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Promoting a reputation for quality

Daniel N Hauser*

I model a firm that invests in the quality of its product and influences how that quality is disclosed. The firm can promote its product; at random intervals it can disclose quality for a cost. At low reputations, promotion allows a firm to reestablish itself and the firm invests to take advantage of this. However, the ability to promote crowds out incentives for investment at high reputations generated by other information sources, in particular information that is generated whenever the firm is selling a low quality product, leading to reputation cycles in settings where high reputations would have otherwise persisted.

1. Introduction

■ The information a firm discloses to consumers is a crucial part of how it manages its reputation. A restaurant chain advertises better ingredients and more stringent quality standards; a researcher decides how much time to spend going to conferences; a software company demonstrates a new program at a trade show. In this article, I study a firm that can invest in the quality of its product—which is not observed by consumers at the point of purchase—and influence the information consumers observe through promotion; costly, stochastic disclosures of product quality. I explore how this ability to promote impacts the firm's incentives for unobservable investment in quality.

To study this question, I build on the reputation for quality model of Board and Meyer-ter-Vehn (2013). A monopolist is selling a product to consumers who don't observe quality directly. Instead, consumers observe news, signals that arrive at a Poisson rate that depends on the current quality of the firm's product. This quality is persistent and evolves stochastically based on the firm's level of investment. This persistence means that knowing quality today is informative, but not perfectly informative, about quality tomorrow. So the firm can build reputation, its past investment influences the market's current beliefs about quality. I augment this model by giving the firm the ability to promote. When the firm is selling a high-quality product, promotional opportunities arrive at a Poisson rate. Whenever one of these opportunities arises, the firm

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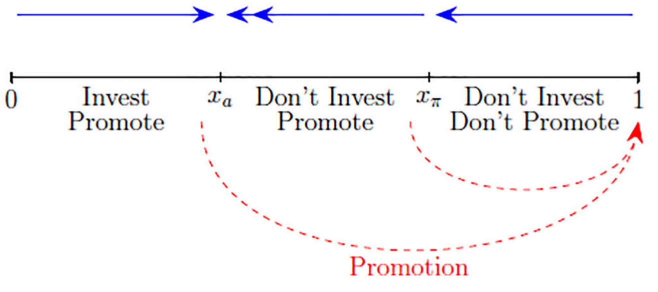
FIGURE 1

STRUCTURE OF AN MPE



can pay a cost to disclose its current quality to consumers. Promotion provides the firm with a tool to endogenously renew its reputation, which dramatically impacts investment incentives and reputation dynamics. Figure 1.

I examine the role of promotion in two settings. In the first, the only source of information is promotion. In this setting, promotion is a complement for investment. The firm wants to invest in quality, as it can then tell the market about that investment through promotion. Promotion generates persistent incentives to invest in quality, but leads to reputation cycles – periods of low investment when the firm's reputation decays followed by high investment to rebuild. I then introduce a second source of information, exogenous bad news. With this additional information source, promotion can lead to long-run investment in situations where otherwise the firm would stop investing with positive probability. However, promotion is no longer a complement to investment. It can crowd out investment incentives at high reputations that bad news would otherwise create.

Promotion naturally creates incentives for investment. The firm invests in unobserved quality so that it can then promote and charge a higher price. This drives the complementarity between investment and promotion and creates persistent incentives for investment. However, at high reputations, the firm does not have incentive to promote, and thus in equilibrium it does not invest. Although promotion in some ways resolves the original moral hazard problem—the unobservability of investment—it introduces a second moral hazard problem. The firm has to pay to generate the information the market uses to monitor its investment, and this expenditure is also unobserved by the market. The induced information structure leads to reputation cycles, periods of low investment when the firm allows quality to degrade followed by periods of high investment where the firm renews the quality of its product and then advertises that to consumers. These cycles persist, even as investment in quality becomes arbitrarily inexpensive.

To further highlight the role of promotion in driving incentives, I then consider a setting where, in addition to promotion, consumers learn through bad news that arrives when the firm is selling a low-quality product. This setting is a natural environment to examine the role of promotion. Allowing for bad news relaxes some of the tension in the previous setting. The market now has an additional way to monitor and potentially incentivize investment, which helps create incentives for the firm at high reputations. Promotion also changes investment incentives relative to a benchmark game where the firm cannot promote and the only source of information is bad news. In a setting without promotion, low reputations are self-perpetuating. The threat of bad news is lowest at low reputations, so the firm naturally does not invest, driving its reputation even lower. Promotion allows the firm to reestablish a high reputation, no matter what the market believes about investment. Unfortunately, promotion may crowd out these incentives for investment at high reputations. Without promotion, bad news is especially damaging for the firm, as it leads to a persistent low reputation in the future. Removing this threat of reputation traps can crowd out the incentives for investment at the top, creating reputation cycles.

The rest of the article proceeds as follows. First I present the general model, and then focus on two specific cases: the case where the only source of information is promotion, and the case where consumers learn from promotion and exogenous bad news. I conclude with a discussion of some modeling assumptions and related literature. All proofs are in the appendices.

2. Model

□ **The firm.** Time is continuous, $t \in [0, \infty)$. There is a single long-lived firm with stochastic quality $\theta_t \in \{L, H\}$, $\theta_0 = H$ with probability $x_0 \in (0, 1)$. The firm has discount rate r. At each instant of time, the firm chooses a level of investment $a_t \in [0, 1]$ for cost ca_t. In addition, whenever $\theta_t = H$ the firm chooses a level of promotion $\pi_t \in [0, 1]$ for cost $k\pi_t$. Quality evolves via Poisson shocks, as in Board and Meyer-ter-Vehn (2013). Specifically, there is a Poisson process with intensity $\lambda > 0$. Whenever there is an arrival of this process, θ_t becomes H with probability a_t and L with probability $1 - a_t$, and is fixed between arrivals.

□ **Consumers.** Consumers do not observe θ_t, a_t or π_t directly or the histories of these processes. Consumers observe a pair of Poisson processes, which provide noisy signals of quality which they use to form beliefs about θ_t. The first process, the exogenous news process, is a "bad news" process; it has arrival rate μ if $\theta_t = L$ and 0 otherwise, so exogenous news arrives only when the firm is selling a low quality product.

In addition, consumers observe promotional campaigns generated by the firm. Promotional opportunities arrive at rate γ. Although arrivals of opportunities are not observed by consumers, whenever one of these opportunities arrives quality is revealed publicly with probability π_t. So successful promotion perfectly reveals that the firm is selling a high-quality product. All these processes are independent, conditional on the state.[1] I initially analyze the case where $\mu = 0$, so all information comes from promotion, and then analyze the case where $\mu > 0$.

Finally, the public history h_t^c is a record of past arrival times of news $0 \le t_1^N \le t_2^N \dots t_n^N \le t$ and the past arrival times of promotion $0 \le t_1^P \le t_2^P \dots \le t_m^P \le t$. The firm's private history h_t^f in addition contains the history of the θ_t process as well as the choices up to time t of π and a.

□ **Strategies, beliefs, and reputation.** A firm's strategy is an investment and promotion plan, a stochastic process $(a_t, \pi_t)_{t \ge 0}$ that determines the investment and promotion choice at each instant of time. Strategies are predictable processes with respect to the filtration generated by the quality Poisson process and the information Poisson processes. Informally, for each time t, the strategy processes are mappings from the realization of the private history up to, but not including, time t to the firm's choice of level of promotion conditional on having a high quality product and the firm's level of investment.

The reputation process is a stochastic process $(\boldsymbol{x}_t)_{t \ge 0}$, $\boldsymbol{x}_t := \Pr(\theta_t = H | h_t^c)$, where the probability measure is the measure induced by the consumers' beliefs about the firm's strategy. These believed strategies are a pair of processes $(\tilde{a}_t, \tilde{\pi}_t)$, predictable with respect to the public history, and $\boldsymbol{x}_t$ is determined after each realized history using the probability distribution that would be induced if the investment and promotion decisions were determined by the $\tilde{a}_t$ and $\tilde{\pi}_t$ processes. Let $\boldsymbol{x}_{t-}$ denote the limit from the left of this process $\boldsymbol{x}_{t-} = \lim_{s \to t-} \boldsymbol{x}_s$.

In addition, I maintain the following assumption on rates.

Assumption 1. Throughout this article, assume that $\lambda \ge \max\{\mu, \gamma - \mu\}$.

[1] As in Board and Meyer-ter-Vehn (2013), formally there is a probability space $(\Omega, \mathcal{F}, P)$, (i) a random variable θ_0 that determines initial quality, (ii) a sequence of uniform [0,1] random variables that control investment, (iii) a sequence of uniform random variables that control promotion, and (iv) the quality, promotional opportunities, and news Poisson processes.

This ensures that quality adjustments occur faster than the arrival rate of information, which greatly simplifies the analysis (for instance, this is necessary even for establishing existence in the simplest version of the model considered, Proposition 1). This assumption is discussed more in subsequent sections.

□ **Payoffs.** The firm receives a flow payoff of $\boldsymbol{x}_t$. This can be motivated either as the willingness to pay of consumers with utility $1_{\theta_t=H}$ or as consumers who are willing to pay 1 arriving at some rate proportional to the public belief. The firm maximizes

$$\sup_{a,\pi} E_{a,\pi}\left(\int_0^\infty e^{-rt}[\boldsymbol{x}_t - ca_t - k\pi_t]\, dt\right)$$

where the expectation is taken with respect to the actual probability measure induced by the firm's chosen investment and promotion level, whereas $\boldsymbol{x}_t$ is determined by what consumers believe about the firm's investment and promotion choice.

□ **Solution concept.** I characterize Markov Perfect Equilibrium. These are Perfect Bayesian Equilibria where the firm's investment and promotion choice at each moment of time only depends on the the current market belief about quality, $\boldsymbol{x}_t$.

Definition 1. A pure strategy **Markov Perfect Equilibrium** (MPE) consists of a Markov investment strategy $a : [0, 1] \to [0, 1]$ and a Markov promotion strategy $\pi : [0, 1] \to [0, 1]$, which maps market beliefs to corresponding investment strategy and promotion strategy, $a_t = a(\boldsymbol{x}_{t-})$ and $\pi_t = \pi(\boldsymbol{x}_{t-})$, respectively, such that, given any $x_0 \in [0, 1]$:

1. Beliefs $\boldsymbol{x}_t$ are formed through Bayes Rule, when possible, given believed strategies (a, π).
2. (a, π) are sequentially rational; they maximize payoffs following any realized private history.
3. If promotion arrives at time t, beliefs at any time $s > t$ are consistent with Bayes rule given the realized history, believed strategies, and belief $\boldsymbol{x}_t = 1$. Similarly, if bad news arrives at time t, beliefs are consistent with Bayes rule at any time $s > t$ given the realized history, believed strategies and belief $\boldsymbol{x}_t = 0$.

Bayes rule is quite complicated in this setting, but, for well behaved believed strategies, it can be compactly described with an ordinary differential equation. In this context, given any Markov believed strategies (a, π), Bayes rule implies on path that if a and π are continuous at x_t, and no signal arrives between time t and $t + \delta$ for small enough δ then beliefs are

$$\boldsymbol{x}_{t+\delta} = \lambda a(\boldsymbol{x}_t)\delta + (1 - \lambda\delta)\frac{x_t(1 - \gamma\pi(\boldsymbol{x}_t)\delta)}{\boldsymbol{x}_t(1 - \pi(\boldsymbol{x}_t)\delta) + (1 - \boldsymbol{x}_t)(1 - \mu\delta)} + o(\delta).$$

So, taking $\delta \to 0$, Bayes rule implies that between arrivals of news the reputation process must evolve according to the differential equation

$$\dot{\boldsymbol{x}}_t = \lambda(a(\boldsymbol{x}_t) - \boldsymbol{x}_t) + (\mu - \gamma\pi(\boldsymbol{x}_t))\boldsymbol{x}_t(1 - \boldsymbol{x}_t).$$

Moreover, beliefs jump to either 1 if promotion arrives or 0 if bad news arrives.

The final condition in the definition ensures that the market always interprets promotion as proof that the firm is selling a high-quality product (and bad news as proof that the firm is selling a low-quality product), and from then on continues using Bayes rule with respect to the (a, π). These updates are directly implied by Bayes rule on-path. This is a natural restriction, given the focus on Markov strategies, in the sense that these conditions imply that (i) the market believes that any realized news perfectly reveals the firm's current quality, as promotion can only be realized when the firm is selling a high quality product, and bad news can only be realized when the firm is selling a low quality product and (ii) the Markov restriction on strategies holds not only on-path but also off-path. Analogous restrictions are used in other, similar settings, including Marinovic et al. (2018).

For fixed Markov believed strategies, $(\tilde{a}(\cdot), \tilde{\pi}(\cdot))$, let $V(x, \theta)$ denote the firm's value function. That is

$$V(x, \theta) := \sup_{a,\pi} E_{a,\pi}\left(\int_0^\infty e^{-rt}[\boldsymbol{x}_t - ca_t - k\pi_t]\,dt \middle| \boldsymbol{x}_0 = x, \theta_0 = \theta\right)$$

where $\boldsymbol{x}_t$ is formed by $(\tilde{a}(\cdot), \tilde{\pi}(\cdot))$. Suppressing the initial condition, let $(x_t)_{t\geq 0}$ denote the trajectory of the beliefs conditional on no news arriving by time t, that is

$$x_t = \Pr_{\tilde{a},\tilde{\pi}}(\theta = H \mid \text{no arrivals of the news or promotion process for any } t \in [0, t]).$$

This process is deterministic and describes how beliefs evolve between arrivals. As discussed previously, for well-behaved believed strategies, it must solve the differential equation

$$\dot{x}_t = \lambda(\tilde{a}(x_t) - x_t) + (\mu - \gamma\tilde{\pi}(x_t))x_t(1 - x_t).$$

There may be Markov believed strategies where this process is ill-defined (see Klein and Rady (2011) for a discussion of this issue), so I impose an admissibility restriction on believed strategies. This condition is essentially identical to the restriction from Board and Meyer-ter-Vehn (2013).

Definition 2. Fix Markov believed strategies $(\tilde{a}, \tilde{\pi})$ and let $g(x) := \lambda(\tilde{a}(x) - x) + (\mu - \gamma\tilde{\pi}(x))x(1 - x)$. These believed strategies are said to be **admissible** if they satisfy one of the following conditions at any point of discontinuity of $g(x)$:

1. $g(x) = 0$
2. $g(x) > 0$ and $\tilde{a}(x), \tilde{\pi}(x)$ are right continuous at x,
3. $g(x) < 0$ and $\tilde{a}(x), \tilde{\pi}(x)$ are left continuous at x,

and [0,1] can be partitioned into a finite set of intervals $[0, x_1^*), \ldots, (x_i^*, x_{i+1}^*), \ldots, (x_n^*, 1]$ such that both the $\tilde{a}$ and $\tilde{\pi}$ are Lipschitz continuous on the interior of all these intervals.

These conditions guarantee a solution to the differential equation exists. When there are multiple solutions, I select the one consistent with a discrete-time approximation. As shown in appendix A.1 of Board and Meyer-ter-Vehn (2013) these conditions pin down a unique continuous solution to the belief law of motion and ensure that the mapping $t \mapsto (\tilde{a}(x_t), \tilde{\pi}(x_t))$ is unique and right continuous.[2,3]

3. Preliminary analysis

□ **Incentives.** The value functions provide a convenient tool to express equilibrium incentives. Define the value of quality as $D(x) := V(x, H) - V(x, L)$ and marginal value of promotion as $\Delta(x) := V(1, H) - V(x, H)$.

Lemma 1. (**Sequential rationality**) For admissible Markov believed strategies $(\tilde{a}, \tilde{\pi})$ and the corresponding value functions, strategies $(a_t, \pi_t)_{t=0}^\infty$ are sequentially rational if and only if after every private history, except on a measure 0 set of times $t \in [0, \infty)$, a_t solves

$$\max_{a\in[0,1]} \lambda D(x)a - ca, \tag{1}$$

[2] This selection is unique. Without it, there can be multiplicities caused by the discontinuity in the drift $g(x)$. For instance, consider the differential equation described by $g(x) = 1_{\{x>.5\}} - 1_{\{x<.5\}}$ and $x_0 = .5$. The equations $x_t = .5 + t$, $x_t = .5 - t$ and $x_t = .5$ all solve the equation $x_t = x_0 + \int_0^t g(x_s)\,ds$, but only $x_t = .5$ is consistent with a discrete time approximation.

[3] The Picard-Lindelöf theorem implies that a unique solution exists for any $x_0 \in (x_i^*, x_{i+1}^*)$ until the trajectory reaches the boundary, and the selection rule pins fixes behavior at the boundaries. See Klein and Rady (2011) for a detailed discussion. The argument from Appendix A.1 of Board and Meyer-ter-Vehn (2013) applies to this setting mutatis mutandis.

and π_t solves

$$\max_{\pi \in [0,1]} \gamma \Delta(x)\pi - k\pi \tag{2}$$

for $x = \boldsymbol{x}_t$.

This immediately implies that for any admissible Markov believed strategies, there exist sequentially rational Markov strategies. From now on, in a slight abuse of language, I say a Markov strategy is sequentially rational if it satisfies equations (1) and (2) for all $x \in [0, 1]$.

The firm's incentive to invest is driven by $D(x)$, the change in the firm's expected payoffs if a quality change arrives at that instant. The firm's incentive to promote is determined by $\Delta(x)$; the change in payoffs if beliefs jumped to 1 at that moment. Note that the investment choice is independent of the firm's current type. In addition, due to the linearity of costs, the sequential rationality conditions are linear, which in most cases implies that the optimal control is bang-bang.

$D(x)$ and $\Delta(x)$ have a tight relationship, the increase in a firm's payoff from having high quality is due to the potential reputational benefits the firm receives in the future once it is selling a high-quality product. This is captured in the following proposition.

Lemma 2. For admissible Markov believed strategies $(\tilde{a}, \tilde{\pi})$ and corresponding value functions, the value of quality satisfies

$$D(x) = \int_0^\infty e^{-(r+\lambda)t}[\pi(x_t)(\gamma \Delta(x_t) - k) + \mu(V(x_t, L) - V(0, L))]\, dt, \tag{3}$$

where x_t solves $\dot{x}_t = \lambda(\tilde{a}(x_t) - x_t) + (\mu - \gamma\tilde{\pi}(x_t))x_t(1 - x_t)$ with $x_0 = x$.

Moreover, payoffs satisfy the following properties for any believed strategies

1. $V(\cdot, H)$ and $V(\cdot, L)$ are strictly increasing.
2. $\Delta(x)$ is strictly decreasing.
3. $V(x, H) \geq V(x, L)$ for all x.
4. If $\mu = 0$, $D(x)$ is weakly decreasing, strictly so unless there exists a $\pi(x)$ that solves Eq. (2) and satisfies $\Pr(\pi(x_t) = 0 \text{ for almost all } t \geq 0 | \theta_0 = H, x_0 = x) = 1$, that is, the firm never promotes again.

The value functions are increasing. Given any two initial conditions x_0 and x'_0, if $x_0 > x'_0$ then $x_t > x'_t$ until the first arrival of news. A firm with initial condition x_0 can always play the strategies that a firm with initial condition x'_0 would play and induce the same probability measure over θ_t and arrivals at all points whereas receiving a strictly higher flow payoff. Therefore, its payoff must be higher. Property 2 immediately follows from property 1. Finally, it follows equation (3), the observation that $\Delta(x_t)$ decreasing, and the sequential rationality conditions that $D(x) \geq 0$ and is increasing, implying properties 3 and 4.

The value of quality, $D(x)$ can be expressed as the sum of the discounted benefit from future promotion, and future losses due to bad news. The firm's continuation $V(x, \theta)$ is increasing in the firm's reputation, so promotion is less valuable for a firm with a higher reputation. Thus the investment incentives generated by promotion are lowest at high reputations and highest at low reputations.

4. The no exogenous news case

■ To isolate the role of promotion in driving reputation dynamics, I first focus on an environment where there is no exogenous news, $\mu = 0$, so consumers learn only through promotion generated by the firm.

□ **Beliefs.** In an environment with only promotion, arrivals of the information process can only occur when the firm has a high quality product. Whenever the firm successfully promotes beliefs jump to 1. Between arrivals, given Markov believed strategies $(\tilde{a}, \tilde{\pi})$, beliefs follow the law of motion

$$\dot{x}_t = \underbrace{\lambda(\tilde{a}(x_t) - x_t)}_{\text{Quality breakthroughs}} - \underbrace{\gamma\tilde{\pi}(x_t)x_t(1 - x_t)}_{\text{Absence of signals}}.$$

Between arrivals, the drift of consumers' beliefs can be expressed as the sum of two terms. The firm's believed investment choice determines the first term $\lambda(\tilde{a}(x_t) - x_t)$. This term captures how consumers account for unobserved quality shocks. Depending on how much the firm is believed to be investing this term is either positive or negative. If consumers believe it is more likely that high-quality products are switching to low-quality products than low-quality products are switching to high-quality products, then this term is negative; otherwise, it is positive. The second term, $-\gamma\tilde{\pi}(x_t)x_t(1 - x_t)$, is determined by the firm's choice of promotion. This term is always non-positive; no news is always (weakly) a signal that the firm is selling a low-quality product.

□ **Incentives.** In the setting with only promotion, the firm's investment is driven by how much it expects to gain from promotion. A natural first step in this analysis is asking how promotion drives investment incentives, fixing believed strategies.

Definition 3. Fix any pair of admissible Markov believed strategies $(\tilde{a}, \tilde{\pi})$. Promotion is a **complement** for investment if for any promotion costs k and k', $k > k'$ and for any pair of sequentially rational Markov investment strategies, $a_k(x)$ and $a_{k'}(x)$ in the respective games, the firm's optimal level of investment $a(x)$ is decreasing in k, $a_k(x) \geq a_{k'}(x)$ for all $x \in [0, 1]$.

Promotion is a complement for investment if, fixing market believed strategies, making promotion cheaper increases investment (weakly) at every belief. This does not necessarily imply that reducing the cost of promotion increases the equilibrium level of investment. The new equilibrium believed strategies could give lower incentives for investment.

Lemma 3. Suppose $\mu = 0$, then promotion and investment are complements.

The cheaper promotion becomes, the more the firm benefits from creating a high-quality product. So, fixing believed strategies, as promotion becomes cheaper the firm's incentives to invest become stronger.

□ **Equilibrium.** Lemma 2 implies that the optimal promotion strategy is a cutoff strategy when there is no exogenous news. Moreover, $D(x)$ is strictly decreasing in x as long as the firm plans to promote at some time in the future. So both the optimal promotion and investment strategies are bang-bang and can be characterized by two cutoffs x_a and x_π. The firm shirks and doesn't promote at high beliefs. As beliefs drift down eventually the firm starts promoting and investing. This implies that in any equilibrium beliefs can be categorized into four different regions; the region where the firm is neither promoting nor investing, the region where the firm is promoting but not investing, the region where the firm is investing but not promoting, and the region where the firm is investing and promoting. The following proposition establishes that, in equilibrium, the region where the firm is promoting but not investing never arises. Although the game may have multiple equilibria, the following proposition establishes the unique structure of any equilibrium.

Proposition 1. A Markov Perfect Equilibrium exists.

Fix a Markov Perfect Equilibrium. There exist two cutoffs $x_a, x_\pi \in [0, 1)$, such that equilibrium strategies satisfy

$$a(x) = \begin{cases} 1 \text{ if } x < x_a \\ 0 \text{ if } x > x_a \end{cases},$$

$$\pi(x) = \begin{cases} 1 \text{ if } x < x_\pi \\ 0 \text{ if } x > x_\pi \end{cases},$$

and either $x_a = x_\pi = 0$ or $x_a < x_\pi$.

As beliefs decrease, the firm begins promoting before it starts investing. This is a consequence of the endogenous monitoring. If the firm was ever at a reputation where the market believes it is investing but not promoting, then its reputation must be drifting up. From then on, its incentives to promote only get weaker over time, so the firm never promotes. However, if this was the case, then the firm would have no incentive to work, as the information consumers expect to see wouldn't change if the firm shirked. Therefore, in any equilibrium, the firm must start promoting before it starts investing.

In equilibrium there are three regions defined by the cutoffs, and the drift of beliefs changes discontinuously at these cutoffs. At the lower extreme, $[0, x_a)$, the firm is both investing and promoting and beliefs are drifting up, as the market believes the firm is investing. In the upper extreme, $(x_\pi, 1]$, beliefs are drifting down. Although the market believes that no news is equally likely under both high and low quality, the market also believes the firm is not investing, so reputation gradually decays. Finally there is a region (x_a, x_π) where the firm is promoting but not investing. In this region, beliefs are still drifting down, and are drifting down at a faster rate, as consumers update from the lack of promotion and investment. After a quality breakthrough and successful promotion, the firm collects reputational dividends and uses its ability to promote to enhance the dividend it collects from that original successful investment. If the firm is fortunate, news arrives before it has to invest again. In this region, the firm tries to use promotion to renew it's reputation multiple times as a result of a single initial success.

The existence of Markov Perfect Equilibrium is established using Kakutani's fixed point theorem on an operator that maps believed cutoffs to the firm's optimal cutoffs given those believed strategies. The discontinuous drift of beliefs due to cutoff strategies complicates this argument. This is simplified to some extent by the assumption on the rates, $\lambda \geq \gamma$, which implies that market beliefs about investment determine the direction of the drift of reputation. Markov Perfect Equilibria always exist if $\lambda \geq \gamma$.

Without Assumption 1, that is, if $\lambda < \gamma$, existence would be more difficult to establish. The main technical difficultly lies in the discontinuity of value functions when viewed as a function of believed strategies. This difficulty arises from the additional discontinuity in the drift of beliefs. Consider the drift of reputation at some x slightly below some believed investment cutoff x_a. If $x_\pi < x$, beliefs are drifting up at this point, whereas if $x_\pi > x$ then beliefs may be drifting down, which Assumption 1 ruled out. So small perturbations in x_π can lead to large changes in the trajectory of beliefs, and thus in the induced value functions. This logic leads to a discontinuity in the value functions around the beliefs where $x_a = x_\pi$, which in turn leads to the potential non-existence of MPEs.

The region where the firm promotes but does not invest does not exist in Marinovic et al. (2018). They consider a similar model, where the firm can deterministically reveal the quality of its product at any time for a cost. In that setting, in any MPE the firm is either both investing and disclosing as soon as it produces a high-quality product or neither investing nor disclosing. As a result, the marginal value of disclosures never exceeds the cost of disclosing in equilibrium. In contrast, the noisiness of promotion in the present model allows the firm to benefit from its disclosures. Once beliefs drop below x_π then $\gamma\Delta(x) > k$; the marginal value of promotion for the firm is positive at almost all times when the firm is promoting.

□ **Dynamics and comparison to exogenous news.** In this setting, investment is driven by the gains the firm anticipates receiving from promotion. These incentives are strongest at low reputations, where the gains from promotion are largest. This leads to reputation cycles.

Definition 4. An MPE exhibits **reputation cycles** if for all $t \geq 0$, $\Pr(\boldsymbol{x}_s = 1 \text{ for some } s > t) = 1$ and $\Pr(\boldsymbol{x}_s < 1 \text{ for some } s > t) = 1$.

To describe these cycles in more detail, I use the exogenous good news model from Board and Meyer-ter-Vehn (2013) as a benchmark. In their setting, exogenous news arrives at rate $\gamma 1_{\theta_t = H}$ and the firm cannot promote.[4] Throughout this section, I call this the game *without promotion* and I call the baseline game the game *with promotion*. Board and Meyer-ter-Vehn (2013) shows that in the game without promotion, all MPEs have the following form

$$a(x) = \begin{cases} 1 \text{ if } x < x_{a,EX} \\ 0 \text{ if } x > x_{a,EX} \end{cases}$$

for some cutoff $x_{a,EX} \in [0, 1]$.

Reputation cycles are a long-run feature of the firm's reputation dynamics. No matter how much time has passed, the firm will eventually renew its reputation and then allow its reputation to decay. This dynamic manifests slightly differently in the game with promotion than in the game without promotion. This is due to the additional moral hazard problem, which leads to both a rather direct increase in costs – information costs k when it was free before – and the existence of a region where no information arrives ($(x_\pi, 1]$). The latter is straightforward to see from the equilibrium characterization, the former is summarized in the Lemma 4

Lemma 4. Let (x_a, x_π) be cutoffs corresponding to an MPE in the game with promotion and $x_{a,EX}$ be the MPE cutoff in the game without promotion. Then in those MPEs, either the firm doesn't invest at any $x \in [0, 1]$ or

$$\gamma\Delta(x_a) = (1 + r/\lambda)c + k$$

in the game with promotion and

$$\gamma\Delta_{EX}(x_{a,EX}) = (1 + r/\lambda)c$$

in the game without promotion.

The marginal value of news must be higher to incentivize investment in this setting, relative to that of Board and Meyer-ter-Vehn (2013). These differences manifest in the qualitative features of the reputation dynamics.

Definition 5. Fix initial reputation $x_0 = 1$. A reputation process $\boldsymbol{x}_t$ has **longer** reputation cycles than a process $\boldsymbol{x}'_t$ if the stopping times $\tau := \inf\{t > 0 : \boldsymbol{x}_t = 1\}$, and $\tau' = \inf\{t > 0 : \boldsymbol{x}'_t = 1\}$ satisfy $E(\tau) \geq E(\tau')$.

Definition 6. A reputation process $\boldsymbol{x}_t$ has **larger** reputation cycles than $\boldsymbol{x}'_t$ if $supp(\mathcal{X}') \subseteq supp(\mathcal{X})$ where $\mathcal{X}$ and $\mathcal{X}'$ are $\boldsymbol{x}_t$ and $\boldsymbol{x}'_t$'s respective stationary distributions.[5]

[4] The exogenous good news model can equivalently be viewed as the equilibria of the game where $k = 0$.

[5] In any equilibrium of this game and the game from Board and Meyer-ter-Vehn (2013), the distribution of beliefs converges to a unique stationary distribution. The latter follows from lemma 4 in Board and Meyer-ter-Vehn (2013), the argument for former is shown in the online appendix (Lemma 15).

Definition 7. Fix initial reputation $x_0 = 1$. A reputation process $\boldsymbol{x}_t$ has **slower** reputation cycles than a process $\boldsymbol{x}'_t$ if at all $t \geq 0$, the reputation processes conditional on no news, x_t and x'_t, satisfy $x_t - x'_t \geq 0$.

Given a pair of possible reputation processes, one process exhibits longer reputation cycles than the other if expected time it takes for the firm's reputation to return to 1 is larger under that process than it is under the other process. This time until renewal depends not only on the monitoring structure but also on a firm's investment decisions; a firm that never invests in quality may face longer reputation cycles than a firm that does, even if news arrives much faster for the firm that doesn't invest. Reputation cycles are larger if, in the long run, consumers can become more pessimistic about quality than they ever would be in the alternative reputation process. Reputation cycles are slower if in the absence of news reputation decays more slowly.

Corollary 1. Fix an MPE of the game with promotion and an MPE of the game without promotion. Reputation cycles are longer, larger, and slower in the game with promotion.

The expected time it takes for reputation to return to 1 is shorter in the game with exogenous news. Reputation is renewed more frequently, but drifts down faster. The slower, longer cycles in the model with promotion are partially driven by the existence of the region $(x_\pi, 1]$, where the market does not expect any information. Moreover, there is effectively an increased cost of investment, as the firm needs to be compensated for the costs of promotion. These forces together lower the investment cutoff, x_a, and slow both the rate that reputation decays and the rate that reputation is renewed.

To further explore how the addition of promotion, a second unobserved action influences incentives, consider the case where $c \to 0$, so any distortion in the level of investment is driven by the endogenous monitoring. In this limit, the firm's payoffs are lower in the game with promotion. The firm's losses exceed the loss due to the cost of generating information, and the equilibrium market beliefs that the presence of that cost induces.

Proposition 2. (Limiting properties as $c \to 0$)

- *Reputation cycles persist as the cost of investment goes to 0*: There exists a $\bar{x} < 1$ such that for any c and any MPE of the game with promotion, $a(x) = 0$ if $x > \bar{x}$
- *Cycles vanish in the game without promotion*: In the game without promotion for any $\varepsilon > 0$, there exists a $\bar{c}$ such that if $c < \bar{c}$ then $a(x) = 1$ if $x < 1 - \varepsilon$.
- There exists a $\hat{c}$ such that if $c < \hat{c}$ then the firm's losses from endogenous promotion exceed the firm's expenditure on promotion:

$$V_{EX}(x, \theta) - V(x, \theta) > E\left(\int_0^\infty e^{-rt} k\pi(\boldsymbol{x}_t) 1_{\{\theta_t = H\}}\, dt | \boldsymbol{x}_0 = x, \theta_0 = \theta\right)$$

where $\pi(x)$ is the promotion strategy in the game with promotion and $\boldsymbol{x}_t$ is the corresponding reputation process

Even as investment becomes arbitrarily cheap, the firm still doesn't invest everywhere. In order for reputation to be an effective force in disciplining the firm, the firm needs to be monitored. The firm still has to pay for promotion to make investment worthwhile, which indirectly makes investment costly. As a result of learning through promotion and not exogenous news, reputation cycles persist even as $c \to 0$. To get rid of these cycles, a firm with a low cost of investment would be willing to pay more than it expects to spend on promotion in order to commit to releasing information independent of its reputation.

5. Promotion and exogenous bad news

■ In the previous section, promotion generated reputational incentives. In equilibrium, the firm paid for information that allowed the market to monitor what would otherwise be an unobservable investment. Promotion introduced a tension, investment is only credible when promotion is valuable to the firm, but the gains from promotion only cover its cost when the firm's reputation is very low. Although promotion created incentives for investment, it also led to reputation cycles; repeated periods of reputational collapse followed by renewals through investment and promotion. Perhaps the firm would have stronger incentives to invest at high reputations if the market could monitor the firm at high reputations. It is natural to ask how promotion interacts with the reputational incentives generated by other, exogenous, sources of information.

Reputation is often driven by bad news that reveals low quality. A restaurant's reputation can be ruined by a bad health report, a phone manufacturer's reputation by an article about faulty hardware, a politician's career can be ended by bad press. On its own, the threat of bad news delivers powerful incentives at high reputations. A firm with a high reputation has the most to lose from bad news and thus has strong incentives to invest in quality.[6] As a result, when consumers learn about quality only through observing bad news, once a firm has established a high reputation it is likely to keep it. However, at low reputations, bad news is not threatening to the firm, reducing investment incentives. The market's equilibrium beliefs about strategies amplify this effect. If the market does not believe that low-reputation firms are investing then the firm has less to lose from future bad news, weakening incentives to invest at low reputations further. This effect leads to "reputation traps," a low-reputation firm is likely to be a low-reputation firm forever.

A firm's ability to promote fundamentally changes these dynamics. In this section, I explore the impact of promotion on the incentives generated by bad news. As the incentives for investment generated by promotion are highest at low reputations, allowing the firm to promote can dramatically change reputation dynamics. Promotion gives the firm a tool to renew its reputation, regardless of what the market believes at low reputations, so promotion can eliminate these reputation traps. However, promotion can also crowd out the incentives for investment at high reputations that bad news generates.

□ **Promotion and reputation traps.** Consumers now learn from promotion and exogenous bad news. Specifically, consumers observe the bad news process, a Poisson process with arrival rate $\mu 1_{\{\theta_t = L\}}$ in addition to the promotion process, which behaves as before. Exogenous news only arrives when the firm is selling a low-quality product, so whenever exogenous news arrives the market belief jumps to 0. Promotion reveals that the firm is selling a high-quality product, so whenever promotion arrives the firm's reputation jumps to 1. In the absence of news, given believed strategies $(\tilde{a}, \tilde{\pi})$, beliefs follow the law of motion

$$\dot{x}_t = (\tilde{a}(x_t) - x_t)\lambda + (\mu - \tilde{\pi}(x_t)\gamma)x_t(1 - x_t).$$

As before, the incentive to promote is driven by the marginal value of having the firm's reputation jump to 1, and investment is driven by the difference between the value of having a high and low-quality product at the current reputation. There still is a close link between these two objects, but the addition of bad news complicates things. In particular, the monotonicity properties of

$$D(x) = \int e^{-(r+\lambda)t}[\pi(x_t)(\gamma(V(1, H) - V(x_t, H)) - k) + \mu(V(x_t, L) - V(0, L))]\, dt$$

are more complex; it is no longer unambiguously decreasing. With endogenous promotion, at high reputations the only source of monitoring is bad news, which drives incentives, whereas at low reputations both bad news and promotion together determine the incentives for investment.

[6] In Hauser (2023) I consider a setting where a firm can censor bad news.

This non-monotonicity is not present in settings with only exogenous bad news or exogenous good news; moreover it is also not present in setting with both exogenous bad and good news.[7] This allows for the possibility of equilibria where at high reputations the firm invests to prevent bad news, and at low reputations the firm invests to create something to promote, but has no incentive to invest at intermediate reputations.

Proposition 3. Suppose $\mu \neq \gamma$. Fix a Markov Perfect Equilibrium. In this equilibrium, there exist three cutoffs, $x_\pi \in [0, 1)$, $\underline{x}_a \in [0, x_\pi]$, $\overline{x}_a \in [\underline{x}_a, 1]$, such that

$$a(x) = \begin{cases} 1 \text{ if } x > \overline{x}_a \text{ or } x < \underline{x}_a \\ 0 \text{ if } \overline{x}_a > x > \underline{x}_a \end{cases}$$

$$\pi(x) = \begin{cases} 1 \text{ if } x < x_\pi \\ 0 \text{ if } x > x_\pi \end{cases}$$

Moreover, if $\mu > \gamma$ then $\underline{x}_a = 0$.

In any equilibrium, the firm invests at extreme reputations, where either the gains from promotion or the threat of bad news are strongest, and shirks at intermediate reputations. When bad news arrives sufficiently rapidly, $\mu > \gamma$, the firm only invests at high reputations. In this case, the threat of bad news dominates, driving investment more than the potential gains from promotion. However, once promotional opportunities arrive sufficiently rapidly, investment incentives start building up again at low reputations. This allows for equilibria where the firm invests at high and low reputations and does not at intermediate reputations.

If the firm was unable to promote, $\gamma = 0$, all equilibria would feature at most one investment cutoff, $\overline{x}_a$. The firm would invest at high reputations, above $\overline{x}_a$, and not invest at low reputations. In this game, there always exist equilibria where $a(0) = 0$; equilibria where after bad news the firm stops working. Of particular interest are the equilibria where $a(0) = 0$ and $a(1) = 1$.[8] In these equilibria high reputations are persistent, a firm that has established a high reputation keeps it. The firm continues to invest in quality, knowing that if it stopped and produced a bad product it would be punished with bad news and a bad reputation forever. This threat of punishment, provides strong incentives at the top. However, bad news also means that firms that either start with a low reputation or generate bad news have low incentives to continue to invest. A low-reputation firm never invests in quality or establishes a reputation. In these equilibria, *ex ante*, the firm always risks falling into a reputation trap.

Definition 8. An MPE has a **reputation trap** if with positive probability ex ante the firm's long-run reputation reaches and remains at 0, that is, $\Pr(\underline{\tau} < \infty) > 0$ where $\underline{\tau} = \inf\{t : \boldsymbol{x}_s = 0 \, \forall s > t\}$.

Definition 9. An MPE has **persistent reputation** if there exists an $\underline{x}_0 \in [0, 1)$ s.t. if $x_0 \geq \underline{x}_0$ then $\Pr(\lim_{t\to\infty} x_t = 1) > 0$.

Promotion allows the firm to reestablish itself after bad news. Without promotion, if consumers believe that $a(0) = 0$, then it is always sequentially rational for the firm to play $\tilde{a}(0) = 0$. Now the low-reputation firm may have an incentive to invest, even in the face of these pessimistic

[7] In these settings

$$D(x) = \int_0^\infty e^{-(r+\lambda+\mu)t}[\gamma V(1, H) - \mu V(0, L) - (\gamma - \mu)V(x_t, H)]\, dt,$$

which is increasing if $\mu > \gamma$ and decreasing if $\mu < \gamma$, so there is at most 1 investment cutoff in any equilibrium.

[8] In the game where $\gamma = 0$, all equilibria are either of this form or have either no investment at any belief or investment at all beliefs. At least one equilibria where $a(0) = 0$ exists in the game where $\gamma = 0$ for all parameter values.

believed strategies. It knows that it can always credibly signal investment through promotion and gain a high reputation. Moreover, unlike in the setting with only promotion, the firm still has an incentive to invest at high reputations; it wants to avoid bad news. If the firm is investing at $x = 1$, it is an absorbing state. Eventually, if a firm establishes a high reputation, it can invest in quality to avoid the potential for future bad news. In the long run, the firm is always selling a high-quality product and receives the same continuation payoff it would receive if investment was observable. The interaction between promotion and bad news can rule out MPEs where the firm doesn't invest at low reputations, which would not exist in the game without promotion.

Proposition 4. For sufficiently low cost of investment and promotion, in any MPE, $a(0) > 0$, that is, the MPE does not have a reputation trap.[9]

□ **Incentives at high reputations.** In the presence of exogenous bad news, the ability to promote is potentially a double-edged sword. By creating credible investment incentives at low reputations, promotion can crowd out the incentives for investment at high reputations. The penalty a firm faces for failing to maintain a high reputation is less severe if it can promote to escape a reputation trap. In fact, with exogenous bad news, promotion need not be a complement for investment; reducing the cost of promotion can increase investment at low reputations and decrease investment at high reputations. For intermediate costs of investment, when $\gamma > \mu$, the ability to promote can crowd out equilibria where the firm invests at high reputations, which existed in the game without promotion.

Proposition 5. Suppose the rate of promotion is sufficiently high relative to the rate of bad news, $\frac{\gamma-\mu}{r+\mu} > k$. Then

1. For sufficiently high costs of investment, that is, there exists a $\underline{c}$ such that if $\frac{\mu\lambda}{(r+\mu)(r+\lambda)} > c > \underline{c} > 0$, an MPE exists and no MPEs have persistent reputation.
2. If the firm can not promote, $\gamma = 0$, if $\frac{\mu\lambda}{(r+\mu)(r+\lambda)} > c$ there exists an MPE with persistent reputation.

In fact, if $\gamma > \mu$ and $\frac{\mu\lambda}{(r+\mu)(r+\lambda)} > c$, then in the game where the firm cannot promote there exists an MPE with persistent reputation. So, if $\gamma > \mu$ then if there is an equilibrium in the game with promotion with persistent reputation, then there is an equilibrium in the game with only bad news has persistent reputation.[10]

When promotion is sufficiently effective, both in terms of the rate of promotional opportunities and the cost of promotion, then it has a perverse impact on incentives. Bad news no longer effectively disciplines the firm at high reputations, because the firm knows if it stops investing it can always renew its reputation through promotion. Moreover, if $\gamma > \mu$, it is never the case that the ability to promote can create incentives to invest at high reputations that wouldn't also exist in the game without promotion. As firms gain access to cheaper and more effective advertising channels, like internet advertising through Facebook or Google, the long-run dynamics of reputation and product quality become more volatile.

6. Discussion and conclusion

■ To conclude, I briefly discuss some relevant modeling assumptions and related literature.

□ **Discussion of the model.** I augmented the reputation for quality model from Board and Meyer-ter-Vehn (2013) by adding an endogenous source of information. In doing this, I made some assumptions and modeling choices that are worth some discussion.

[9] $c((r+\gamma)/\lambda+1)+k < \frac{\gamma(r+2\lambda+\mu)}{(r+\mu+\gamma)(r+2\lambda+\gamma-\mu)}$ if $\mu \leq \gamma$ and $c((r+\gamma)/\lambda+1)+k < \frac{\gamma(r+2\lambda+\gamma)}{(r+\lambda+\mu)(r+2\lambda)}$ if $\mu > \gamma$.

[10] If $c > \frac{\mu\lambda}{(r+\mu)(r+\lambda)}$, then $a(1) = 1$ is never sequentially rational in either game. See Lemma B5.

Promotion. A key feature of the model is the stochastic arrival of promotional opportunities. Conceptually, this noise and the maximal rate of promotion, γ, can be viewed either as a technological constraint—for example, the uncertain amount of time to produce a successful ad campaign, the time it takes for a reviewer to want to do a write-up of your restaurant, the rate that opportunities to promote at a convention or trade show arrive. An alternate interpretation is that this process captures market inattention. Whereas a firm with a high-quality product is constantly releasing information, consumers notice these advertisements at rate $\gamma\pi_t$. From a technical perspective, this noise ensures that the absence of good news when the market expects the firm to promote does not perfectly reveal that the firm is selling a low-quality product. Without this friction, as soon as consumers believe the firm has a strict incentive to disclose, they immediately deduce the firm's type. Parallel work in Marinovic et al. (2018) studies a model where the firm can deterministically disclose quality and show that this unraveling of private information leads counter-intuitive features in MPEs, for instance, the firm is always made better by making disclosure more expensive. This unraveling doesn't occur here, as the market can never be sure if it is not seeing news due to a low quality or simply that the firm had no opportunity to promote. An analogous approach is commonly used in the static disclosure literature to eliminate this unraveling (Dye, 1985).

Rates. Assumption 1 restricts attention to environments where λ, the arrival rate of quality shocks, is larger than the maximum net arrival rates of news. This implies that, although quality is persistent, the firm is able to adjust quality relatively flexibly. I make this assumption primarily to simplify the evolution of beliefs. Beliefs are always drifting up when the firm is believed to be investing in quality ($a = 1$) and down otherwise ($a = 0$). Although some results on the structure of MPEs hold without this assumption, establishing the existence of MPEs even in the setting with only promotion becomes significantly more complex. This assumption also guarantees equilibrium uniqueness in the setting from Board and Meyer-ter-Vehn (2013), which helps facilitate comparisons between the two games.

Although I primarily make this assumption for technical convenience, this assumption is implicit in many games with Poisson information where the firm's quality is not persistent; for example, Faingold and Sannikov (2011). It is a natural assumption for many settings where quality is easy to adjust but not that visible. It may be hard to promote that a firm has hired a valuable employee, or that a firm has stopped cutting corners on some safety measures, compared to how easy it is for a firm to hire/fire that employee or begin enforcing stricter policies.

□ **Literature review.** This article builds on the approach to modeling reputation developed in Board and Meyer-ter-Vehn (2013), in which a firm builds a reputation through unobserved investments that have a persistent effect on quality. Their results, and relation to mine, are discussed in detail in previous sections. This framework is also used in work by Halac and Prat (2016) to study a manager trying to build a reputation for being attentive, Board and Meyer-ter Vehn (2022) who study a firm's life-cycle when it faces these reputational concerns. Most notably, this setting is used in closely related work by Marinovic et al. (2018), who allows the firm to deterministically disclose quality for a cost and show that the firm does not benefit from disclosure in any MPE. This is discussed in more detail in section 2.

There is also a large literature that focuses on information transmission via the disclosure of hard evidence (Acharya et al., 2011; Dye, 1985; Grossman, 1981; Jovanovic, 1982; Milgrom, 1981; Song Shin, 2003; Verrecchia, 1983). These articles generally consider games without any sort of moral hazard or reputational concerns, focusing on the properties of communication games where a privately informed receiver can choose whether or not to reveal hard evidence to the receiver.[11] In this literature, the closest article is Van Der Schaar and Zhang (2015). They

[11] Other somewhat related work in the literature on disclosure and signaling games includes principal-agent models with costly state-verification (Border and Sobel, 1987; Monnet and Quintin, 2005; Townsend, 1979), the literature on the incentives of certification agencies (Farhi et al., 2013; Lizzeri, 1999; Stahl and Strausz, 2017), and the literature on

study a worker with a fixed quality that has the option to certify that quality is above a given level. In addition to certification, the market gradually learns through exogenous information, generated whenever the worker works. They highlight some features of certification that are also present here, in particular how this ability to certify can eliminate reputation traps, where high quality but low reputation workers stop working.

This work fits into the literature on reputation in games, beginning in Kreps and Wilson (1982) and Milgrom and Roberts (1982), and surveyed in Mailath and Samuelson (2015). These articles, for the most part, take a different approach to modeling reputation. The player with reputational concerns has a fixed unobserved type, either strategic or a commitment type, and the strategic types distort their actions to try to mimic commitment types. These articles explore the strengths of these incentives—for instance, they can allow a patient firm to achieve a payoff close to the Stackelberg bound (Fudenberg and Levine, 1989, 1992)—and limitations (e.g., these effects vanish in the long run (Cripps et al., 2004)). Most of these articles treat the types as exogenous and fixed – unlike Board and Meyer-ter-Vehn (2013). Beyond the different approaches to modeling reputation, these articles, in general, treat the monitoring structure as given, whereas I endogenize it. A notable exception is Mailath and Samuelson (2001), which both allow the firm's underlying type (either strategic or an inept type) to change as a result of endogenous changes in ownership, which somewhat mimics the persistent quality used here. They provide a brief informal discussion of how allowing the firm to signal their type could change/preserve features of the equilibrium set.

Several articles introduce more complex monitoring structures to reputation models with commitment types. Some do this directly as a design question—asking what sort of information structures a designer would design to encourage reputation building (Hörner and Lambert, 2021; Knoepfle and Wagner, 2021; Pei, 2018; Salmi, 2023; Varas et al., 2020)– and others introduce features to the game to mimic some conceptual features of advertising (Dellarocas, 2005; Ekmekci, 2011; Liu and Skrzypacz, 2014; Liu, 2011). Of these, Liu (2011) is closest to this article, they consider a setting where consumers can choose how much information to acquire about past firm behavior and shows that this leads to reputation cycles, whereas I consider how the firm discloses information to the market.

7. Conclusion

■ This article considers how allowing a firm to promote influences how it builds and maintains its reputation. The ability to promote creates incentives for the firm to invest in quality, allowing consumers to better monitor the firm's privately known quality and investment decisions. When the only source of information is promotion, promotion is a complement for investment and drives incentives. This force is particularly powerful when the firm's reputation is very low; the firm then has strong incentives to produce a high-quality product because it knows it will then be able to promote it and charge a higher price in the future.

When consumers also learn from bad news about the product, promotion allows the firm to escape from reputation traps. Without promotion, low reputations are self-perpetuating; once the firm can promote, it has a mechanism to restore incentives and recover from reputation traps. When promotion and investment are sufficiently cheap, in every equilibrium the firm will eventually establish and maintain a high reputation. However, promotion and investment are no longer complements when the market also learns through other sources of information, that is, exogenous bad news. The ability to promote reduces the cost of letting the market see bad news, and can crowd out the incentives for investment at high reputations created by bad news. As a result, the ability to promote positive information about a product can make long-run reputations more volatile, leading to reputation cycles in situations where if the only source of information was bad news, reputation would eventually stabilize at either the highest or lowest level.

dynamic noisy signaling (Daley and Green, 2012; Dilmé and Li, 2016; Kremer and Skrzypacz, 2007; Noldeke and Van Damme, 1990; Swinkels, 1999).

Future research could explore other aspects of advertisement and reputation. In a companion article, Hauser (2023), I explore the role of censorship, a firm's ability to hide bad news about its product, in driving reputation dynamics. This ability to censor can completely destroy reputational incentives if it is too inexpensive, but it also can greatly enhance them if it is sufficiently expensive.

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Appendix A: Preliminaries

In this section, I characterize sequential rationality and it's implications for the value function for well-behaved Markov believed strategies.

Proof of Lemma 1. The firm's payoffs depend on a state $(\boldsymbol{x}_t, \theta_t)$ that evolves deterministically between Poisson arrivals. The standard approach, setting up an HJB equation and solving it, is difficult here as there is no clear initial condition. However, the piece-wise deterministic structure allows the Bellman equation for the problem to take on a relatively tractable form. For a general treatment of stochastic optimal control problems where the underlying process consists of deterministic drift and stochastic jumps, see Davis (1993).[12]

Let $\mathcal{A}$ and $\mathcal{P}$ denote the set of investment and promotion strategies and let A and Π denote the set of Markov investment and promotion strategies, measurable functions from [0,1] to [0,1]. Fix admissible Markov believed strategies $(\tilde{a}, \tilde{\pi}) \in A \times \Pi$ and corresponding value function $V : [0, 1] \times \{L, H\} \to \mathbb{R}$. Let $D(x) := V(x, H) - V(x, L)$ and $\Delta(x) := V(1, H) - V(x, H)$. Truncating at the first arrival of either news or a quality shock, the value function must satisfy

$$\begin{aligned} V(x, H) = \sup_{a\in\mathcal{A},\pi\in\mathcal{P}} \int_0^\infty & e^{-(r+\lambda+\gamma)t}[x_t + a_t(\lambda D(x_t) - c) \\ & + (\pi_t(\gamma\Delta(x_t) - k) + \gamma V(x_t, H) + \lambda V(x_t, L)]\, dt, \\ V(x, L) = \sup_{a\in\mathcal{A}} \int_0^\infty & e^{-(r+\lambda+\mu)t}[x_t + a_t(\lambda D(x_t) - c) \\ & + \mu V(0, L) + \lambda V(x_t, L)]\, dt, \end{aligned}$$

where x_t solves $\dot{x}_t = \lambda(\tilde{a}(x_t) - x_t) - (\gamma\tilde{\pi}(x_t) - \mu)x_t(1 - x_t)$ with $x_0 = x$. Therefore the optimal $a(x_t)$ solves

$$\max_{a\in[0,1]} \lambda D(x_t)a - ca$$

and the optimal $\pi(x_t)$ solves

$$\max_{\pi\in[0,1]} \gamma\Delta(x_t)\pi - k\pi$$

[12] Davis (1993) Chapter 4, notably Section 42, formally lays out the Bellman equation approach, and corresponding verification theorem, for a general class of problems with the structure considered here. Unlike in Davis, the vector field of the deterministic part of the process is not continuous in the problem considered here. However, admissibility ensures that the process has a unique, well-behaved solution, which allows his arguments to go through unchanged.

because these maximize the integrand pointwise. Any selection of functions $a(x)$ and $\pi(x)$ that solve these equations on [0,1] are Markovian, and are optimal by the principle of optimality.

Similarly, if strategies (a_t, π_t) are sequentially rational they must solve the above maximization problems after every history and every initial state except on an measure 0 set of times. Otherwise there would be some (x_0, θ) where, conditional on no arrival, strategies differ from these on a set of positive measure. However, as these strategies are optimal they must satisfy the recursion

$$V(x_0, H) = \int_0^\infty e^{-(r+\lambda+\gamma)t}[x_t + a_t(\lambda D(x_t) - c) + (\pi_t(\gamma\Delta(x_t) - k) + \gamma V(x_t, H) + \lambda V(x_t, L)]\, dt,$$

$$V(x_0, L) = \int_0^\infty e^{-(r+\lambda+\mu)t}[x_t + a_t(\lambda D(x_t) - c) + \mu V(0, L) + \lambda V(x_t, L)]\, dt.$$

However, as a_t and π_t do not satisfy equations (1) and (2) on a set of positive measure, they don't maximize this integral, contradicting the optimality of these strategies. □

Lemma A1. Fix any two initial conditions, x^a, x^b with $x^a > x^b$. Given a pair of admissible Markov believed strategies $(\tilde{a}, \tilde{\pi})$, let $g(x) := \lambda(\tilde{a}(x) - x) - (\pi(x)\gamma - \mu)x(1 - x)$. If $(x_t^a)_{t\geq 0}$ solves $\dot{x}_t^a = g(x_t^a)$ for $x_0^a = x^a$ and $(\dot{x}_t^b)_{t\geq 0}$ solves $(\dot{x}_t^b)_{t\geq 0} = g(x_t^b)$ with $x_0^b = x^b$ then for all $t \geq 0$, $x_t^a \geq x_t^b$, strictly so on an interval $(0, \tau)$ for some $\tau > 0$.

Proof. The solutions to the differential equations x_t^a and x_t^b are continuous and are not equal at time $t = 0$, so the strict inequality on an open interval of times is immediate.

By way of contradiction, suppose that $x_0^a > x_0^b$ and that there exists a time s such that $x_s^a < x_s^b$. As x_t^a and x_t^b are continuous, $s > 0$ and there exists a time $\tau \in (0, s)$ s.t. $x_\tau^a = x_\tau^b$. Then, there is a unique trajectory $(x_t^c)_{t\geq\tau}$ that solves the differential equation $\dot{x}_t^c = g(x_t^c)$ with initial condition $x_\tau^c = x_\tau^a$. As the solution to the belief differential equation that's consistent with the discrete time approximation from time τ onward is unique, it must be that for all $t \geq \tau$, $x_t^a = x_t^b$, which is a contradiction. □

The next result, and the much of the analysis that follows, takes advantage of the following technical result from Board and Meyer-ter-Vehn (2013). This result, lemma 5 in Board and Meyer-ter-Vehn (2013), provides an integral equation that is analogous to the standard HJB equation for these problems with Poisson information.

*Lemma A1*Board and Meyer-ter-Vehn (2013). For any $\rho > 0$ and any bounded, measurable function $\phi : [0, \infty) \to \mathbb{R}$ the function $\psi(t) = \int_t^\infty e^{-\rho(s-t)}\phi(s)\, ds$ is the unique bounded solution to the integral equation

$$f(t) - f(t') = \int_t^{t'} (\phi(s) - \rho f(s))\, ds.$$

Lemma A2. Fix any admissible Markov believed strategies $(\tilde{a}, \tilde{\pi})$. For any $x_0 \in [0, 1]$ and $\theta \in \{L, H\}$, the function $t \mapsto V(x_t, \theta)$ is continuous, and $V(x, \theta)$ is continuous in x when the domain is restricted to the set $\{x : x_t = x \text{ for some } t \geq 0\}$.

Proof. This is an immediate consequence of Board and Meyer-ter-Vehn (2013) Lemma A1. Fix two times t, t', $t' > t$. Board and Meyer-ter-Vehn (2013) Lemma A1 establishes that for any admissible Markov believed strategies $(\tilde{a}, \tilde{\pi})$, corresponding value function V and sequentially rational strategies (a, π) described by the equations in Lemma 1 the value function can be expressed as

$$V(x_t, H) - V(x_{t'}, H) = \int_t^{t'} [x_s + a(x_s)(\lambda D(x_s) - c) + \pi(x_s)(\gamma\Delta(x_s) - k) - rV(x_s, H) - \lambda D(x_s)]\, ds,$$

$$V(x_t, L) - V(x_{t'}, L) = \int_t^{t'} [x_s + a(x_s)(\lambda D(x_s) - c) - \mu(V(x_s, L) - V(0, L)) - rV(x_s, L)]\, ds,$$

The integrand in both these expressions is uniformly bounded above, so for any $\varepsilon > 0$, there exists a δ s.t. if $|t - t'| < \delta$ then $|V(x_t, \theta) - V(x_{t'}, \theta)| < \varepsilon$. For any admissible Markov believed strategies and x_0, the reputation process conditional on no news $(x_t)_{t\geq 0}$ is continuous and must always be increasing or decreasing. Moreover, it must be strictly increasing or strictly decreasing until reaching the first time where it has 0 drift, after which it is constant, by admissibility and that the process was chosen to be consistent with the discrete time approximation. Thus, the function $T(x) = \inf\{t : x_t = x\}$ is continuous. Finally, $V(x_{T(x)}, \theta)$ is a continuous function defined over the set $\{x : x_t = x \text{ for some } t \geq 0\}$ and $V(x_{T(x)}, \theta) = V(x, \theta)$. □

This result implies that under the cutoff strategies I consider for most the article, the value functions is continuous in the firm's initial reputation except at any points x^* where the drift of beliefs is negative as $x \to x^*$ from below and positive as $x \to x^*$ from above. With these continuity properties and the characterization of sequential rationality, I can establish various monotonicity properties of the value functions.

Proof of Lemma 2. Fix two times $t, t', t' > t$. Board and Meyer-ter-Vehn (2013) Lemma A1 establishes that for any admissible Markov believed strategies $(\tilde{a}, \tilde{\pi})$, corresponding value function $V(x, theta)$ and sequentially rational strategies (a, π) described by the equations in Lemma 1 the value function can be expressed as

$$V(x_t, H) - V(x_{t'}, H) = \int_t^{t'} [x_s + a(x_s)(\lambda D(x_s) - c) + \pi(x_s)(\gamma\Delta(x_s) - k)$$

$$- rV(x_s, H) - \lambda D(x_s)]\, ds,$$

$$V(x_t, L) - V(x_{t'}, L) = \int_t^{t'} [x_s + a(x_s)(\lambda D(x_s) - c)$$

$$- \mu(V(x_s, L) - V(0, L)) - rV(x_s, L)]\, ds.$$

Subtracting the two value functions gives

$$D(x_t) - D(x_{t'}) = \int_t^{t'} \pi(x_s)(\gamma\Delta(x_s) - k) + \mu(V(x_s, L) - V(0, L)) - (r + \lambda)D(x_s)\, ds.$$

Applying the Board and Meyer-ter-Vehn (2013) Lemma A1 again gives

$$D(x_t) = \int_t^{\infty} e^{-(r+\lambda)(s-t)}[\mu(V(x_s, L) - V(0, L)) + \pi(x_s)(\gamma\Delta(x_s) - k)]\, ds.$$

With these expressions in hand, I can establish the 4 desired properties. □

Property 1. Consider any pair of initial conditions $x, x', x > x'$. Fix initial quality at θ_0 and let $(\boldsymbol{x}'_{t-})_{t=0}^{\infty}$ denote the belief process starting at $x_0 = x'$. Let $\boldsymbol{a}_t = a(\boldsymbol{x}'_{t-})$ and $\boldsymbol{\pi}_t = \pi(\boldsymbol{x}'_{t-})$. These strategies are entirely determined by the firm's private history, the set of realized quality shocks, promotion and news. A firm that has initial quality θ_0 and initial reputation $x_0 = x$ can always follow the strategies $\boldsymbol{a}_t$ and $\boldsymbol{\pi}_t$. This induces the exact same measure over private and public histories. By Lemma A1, the firm receives weakly higher reputational payoff at every instant of time, strictly higher for a positive interval of time, and pays exactly the same costs as a firm with initial condition $x_0 = x'$ by following these strategies. Therefore, $V(x, \theta) > V(x', \theta)$.

Property 2. As $\Delta(x) = V(1, H) - V(x, H)$, this is an immediate consequence of property 1.

Property 3. Recall that

$$D(x) = V(x, H) - V(x, L) = \int_0^{\infty} e^{-(r+\lambda)s}[\mu(V(x_s, L) - V(0, L)) + \pi(x_s)(\gamma\Delta(x_s) - k)]\, ds.$$

By sequential rationality $\pi(x_s)(\gamma\Delta(x_s) - k) \geq 0$ and by property 1 $\mu(V(x_s, L) - V(0, L)) \geq 0$ for all $s \in [0, \infty)$. Therefore, $V(x, H) - V(x, L) \geq 0$.

Property 4. Setting $\mu = 0$, the difference $D(x)$ becomes

$$D(x) = \int_0^{\infty} e^{-(r+\lambda)(s)}[\pi(x_s)(\gamma\Delta(x_s) - k)]\, ds.$$

where $\pi(x_t)$ maximizes $\pi(x_t)[\gamma\Delta(x_t) - k]$.

By property 3, $\gamma\Delta(x) - k$ is decreasing in x. So for initial conditions $x, x', x > x'$, it follows from this observation and Lemma A1 that the integrand is smaller at every $t \in [0, \infty)$ when $x_0 = x$ relative to initial condition $x_0 = x'$, and is strictly decreasing if the firm promotes at any positive measure set of times. Thus $D(x)$ is decreasing in x, strictly so unless a solution to equation (2) sets $\pi(x_t) = 0$ a.e.

It follows from Lemma 1 and Lemma 2 that, at least in the game without exogenous news, given any Markov believed strategies the firm's optimal strategy is a cutoff strategy. Formally, a Markov strategy has cutoffs $(x_a, x_\pi) \in [0, 1]^2$ if

$$\pi(x) = \begin{cases} 1 \text{ if } x < x_\pi \\ 0 \text{ if } x > x_\pi \end{cases} \qquad a(x) = \begin{cases} 1 \text{ if } x < x_a \\ 0 \text{ if } x > x_a \end{cases}.$$

Given the admissibility restrictions and Assumption 1, admissible cutoff beliefs must also satisfy

$$g(x) = \begin{cases} 0 \text{ if } x = x_a \\ \lambda(1_{\{x \le x_a\}} - x) - (\gamma 1_{\{x > x_a\}} - \mu)x(1-x) \text{ if } x = x_\pi \text{ and } x_\pi \neq x_a \\ \lambda(1_{\{x \le x_a\}} - x) - (\gamma 1_{\{x \le x_\pi\}} - \mu)x(1-x) \text{ otherwise.} \end{cases} \quad \text{(A1)}$$

For any pair of cutoffs (x_a, x_π), there clearly exist a pair of strategies that satisfy these cutoffs and induce exactly this drift, and thus this pair of strategies is admissible (the other parts of admissibility vacuously hold for cutoff strategies). Now I can show that promotion and investment are complements in the game without exogenous news.

Proof of Lemma 3. Fix a pair of admissible strategies $(\tilde{a}, \tilde{\pi})$ and let V_k denote the value function for these believed strategies given promotion cost k. Define D_k and Δ_k analogously. Consider any two costs of promotion k and k', $k < k'$. Note that for any initial (x, θ) it must be that $V_k(x, \theta) \geq V_{k'}(x, \theta)$, as a firm with costs k can always play the strategy it would have played with cost k' and receive higher payoffs than a firm with costs k'.

It follows from Lemma 1 and Lemma 2 the optimal strategies are Markovian, are cutoff strategies and are unique up to behavior at possibly a single belief where the firm is indifferent. Define investment cutoff $x_{a,k} = \sup\{x : \lambda D_k(x) > c\} \cup \{0\}$ and $x_{a,k'}$ analogously.

By way of contradiction, suppose that $x_{a,k} < x_{a,k'}$. Then by construction $\lim_{x \to x_{a,k}+} D_{k'}(x) - D_k(x) > 0$. Let

$$g(x) := \lambda(\tilde{a}(x) - x) - \tilde{\pi}(x)\gamma x(1-x)$$

denote the drift of beliefs – that is, the law of motion beliefs follow in between arrivals of news or promotion. Let $\tau_x = \inf\{\tau : x_\tau = x_{a_k} \text{ if } x_0 = x\}$ where x_t is the solution to the differential equation $\dot{x}_t = g(x_t)$ with initial condition $x_0 = x$.

Suppose the drift of beliefs is such that for all $x > x_{a,k}$, $\tau_x = \infty$. As $D_k(x)$ is monotone, it is optimal for the firm to never invest in quality if it faces costs k and has initial reputation above $x_{a,k}$. So, a firm that starts with low quality never promotes if the cost of promotion is k. No matter the promotion cost, for any $x_0 > x_{a,k}$, if the initial state is (x_0, L) then playing $a_t = 0$ for all $t \geq 0$ gives the same payoffs as a firm with cost k would receive, so $V_k(x, L) \leq V_{k'}(x, L)$. Therefore $V_k(x, L) = V_{k'}(x, L)$. However, then $D_k(x) = V_k(x, H) - V_k(x, L) \geq V_{k'}(x, H) - V_{k'}(x, L) = D_{k'}(x)$ for all $x > x_{a,k}$. By the monotonicity of $D_{k'}(x)$ and $D_k(x)$, for any $x \in (x_{a,k}, x_{a,k'})$ this difference in payoffs satisfies $D_k(x) \leq c/\lambda < D_{k'}(x)$, which is a contradiction.

Now suppose there exists some $x > x_{a,k}$ s.t. $\tau_x < \infty$. Then there exists $\varepsilon > 0$ such that, for all $x \in (x_{a,k}, x_{a,k} + \varepsilon)$, $\tau_x < \infty$ and $\lim_{x \to x_{a,k}+} \tau_x = 0$. It must be that (i) $g(x) < 0$ for all $x \in (x_{a,k}, x_{a,k} + \varepsilon)$ and by admissibility $g(x_{a,k}) \leq 0$. First, I show that below $x_{a,k}$, the low cost firm is promoting whenever the high cost firm is. If not, the k' cost firm promotes at some reputation $x < x_{a,k}$ where the low cost firm does not. This implies that

$$\begin{aligned} &\gamma(\Delta_k(x) - \Delta_{k'}(x)) - (k - k') \\ &\geq \gamma(V_k(1, H) - V_{k'}(1, H)) + (k' - k) \\ &- \gamma \int_{T_k}^{\infty} e^{-(r+\lambda+\gamma)t}[\gamma(V_k(1, H) - V_{k'}(1, H)) + (k' - k)]\, dt \\ &> 0, \end{aligned}$$

where $T_k \in [0, \infty) \cup \{\infty\}$ is the first time the firm with costs k starts promoting conditional on no news and the first inequality follows from k' firm with initial reputation x deviating to follow the k firm's promotion strategy. So the firm with cost k also has incentive to promote.

Also, note that the high cost firm has incentive to promote at some reputation reached with positive probability in the absence of news from initial reputation $x_{a,k}$, because otherwise $D_{k'}(x_{a,k}) = 0$, contradicting $x_{a,k} < x_{a,k'}$. Thus, both firms have incentive to invest at some time, and payoffs at $x_{a,k}$ satisfy

$$\begin{aligned} V_k(x_{a,k}, H) - V_{k'}(x_{a,k}, H) = &\int_T^{T'} e^{-rt}[\gamma\Delta_k(x_t) - k]\, dt \\ &+ \int_{T'}^{\infty} e^{-(r+\gamma)t}[(k' - k) + \gamma(V_k(1, H) - V_{k'}(1, H))]\, dt \end{aligned}$$

where T is the first time where the firm with costs k promotes and T' is the first time where the firm with costs k' promotes. All the integrands are positive. On the other hand, if quality is initially low, let $\tilde{t} = \inf\{t : \theta_t = H\}$, then

$$\begin{aligned} V_k(x_{a,k}, L) - V_{k'}(x_{a,k}, L) = E\Bigg(&\int_T^{T'} e^{-rt}[\gamma\Delta_k(x_t) - k]1_{\{t > \tilde{t}\}}\, dt \\ &+ \int_{T'}^{\infty} e^{-(r+\gamma)t}[(k' - k) + \gamma(V_k(1, H) - V_{k'}(1, H))]1_{\{t > \tilde{t}\}}\, dt\Bigg) \end{aligned}$$

Clearly this implies that $D_k(x_{a,k}) - D_{k'}(x_{a,k}) > 0$, as $\tilde{\tau}$ is bounded away from 0 with positive probability. This must also hold for $x > x_{a,k}$, sufficiently close to $x_{a,k}$.[13] So $\lim_{x \to x_{a,k}+} D_k(x) - D_{k'}(x) > 0$. This is a contradiction. Thus $x_{a,k} \geq x_{a,k'}$, so promotion and investment are complements. □

Appendix B: Appendix B: Equilibrium

Promotion only model. First I establish existence of an MPE in the case where $\mu = 0$. Let $U(x_0, \theta; x_a, x_\pi)$ denote the value function with initial conditions x_0, θ and believed strategies determined by cutoffs $x_a, x_\pi \in [0, 1]^2$ (so the drift satisfies equation (A1)). Define the correspondence $\Gamma : [0, 1] \times [0, 1] \rightrightarrows [0, 1] \times [0, 1]$ as follows

$$\Gamma_1(x_a, x_\pi) = \arg\min_{x \in [0,1]} \int_x^1 \lambda[U(s, H; x_a, x_\pi) - U(s, L; x_a, x_\pi)] - c\, ds$$

$$\Gamma_2(x_a, x_\pi) = \arg\min_{x \in [0,1]} \int_x^1 \gamma[U(1, H; x_a, x_\pi) - U(s, H; x_a, x_\pi)] - k\, ds$$

where $\Gamma(x_a, x_\pi) = \Gamma_1(x_a, x_\pi) \times \Gamma_2(x_a, x_\pi)$. As $U(s, H; x_a, x_\pi) - U(s, L; x_a, x_\pi)$ and $U(1, H; x_a, x_\pi) - U(s, H; x_a, x_\pi)$ are monotone, this operator fixes the believed cutoffs, and identifies the set of cutoffs consistent with the sequential rationality conditions. So, a fixed point of this operator is an equilibrium. In the online appendix, I establish the that the function $(x_a, x_\pi) \mapsto U(x_0, \theta; x_a, x_\pi)$ is continuous for each $x_0 \in [0, 1]$ and $\theta \in \{L, H\}$ (Lemma 14), which is important for the existence argument.

Theorem B1. Assume Assumption 1 holds and $\mu = 0$. Then the operator $\Gamma(x_a, x_\pi)$ has a fixed point. This fixed point describes a Markov Perfect Equilibrium.

Proof. By the dominated convergence theorem and Lemma 14, $(x, x_a, x_\pi) \mapsto \int_x^1 \lambda[U(s, H; x_a, x_\pi) - U(s, L; x_a, x_\pi)] - c\, ds$ and $(x, x_a, x_\pi) \mapsto \int_x^1 \gamma[U(1, H; x_a, x_\pi) - U(s, H; x_a, x_\pi)] - k\, ds$ are continuous.[14] So, each component of Γ is the minimum of a continuous function over a compact space. A minimum exists and by the maximum theorem $\Gamma_1(x_a, x_\pi)$ and $\Gamma_2(x_a, x_\pi)$ are upper hemicontinous. Thus, $\Gamma(x_a, x_\pi) = \Gamma_1(x_a, x_\pi) \times \Gamma_2(x_a, x_\pi)$ is upper hemicontinuous and non-empty.

Finally, Γ must also be convex-valued. If $x, y \in \Gamma_1(x_a, x_\pi)$, $x < y$, then any $z \in (x, y)$ also is an element of correspondence, $z \in \Gamma_1(x_a, x_\pi)$. This follows from the observation that as the integrand is decreasing, the integrand must be 0 a.e. between x, y. To see this, note that if the integrand is negative on a set of positive measure contained in (x, y), then it is negative on a subinterval of $[x, y]$ that includes y, so y would not be a minimizer. If the integrand is positive on a set of positive measure between x and y, than it would be positive over an interval that includes x, so x would not be a minimizer. Therefore, $z \in \Gamma_1(x_a, x_\pi)$ and $\Gamma_1(x_a, x_\pi)$ is convex valued. The same argument applies to $\Gamma_2(x_a, x_\pi)$. Finally, Γ is the Cartesian product of two convex sets, so it is also convex.

Therefore by Kakutani's fixed point theorem, $\Gamma(x_a, x_\pi)$ has a fixed point. Denote a fixed point of this operator by (x_a^*, x_π^*). I'll now show that the admissible strategies determined by this cutoff constitute an equilibrium.

I show that for all $x > x_a^*$, $U(x, H; x_a^*, x_\pi^*) - U(x, L; x_a^*, x_\pi^*) \geq c/\lambda$ and for all $x < x_a^*$, $U(x, H; x_a^*, x_\pi^*) - U(x, L; x_a^*, x_\pi^*) \leq c/\lambda$. As this difference is monotone for any cutoffs, if this wasn't true, then either increasing or decreasing the value of x would decrease the integral. So the investment strategy satisfies equation (1). An analogous argument holds for x_π^*. Therefore this is a Markov Perfect equilibrium. □

Theorem B2. Suppose Assumption 1 holds and $\mu = 0$. In any MPE, $1 > x_\pi > x_a$ or $x_a = x_\pi = 0$.

Proof. Fix an MPE (x_a, x_π). It follows from the cutoff structure of strategies that the drift at x_a must be non-negative (in fact, it must be 0). Therefore, if $x_a \geq x_\pi$ by Lemma 2

$$D(x_a) = \int_0^\infty \pi(x_a)[\gamma \Delta(x_a) - k]\, dt$$

so if $x_\pi \leq x_a$, then investment cannot be sequentially rational. Moreover, if $x_a = 1$, then $D(x_a) = \int_0^\infty e^{-(r+\lambda)t} \pi(1)(-k)\, dt \leq 0$, so the investment cutoff must lie below 1.

[13] $D_k(x) = \int_0^{\tau_x} e^{-(r+\lambda)s}[\pi(x_s)(\gamma \Delta(x_s) - k)]\, ds + e^{-(r+\lambda)\tau_x} D_k(x_{a,k})$, and $\tau_x \to 0$ as $x \to x_{a,k}$ from the right. As the integrand is bounded, $\lim_{x \to x_{a,k}+} D_k(x) = D_k(x_{a,k})$. The same logic holds for the other cost.

[14] Take any convergent sequence $(x_n, x_{a,n}, x_{\pi,n}) \to (x, x_a, x_\pi)$. The function $f_n(s) = 1_{\{s \geq x_n\}}(\lambda[U(s, H; x_{a,n}, x_{\pi,n}) - U(s, L; x_{a,n}, x_{\pi,n})] - c)$ converges pointwise a.e. to $f(s) = 1_{\{s \geq x\}}(\lambda[U(s, H; x_a, x_\pi) - U(s, L; x_a, x_\pi)] - c)$ by Lemma 14 and is bounded for all n. Therefore $\int_0^1 f_n(s)\, ds \to \int_0^1 f(s)\, ds$, and thus objective is a continuous function of (x, x_a, x_π) by the sequential characterization of continuity. An analogous argument applies to the other integral.

It remains to show that $x_\pi \neq x_a$. Suppose $x_\pi = x_a \neq 0$. Fix any $x_0 > x_a$ and let $\tau = \inf\{t : x_t = x_a\}$. As for any $x_0 > x_a$, $\dot{x}_t = -\lambda x_t$, x_a is reached in finite time, $\tau < \infty$. If $\Delta(x_a) > k/\gamma$, then for any $x_0 > x_a$

$$\Delta(x_0) = -\int_0^\tau e^{-(r+\lambda)t}[x_t + \lambda V(x_t, L)]\,dt + (-e^{-(r+\lambda)\tau} + 1)V(x_a, H) + \Delta(x_a)$$

The first two terms go to 0 as $\tau \to 0$, so by the continuity of the belief trajectory, there exists an x_0 such that $\Delta(x_a) - \Delta(x_0) < (D(x_a) - k/\gamma)/2$, so $\Delta(x_0) > k/\gamma$, which is a contradiction. □

Now I compare the equilibrium in the game with promotion to the equilibrium in a game with exogenous good news and no promotion studied in Board and Meyer-ter-Vehn (2013).

Proof of Lemma 4. Fix an equilibrium with cutoffs (x_a, x_π). Let $x_{a,EX}$ be an equilibrium cutoff in the game without promotion. At the investment cutoff x_a, either $x_a = 0$ or

$$\lambda D(x_a) = c.$$

The cutoff structure of the equilibrium and Assumption 1 imply that the drift at x_a must be 0 and therefore

$$D(x_a) = \int_0^\infty e^{-(r+\lambda)t}[\gamma\Delta(x_a) - k]\,dt$$

so $\gamma\Delta(x_a) = (1 + r/\lambda)c + k$. The same logic, applied to the game with exogenous news implies that at $x_{a,EX}$

$$\frac{c}{\lambda} = D_{EX}(x_{a,EX}) = \int_0^\infty e^{-(r+\lambda)t}\gamma\Delta(x_{a,EX})\,dt,$$

so $\gamma\Delta(x_{a,EX}) = (1 + r/\lambda)c$. □

Lemma B1. Suppose $\lambda \geq \gamma$, $\mu = 0$ and fix an equilibrium of the game with and the game without promotion. The investment cutoff is higher in the game without promotion.

Proof. Fix an equilibrium in the game with promotion (x_a, x_π). Let $x_a > 0$ be the work/shirk cutoff in an equilibrium of the game with promotion (in the case where this is equal to 0, the argument is trivial). Let $V_{EX}(x_0, \theta; x)$ be the value function when the cutoff is x and let $D_{EX}(x_0; x)$ be the corresponding difference between high and low quality payoffs at that point in the game without promotion where the firm plays according to investment cutoff x. As shown in Board and Meyer-ter-Vehn (2013), the equilibrium in the game without promotion is unique, the function $x \mapsto D_{EX}(x; x)$ is continuous in x and $D_{EX}(1; 1) = 0$, so it suffices to show that $D_{EX}(x_a; x_a) > D(x_a) = \frac{c}{\lambda}$.

By way of contradiction, suppose that $D_{EX}(x_a; x_a) \leq D(x_a)$. This implies that the firm is prefers (weakly) not working to working at all beliefs $x \geq x_a$ in the game without promotion.

Consider the payoffs under the following auxiliary process. Let $\boldsymbol{y}_0 = x_0$, between arrivals $\boldsymbol{y}_t$ drifts according to $\dot{y}_t = \lambda(a(y_t) - y_t) - \gamma\pi(y_t)y_t(1 - y_t)$ and $\boldsymbol{y}_t = x_\pi$ after whenever promotion arrives. So $\boldsymbol{y}_t$ behaves exactly like $\boldsymbol{x}_t$, except it jumps to x_π instead of 1 after news. Fix firm strategies as the actions they would have played in the equilibrium under the modified reputation process, with the modification that $a(x_a) = 0$ (as the firm is indifferent at x_a, this would not change payoffs in the original game), that is, $a_t = a(\boldsymbol{y}_{t-})$ and $\pi_t = \pi(\boldsymbol{y}_{t-})$. Finally, assume promotion is costless. The firm payoffs under this modified investment, promotion and reputation process problem can be written recursively as

$$\begin{aligned} W(x, \theta) = \int_0^\infty e^{-(r+\lambda+\gamma)t}[x_t + \gamma(\pi(x_t)W(x_\pi, H) \\ + (1 - \pi(x_t))W(x_t, H)) + \lambda(a(x_t)(W(x_t, H) - c) + (1 - a(x_t))W(x_t, L)))]\,dt. \end{aligned}$$

Now return to the original game. In the game, for any initial belief $x_0 \in (x_a, x_\pi]$ the $\boldsymbol{x}_t$ belief process follows the same law of motion as $\boldsymbol{y}_t$ until news arrives. Moreover, payoffs in the game with promotion satisfy

$$\gamma\Delta(x) - k = \gamma(V(x_\pi, H) - V(x, H)).$$

by the sequential rationality condition. So, for any $x_0 \in (x_a, x_\pi]$ the value function can be rewritten as

$$V(x_0, H) = \int_0^\infty e^{-(r+\lambda+\gamma)t}[x_t + \gamma V(x_\pi, H) + \lambda V(x_t, L)]dt,$$

which is exactly the equation for the value function under the modified belief process, and thus $V(x_0, H) = W(x_0, H)$ for any $x_0 \in [x_a, x_\pi]$.[15] This means that at every instant of time, the firm receiving the modified payoffs where beliefs jump

[15] The uniqueness of the solution to this equation is immediate. For any $x_0 \in (x_a, x_\pi]$, $V(x_t, L) = W(x_t, L) = \int_0^\infty e^{-rt}x_t\,dt$. Setting $\xi(t) = V(x_t, H) - W(x_t, H)$ where $x_0 = x_\pi$ is the solution to the integral equation $\xi(t) = \int_t^\infty e^{-(r+\lambda+\gamma)(s-t)}\gamma\xi(0)\,ds$ which, by setting $t = 0$, implies that $\xi(0) = 0$.

to x_π instead of 1 is receiving weakly lower flow payoffs than the firm who has committed to promoting everywhere, and the low quality firm is receiving exactly the same payoffs in either case, as it is never working and news never arrives. In other words

$$V_{EX}(x_a, L; x_a) = V(x_a, L)$$

$$V_{EX}(x_a, H; x_a) > V(x_a, H)$$

(the strict inequality comes from $x_\pi < 1$), which contradicts $D_{EX}(x_a; x_a) \le D(x_a)$. So $D_{EX}(x_a; x_a) > D(x_a)$ which means the firm starts investing at a higher belief in the game without promotion. □

Corollary 1 then follows directly from this observation and the fact that the stationary distribution has support $[x_a, 1]$ in any MPE.[16] Cycles are larger because the investment cutoff is lower and the drift must be 0 at the investment cutoff by Assumption 1, so in the long-run stationary distribution beliefs are bounded below by the investment cutoff. Cycles are longer as beliefs drift down slower in the game with promotion, the arrival rate of information is weakly lower and the time until the firm starts investing given no news is larger.

Proof of Proposition 2

Reputation cycles. The existence of cycles, even in the limit, directly follows from Proposition 1 and Lemma 12, which shows that x_π (and thus x_a) are bounded away from 1 in equilibrium and that bound is independent of the cost of investment c.

Cycles vanish without promotion. Suppose the second claim was false for some $\varepsilon > 0$; for arbitrarily small $\bar{c} > 0$ there exists a $c < \bar{c}$ and an equilibrium of the game without promotion with investment cutoff x_a^c where $x_a^c < 1 - \varepsilon$. Let $V_{EX,c}$ denote the value function in this equilibrium and define $D_{EX,c}$ and $\Delta_{EX,c}$ similarly. In such an equilibrium $D_{EX,c}(x_a^c) = \int_0^\infty e^{-(r+\lambda)t}\gamma(\Delta_{EX,c}(x_a^c))\,dt$. As $c \to 0$, $D_{EX,c}(x_a^c)$ must go to 0, so $\Delta_{EX,c}(x_a^c)$ goes to 0. However, if there existed arbitrarily small c such that $x_a^c < 1 - \varepsilon$, at any such c

$$\Delta_{EX,c}(x_a^c) = \int_0^\infty e^{-(r+\lambda+\gamma)t}[x_t - x_a^c + \lambda(V_{EX,c}(x_t, L) - V_{EX,c}(x_a^c, L))]\,dt \ge \int_0^T e^{-(r+\lambda+\gamma)t}[x_t - (1-\varepsilon)]\,dt$$

where x_t is the belief process with initial condition $x_0 = 1$ and $T = \inf\{t : x_t \ge 1 - \varepsilon\}$, the time it takes the x_t process to reach $1 - \varepsilon$, which is at least $\log((1-\varepsilon)/(\lambda+\gamma))$. As $c \to 0$, $D_{EX}(x_a^c)$ must go to 0 along any sequence of equilibria, but the right hand side of this inequality is bounded away from 0. So for small enough c the firm must invest at reputation $x = 1 - \varepsilon$ in any equilibrium, and thus the firm invests at all lower reputations. Therefore, for any $\varepsilon > 0$ there is a $\bar{c}$ such that $x_a^c \ge 1 - \varepsilon$ for any $c < \bar{c}$.

Benefits from commitment. The intuition of this result is straightforward. From the previous result, in the game without promotion, as $c \to 0$, $x_a^c \to 1$, whereas in the game with promotion the starts never invests above the B from Lemma 12. So, if c is small enough, then in the game with promotion the firm's payoffs are lower not only due to having to pay the cost of promotion, but also from the reputation cycles which no longer vanish as $c \to 0$.

Formally, Lemma 12 shows that x_π is bounded above by some $B < 1$ that is independent of c. Let $\hat{c}$ be such that if $c < \hat{c}$ then the investment cutoff $x_a^{EX,c} > B$ in any equilibrium of the game without promotion. Let $T_B := \log(B/\lambda)$, the time it takes for the solution to the differential equation $\dot{y}_t = -\lambda y_t$, $y_0 = 1$ to reach B. For any $\theta \in \{H, L\}$, and any x

$$\begin{aligned}
&V(x,\theta) + E\left(\int_0^\infty e^{-rt} k\pi_t\,dt \,\middle|\, x_0 = x, \theta_0 = \theta\right) \\
&\le \int_0^{T_B} e^{-rt}\,dt + \int_{T_B}^\infty e^{-(r+\gamma)t}\left(B + \gamma\frac{1}{r}\right)dt \\
&= \frac{1}{r}\left((1 - e^{-rT_B}) + \frac{rB+\gamma}{r+\gamma}e^{-(r+\gamma)T_B}\right) \\
&< \frac{1}{r}
\end{aligned}$$

where the first term is a bound on the payoff the firm receives before it starts promoting and has reputation above B, as the amount of time the firm has both a reputation above B and is not promoting is at most T_B. The reputation after the time when the firm starts promoting is at most B until promotion arrives which happens at most at rate γ and once promotion arrives, the highest possible payoff the firm can receive is $1/r$. Let $\bar{V} := \frac{1}{r}((1 - e^{-rT_B}) + \frac{rB+\gamma}{r+\gamma}e^{-(r+\gamma)T_B})$ be this upper bound.

[16] The existence of a stationary distribution in any MPE, with support equal to the interval $[x_a, 1]$, is established Lemma 15 in the online appendix.

Fix ε s.t. $1 - 2\varepsilon > r\bar{V}$. Take any $\hat{c}$ so that the investment cutoff in the game without promotion lies above $1 - \varepsilon$ in all equilibrium. In any such equilibrium, the firm receives $rV_{EX}(x, \theta) > 1 - 2\varepsilon > r\bar{V}$ for any $x > 1 - \varepsilon$. For any $x < 1 - \varepsilon$, until news arrives, the firm's reputation in the game without promotion lies above the reputation in the game with promotion, the firm in the game without promotion is always selling a weakly higher quality product and news arrives weakly faster. After news arrives, the firm in the game without promotion's reputation jumps to 1, whereas the firm in the game with promotion's reputation is at most 1 and $V_{EX}(1, H) > \bar{V}$. Therefore, for any $c < \hat{c}$

$$V_{EX}(x, \theta) > V(x, \theta) + E\left(\int_0^\infty e^{-rt} k\pi_t \, dt \middle| x_0 = x, \theta_0 = \theta\right),$$

so the firm benefits from commitment.

□

Bad news case.

Proof of Proposition 3. I establish this result separately for the $\mu < \gamma$ and the $\mu > \gamma$ case.

Lemma B2. Suppose $\mu > \gamma$, and fix an MPE (a, π). In this MPE there exist two cutoffs $x_\pi \in [0, 1)$, and $x_a \in [0, 1]$ such that

$$\pi(x) = \begin{cases} 1 \text{ if } x < x_\pi \\ 0 \text{ if } x > x_\pi, \end{cases} \qquad a(x) = \begin{cases} 0 \text{ if } x < x_a \\ 1 \text{ if } x > x_a. \end{cases}$$

Proof. By Lemma 1, $\pi(x)$ solves

$$\pi(x) \in \arg\max_{\pi \in [0,1]} \pi(\gamma \Delta(x) - k)$$

and as $\Delta(x)$ is strictly decreasing in x by Lemma 2, a cutoff x_π exists.

Now consider the drift of beliefs at x_π, $\lambda(a(x_\pi) - x_\pi) - (\pi(x_\pi)\gamma - \mu)x_\pi(1 - x_\pi)$. There are two cases to consider, either (i) the drift is positive at x_π and there exists an $x_0 < x_\pi$ such that, starting from x_0, there exists a $t < \infty$ such that $x_t > x_\pi$, that is, beliefs cross x_π in finite time if no news arrives, or (ii) beliefs that start at or below x_π only ever exceed x_π if promotion arrives, that is, for any $x_0 < x_\pi$ $x_t < x_\pi$ for all $t > 0$. First consider *case i*. For any $x_0 \geq x_\pi$,

$$D(x_0) = \int_0^\infty e^{-(r+\lambda)t} \mu(V(x_t, L) - V(0, L)) \, dt$$

Increasing x_0 increases the integrand pointwise, strictly so in an open interval of time around $t = 0$, so $D(x_0)$ is increasing on $[x_\pi, 1]$.

As the drift is positive at x_π, the firm must have incentive to work at x_π, $\lambda D(x_0) > \lambda D(x_\pi) \geq c$, so the firm invests at every x_0 above the promotion cutoff. Moreover, $V(x_t, \theta)$ is continuous at the promotion cutoff, so $V(x_\pi, H) = V(1, H) - k$.

Below the promotion cutoff, for any $x_0 < x_\pi$,

$$\begin{aligned} D(x_0) &= \int_0^\infty e^{-(r+\lambda)t} [\pi(x_t)(\gamma(V(1, H) - V(x_t, H)) - k) + \mu(V(x_t, L) - V(0, L))] \, dt \\ &= \int_0^\infty e^{-(r+\lambda+\gamma)t} [\pi(x_t)(\gamma(V(1, H) - V(x_t, H)) - k) + \mu(V(x_t, L) - V(0, L)) + \gamma D(x_t)] \, dt \\ &= \int_0^\infty e^{-(r+\lambda+\gamma)t} [\gamma(V(\max(x_t, x_\pi), H) - V(x_t, L))) + \mu(V(x_t, L) - V(0, L))] \, dt \end{aligned}$$

As $\mu > \gamma$ and $D(x_0)$ is strictly increasing above x_π the integrand here is a strictly increasing function of x, so $D(x_0)$ is strictly increasing. Therefore, there is a single investment cutoff.

Now consider *case ii*. There are two possibilities, either the drift is negative at the promotion cutoff, then by the logic above, below the promotion cutoff

$$\begin{aligned} D(x_0) &= \int_0^\infty e^{-(r+\lambda)t} [\gamma(V(1, H) - V(x_t, H)) - k + \mu(V(x_t, L) - V(0, L))] \, dt \\ &= \int_0^\infty e^{-(r+\lambda+\gamma)t} [\gamma V(1, H) - k - \mu V(0, L) + (\gamma - \mu)V(x_t, L)] \, dt. \end{aligned}$$

If the firm invests at some $x_0 < x_\pi$, the firm must invest at all $x \in (x_0, x_\pi)$ and this incentive is strict at all these points. Therefore, the drift must be 0 at x_π. However, then

$$D(x_\pi) \geq D(x_0) = \int_0^\infty e^{-(r+\lambda)t}[\pi(x_t)(\gamma(V(1,H) - V(x_t,H)) - k) + \mu(V(x_t,L) - V(0,L))]\,dt$$
$$= \lim_{x \to x_\pi -} D(x) > c/\lambda$$

which is a contradiction. Therefore x_a exists in any MPE. □

Lemma B3. Suppose $\mu < \gamma$. Fix an MPE (a, π). In any MPE there exist three cutoffs $x_\pi \in [0,1)$, $\underline{x}_a \in [0, x_\pi]$, $\bar{x}_a \in [\underline{x}_a, 1]$ such that a.e.

$$\pi(x) = \begin{cases} 1 \text{ if } x < x_\pi \\ 0 \text{ if } x > x_\pi \end{cases} \qquad a(x) = \begin{cases} 0 \text{ if } x \in (\underline{x}_a, \bar{x}_a) \\ 1 \text{ if } x \in [0, \underline{x}_a) \cup (\bar{x}_a, 1] \end{cases}$$

Proof. By Lemma 1, a promotion cutoff x_π exists. There are two possible cases in equilibrium, either (i) there exists an $\varepsilon > 0$ s.t. if $x_0 < x_\pi$ is such that $x_\pi - x_0 \leq \varepsilon$ then x_t hits x_π in finite time—i.ethat is, there exists a $t < \infty$ such that $x_t = x_\pi$ (and thus $a(x) > 0$ for all $x \in [x_\pi - \varepsilon, x_\pi]$)—or (ii) no such ε exists and for any $x_0 \leq x_\pi$, $x_t < x_\pi$ for all $t > 0$.

Consider *case i*. As the firm is investing at x_π, it must be that $\lambda D(x_\pi) \geq c$. So for any $x_0 > x_\pi$, the firm is at (weakly) prefers not promoting to promoting at every instant of time and thus

$$D(x_0) = \int_0^\infty e^{-(r+\lambda)t}\mu(V(x_t,L) - V(0,L))\,dt$$

so $D(x_0)$ is increasing, and thus the firm has strict incentive to invest above the promotion cutoff.

Fix any any $x_0 < x_\pi$ such that beliefs that start at x_0 cross x_π in finite time the absence of news, that is, there exists a $t < \infty$ s.t. $x_t > x_\pi$. Let T be the time such that $T = \inf\{t : x_t = x_\pi\}$. I argue that there can be most one $\tau \in (0, T)$ s.t. $x_\tau < x_\pi$ and $\lambda D(x_\tau) = c$. Suppose there exist two times, τ_1, τ_2 that satisfy this. If there exists an $\varepsilon > 0$ such that $t \mapsto D(x_t)$ is strictly increasing or strictly decreasing on $t \in (\tau - \varepsilon, \tau + \varepsilon)$ then the firm would have strict inventive to not invest before reaching x_π, which contradicts the assumption that x_t hits x_π. So, at any τ, $D(x_t)$ cannot be monotone in a neighborhood. Note that

$$D(x_t) = \int_t^T e^{-(r+\lambda+\gamma)(s-t)}[\gamma V(1,H) - k - (\gamma - \mu)V(x_s,L) - \mu V(0,L)]\,ds + e^{-(r+\lambda+\gamma)(T-t)}D(x_\pi)$$

which is differentiable in a neighborhood of τ with derivative

$$-\gamma V(1,H) + k + (\gamma - \mu)V(x_t,L) + \mu V(0,L) + (r + \lambda + \gamma)D(x_t).$$

However, then, as the derivative must be equal to 0 at both τ_1 and τ_2, $V(x_{\tau_1}, L) = V(x_{\tau_2}, L)$ which is impossible as $V(x_t, L)$ is strictly increasing.

So $a(x_t) = 1$ for almost all times in $[0, T)$ if the x_t process hits x_π. Moreover, this holds for all such x_0's. So, let $\bar{x}_a = \inf\{x : x_0 = x \text{ and } \exists T \text{ s.t. } x_T = x_\pi\}$. It remains to construct the lower cutoff.

Consider any $x_0 < \bar{x}_a$. As beliefs only exceed x_π if promotion arrives, at x_0

$$D(x_0) = \int_0^\infty e^{-(r+\lambda+\gamma)t}[\gamma V(1,H) - k - \mu V(0,L) - (\gamma - \mu)V(x_t,L)]\,dt$$

So $D(x_0)$ is strictly decreasing in x_0 below $\bar{x}_a$. So there can be at most one more investment cutoff where the firm has strict incentive to invest below this cutoff and strict incentive not to invest between this cutoff and $\bar{x}_a$. This cutoff is $\underline{x}_a$.

Now consider *case ii*. Similarly to case i if $x_0 \leq x_\pi$

$$D(x_0) = \int_0^\infty e^{-(r+\lambda+\mu)t}[\gamma V(1,H) - k - \mu V(0,L) - (\gamma - \mu)V(x_t,H)]\,dt$$

which is strictly decreasing on $[0, x_\pi]$. So there is at most one investment cutoff $\underline{x}_a \in [0, x_\pi]$ and it behaves as desired. Finally, suppose that $\{x : a(x) > 0, x > x_\pi\}$ is non-empty. At any $x_0 > \inf\{x : a(x) > 0, x > x_\pi\}$, if beliefs never reach x_π from x_0 then

$$D(x_0) = \int_0^\infty e^{-(r+\lambda)t}\mu(V(x_t,L) - V(0,L))\,dt$$

so $D(x_0)$ is strictly increasing at this point, so the firm must be investing at any higher point.

I now show, by way of contradiction, that if there is some time $t < \infty$ when beliefs reach the promotion cutoff, $x_t = x_\pi$, given initial belief x_0 then there cannot be two times τ_1, τ_2 s.t. both $x_{\tau_1}, x_{\tau_2} > x_\pi$ and $\lambda D(x_{\tau_1}) = \lambda D(x_{\tau_2}) = c$.

Suppose not, that two such times, τ_1 and τ_2, exist. Similar to case 1, it must be that the derivative at both these points is 0. However, the derivative satisfies

$$-\mu(V(x_{\tau_1}, L) - V(0, L)) + (r + \lambda)\frac{c}{\lambda} = -\mu(V(x_{\tau_2}, L) - V(0, L)) + (r + \lambda)\frac{c}{\lambda}$$

which is impossible as $V(x_t, L)$ is strictly increasing. So the firm is at most indifferent between investing and not investing on a set of measure 0 of beliefs that reach x_π with in finite time in the absence of news. □

Proof of Proposition 4

Proof. First suppose $\mu > \gamma$ and $c(\frac{r+\gamma}{\lambda} + 1) + k < \frac{\gamma(r+2\lambda+\mu)}{(r+\mu+\lambda)(r+2\lambda)}$

Suppose an equilibrium $(a, \pi) \in A \times \Pi$ with a reputation trap exists. This equilibrium must either be have no investment or have investment at $x = 1$ by Proposition 3.

The equilibrium value $V(1, H)$ is bounded below by the payoffs the firm receives under optimal play when the firm is believed to be playing according to $\tilde{a}(x) = 0, \tilde{\pi}(x) = \pi(x)$ as these beliefs always lie below the true equilibrium beliefs. Further bound the value below by the value the firm receives instead of playing optimally, it never invests or promotes anywhere. Let y_t denote the modified belief process conditional on no news. Combining these two modifications

$$\begin{aligned}
V(1, H) &\geq \int_0^\infty e^{-(r+\lambda)t}[x_t + \lambda V(x_t, L)]\, dt \\
&\geq \int_0^\infty e^{-(r+\lambda)t}\left[x_t + \lambda \int_t^\infty e^{-(r+\mu)(s-t)} x_s\, ds\right] dt \\
&\geq \int_0^\infty y_t\left[\frac{\lambda}{\lambda - \mu}e^{-(r+\mu)t} + \frac{\mu}{\lambda - \mu}e^{-(r+\lambda)t}\right] dt \\
&\geq \int_0^\infty \left[\frac{\lambda}{\lambda - \mu}e^{-(r+\mu+\lambda)t} - \frac{\mu}{\lambda - \mu}e^{-(r+2\lambda)t}\right] dt \\
&= \frac{r + 2\lambda + \mu}{(r + \mu + \lambda)(r + 2\lambda)}
\end{aligned}$$

Where the fourth line comes from Grönwall's inequality and the observation that $\dot{y}_t \geq -\lambda y_t$. Moreover,

$$V(0, H) \geq \int_0^\infty e^{-(r+\gamma+\lambda)t}[\gamma V(1, H) - k]\, dt,$$

which implies that

$$\begin{aligned}
D(0) &= V(0, H) \\
&\geq \frac{\gamma(r + 2\lambda + \mu)}{(r + \mu + \lambda)(r + 2\lambda)(r + \gamma + \lambda)} - k\frac{1}{r + \gamma + \lambda}.
\end{aligned}$$

So if

$$c\left(\frac{r + \gamma}{\lambda} + 1\right) + k < \frac{\gamma(r + 2\lambda + \mu)}{(r + \mu + \lambda)(r + 2\lambda)}$$

the firm must invest at $x = 0$.

Similarly, if $\mu \leq \gamma$, if $c((r + \gamma)/\lambda + 1) + k < \frac{\gamma(r+2\lambda+\gamma)}{(r+\lambda+\gamma)(r+2\lambda+\gamma-\mu)}$ then

$$\begin{aligned}
V(1, H) &\geq \int_0^\infty x_t\left[\frac{\lambda}{\lambda - \mu}e^{-(r+\mu)t} - \frac{\mu}{\lambda - \mu}e^{-(r+\lambda)t}\right] dt \\
&\geq \int_0^\infty \left[\frac{\lambda}{\lambda - \mu}e^{-(r+\lambda+\gamma)t} - \frac{\mu}{\lambda - \mu}e^{-(r+2\lambda+\gamma-\mu)t}\right] dt \\
&= \frac{r + 2\lambda + \gamma}{(r + \lambda + \gamma)(r + 2\lambda + \gamma - \mu)},
\end{aligned}$$

implying that

$$D(0) \geq \frac{\gamma(r + 2\lambda + \gamma)}{(r + \lambda + \gamma)^2(r + 2\lambda + \gamma - \mu)} - k\frac{1}{r + \gamma + \lambda}.$$

So this can't be an equilibrium if

$$c\left(\frac{r+\gamma}{\lambda}+1\right)+k<\frac{\gamma(r+2\lambda+\gamma)}{(r+\lambda+\gamma)(r+2\lambda+\gamma-\mu)}.$$

□

Proof of Proposition 5

Lemma B4. Suppose $\lambda > \gamma - \mu$, $\gamma > \mu$ and $\frac{\gamma-\mu}{r+\mu} > k$. Then there exists a $\underline{c}$, $\frac{\mu\lambda}{(r+\mu)(r+\lambda)} > \underline{c}$, such that if $c > \underline{c}$ in the game with promotion $a(1) = 0$ in any equilibrium, whereas in the game without promotion there exist MPEs where the firm invests at $x = 1$.

Proof. Fix an equilibrium $(a, \pi) \in A \times \Pi$ such that $a(1) > 0$. Then the payoff of the firm with a low quality product at $x = 0$ is bounded below by the payoff the firm would receive if:

(1) It instead faced market beliefs where the drift of beliefs is 0 at $x = 0$ (otherwise maintain the equilibrium law of motion for beliefs).
(2) Instead of best responding to these beliefs, it played strategies $\hat{a}(0) = 1$, $\hat{\pi}(0) = 1$, otherwise $\hat{a}(x) = a(x)$ and $\hat{\pi}(x) = \pi(x)$.

This bound is

$$\begin{aligned}
V(0,L) &\geq \int_0^\infty e^{-(r+\lambda)t}(\lambda V(0,H)-c)\,dt\\
&\geq \frac{1}{r+\lambda}\left(\lambda\int_0^\infty e^{-(r+\gamma)t}\left[\gamma\frac{1-c}{r}-k-c\right]dt-c\right).\\
&=\frac{\lambda\gamma}{(r+\lambda)(r+\gamma)}\left(\frac{1}{r}-\frac{k}{\gamma}\right)-\frac{c}{r}.
\end{aligned}$$

Denote this lower bound by $\underline{V}$. Moreover, $D(1)$ is bounded above by

$$\begin{aligned}
D(1) &= \int_0^\infty e^{-(r+\lambda)t}[\mu(V(1,L)-V(0,L))]\,dt\\
&= \int_0^\infty e^{-(r+\lambda)t}\left[\mu\left(\int_0^\infty e^{-(r+\mu+\lambda)s}\left[1-c+\mu V(0,L)+\lambda\frac{1-c}{r}\right]ds-V(0,L)\right)\right]dt\\
&\leq \int_0^\infty e^{-(r+\lambda)t}\mu\left(\frac{1}{r+\mu+\lambda}\left[1-c+\lambda\frac{1-c}{r}\right]-\frac{r+\lambda}{r+\mu+\lambda}\underline{V}\right)dt\\
&= \frac{\mu}{r+\mu+\lambda}\left(\frac{1-c}{r}-\underline{V}\right)\\
&= \frac{\mu}{(r+\mu+\lambda)(r+\lambda)(r+\gamma)}(r+\lambda+\gamma+\lambda k)
\end{aligned}$$

As long as c/λ is greater than that bound, then it is impossible for the firm to be investing at $x = 1$ in equilibrium. Moreover, if $k\gamma < \frac{\gamma-\mu}{r+\mu}$ then this bound on $D(1)$ lies below $\frac{\mu}{(r+\mu)(r+\lambda)}$. So

$$\underline{c} := \frac{\lambda\mu}{(r+\mu+\lambda)(r+\lambda)(r+\gamma)}(r+\lambda+\gamma+\lambda k)$$

satisfies the statement of the lemma.

In the game with only exogenous news, there exist equilibria where the firm invests at 1 whenever there are equilibria where 1 and 0 are both absorbing states. If such an equilibria exists

$$D_{NP}(1) = \int_0^\infty e^{-(r+\lambda)t}\mu(V(1,L)-0)\,dt.$$

Moreover,

$$V(1,L) = \int_0^\infty e^{-(r+\lambda+\mu)t}\left[1-c+\lambda\frac{1-c}{r}\right]dt.$$

So $D_{NP}(1) = \frac{\mu}{r+\lambda+\mu}\frac{1-c}{r}$, and this is an equilibrium as long as

$$c \leq \frac{\lambda\mu}{(r+\lambda)(r+\mu)}.$$

Finally, there exists an interval of costs where there are equilibria where the firm invests at 1 in the game without promotion, but no such equilibrium exist with promotion if $(\gamma - \mu)/(r + \mu) > k$. As long as this inequality holds, the firm's gain from investment at 1 in the game with promotion is strictly below $\frac{\lambda\mu}{(r+\lambda)(r+\mu)}$, the highest cost where the firm would be willing to invest at 1 in the game without promotion. So as long as

$$\frac{\lambda\mu}{(r+\lambda)(r+\mu)} > c > \frac{\lambda\mu}{(r+\mu+\lambda)(r+\lambda)(r+\gamma)}(r+\lambda+\gamma+\lambda k),$$

these bounds imply there exist MPE in the game without promotion where the firm invests at 1, but no MPE in the game with promotion where the firm invests at 1. □

The previous lemma does not establish equilibrium existence in the game with promotion and exogenous bad news, which is part of Proposition 5. In the online appendix, Lemma 17, I show that equilibrium exist for any $c > \underline{c}$. The final part of the argument establishes that there can never be an equilibrium in the game with promotion that has persistent reputation unless there is also an equilbrium in the game wihout promotion that has persistent reputation.

Lemma B5. In both the game with and without promotion $a(1) = 0$ in any MPE if $c > \frac{\lambda\mu}{(r+\lambda)(r+\mu)}$. If $\frac{\lambda\mu}{(r+\lambda)(r+\mu)} > c$, then otherwise there exists an MPE in the game with only bad news with persistent reputation.

Proof. The first part is immediate. Suppose there was an equilibrium (a, π) where $a(1) = 1$. Let $V(x, \theta)$ denote the value function in this game. Then

$$V(1, H) = \frac{1-c}{r}$$

$$V(1, L) \geq \int_0^\infty e^{-(r+\lambda+\mu)t}[1 - c + \lambda V(1, H)]\, dt.$$

Evaluating the integrals gives that $\lambda D(1) \geq c$ holds if and only if $c \leq \frac{\lambda\mu}{(r+\lambda)(r+\mu)}$, so if c exceeds this bound then $a(1) = 0$.

Now, consider the game with only bad news. Let $U_{NP}(x, \theta; x_a)$ denote the value function for this game when the firm is believed to be playing according to investment cutoff x_a, that is, $a(x) = 1$ if $x > x_a$, and $a(x) = 0$ if $x < x_a$. Define D_{NP} analogously. Then

$$D_{NP}(1; 1) = \int_0^\infty e^{-(r+\lambda)t}[\mu U_{NP}(1, L; 1)]\, dt \geq \frac{1-c}{r}\frac{\mu}{r+\lambda+\mu} > \frac{c}{\lambda}$$

where the first inequality comes from the possibility playing the strategy of investing until bad news arrives.

This observation, combined with the existence argument from Board and Meyer-ter-Vehn (2013) implies the desired result. Let $D_{NP+}(x; x_a) := \lim_{s\to x+} D_{NP}(s; x_a)$. Board and Meyer-ter-Vehn (2013) show that the function $x_a \mapsto D_{NP+}(x_a; x_a)$ is continuous and strictly increasing on [0,1] with $\lim_{x_a\to 1} D_{NP+}(x_a; x_a) = D_{NP}(1; 1)$. Then, by the intermediate value theorem, there's an interior point where $D_{NP+}(x_a; x_a) = c/\lambda$ or $D_{NP+}(x_a; x_a) > c/\lambda$ everywhere. In the former case, there's an equilibrium where the investment cutoff, x_a, is that point, in the latter case $a(x) = 1$ if $x > 0$ is an equilibrium. These both have persistent reputation. □

Supporting information

Additional supporting information may be found online in the Supporting Information section at the end of the article.

Data S1
Figure 2: Continuity when $x_0, x'_\pi < \underline{x}_A - \varepsilon/4$
Figure 3: Continuity when $x_0, x'_\pi > \bar{x}_A + \varepsilon/4$