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Necessary and sufficient conditions for continuity of hypercontractive processes and fields[☆]

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A B S T R A C T

Sample path properties of random processes are an interesting and extensively studied topic, especially in the case of Gaussian processes. In this article, we study the continuity properties of hypercontractive fields, providing natural extensions for some known Gaussian results beyond Gaussianity. Our results apply to both random processes and random fields alike.

1. Introduction

Sample path regularity of random processes and fields is an important and extensively studied topic in the literature, and can be used as a tool, for example, to study convergence of certain statistical estimators or to study convergence of numerical schemes for stochastic partial differential equations. Indeed, various application areas arise from the fact that sharp modulus of continuity estimates can be translated to supremum tail bounds, providing sharp tail behaviour of the supremum.

The problem is particularly well-studied in the case of Gaussian processes and fields. One of the earliest results in this direction is a sufficient condition due to Fernique (1964) who provided a sufficient condition for the sample path continuity involving the increment metric $d_X(s,t) = \left[\mathbb{E}(X_s - X_t)^2\right]^{1/2}$. Later on, Dudley (1967, 1973) established a necessary condition for the continuity of a Gaussian process X by using a metric entropy. While in general Dudley's condition is not sufficient, in the case of stationary Gaussian processes it turns out to be necessary and sufficient. Finally, general necessary and sufficient conditions for general Gaussian processes X were obtained by Talagrand (1987), in terms of metric entropies. Finally, while Talagrand's necessary and sufficient condition is rather complicated, a simple necessary and sufficient condition for Hölder continuity of Gaussian processes in terms of increment metric $d_X(s,t) = \left[\mathbb{E}(X_s - X_t)^2\right]^{1/2}$ was obtained in Azmoodeh et al. (2014), where the authors proved that the celebrated general Kolmogorov–Chentsov criterion for Hölder continuity is both necessary and sufficient condition for Gaussian processes.

While the topic is widely studied in the Gaussian case, the literature on modulus of continuity beyond Gaussianity is more limited. One general approach to obtain modulus of continuity for processes is to use Garsia–Rodemich–Rumsey lemma (Garsia et al., 1970). A multiparameter version was provided in Hu and Le (2013) where the authors obtained modulus of continuity estimates and joint Hölder continuity of solutions to certain stochastic partial differential equations driven by Gaussian noise. Finally, we mention a

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closely related article (Viens and Vizcarra, 2007), where the authors studied the case of so-called sub-nth Gaussian processes or Wiener processes, where n is an arbitrary integer. Such processes arise, roughly speaking, from processes of form $H_n(X_t)$, where X is a Gaussian process and H_n is the nth order Hermite polynomial. More precisely, in Viens and Vizcarra (2007) the authors provided sufficient conditions for continuity and, in the spirit of Talagrand, estimates for the tail of the supremum via metric entropy by using Malliavin calculus which is well-suited to the setting of a Gaussian Wiener space.

In this article we study sample path continuity for general hypercontractive processes and fields. That is, we assume that the higher-order moments of the increments satisfy

$$\mathbb{E}|X_t - X_s|^p \leq C(p) \left[\mathbb{E}|X_t - X_s|^2\right]^{p/2},$$

which allows us to deduce conditions in terms of the simple increment metric $\mathbb{E}|X_t - X_s|^2$. We show how the growth of the constant $C(p)$ translates into certain exponential moment bounds and tail estimates for the supremum. As a corollary, we extend the necessary and sufficient Kolmogorov–Chentsov criterion for the Hölder continuity beyond the Gaussian case studied in Azmoodeh et al. (2014). It is worth emphasising that, while our results can be used to cover and extend the known results on the Gaussian case (Azmoodeh et al., 2014) and in the case of sub-nth processes (Viens and Vizcarra, 2007), our results require only hypercontractivity and thus can be used in a more general framework where tools such as Malliavin calculus are not available. In our setting, our moment growth condition, Assumption 2.1, corresponds to the sub-Weibull distributions (see Vladimirova et al. (2020)) which generalise the widely studied and applied sub-Gaussian and sub-exponential distributions. Finally, our results extend naturally to random fields, and can be used to study joint Hölder continuity, akin to Hu and Le (2013), for instance.

The rest of the article is organised as follows. In Section 2 we introduce our notation and main results, while all the proofs and auxiliary lemmas are postponed to the supplementary material.

2. Necessary and sufficient conditions for continuity of hypercontractive processes and fields

We consider stochastic processes and fields, respectively, given by $X = (X_t)_{t\in K}$, where $K = [0, 1]$ or $K = [0, 1]^n$. We make use of the following hypercontractivity assumption:

Assumption 2.1. We suppose that for all $p \geq 1$ we have

$$\mathbb{E}|X_t - X_s|^p \leq C_0^p p^{p\iota} \left[\mathbb{E}|X_t - X_s|^2\right]^{\frac{p}{2}}, \tag{2.1}$$

where $C_0 > 0$ is a generic fixed constant and $\iota \geq 0$ is a given parameter.

Remark 2.2. Condition (2.1) implies that the increment $X_t - X_s$ is a sub-Weibull random variable with parameter ι; we recall that a random variable Z follows a sub-Weibull distribution with parameter $\iota \geq 0$ if there exists $c > 0$ such that $(\mathbb{E}[Z^p])^{1/p} \leq cp^{\iota}$ for all $p \geq 1$, see Vladimirova et al. (2020). This class is a generalisation of sub-Gaussian distributions ($\iota = 1/2$) and sub-exponential distributions ($\iota = 1$), and as such also contains random variables with bounded support. We note that a sub-exponentially distributed random variable is essentially a sub-Gaussian random variable squared. Both of these classes are studied extensively in the literature and have numerous applications; for details, we recommend the excellent textbook (Vershynin, 2018), and references therein. The sub-Weibull distribution has recently found widespread application within data science, for instance in dynamic factor models in econometrics (Barigozzi and Hallin, 2020), deep learning and stochastic optimisation (Wood and Dall'Anese, 2023), and in Bayesian neural networks (Vladimirova et al., 2020).

Remark 2.3. In general we could state hypercontractivity as

$$\mathbb{E}|X_t - X_s|^p \leq C(p) \left[\mathbb{E}|X_t - X_s|^2\right]^{\frac{p}{2}},$$

where $C(p)$ is a constant depending solely on p. We then could state our results by using a more general form of $C(p)$ with suitable growth in p, but this would result in an additional layer of notational complexity. For simplicity and to make the connection to sub-Weibull distributions more explicit, we restrict ourselves to the case $C(p) \leq C^p p^{p\iota}$.

Remark 2.4. It is worth to note that our assumption implies the exponential moment assumption (up to an unimportant constant) of Viens and Vizcarra (2007), in the case $\iota = n$ is an integer, cf. proof of Theorem 2.9.

The following examples cover a large spectrum of situations that arise in mathematical modelling and should convince the reader that the class of processes satisfying Assumption 2.1 is large. The first three examples cover various processes arising naturally in Gaussian analysis, while Example 2.8 reveals that hypercontractivity can be achieved in many situations, whether or not one can apply Gaussian analysis and Malliavin calculus.

Example 2.5. If X is Gaussian, then it is well-known that

$$\mathbb{E}|X_t - X_s|^p = \frac{1}{\sqrt{\pi}} 2^{\frac{p}{2}} \Gamma\left(\frac{p+1}{2}\right) \left[\mathbb{E}|X_t - X_s|^2\right]^{\frac{p}{2}},$$

where Γ is the Euler Gamma function. By Stirling's approximation, we have

$$\Gamma(x+1) \sim \sqrt{2\pi x}e^{-x}x^x$$

for large x, and hence we may choose $\iota = \frac{1}{2}$ in Assumption 2.1.

Example 2.6. Let $\mathcal{H}$ be a separable, real Hilbert space, and $Z = \{Z(h) : h \in \mathcal{H}\}$ an isonormal Gaussian process on $\mathcal{H}$. We define the nth Hermite polynomial as $H_0(x) = 1$ and for $n \geq 1$ by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d}{dx} e^{-x^2/2}.$$

Denote by $\mathcal{H}_p$ the linear space generated by the class $\{H_p(Z(h)) : p \geq 0,\, h \in \mathcal{H}, \|h\|_{\leqq} = 1\}$. This linear space is called the pth Wiener chaos of Z. Then it is known (see, e.g. Nourdin and Peccati (2012)) that

$$\left[\mathbb{E}|F|^q\right]^{1/q} \leq \left[\mathbb{E}|F|^r\right]^{1/r} \leq \left(\frac{r-1}{q-1}\right)^{p/2} \left[\mathbb{E}|F|^q\right]^{1/q}.$$

Hence if X is a process living in a fixed Wiener chaos of order p (or in a finite linear combination of them with p as the highest chaos), we have (2.1) with $\iota = \frac{p}{2}$ and recover the processes studied in Viens and Vizcarra (2007). In particular, we recover the Gaussian case by setting $p = 1$.

Example 2.7. Solutions to several kinds of stochastic differential equations driven by Gaussian processes are expected to satisfy Assumption 2.1, while at the same these objects live in infinite amount of Wiener chaoses (that is, have non-finite chaos decompositions). For example, under certain technical conditions, solutions to certain stochastic differential equations driven by the fractional Brownian motion satisfies Assumption 2.1 with $\iota = \frac{1}{2}$, see Baudoin et al. (2016, Condition (ii) of Theorem 5.15, Proof of Theorem 5.16, and the references therein).

Example 2.8. Next, consider $Z_t = X_t Y_t$, where X and Y both satisfy Assumption 2.1 with parameters $C_{0,X}$ (resp. $C_{0,Y}$) and ι_X (resp. ι_Y), and for simplicity assume that both X and Y start at zero. Then it follows from a straightforward application of the Cauchy–Schwartz inequality that Z also satisfies Assumption 2.1 with $C_{0,Z} = 2^{\iota_X+\iota_Y} C_{0,X} C_{0,Y}$, and $\iota_Z = \iota_X + \iota_Y$. Moreover, a linear combination of processes satisfying Assumption 2.1 also satisfies the said assumption, see Vladimirova et al. (2020, Proposition 3.1). This covers, for instance, stochastic examples related to fractional splines, see Unser and Blu (2000); see also Monje et al. (2010), Sheng et al. (2011) for applications in signal processing.

We remark that these examples extend naturally to the case of random fields $X = (X_t)_{t\in[0,1]^n}$ by considering Gaussian fields (cf. Example 2.5), fields living in a fixed Wiener chaos (cf. Example 2.6), solutions to certain stochastic partial differential equation models as in Hu and Le (2013) (cf. Example 2.7), or spline smoothing in multiparameter setting (cf. Example 2.8).

We recall that if X is a stochastic process (resp. random field) on a probability space $(\Omega, \mathcal{F}, P)$ with an index set T, for any fixed random parameter $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega) =: X_t$ defines the sample path. If $t \to t_0$ implies that $X_t \to X_{t_0}$ almost surely, we say that X is continuous at t_0. As usual, when this property holds everywhere on the index set T, we simply say that X is a continuous stochastic process (resp. random field).

We begin with the following general result providing the modulus of continuity and certain moment estimates.

Theorem 2.9. *Suppose that a continuous $X = (X_t)_{t\in[0,1]}$ satisfies Assumption 2.1 and suppose that $\mathbb{E}(X_t - X_s)^2 \leq \rho(|t-s|)^2$ for some non-decreasing, non-negative continuous function ρ, with $\rho(0) = 0$. Then*

$$|X_t - X_s| \leq 8 \int_0^{|s-t|} \beta^{-\iota} \left(\log\left(\frac{4B}{u^2}\right)\right)^{\iota} \mathrm{d}\rho(u) \tag{2.2}$$

for any $\beta > 0$, where

$$B = \int_0^1 \int_0^1 \exp\left(\beta \left(\frac{|X_t - X_s|}{\rho(|t-s|)}\right)^{\frac{1}{\iota}}\right) \mathrm{d}s\, \mathrm{d}t.$$

Moreover, for $\beta \in \left(0, \frac{e\iota}{C_0^{\frac{1}{\iota}}}\right)$, the random variable B has finite p-moments for all p satisfying

$$1 \leq p < \frac{e\iota}{\beta C_0^{\frac{1}{\iota}}}.$$

Remark 2.10. As in Viens and Vizcarra (2007), one could first study the existence of a continuous version in terms of metric entropies. That is, by considering the number N_ϵ of balls required to cover the interval $[0, T]$ with respect to the metric $d_X(s,t) = \left[\mathbb{E}(X_t - X_s)^2\right]^{1/2}$. Then one obtains continuity provided that

$$\int_0^\infty |\log N_\epsilon|^\iota \, \mathrm{d}\epsilon < \infty.$$

As we are interested in the modulus of continuity, we assume continuity a priori.

We obtain immediately the following corollaries.

Corollary 2.11. *Suppose that $X = (X_t)_{t\in[0,1]}$ satisfies Assumption 2.1 and suppose that $\mathbb{E}(X_t - X_s)^2 \le |t-s|^{2\alpha}$, for $\alpha \in (0,1]$. Let $\beta \in \left(0, \frac{e\iota}{C_0^{\frac{1}{\iota}}}\right)$. Then there exists a random variable $C(\omega) = C(\beta,\omega)$ satisfying $\mathbb{E}\exp(\beta_0 C(\omega)^{1/\iota}) < \infty$ for any β_0 satisfying*

$$\beta_0 < \frac{e\iota}{\left(8C_0 \cdot 3^{\max(\iota-1,0)}\right)^{\frac{1}{\iota}}} \tag{2.3}$$

and a deterministic constant $C_d = C_d(\beta)$ such that

$$|X_t - X_s| \le C(\omega)|t-s|^\alpha + C_d|t-s|^\alpha\left(\log\frac{1}{|t-s|}\right)^\iota. \tag{2.4}$$

In particular, we have

$$\limsup_{|t-s|\to 0} \frac{|X_t - X_s|}{|t-s|^\alpha\left(\log\frac{1}{|t-s|}\right)^\iota} \le C_d. \tag{2.5}$$

Corollary 2.12. *Suppose $X = (X_t)_{t\in[0,1]}$ satisfies Assumption 2.1 and assume that $\mathbb{E}(X_t - X_s)^2 \le |t-s|^{2\alpha}$, for $\alpha \in (0,1]$. Let $\beta \in \left(0, \frac{e\iota}{C_0^{\frac{1}{\iota}}}\right)$. Then for any interval $I \subset [0,1]$ and any $s \in I$ we have*

$$P\left(\sup_{t\in I}|X_t - X_s| \ge u|I|^\alpha + C_d e^{-\alpha\iota}\iota^\iota\right) \le C(\beta_0)e^{-\beta_0 u^{1/\iota}},$$

where

$$C(\beta_0) = C(\beta_0,\beta) = \mathbb{E}\left(4B^{\beta_0\beta^{-1}\left(8\cdot 3^{\max(\iota-1,0)}\right)^{1/\iota}}\right) < \infty$$

for any β_0 satisfying (2.3).

Remark 2.13. The above result is close in spirit to Viens and Vizcarra (2007, Theorem 3.1), and generalises naturally the well-known exponential decay of the supremum of Gaussian processes, in which case we would obtain

$$P(\sup_{t\in I}|X_t - X_s| \ge u|I|^\alpha + C_1) \le C_2 e^{-C_3u^2}$$

for constants C_1, C_2, and C_3.

As our final main theorem, we obtain the following characterisation of Hölder continuity: under Assumption 2.1, Kolmogorov continuity criterion is a necessary and sufficient condition for Hölder continuity. This extends the results of Azmoodeh et al. (2014) beyond Gaussian processes and covers, in particular, processes living in a finite sum of Wiener chaoses, cf. Example 2.6.

Theorem 2.14. *Suppose that $X = (X_t)_{t\in[0,1]}$ satisfies Assumption 2.1. Then X is Hölder continuous of any order $\gamma < \alpha$, i.e. for any $\epsilon > 0$*

$$|X_t - X_s| \le C_\epsilon(\omega)|t-s|^{\alpha-\epsilon},$$

if and only if for any $\epsilon > 0$ we have

$$\mathbb{E}(X_t - X_s)^2 \le C_\epsilon|t-s|^{2\alpha-\epsilon}. \tag{2.6}$$

Moreover, in this case the Hölder constant $C_\epsilon(\omega)$ of X satisfies

$$\mathbb{E}\exp\left(\beta C_\epsilon(\omega)^{\frac{1}{\iota}}\right) < \infty \tag{2.7}$$

for small enough $\beta > 0$ which depends only on C_0, α, ι, and ϵ.

Our results extend naturally to the case of random fields $X = (X_t)_{t\in[0,1]^n}$. We begin with the following result, which extends Theorem 2.14 to the case of fields in a natural way.

Proposition 2.15. *Suppose that $X = (X_t)_{t\in[0,1]^n}$ satisfies Assumption 2.1. Then X is Hölder continuous of any order $\gamma < \alpha$, i.e. for any $\epsilon > 0$,*

$$|X_t - X_s| \le C_\epsilon(\omega)\|t-s\|^{\alpha-\epsilon},$$

if and only if for any $\epsilon > 0$ we have

$$\mathbb{E}(X_t - X_s)^2 \le C_\epsilon\|t-s\|^{2\alpha-\epsilon}. \tag{2.8}$$

Moreover, in this case the Hölder constant of X satisfies

$$\mathbb{E}\exp\left(\beta C_\epsilon(\omega)^{\frac{1}{\iota}}\right) < \infty$$

for small enough $\beta > 0$ which depends only on C_0, α, ι, and ϵ.

We next consider rectangular increments and joint continuity. We first introduce some notation, taken from Hu and Le (2013). Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two elements in $\mathbb{R}^d$. For each integer $k = 1, 2, \dots, n$, we define

$$V_{k,y}x := (x_1, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n).$$

Let f be a $\mathbb{R}^m$-valued map on $\mathbb{R}^n$. We define the operator $V_{k,y}$ acting on f as follows:

$$V_{k,y}f(x) := f(V_{k,y}x).$$

It is simple to verify that $V_{k,y}V_{k,y}f(x) = V_{k,y}f(x)$ and that $V_{k,y}V_{l,y}f(x) = V_{l,y}V_{k,y}f(x)$ for any f.

Next, we define the joint (rectangular) increment of a function f on an n-dimensional rectangle,

$$\square_y^n f(x) = \prod_{k=1}^n (I - V_{k,y})f(x), \tag{2.9}$$

where I denotes the identity operator.

For a random field $X = (X_t)_{t\in[0,1]^n}$, let $d_X(t,s) := \sqrt{\mathbb{E}|\square_t^n X(s)|^2}$. We assume that the following condition, analogous to Assumption 2.1, is satisfied.

Assumption 2.16. We suppose that for all $p \geq 1$ we have

$$\mathbb{E}|\square_t^n X(s)|^p \leq C_0^p p^{p\iota} d_X^p(t,s), \tag{2.10}$$

where $C_0 > 0$ is a generic fixed constant and $\iota \geq 0$ is a given parameter.

Theorem 2.17. *Suppose that $X = (X_t)_{t\in[0,1]^n}$ is continuous and satisfies Assumption 2.16 and suppose that $d_X(t,s) \leq \prod_{j=1}^n \rho_j(|t_j - s_j|)$. Then*

$$|\square_t^n X(s)| \leq 8^n \int_0^{|s_1-t_1|} \dots \int_0^{|s_n-t_n|} \beta^{-\iota}\left(\log\left(\frac{4^n B}{u_1^2 \dots u_n^2}\right)\right)^{\iota} \mathrm{d}\rho_1(u_1)\dots \mathrm{d}\rho_n(u_n)$$

for any $\beta > 0$, where

$$B = \int_{[0,1]^n}\int_{[0,1]^n} \exp\left(\beta\left(\frac{|\square_t^n X(s)|}{\prod_{j=1}^n \rho_j(|t_j - s_j|)}\right)^{\frac{1}{\iota}}\right) \mathrm{d}s\,\mathrm{d}t.$$

Moreover, for $\beta \in \left(0, \frac{e\iota}{C_0^{\frac{1}{\iota}}}\right)$, the random variable B has finite p-moments for all p satisfying

$$1 \leq p < \frac{e\iota}{\beta C_0^{\frac{1}{\iota}}}.$$

The following corollary is analogous to Corollary 2.11, and extends some of the results in Hu and Le (2013) beyond Gaussianity.

Corollary 2.18. *Suppose that $X = (X_t)_{t\in[0,1]^n}$ satisfies 2.16 and suppose that $d_X^2(t,s) \leq \prod_{j=1}^n |t_j - s_j|^{2\alpha_j}$. Let $\beta \in \left(0, \frac{e\iota}{C_0^{\frac{1}{\iota}}}\right)$. Then there exists a random variable $C(\omega) = C(\beta,\omega)$ satisfying $\mathbb{E}\exp(\beta_0 C^{1/\iota}(\omega)) < \infty$ for any β_0 satisfying*

$$\beta_0 < \frac{e\iota}{\left(8^n C_0 \cdot 3^{\max(\iota-1,0)}\right)^{\frac{1}{\iota}}} \tag{2.11}$$

and a deterministic constant $C_d = C_d(\beta)$, such that

$$|\square_t^n X(s)| \leq C(\omega)\prod_{j=1}^n |t_j - s_j|^{\alpha_j} + C_d \prod_{j=1}^n |t_j - s_j|^{\alpha_j}\left(\log\frac{1}{\prod_{j=1}^n |t_j - s_j|}\right)^{\iota}.$$

In particular, we have

$$\limsup_{\max_j |t_j - s_j| \to 0} \frac{|\square_t^n X(s)|}{\prod_{j=1}^n |t_j - s_j|^{\alpha_j}\left(\log\frac{1}{\prod_{j=1}^n |t_j-s_j|}\right)^{\iota}} \leq C_d.$$

Corollary 2.19. *Suppose that $X = (X_t)_{t\in[0,1]^n}$ satisfies 2.16 and suppose that $d_X^2(t,s) \le \prod_{j=1}^n |t_j - s_j|^{2\alpha_j}$. Let $\beta \in \left(0, \frac{e\iota}{C_0^{\frac{1}{\iota}}}\right)$. Then for any intervals $I_j \subset [0,1]$ and any $s \in I = I_1 \times \cdots \times I_n$, we have*

$$P\left(\sup_{t\in I} |\Box_t^n X(s)| \ge u \prod_{j=1}^n |I_j|^{\alpha_j} + \widetilde{C}\right) \le C(\beta_0) e^{-\beta_0 u^{1/\iota}},$$

where

$$C(\beta_0) = C(\beta_0,\beta) = \mathbb{E}\left[4B^{\beta_0\beta^{-1}\left(8\cdot 3^{\max(\iota-1,0)}\right)^{1/\iota}}\right] < \infty$$

for any β_0 satisfying (2.11), and

$$\widetilde{C} = C_d \max_{0\le x_j\le 1} \prod_{j=1}^n |x_j|^{\alpha_j}\left(\log \frac{1}{\prod_{j=1}^n |x_j|}\right)^{\iota}. \tag{2.12}$$

Similarly, Theorem 2.14 extends in a natural manner to the multiparameter case. Again, the proof is analogous to the proof of Theorem 2.14 and is thus left to the reader.

Theorem 2.20. *Suppose that $X = (X_t)_{t\in[0,1]^n}$ satisfies Assumption 2.16. Then X is jointly Hölder continuous of any order $\gamma = (\gamma_1,\dots,\gamma_n)$ with $\gamma_j < \alpha_j$, i.e. for any $\epsilon = (\epsilon_1,\dots,\epsilon_n)$ with $\epsilon_i > 0$,*

$$|\Box_t^n X(s)| \le C(\omega) \prod_{j=1}^n |t_j - s_j|^{\gamma_j - \epsilon_j},$$

if and only if for any $\epsilon = (\epsilon_1,\dots,\epsilon_n)$ with $\epsilon_i > 0$ we have

$$d_X^2(t,s) \le C_\epsilon \prod_{j=1}^n |t_j - s_j|^{2\alpha_j - \epsilon_j}.$$

Moreover, in this case the Hölder constant of X satisfies

$$\mathbb{E}\exp\left(\beta C_\epsilon(\omega)^{\frac{1}{\iota}}\right) < \infty$$

for small enough $\beta > 0$ which depends only on C_0, α, ι, and ϵ.

Data availability

No data was used for the research described in the article.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.spl.2024.110049.

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