with $\phi \geq 0$, i.e.,

$$-\Delta_m u(z) \geq \left(\frac{N\omega_N A_\alpha}{\alpha}\right)^\beta g(m_{x_0}(r))f(m_{x_0}(r))^\beta |z-x_0|^\gamma (r-|z-x_0|)^{\alpha\beta} \quad \text{in } B_r(x_0). \tag{2.10}$$

We consider an auxiliary function

$$\Phi(z) = \Phi(|z|) = \int_{|z|}^1 \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} t^\gamma (1-t)^{\alpha\beta}\mathrm{d}t\right)^{\frac{1}{m-1}} \mathrm{d}s,$$

which is the unique radial solution of

$$\begin{cases} -\Delta_m\Phi(z) = |z|^\gamma(1-|z|)^{\alpha\beta} & \text{in } B_1(0),\\ \Phi'(0) = \Phi(1) = 0. \end{cases}$$

We observe that the function

$$\Psi(z) = r^{\frac{m+\gamma+\alpha\beta}{m-1}}\Phi\left(\frac{|z-x_0|}{r}\right)$$

is a solution to

$$\begin{cases} -\Delta_m\Psi(z) = |z-x_0|^\gamma(r-|z-x_0|)^{\alpha\beta} & \text{in } B_r(x_0),\\ \Psi = 0 & \text{on } \partial B_r(x_0). \end{cases}$$

Let $v(z) = u(z) - m_{x_0}(r)$, $z \in B_r(x_0)$, and

$$w(z) = A(g(m_{x_0}(r)))^{\frac{1}{m-1}} f(m_{x_0}(r))^{\frac{\beta}{m-1}}\Psi(z), \quad z \in B_r(x_0),$$

with A as in (2.5). From (2.10), we obtain

$$-\Delta_m v(z) \geq -\Delta_m w(z) \quad \text{in } B_r(x_0),$$

i.e.,

$$\int_{B_r(x_0)} |\nabla v(z)|^{m-2}\nabla v(z)\cdot\nabla\phi(z)\mathrm{d}z \geq \int_{B_r(x_0)} |\nabla w(z)|^{m-2}\nabla w(z)\cdot\nabla\phi(z)\mathrm{d}z$$

for every $\phi \in C_c^\infty(B_r(x_0))$, with $\phi \geq 0$. The rest of the proof is quite similar to the proof of Theorem 2.3; using the comparison principle and the fact that this time we have

$$\lim_{h\to r}\frac{\Phi\left(\frac{h}{r}\right)}{r-h} = \frac{-\Phi'(1)}{r} = \frac{1}{r}\left(\int_0^1 t^{N-1+\gamma}(1-t)^{\alpha\beta}\mathrm{d}t\right)^{\frac{1}{m-1}},$$

we arrive at

$$\begin{cases} -m'_{x_0}(r) \geq Ag\big(m_{x_0}(r)\big)^{\frac{1}{m-1}} f\big(m_{x_0}(r)\big)^{\frac{\beta}{m-1}} r^{\frac{\gamma+\alpha\beta+1}{m-1}} \\ \quad \cdot \left(\int\limits_0^1 t^{N-1+\gamma}(1-t)^{\alpha\beta}\mathrm{d}t\right)^{\frac{1}{m-1}} \quad \text{a.e} \quad r \in (0, d_\Omega(x_0)), \\ m_{x_0}(0) = u(x_0). \end{cases} \tag{2.11}$$

3 Estimates for supersolutions and extremal parameters

Theorem 2.3 leads to explicit estimates on supersolutions to (1.1). Assume that f and g satisfy (C). Let

$$J(t) = \int\limits_0^t (f^\beta(s)g(s))^{-\frac{1}{m-1}}\mathrm{d}s, \tag{3.1}$$

and let J^{-1} be the inverse function of J. In the next result, we are interested in the case where $J(t) < \infty$ for some $0 < t \leq \infty$, and the other case will be considered later in Section 4. As an immediate consequence of Theorem 2.3, we have the following.

Proposition 3.1. *Let $1 < m < \infty$, $\beta \geq 0$, $0 < \alpha < N$ and assume that $\Omega \subset \mathbb{R}^N$ is a domain. Assume that f, g, H, and Q satisfy* (C) *and let u be a positive supersolution to* (1.1) *in Ω. Then,*

$$C_{\alpha,\beta,N}^{\frac{1}{m-1}} \int\limits_0^{d_\Omega(x)} (s^{\alpha\beta+1}H_x(s)Q_x(s)^\beta)^{\frac{1}{m-1}}\mathrm{d}s \leq J(u(x)) \tag{3.2}$$

for every $x \in \Omega$. Here, $C_{\alpha,\beta,N}$ is the constant in (2.3).

We point out that Theorem 3.1 gives pointwise estimates for positive supersolutions. It follows from (3.2) that

$$u(x) \geq J^{-1}\left(C_{\alpha,\beta,N}^{\frac{1}{m-1}} \int\limits_0^{d_\Omega(x)} (s^{\alpha\beta+1}H_x(s)Q_x(s)^\beta)^{\frac{1}{m-1}}\mathrm{d}s\right)$$

for every $x \in \Omega$. In particular, if $H \equiv Q \equiv 1$, then (3.2) implies that

$$J(u(x)) = \int\limits_0^{u(x)} (f^\beta(s)g(s))^{-\frac{1}{m-1}}\mathrm{d}s \geq \frac{m-1}{\alpha\beta+m} C_{\alpha,\beta,N}^{\frac{1}{m-1}} d_\Omega(x)^{\frac{\alpha\beta+m}{m-1}},$$

for every $x \in \Omega$, from which we obtain

$$u(x) \geq J^{-1}\left(\frac{m-1}{\alpha\beta+m} C_{\alpha,\beta,N}^{\frac{1}{m-1}} d_\Omega(x)^{\frac{\alpha\beta+m}{m-1}}\right) \tag{3.3}$$

for every $x \in \Omega$.

Next, we discuss estimates for the extremal parameter defined in (1.4).

Corollary 3.2. *The extremal parameter λ^* of Problem* (1.3) *with $\gamma = 0$ satisfies*

$$\lambda^* \leq C_{\alpha,\beta,N}^{-1}\left(\frac{\alpha\beta+m}{m-1}\right)^{m-1} J(a)^{m-1}\left(\sup_{x\in\Omega} d_\Omega(x)\right)^{-(\alpha\beta+m)}. \tag{3.4}$$

Moreover, if u_λ is a positive supersolution to (1.3) *for some $0 < \lambda \leq \lambda^*$, then*

$$u_\lambda(x) \geq J^{-1}\left(\frac{m-1}{\alpha\beta+m}(\lambda C_{\alpha,\beta,N})^{\frac{1}{m-1}} d_\Omega(x)^{\frac{\alpha\beta+m}{m-1}}\right)$$

for every $x \in \Omega$. Here, $C_{\alpha,\beta,N}$ is the constant in (2.3).

Proof. Proposition 3.1 with $H \equiv Q \equiv 1$ and replacing $g(s)$ with $\lambda g(s)$ imply that

$$\begin{aligned} C_{\alpha,\beta,N}^{\frac{1}{m-1}} \frac{m-1}{\alpha\beta+m} d_\Omega(x)^{\frac{\alpha\beta+m}{m-1}} &= C_{\alpha,\beta,N}^{\frac{1}{m-1}} \int_0^{d_\Omega(x)} s^{\frac{\alpha\beta+1}{m-1}} \mathrm{d}s \\ &\leq \lambda^{-\frac{1}{m-1}} \int_0^{u(x)} (f^\beta(s) g(s))^{-\frac{1}{m-1}} \mathrm{d}s \\ &= \lambda^{-\frac{1}{m-1}} J(u_\lambda(x)) \end{aligned}$$

for every $x \in \Omega$. By taking supremum over $x \in \Omega$ on both sides and rearranging the terms, we obtain

$$\lambda \leq C_{\alpha,\beta,N}^{-1}\left(\frac{\alpha\beta+m}{m-1}\right)^{m-1} J\left(\sup_{x\in\Omega} u_\lambda(x)\right)^{m-1}\left(\sup_{x\in\Omega} d_\Omega(x)\right)^{-(\alpha\beta+m)}.$$

The lower bound for u_λ follows from (3.3). □

Example 3.3. Let $\Omega = B_R(0)$ in Corollary 3.2, and note that in this case, $\sup_{x\in\Omega} d_\Omega(x) = R$.

(i) If $f(s) = g(s) = e^s$, then

$$J(\infty) = \int_0^\infty (f^\beta(s) g(s))^{-\frac{1}{m-1}} \mathrm{d}s = \int_0^\infty e^{-\frac{\beta+1}{m-1}s} \mathrm{d}s = \frac{m-1}{\beta+1}.$$

By (3.4), we have

$$\lambda^* \leq C_{\alpha,\beta,N}^{-1}\left(\frac{\alpha\beta+m}{\beta+1}\right)^{m-1} R^{-(\alpha\beta+m)}.$$

(ii) If $f(s) = (1+s)^p$ and $g(s) = (1+s)^q$ with $p, q > 0$ and $\beta p + q > m-1$, then

$$J(\infty) = \int_0^\infty (1+s)^{-\frac{\beta p+q}{m-1}s} \mathrm{d}s = \frac{m-1}{\beta p + q - m + 1}.$$

By (3.4), we have

$$\lambda^* \leq C_{\alpha,\beta,N}^{-1}\left(\frac{\alpha\beta+m}{\beta p+q-m+1}\right)^{m-1} R^{-(\alpha\beta+m)}.$$

(iii) If $f(s) = \max\{s^{p_1}, s^{p_2}\}$ and $g(s) = \max\{s^{q_1}, s^{q_2}\}$ with

$$p_1, q_1 < 1 < p_2, q_2 \quad \text{and} \quad p_1\beta + q_1 < m - 1 < p_2\beta + q_2,$$

then

$$\begin{aligned} J(\infty) &= \int_0^1 s^{-\frac{\beta p_1+q_1}{m-1}s} \mathrm{d}s + \int_1^\infty s^{-\frac{\beta p_2+q_2}{m-1}s} \mathrm{d}s \\ &= \frac{m-1}{m-1-\beta p_1 - q_1} + \frac{m-1}{\beta p_2 + q_2 - m + 1} \\ &= \frac{\beta(p_2-p_1) + q_2 - q_1}{(\beta p_2 + q_2 - m + 1)(m - 1 - \beta p_1 - q_1)}. \end{aligned}$$

By (3.4), we have

$$\lambda^* \le C^{-1}_{\alpha,\beta,N}\left(\frac{(\alpha\beta+m)(\beta(p_2-p_1)+q_2-q_1)}{(\beta p_2+q_2-m+1)(m-1-\beta p_1-q_1)}\right)^{m-1} R^{-(\alpha\beta+m)}.$$

Next, we consider the eigenvalue Problem (1.6) with singular nonlinearities. For the sake of simplicity, we only consider the case when $\Omega = B_R(0) = B_R$. We obtain the following bounds for solutions to (1.6) and the related extremal parameter.

Corollary 3.4. *The extremal parameter of Problem* (1.6) *with* $\Omega = B_R$ *satisfies*

$$\lambda^* \le C^{-1}_{\gamma,\alpha,\beta,N}\left(\frac{\alpha\beta+m+\gamma}{\beta p+q+m-1}\right)^{m-1} R^{-(\alpha\beta+m+\gamma)}. \tag{3.5}$$

Moreover, for any solution u_λ *to* (1.6), *we have*

$$\|u_\lambda\|_\infty \ge u_\lambda(0) \ge 1-(1-\Lambda)^{\frac{m-1}{\beta p+q+m-1}}, \quad 0<\lambda\le\lambda^*,$$

where

$$\Lambda = (\lambda C_{\gamma,\alpha,\beta,N})^{\frac{1}{m-1}}\frac{\beta p+q+m-1}{\gamma+\alpha\beta+m}R^{\frac{\gamma+\alpha\beta+m}{m-1}}.$$

Here, $C_{\gamma,\alpha,\beta,N}$ *is the constant in* (2.9).

Proof. Let $f(s) = (1-s)^{-p}$ and $g(s) = \lambda(1-s)^{-q}$, $p, q > 0$, and u_λ be a solution to (1.6) in $\Omega = B_R$. By Theorem 2.5, we obtain

$$\begin{aligned}\int_0^{u_\lambda(0)} (f^\beta(s)g(s))^{-\frac{1}{m-1}}ds &= \lambda^{\frac{-1}{m-1}}\int_0^{u_\lambda(0)}(1-s)^{\frac{\beta p+q}{m-1}}ds\\ &= \frac{(m-1)\lambda^{\frac{-1}{m-1}}}{\beta p+q+m-1}\left(1-(1-u_\lambda(0))^{\frac{\beta p+q+m-1}{m-1}}\right)\\ &\ge C^{\frac{1}{m-1}}_{\gamma,\alpha,\beta,N}\frac{(m-1)}{\gamma+\alpha\beta+m}R^{\frac{\gamma+\alpha\beta+m}{m-1}}.\end{aligned}$$

This implies that

$$\|u_\lambda\|_\infty \ge u_\lambda(0) \ge 1-(1-\Lambda)^{\frac{m-1}{\beta p+q+m-1}},$$

where

$$\Lambda = (\lambda C_{\gamma,\alpha,\beta,N})^{\frac{1}{m-1}}\frac{\beta p+q+m-1}{\gamma+\alpha\beta+m}R^{\frac{\gamma+\alpha\beta+m}{m-1}}.$$

Moreover, since

$$J(1) = \int_0^1 (f^\beta(s)g(s))^{-\frac{1}{m-1}}ds = \int_0^1 (1-s)^{\frac{\beta p+q}{m-1}}ds = \frac{m-1}{\beta p+q+m-1},$$

we obtain from Theorem 2.5 that

$$\lambda^* \le (C_{\gamma,\alpha,\beta,N})^{-1}\left(\frac{\gamma+\alpha\beta+m}{\beta p+q+m-1}\right)^{m-1}R^{-(\gamma+\alpha\beta+m)}.$$ □

Remark 3.5. Corollary 3.4 extends several known bounds for the extremal parameter.

(1) By applying Corollary 3.4 with $\beta = 0$, we obtain the bound

$$\lambda^* \le \frac{m^{m-1}N}{R^m}$$

for the extremal parameter λ^* of the eigenvalue Problem (1.5) in B_R with $g(s) = e^s$ and the bound

$$\lambda^* \le \left(\frac{m}{q-m+1}\right)^{m-1} \frac{N}{R^m},$$

with $g(s) = (1+s)^q$. For these bounds, see [2,3,10,22].

(2) Corollary 3.4 with $\beta = 0$ implies the bounds

$$\lambda^* \le (N+\gamma)\left(\frac{m+\gamma}{q+m-1}\right)^{m-1} R^{-(m+\gamma)}$$

and

$$\|u^*\|_{L^\infty} \ge 1 - \left(1 - \frac{q+m-1}{\gamma+m}\left(\frac{\lambda^*}{N+\gamma}\right)^{\frac{1}{m-1}} R^{\frac{\gamma+m}{m-1}}\right)^{\frac{m-1}{q+m-1}}$$

for the pull-in voltage λ^* and pull-in distance $\|u^*\|_{L^\infty}$ of the eigenvalue Problem (1.8). In particular, when $m = 2 = q$, we have

$$\lambda^* \le \frac{(N+\gamma)(2+\gamma)}{3} R^{-(2+\gamma)}$$

and

$$\|u^*\|_\infty \ge 1 - \sqrt[3]{1 - \frac{3\lambda^*}{(N+\gamma)(N+2)} R^{\gamma+2}}.$$

For these bounds, see [18,20,27,29].

Next, we discuss a Liouville-type result for the autonomous Choquard equation in a general unbounded domain.

Proposition 3.6. *Let $p, q \ge 0$ with $\beta p + q < m-1$ and assume that u is a positive supersolution to*

$$-\Delta_m u = (I_\alpha * u^p)^\beta u^q \quad \text{in } \Omega. \tag{3.6}$$

Then,

$$u(x) \ge C \operatorname{dist}(x, \partial\Omega)^{\frac{m+\alpha\beta}{m-1-\beta p-q}} \quad \text{in } \Omega, \tag{3.7}$$

with

$$C = \left(C_{\alpha,\beta,N}^{\frac{1}{m-1}} \frac{m-1-\beta p-q}{m+\alpha\beta}\right)^{\frac{m-1}{m-1-\beta p-q}}.$$

Here, $C_{\alpha,\beta,N}$ is the constant in (2.3).

Proof. Let $f(u) = u^p$, $g(u) = u^q$, and $H \equiv Q \equiv 1$ in Theorem 2.3. Since $\beta p + q < m-1$, we have

$$\int_{m_x(r)}^{u(x)} (f^\beta(s)g(s))^{-\frac{1}{m-1}} ds = \int_{m_x(r)}^{u(x)} s^{-\frac{\beta p+q}{m-1}} ds \le \frac{m-1}{m-1-\beta p-q} u(x)^{\frac{m-1-\beta p-q}{m-1}}.$$

From (2.2), we obtain

$$\frac{m-1}{m-1-\beta p-q} u(x)^{\frac{m-1-\beta p-q}{m-1}} \ge C_{\alpha,\beta,N}^{\frac{1}{m-1}} \int_0^r s^{\frac{\alpha\beta+1}{m-1}} ds = C_{\alpha,\beta,N}^{\frac{1}{m-1}} \frac{m-1}{m+\alpha\beta} r^{\frac{m+\alpha\beta}{m-1}},$$

for every $0 < r < \operatorname{dist}(x, \partial\Omega)$. This implies that u satisfies (3.7) for every $x \in \Omega$. □

Remark 3.7. The next two results follow immediately from Proposition 3.6.

(i) If

$$\sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega) = \infty, \tag{3.8}$$

then (3.6) does not have any bounded positive solution in Ω. From (3.7), we see that if Ω satisfies (3.8), then u has to be unbounded.

(ii) If $\beta > 0$ and

$$\limsup_{x\in\Omega, |x|\to\infty} \frac{d_\Omega(x)}{|x|^s} = \infty, \quad \text{with } s = \frac{N-\alpha}{N + \frac{p(m+\alpha\beta)}{m-1-\beta p-q}}, \tag{3.9}$$

then (3.6) does not admit any positive solution in Ω. In particular, this is the case if Ω is $\mathbb{R}^N$, $\mathbb{R}^N_+$, an exterior domain or an unbounded cone-like domain $\{(r,\theta) \in \mathbb{R}^N : \theta \in \mathcal{S},\ r > 0\}$, where (r,θ) are the polar coordinates in $\mathbb{R}^N$ and $\mathcal{S} \subset S^{N-1}$ is a subdomain of the unit sphere S^{N-1} in $\mathbb{R}^N$. For the proof, we show that the condition

$$\int_\Omega \frac{f(u(y))}{1+|y|^{N-\alpha}} \mathrm{d}y < \infty$$

does not hold if Ω satisfies (3.9). If (3.9) holds, there exists a sequence of points $x_n \in \Omega$ with $R_n \le |x_n|$ so that $B_{R_n}(x_n) \subset \Omega$ and

$$\lim_{n\to\infty} \frac{R_n}{|x_n|^s} = \infty.$$

Then, by (3.7), for n large, we have

$$\begin{aligned}\int_\Omega \frac{f(u(y))}{1+|y|^{N-\alpha}} \mathrm{d}y &\ge \int_{B_{R_n}(x_n)} \frac{u(y)^p}{1+|y|^{N-\alpha}} \mathrm{d}y \ge C \int_{B_{R_n}(x_n)} \frac{R_n^{\frac{p(m+\alpha\beta)}{m-1-\beta p-q}}}{(|x_n|+R_n)^{N-\alpha}} \mathrm{d}y \\ &\ge C \frac{R_n^{N+\frac{p(m+\alpha\beta)}{m-1-\beta p-q}}}{|x_n|^{N-\alpha}} = C\left(\frac{R_n}{|x_n|^s}\right)^{N+\frac{p(m+\alpha\beta)}{m-1-\beta p-q}} \to \infty\end{aligned}$$

as $n \to \infty$. Hence, if Ω satisfies (3.9), there is no positive solution to (3.6).

Remark 3.8. We remark that the existence and nonexistence of positive supersolutions for Problem (3.6) in the case when $\beta = 1$ and Ω is an exterior domain in $\mathbb{R}^N$ have been investigated very recently in [23], where the authors obtained the optimal ranges of exponents p, q, and α for which positive supersolutions exist. In particular, they showed that if $p + q \ne m - 1$, then Problem (3.6) has a bounded radial supersolution in any bounded open set $\Omega \subset \mathbb{R}^N (N \ge 1)$.

The following result is an immediate consequence of Theorem 2.5.

Corollary 3.9. *Assume that there exists $0 < t \le \infty$ such that $J(t) < \infty$, where $J(t)$ is as in (3.1). The problem*

$$-\Delta_m u = |x|^\gamma (I_\alpha * f(u))^\beta g(u) \quad \text{in } \mathbb{R}^N,$$

where $\gamma, \beta \ge 0$, does not have any positive supersolution.

Proof. By applying Theorem 2.5 with $\Omega = \mathbb{R}^N$, we have

$$\int_{m_0(r)}^{u(0)} (f^\beta(s)g(s))^{-\frac{1}{m-1}} \mathrm{d}s \ge C r^{\frac{\gamma+\alpha\beta+m}{m-1}}, \quad 0 < r < \infty,$$

then by letting $r \to \infty$ in the aforementioned inequality, we obtain

$$\infty > \int_0^{u(0)} (f^\beta(s)g(s))^{-\frac{1}{m-1}} \mathrm{d}s \ge \int_{m_0(r)}^{u(0)} (f^\beta(s)g(s))^{-\frac{1}{m-1}} \mathrm{d}s \to \infty$$

as $r \to \infty$, which is a contradiction. □

4 Applications to problems in exterior domains

In this section, we discuss some applications of our results to obtain nonexistence results for certain problems in unbounded domains and, in particular, we obtain several Liouville-type results. The following well-known auxiliary results will be useful for us (see, e.g., [44, Lemma 2.3] and [6, Lemma 3.7]).

Lemma 4.1. *Assume that Ω is an exterior domain in $\mathbb{R}^N$ and let u be a positive supersolution to*

$$-\Delta_m u = 0 \quad \text{in } \Omega.$$

(i) *If $N > m$, then there exists a constant C, depending only on Ω, N, and u, such that*

$$u(x) \geq C|x|^{-\frac{N-m}{m-1}} \quad \text{in } \Omega \tag{4.1}$$

and

$$\liminf_{|x|\to\infty} u(x) \leq C. \tag{4.2}$$

(ii) *If $N \leq m$, then*

$$\liminf_{|x|\to\infty} u(x) > 0.$$

We also apply the following result (see [9, Proposition 2.7 (ii)] for the first part and [9, Theorem 3.3] for the second part).

Lemma 4.2. *Suppose that $N > m > 1$.*
(i) *If u is a positive supersolution to*

$$-\Delta_m u(x) = C|x|^{-N} \quad \text{in } \mathbb{R}^N \backslash \overline{B_1},$$

for some constant $C > 0$, then there exists a constant $c > 0$ such that

$$u(x) \geq c|x|^{-\frac{N-m}{m-1}} (\ln|x|)^{\frac{1}{m-1}} \quad \text{in } \mathbb{R}^N \backslash B_2.$$

(ii) *The problem*

$$-\Delta_m u(x) = C|x|^{\gamma} u(x)^q \quad \text{in } \mathbb{R}^N \backslash \overline{B_1}$$

does not have any positive supersolution, provided

$$m - 1 < q \leq \frac{(N+\gamma)(m-1)}{N-m}.$$

Consider the problem

$$-\Delta_m u(x) \geq H(x) \left(\int_{\Omega} \frac{Q(y)u(y)^p}{|x-y|^{N-\alpha}} \mathrm{d}y \right)^{\beta} u(x)^q \quad \text{in } \mathbb{R}^N \backslash \overline{B_1}, \tag{4.3}$$

where $p, q \geq 0$, and H, Q satisfy the following condition:
(C′) There exist $x_n \in \Omega$ and $n \in \mathbb{N}$, with $|x_n| \to \infty$ as $n \to \infty$, and there exist $R_n \in \mathbb{R}$, $0 < R_n < d_\Omega(x_n) = |x_n| - 1$, and $n \in \mathbb{N}$, such that

$$\limsup_{n\to\infty} \frac{R_n}{|x_n|^s} > 0, \tag{4.4}$$

for some $0 < s \leq 1$. Moreover,

$$H(x) \geq C|x|^{\gamma} \quad \text{and} \quad Q(x) \geq C|x|^{\sigma} \quad \text{for every } x \in B_{R_n}(x_n), \tag{4.5}$$

where $\gamma, \sigma \in \mathbb{R}$.

Proposition 4.3. *Consider* (4.3) *in the exterior domain* $\mathbb{R}^N\backslash\overline{B_1}$, *with* $p, q \geq 0$, $\beta p + q \geq m - 1$, *and* H *and* Q *satisfy* (C') *for some* $0 < s \leq 1$ *and* $\gamma, \sigma \in \mathbb{R}$. *Problem* (4.3) *does not have any positive solution if one of the following conditions is satisfied:*

(i) $N \leq m$ *and, either* $\sigma + \alpha \geq (1 - s)N$ *or* $\gamma + \sigma\beta > -s(m + \alpha\beta)$.

(ii) $N > m$ *and*

$$m - 1 < \beta p + q < \frac{(m-1)(N + \gamma + \beta\sigma + s(m + \alpha\beta) - m)}{N - m}.$$

Furthermore, Problem (4.3) *does not have any bounded positive solution if*

$$m - 1 = \beta p + q \quad \text{and} \quad \gamma + \sigma\beta > -s(m + \alpha\beta).$$

As a consequence, if H *and* Q *satisfy* (4.5) *for any* $x \in \Omega$, *then* (4.3) *does not have any positive solution if either*

$$m - 1 < \beta p + q \leq \frac{(m-1)(N + \gamma + \beta\sigma + \alpha\beta)}{N - m},$$

or $m - 1 = \beta p + q$ *and* $\gamma + \sigma\beta > -(m + \alpha\beta)$.

Proof. First, assume that Condition (i) holds. We apply Theorem 2.3 with $f(u) = u^p$ and $g(u) = u^q$. If $\beta p + q > m - 1$, we have

$$\int_{m_x(r)}^{u(x)} (f^\beta(s)g(s))^{-\frac{1}{m-1}} ds \leq \frac{m-1}{\beta p + q - m + 1} m_x(r)^{-\frac{\beta p + q - m + 1}{m-1}}. \tag{4.6}$$

Let $x_n \in \Omega$, $n \in \mathbb{N}$, be as in (C'). By (2.1) and (4.5), we have

$$H_{x_n}(r) = \inf_{y \in B_r(x_n)} H(y) \geq C(|x_n| - (\operatorname{sgn}\gamma)r)^\gamma$$

and

$$Q_{x_n}(r) = \inf_{y \in B_r(x_n)} Q(y) \geq C(|x_n| - (\operatorname{sgn}\sigma)r)^\sigma$$

for $0 < r < R_n$, $n \in \mathbb{N}$. These imply that

$$H_{x_n}(r) \geq C|x_n|^\gamma \quad \text{and} \quad Q_{x_n}(r) \geq C|x_n|^\sigma$$

for $0 < r < \frac{R_n}{2}$, $n \in \mathbb{N}$. It follows that

$$\int_0^r (s^{\alpha\beta+1} H_{x_n}(s) Q_{x_n}(s)^\beta)^{\frac{1}{m-1}} ds \geq C|x_n|^{\frac{\gamma+\sigma\beta}{m-1}} R_n^{\frac{m+\alpha\beta}{m-1}} \tag{4.7}$$

for $\frac{R_n}{2} < r < R_n$, $n \in \mathbb{N}$. From (4.6), (4.7), and (2.2), we conclude that

$$m_{x_n}(r)^{-\frac{\beta p + q - m + 1}{m-1}} \geq C|x_n|^{\frac{\gamma+\sigma\beta}{m-1}} R_n^{\frac{m+\alpha\beta}{m-1}},$$

which implies that

$$m_{x_n}(r) \leq C|x_n|^{-\frac{\gamma+\sigma\beta}{\beta p + q - m + 1}} R_n^{-\frac{m+\alpha\beta}{\beta p + q - m + 1}} \tag{4.8}$$

for $\frac{R_n}{2} < r < R_n$, $n \in \mathbb{N}$. By (4.4) and (4.8), we obtain that

$$m_{x_n}(r) \leq C|x_n|^{-\frac{\gamma+\sigma\beta+s(m+\alpha\beta)}{\beta p + q - m + 1}}$$

for n large, which implies that $m_{x_n}(r) \to 0$ as $n \to \infty$ if $\gamma + \sigma\beta + s(m + \alpha\beta) > 0$. Thus, we have

$$\liminf_{|x| \to \infty} u(x) = 0,$$

which contradicts Lemma 4.1 (ii) if $N \le m$. Note that by Lemma 4.1 (ii), we have $u(x) \ge C$ in $B_{R_n}(x_n)$ for n large, we then obtain

$$\begin{aligned}\int_\Omega \frac{Q(y)u(y)^p}{1+|y|^{N-\alpha}}\mathrm{d}y &\ge \int_{B_{R_n}(x_n)} \frac{C|y|^\sigma}{1+|y|^{N-\alpha}}\mathrm{d}y\\ &\ge C\int_{B_{R_n}(x_n)} \frac{|x_n|^\sigma}{(|x_n|+R_n)^{N-\alpha}}\mathrm{d}y\\ &\ge C|x_n|^{\sigma+\alpha-N}R_n^N\\ &= C|x_n|^{\sigma+\alpha-(1-s)N}\bigg(\frac{R_n}{|x_n|^s}\bigg)^N\end{aligned}$$

for n large. By (4.4) and the aforementioned inequality, we conclude that, for $\sigma+\alpha > (1-s)N$, we have

$$\int_\Omega \frac{Q(y)u(y)^p}{1+|y|^{N-\alpha}}\mathrm{d}y = \infty.$$

Therefore, there does not exist any positive solution (4.3) if (i) holds true.

Next, we consider Condition (ii). If $N > m$, by Lemma 4.1, there exists a constant $C > 0$ such that $u(x) \ge C\,|x|^{-\frac{N-m}{m-1}}$ for every $x \in \Omega$, which implies that

$$m_{x_n}(r) \ge C_1(|x_n|+R_n)^{-\frac{N-m}{m-1}} \ge C|x_n|^{-\frac{N-m}{m-1}} \tag{4.9}$$

for $\frac{R_n}{2} < r < R_n$, $n \in \mathbb{N}$. Comparing (4.9) with (4.8), we have

$$\frac{|x_n|^{\frac{\gamma+\sigma\beta}{\beta p+q-m+1}} R_n^{\frac{m+\alpha\beta}{\beta p+q-m+1}}}{|x_n|^{\frac{N-m}{m-1}}} \le C$$

for n large. This can be rewritten as:

$$\bigg(\frac{R_n}{|x_n|^s}\bigg)^{\frac{(m+\alpha\beta)}{\beta p+q-m+1}} \le C|x_n|^{\frac{(N-m)}{m-1}-\frac{\gamma+\sigma\beta+s(m+\alpha\beta)}{\beta p+q-m+1}}$$

for n large. Taking into account (4.4), it follows from the aforementioned inequality that

$$\frac{\gamma+\sigma\beta+s(m+\alpha\beta)}{\beta p+q-m+1} \le \frac{N-m}{m-1}. \tag{4.10}$$

Therefore, if (4.10) does not hold, then there does not exist any positive solution to (4.3), i.e., when

$$\beta p + q < \frac{(m-1)(N+\gamma+\beta\sigma+s(m+\alpha\beta)-m)}{N-m}.$$

To prove the remaining claims, let $\beta p + q = m-1$. An easy computation gives

$$\int_{m_x(r)}^{u(x)} (f^\beta(s)g(s))^{-\frac{1}{m-1}}\mathrm{d}s = \int_{m_x(r)}^{u(x)} \frac{1}{s}\mathrm{d}s = \ln\frac{u(x)}{m_x(r)} \tag{4.11}$$

and as mentioned earlier, using (4.6), (4.7), and (2.2), we obtain

$$\ln\frac{u(x_n)}{m_{x_n}(r)} \ge C|x_n|^{\frac{\gamma+\sigma\beta}{m-1}} R_n^{\frac{m+\alpha\beta}{m-1}}$$

for $\frac{R_n}{2} < r < R_n$, $n \in \mathbb{N}$, or equivalently

$$u(x_n) \ge m_{x_n}(r)e^{|x_n|^{\frac{\gamma+\sigma\beta}{m-1}} R_n^{\frac{m+\alpha\beta}{m-1}}}$$

for $\frac{R_n}{2} < r < R_n$, $n \in \mathbb{N}$. By (4.9) and (4.4), we obtain

$$u(x_n) \geq C_1|x_n|^{\frac{-(N-m)}{m-1}} e^{C|x_n|^{\frac{\gamma+\sigma\beta+s(m+\alpha\beta)}{m-1}}}$$

for n large. If $\gamma + \sigma\beta > -s(m + \alpha\beta)$, from the aforementioned inequality and the fact that for any $\varepsilon, M > 0$, we have

$$\lim_{t\to\infty} \frac{e^{t^{\varepsilon}}}{t^M} = \infty,$$

we conclude that $u(x_n) \to \infty$ as $|x_n| \to \infty$. This implies that u is unbounded.

Assume that H and Q satisfy (4.5) for any $x \in \Omega$. Then, we have Condition (i) with $s = 1$, and hence, (4.3) does not have any positive solution if

$$m - 1 < \beta p + q < \frac{(m-1)(N + \gamma + \beta\sigma + \alpha\beta)}{N - m}.$$

Next, we discuss the case

$$\beta p + q = \frac{(m-1)(N + \beta\alpha + \beta\sigma + \gamma)}{N - m}. \tag{4.12}$$

By (4.8) and (4.9), we obtain

$$C_1|x|^{-\frac{N-m}{m-1}} \leq m_x\left(\frac{|x|}{2}\right) \leq C|x|^{-\frac{N-m}{m-1}} \tag{4.13}$$

for every $x \in \mathbb{R}^N \setminus \overline{B_1}$ with $|x|$ sufficiently large. From (4.3), we obtain

$$\begin{aligned}
-\Delta_m u(x) &\geq |x|^{\gamma}\left(\int_{\Omega} \frac{|y|^{\sigma} u(y)^p}{|x-y|^{N-\alpha}} \mathrm{d}y\right)^{\beta} u(x)^q \\
&\geq C|x|^{\gamma}\left(\int_{\frac{|x|}{2}<|y-x|<|x|} \frac{|y|^{\sigma}|y|^{-\frac{p(N-m)}{m-1}}}{|x-y|^{N-\alpha}} \mathrm{d}y\right)^{\beta} |x|^{-\frac{q(N-m)}{m-1}} \\
&\geq C|x|^{\gamma - \frac{q(N-m)}{m-1} + \beta(\sigma + \alpha - \frac{p(N-m)}{m-1})} \\
&= C|x|^{-N},
\end{aligned}$$

where we also applied (4.12). By Lemma 4.2, we have

$$u(x) \geq c|x|^{-\frac{N-m}{m-1}} (\ln|x|)^{\frac{1}{m-1}}$$

for $|x|$ large, which contradicts (4.13).

Also, if $\beta p + q = m - 1$, as in the proof in the case of Condition (ii) with $s = 1$, we obtain

$$u(x) \geq C_1|x|^{-\frac{N-m}{m-1}} e^{C|x|^{\frac{\gamma+\sigma\beta+m+\alpha\beta}{m-1}}}.$$

Now, if $\gamma + \sigma\beta > -(m + \alpha\beta)$, we then deduce from the aforementioned inequality that $u(x) \to \infty$ as $|x| \to \infty$, which contradicts (4.2) in Lemma 4.1. □

Remark 4.4. The functions H and Q in Proposition 4.3 are allowed to be zero on a subset $\Omega' \subset \Omega$ with $|\Omega'| = \infty$. For example, let

$$H(x) = \begin{cases} 0, & 3^{2n} \leq |x| < 3^{2n+1}, \\ |x|^{\gamma}, & 3^{2n+1} \leq |x| < 3^{2n+2}, \end{cases}$$

with $n \in \mathbb{N}$ and a similar formula for Q with σ instead of γ. By taking $x_n = 2 \cdot 3^{2n+1}$ and $R_n = 3^{2n+1}$ then H and Q satisfy (4.5); also, (4.4) holds with $s = 1$. However, we have $H \equiv Q \equiv 0$ on $\Omega' = \{x \in \Omega : 3^{2n} \leq |x| < 3^{2n+1}\}$ with $|\Omega'| = \infty$.

Remark 4.5. By Proposition 4.3, with $\gamma = \sigma = 0$, we see that the problem

$$-\Delta_m u(x) = \left(\int_\Omega \frac{u(y)^p}{|x-y|^{N-\alpha}}\,\mathrm{d}y\right)^\beta u(x)^q \quad \text{in } \mathbb{R}^N \backslash \overline{B_1} \tag{4.14}$$

does not have any positive supersolution if $N \le m$ or

$$1 < m < N \quad \text{with } m-1 \le \beta p + q \le \frac{(m-1)(N+\alpha\beta)}{N-m}.$$

For $\beta = 1$, this recovers a similar result in [23], which gave a complete classification of existence and non-existence of positive solutions.

We mention that by Proposition 4.3, we may consider Problem (1.1) with functions H and Q such as e^{ax_1}, $|x_1|^\gamma$, or generally $\rho(x_1, \dots, x_k)$, $1 \le k \le N$, with the property that for some $m \in \mathbb{R}$, $\rho(t, \dots, t) \ge C\,|t|^m$ for $|t|$ large. For example, consider the problem

$$-\Delta_m u(x) \ge |x_1|^\gamma \left(\int_\Omega \frac{|y_1|^\sigma u(y)^p}{|x-y|^{N-\alpha}}\,\mathrm{d}y\right)^\beta u(x)^q \quad \text{for } x = (x_1, \dots, x_N) \in \Omega := \mathbb{R}^N \backslash \overline{B_1}, \tag{4.15}$$

where β, p, q, γ, and $\sigma \ge 0$. For any

$$z \in \{(x_1, x_1, \dots, x_1) \in \Omega,\ x_1 > 0\}$$

and $R_z = \frac{z_1}{2}$, by noting that for every $x \in B_{R_z}(z)$, we have $|x_1 - z_1| \le |x - z| < R_z$ and $|z| = \sqrt{N} z_1$, and we easily have $H(x) \ge C\,|x|^\gamma$ and $Q(x) \ge C\,|x|^\sigma$ for $x \in B_{R_z}(z)$. Hence, Condition (4.4) holds with $s = 1$. By Proposition 4.3, we obtain the following result.

Corollary 4.6. *Let $\beta p + q \ge m - 1$. Then, (4.15) does not have any positive solution if $N \le m$, or $1 < m < N$ and*

$$m - 1 < \beta p + q < \frac{(m-1)(N + \gamma + \beta\sigma + \alpha\beta)}{N-m}.$$

Moreover, there exists no bounded positive solution when $m - 1 = \beta p + q$.

As an another example, consider the problem

$$-\Delta_m u(x) \ge e^{ax_1} \left(\int_\Omega \frac{|y_1|^\sigma u(y)^p}{|x-y|^{N-\alpha}}\,\mathrm{d}y\right)^\beta u(x)^q \quad \text{for } x = (x_1, \dots, x_N) \in \mathbb{R}^N \backslash \overline{B_1}, \tag{4.16}$$

where $\beta, p, q, \sigma \ge 0$ and $a \in \mathbb{R}$. Let $a > 0$ (the case $a < 0$ is similar), then note that for any $\gamma > 0$, there exists a constant $C_\gamma > 0$ so that $e^{ax_1} \ge C_\gamma x_1^\gamma$ for $x_1 > 0$ sufficiently large. As mentioned earlier, for $z = z_1 e$, $e = (1, 1, \dots, 1)$, with $z_1 > 0$ and $R_z = \frac{z_1}{2}$, we have $e^{ax_1} \ge C\,|x|^\gamma$ for $x \in B_{R_z}(z)$. Hence, Condition (4.4) holds with $s = 1$ and any $\gamma > 0$. Proposition 4.3 then implies the following result.

Corollary 4.7. *Let $\beta p + q \ge m - 1$. For any $0 \ne a \in \mathbb{R}$ and $\sigma \ge 0$, Problem (4.16) does not have any positive solution if $N \le m$, or $1 < m < N$ and $m - 1 < \beta p + q$. Moreover, there does not exist any bounded positive solution if $m - 1 = \beta p + q$.*

In the next result, we apply our main estimates on Problem (1.1) in the punctured space $\mathbb{R}^N \backslash \{0\}$ and give a nonexistence result for the positive supersolution.

Proposition 4.8. *Consider Problem (1.1) in $\Omega = \mathbb{R}^N \backslash \{0\}$ with $N \ge m$ and $f, g : [0, \infty) \to [0, \infty)$ satisfying the condition (C). If $N \ge m$ and*

$$\lim_{|x|\to 0}\int_0^{\frac{|x|}{2}}(s^{\alpha\beta+1}H_x(s)Q_x(s)^{\beta})^{\frac{1}{m-1}}\mathrm{d}s=\infty, \tag{4.17}$$

then for any positive supersolution u *to* (1.1) *in* Ω, *we have*

$$\limsup_{|x|\to 0}u(x)=\infty.$$

Moreover, if (4.17) *holds true and*

$$\int_{\delta}^{\infty}(f^{\beta}(s)g(s))^{\frac{1}{m-1}}\mathrm{d}s<\infty \tag{4.18}$$

for some $\delta>0$, *then the problem does not admit any positive supersolution.*

Proof. For a contradiction, assume that u is a positive supersolution to (1.1) in Ω, with

$$\limsup_{|x|\to 0}u(x)<\infty.$$

Since u is a weak supersolution to the m-Laplace equation in $\mathbb{R}^N\setminus\{0\}$ and $N>m$, then by Lemma 3.9 in [6], we have

$$\liminf_{|x|\to 0}u(x)>0.$$

Hence, we can find a sequence of points $x_j\in\Omega$ with $|x_j|\to 0$ so that

$$c\le\inf_{y\in B_{\frac{|x_j|}{2}}(x_j)}u(y)=m_{x_j}\left(\frac{|x_j|}{2}\right)\quad\text{and}\quad u(x_j)\le C, \tag{4.19}$$

as in (2.1). Theorem 2.3 implies

$$\int_{m_{x_j}\left(\frac{|x_j|}{2}\right)}^{u(x_j)}(f^{\beta}(s)g(s))^{-\frac{1}{m-1}}\mathrm{d}s\ge C\int_0^{\frac{|x_j|}{2}}(s^{\alpha\beta+1}H_{x_j}(s)Q_{x_j}(s)^{\beta})^{\frac{1}{m-1}}\mathrm{d}s$$

for every $j\in\mathbb{N}$, and from (4.19) together with the assumption (4.17), we obtain

$$\infty>\int_c^C(f^{\beta}(s)g(s))^{-\frac{1}{m-1}}\mathrm{d}s\ge\int_{m_{x_j}\left(\frac{|x_j|}{2}\right)}^{u(x_j)}(f^{\beta}(s)g(s))^{-\frac{1}{m-1}}\mathrm{d}s\to\infty\quad\text{as } j\to\infty.$$

This is a contradiction. Assume that (4.17) and (4.18) hold and u is a positive supersolution in Ω. Since $u(x)\ge C$ in Ω for some positive constant C (by the fact that $\liminf_{x\to 0}u(x)>0$), then from Theorem 2.3, we reach a contradiction similarly as mentioned earlier. □

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