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Majorana STM as a perfect detector of odd-frequency superconductivity

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We propose a novel scanning tunneling microscope (STM) device in which the tunneling tip is formed by a Majorana bound state (MBS). This peculiar bound state emerges at the boundary of a one-dimensional topological superconductor. Since the MBS has to be effectively spinless and local, we argue that it is the smallest unit that shows itself the properties of odd-frequency superconducting pairing. Odd-frequency superconductivity is characterized by an anomalous Green's function, which is an odd function of the time arguments of the two electrons building the Cooper pair. Interestingly, our Majorana STM can be used as the perfect detector of odd-frequency superconductivity. The reason is that a supercurrent between the Majorana STM and any other superconductor can only flow if the latter system exhibits itself odd-frequency pairing. To illustrate our general idea, we consider the tunneling problem of the Majorana STM coupled to a quantum dot placed on a surface of a conventional superconductor.

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I. INTRODUCTION

The phenomenon of superconductivity (SC) comes in different facets. Conventional SC, having granted us a number of exciting microscopic and macroscopic effects, is only a part of the whole manifold of superconducting phenomena. In recent years, unconventional superconducting pairing [1,2] has been proposed to exist in various forms, for instance, as the pairing mechanism for high-T_c SC [3], *p*-wave SC [4], topological SC with Majorana bound states as part of it [5,6], and odd-frequency SC [7–15]. In this article, we will combine the latter two forms of unconventional SC to propose a new device: the Majorana scanning tunneling microscope (STM).

Let us start with describing the different ingredients of the Majorana STM, see Fig. 1 for a schematic. Most importantly, we need a Majorana bound state (MBS), which has been predicted to exist at the boundary of a one-dimensional (1D) topological superconductor [5]. A MBS can be induced into a spin-orbit coupled nanowire under the combined influence of conventional *s*-wave pairing and an external magnetic field [16,17]. Recent experiments on the basis of nanowires and magnetic adatoms on *s*-wave superconductors have, indeed, shown some evidence that these exotic bound states, which constitute their own antiparticles, do exist in nature [18–21]. A MBS should form the tip of our STM, which can, for instance, be achieved by using a corresponding nanowire setup or, likewise, by any other realization of a 1D topological superconductor. Now, the interesting question comes up how this device relates to odd-frequency pairing.

Odd-frequency SC is defined on the basis of the anomalous Green's function that describes the superconducting pairing, cf. Eq. (1) below. This Green's function contains two annihilation operators corresponding to the particles that form the Cooper pair. Due to the Pauli principle, the Green's function has to be odd under the exchange of these operators. In the case of equal-time pairing, this oddness implies that singlet pairing has to be even in space coordinates and odd in triplet pairing. Interestingly, Berezinskii realized already in 1974 [7] that the symmetry of the pairing amplitude becomes richer if pairing at different times is allowed. This was the birth of odd-frequency superconductivity where the oddness in time is transferred to the frequency domain. Then, pairing mechanisms that are odd in frequency, triplet in spin space, and even in spatial parity (OTE) [10] are allowed by symmetry. Exciting new physics is attributed to OTE pairing, e.g., related to a long-range proximity effect in hybrid Josephson junctions based on ferromagnetism and superconductivity [22–25], cross correlations between the end states in a topological wire [26], the interplay of superconductivity and magnetism in double quantum dots [27], or its connection to crossed Andreev reflection at the helical edge of a 2D topological insulator [28].

Remarkably, a single MBS is the prime example for OTE superconductivity. This somewhat provocative statement can be understood by very simple means: The annihilation operator γ of the MBS is Hermitian, i.e., $\gamma = \gamma^\dagger$. Thus, in the case of a MBS, the normal and the anomalous Green's functions coincide (see Appendix A). Moreover, a single MBS has no additional quantum numbers, such as spin, momentum, etc., i.e., it corresponds to a spinless, local object. Thus, the time-ordered Majorana correlator $\langle \mathsf{T}\gamma(t)\gamma(0)\rangle$ has to be antisymmetric in t because of the Pauli principle. This property is, in fact, in one-to-one correspondence to the definition of odd-frequency superconductivity [24].

Therefore, it is natural to use this property of a MBS as building block for our theoretical proposal of the Majorana STM. If the tip of this device is formed by the MBS then a supercurrent from this tip can only flow into any other superconductor if and only if this superconductor also exhibits (at least partly) odd-frequency triplet SC. If not the corresponding supercurrent completely vanishes for symmetry reasons. It should be mentioned that it is rather difficult to detect odd-frequency SC in general. So far, most experiments rely on an indirect probe, for instance, the detection of triplet pairing in a disordered system where spatial parity has to

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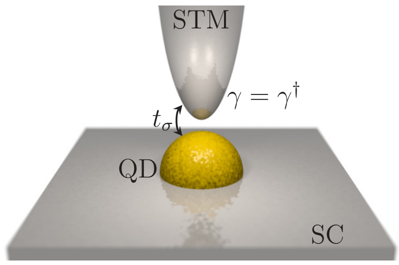

FIG. 1. Majorana STM. The scanning tip contains the MBS described by γ, which is used to probe—via tunneling coupling t_σ—either an unknown superconductor (SC) with odd-frequency pairing or a quantum dot (QD), which can realize odd-frequency pairing by proximity to an s-wave superconductor and an external magnetic field.

be even [13–15]. In that case, the Pauli principle implies that odd-frequency pairing should be present. Our novel idea constitutes a qualitative way of achieving this challenging task instead. The reason is that in our setup the supercurrent through the Majorana STM is really absent in the absence of odd-frequency pairing in the scanned sample.

The article is organized as follows. In Sec. II we discuss the general symmetries of the anomalous correlations functions, which are play an important role in the Majorana STM setup analyzed in Sec. III. Section IV demonstrates the general result on the basis of the superconducting quantum dot, and Sec. V analyzes the results of the previous section within the Green's functions formalism.

II. SYMMETRY CONSIDERATIONS

Odd-frequency SC can be best understood on the basis of the symmetry properties of the Green's function that describes the anomalous (causal) correlation function, i.e.,

$$F^c_{\alpha\beta}(t) = -i\langle \mathsf{T}\psi_\alpha(t)\psi_\beta(0)\rangle = \tfrac{1}{2}\big(F^K_{\alpha\beta} + F^R_{\alpha\beta} + F^A_{\alpha\beta}\big)(t). \quad (1)$$

Here, $\psi_\alpha(t)$ is an annihilation operator for the electron in state α (encoding orbital and/or spin degrees of freedom) at time t; angle brackets denote the averaging over the ground state; and $F^{R/A/K}_{\alpha\beta}(t)$ are the retarded/advanced/Keldysh components [29] of the anomalous correlation function. Due to the Pauli principle, the time-ordered Green's function fulfills the following symmetry condition: $F^c_{\beta\alpha}(-t) = -F^c_{\alpha\beta}(t)$. When we calculate transport properties below, not the time-dependent correlation functions $F^{R/A/K}_{\alpha\beta}(t)$ matter but instead their Fourier transforms $F^{R/A/K}_{\alpha\beta}(\omega)$. Therefore, it is important to state how retarded, advanced, and Keldysh Green's functions behave under a sign change of ω. This behavior is summarized in Table I. In this table, we do not only refer to the symmetry properties of the anomalous Green's functions $F^{R/A/K}_{\alpha\beta}(\omega)$ (relevant for the superconducting properties of the system) but, for completeness, also to the normal Green's functions $G^{R/A/K}_{\alpha\beta}(\omega)$. The detailed analysis of the correlators, their symmetries, and the manifestation of the fermionic anticommutativity is given in the Appendix A.

In thermal equilibrium at temperature T, the Keldysh Green's function can be expressed as a simple function of retarded and advanced Green's function via (with $\hbar = k_B = 1$)

$$X^K(\omega) = \tanh\frac{\omega}{2T}[X^R(\omega) - X^A(\omega)], \quad (2)$$

where X could be the normal (G) or the anomalous (F) Green's function. Evidently, *cf.* Table I, some linear combinations of retarded, advanced, and Keldysh Green's function are even with respect to frequency ω and others are odd. Therefore, we need to carefully address their influence on the current that will flow through the Majorana STM to fully understand why this device functions as a perfect detector for odd-frequency SC. We now develop a general microscopic model for the Majorana STM. Specifically, we derive a formula for the Josephson supercurrent between the superconducting STM tip and an unknown SC as sample.

TABLE I. Symmetry properties of the symmetrized (with respect to α,β space) Green's functions of a general fermionic system: Re / Im denotes whether the functions are real or imaginary; even/odd means whether the functions are even/odd in ω. $R \pm A$ should be understood as the corresponding linear combination of retarded and advanced Green's function.

		$R+A$	$R-A$	K
Normal	$G_{\alpha\beta} + G_{\beta\alpha}$	Re	Im	Im
Anomalous	$F_{\alpha\beta} + F_{\beta\alpha}$	odd	even	odd
Majorana	D	Re, odd	Im, even	Im, odd

III. MAJORANA STM

The coupling between the Majorana state γ and another system can be described by the tunneling Hamiltonian [30]

$$H_t = \gamma \sum_\alpha \mathrm{t}_\alpha(\psi_\alpha - \psi^\dagger_\alpha) = \sum_\alpha \mathrm{t}_\alpha(\gamma\psi_\alpha + \psi^\dagger_\alpha\gamma), \quad (3)$$

where α denotes the different quantum numbers (e.g., spin, momentum, etc.), ψ_α is the annihilation operator of the scanned sample, and t_α is the absolute value of the tunneling amplitude. This representation is valid for the general case (see Appendix B). The current operator can be written as

$$\hat{I} = e\dot{N} = i[H_t, N]_- = i\frac{e}{\hbar}\gamma\sum_\alpha \mathrm{t}_\alpha(\psi_\alpha + \psi^\dagger_\alpha), \quad (4)$$

where N is the number of electrons in the studied superconductor. Hence, the average current is given by

$$I = \frac{e}{\hbar}\sum_\alpha \mathrm{t}_\alpha \,\mathrm{Re}\int\frac{d\omega}{2\pi}W^K_\alpha(\omega), \quad (5)$$

where the integrand is the Fourier transformed Keldysh component of the cross correlator $W^K_\alpha(t) = -i\langle[\psi_\alpha(t),\gamma(0)]_-\rangle$, which can be calculated exactly by means of the Dyson formula

$$W^K_\alpha(\omega) = \sum_\beta \mathrm{t}_\beta\big\{\big[G^{(0)R}_{\alpha\beta}(\omega) - F^{(0)R}_{\alpha\beta}(\omega)\big]D^K(\omega) + \big[G^{(0)K}_{\alpha\beta}(\omega) - F^{(0)K}_{\alpha\beta}(\omega)\big]D^A(\omega)\big\}. \quad (6)$$

In this expression, the functions of ω are the Fourier transforms of the retarded/advanced/Keldysh components of the Majorana Green's function D, and normal (anomalous) Green's function of the lead G (F), which are defined in a standard way (see Appendix A). The superscript $^{(0)}$ in Eq. (6) denotes that the Green's functions are bare with respect to the tunneling Hamiltonian H_t. Substituting the cross correlator from Eq. (6) into Eq. (5) and removing all vanishing terms due to the mismatching symmetries with respect to ω, we obtain

$$\begin{aligned} I = \frac{e}{2\hbar}\sum_{\alpha\beta} \mathrm{t}_\alpha \mathrm{t}_\beta \int \frac{d\omega}{2\pi}\{ &-\mathrm{Im}\left[G^{(0)R}_{\alpha\beta}(\omega) - G^{(0)A}_{\alpha\beta}(\omega)\right] \\ &\times \mathrm{Im}\, D^K(\omega) + \mathrm{Im}\, G^{(0)K}_{\alpha\beta}(\omega)\,\mathrm{Im}[D^R(\omega) - D^A(\omega)] \\ &+ \mathrm{Im}\left[F^{(0)R}_{\alpha\beta}(\omega) + F^{(0)A}_{\alpha\beta}(\omega)\right]\mathrm{Im}\, D^K(\omega) \\ &- \mathrm{Re}\, F^{(0)K}_{\alpha\beta}(\omega)\,\mathrm{Re}[D^R(\omega) + D^A(\omega)]\}. \end{aligned} \tag{7}$$

Note that this current takes into account both normal current and supercurrent contributions. The first two summands in the integrand of Eq. (7) are responsible for the normal current that probes the density of states in the substrate (convoluted with the Majorana Green's functions). We are, however, more interested in the supercurrent determined by the last two summands in the integrand of Eq. (7). This means that we want to run the Majorana STM at zero bias (and zero temperature difference) between the tip and the sample to be able to detect odd-frequency SC. The assumption of thermal equilibrium reduces the number of independent Keldysh Green's functions according to Eq. (2). In that case, the terms proportional to the normal Green's functions drop out, and we are left with the following expression for the supercurrent

$$\begin{aligned} I = \frac{e}{2\hbar}\sum_{\alpha\beta} \mathrm{t}_\alpha \mathrm{t}_\beta \int \frac{d\omega}{2\pi}\tanh\frac{\omega}{2T} &\times \left\{ \mathrm{Im}\left[F^{(0)R}_{\alpha\beta}(\omega) + F^{(0)A}_{\alpha\beta}(\omega)\right]\mathrm{Im}[D^R(\omega) - D^A(\omega)] \right. \\ &\left. - \mathrm{Re}\left[F^{(0)R}_{\alpha\beta}(\omega) - F^{(0)A}_{\alpha\beta}(\omega)\right]\mathrm{Re}[D^R(\omega) + D^A(\omega)]\right\}. \end{aligned} \tag{8}$$

If we now compare the integrand of the latter equation with the symmetry properties of the Green's functions in Table I, we evidently see that only odd contributions to the anomalous correlation functions (that describe the scanned sample) can contribute to the supercurrent. This constitutes the main result of our article. In order to express the current by means of correlators of the investigated superconductor only, we need to know the full Majorana Green's function D, the self-energy of which can be written as

$$\begin{aligned} \Sigma^{R/A}(\omega) = \sum_{\alpha\beta} \mathrm{t}_\alpha \mathrm{t}_\beta \Big\{ &G^{(0)R/A}_{\alpha\beta}(\omega) - G^{(0)A/R}_{\alpha\beta}(-\omega) \\ &- F^{(0)R/A}_{\alpha\beta}(\omega) - \left[F^{(0)A/R}_{\alpha\beta}(\omega)\right]^*\Big\}. \end{aligned}$$

Note that the self-energy obeys the same symmetry relations as the Majorana Green's function itself, i.e., $\Sigma^R(\omega) = [\Sigma^A(\omega)]^* = -\Sigma^A(-\omega)$. The bare Majorana Green's function is $D^{(0)R/A} = 2(\omega \pm i0)^{-1}$, so the full Majorana Green's function becomes $D^{R/A} = 2[\omega - 2\Sigma^{R/A}(\omega)]^{-1}$. These expressions can be plugged into Eq. (8) to further evaluate the supercurrent for a particular sample.

In the case of weak tunnel coupling, the self-energy of the MBS can be written as $\Sigma^{A/R}(\omega) \approx \mp i\Gamma/2$ where the rate $\Gamma = 4\pi\sum_\alpha \nu_\alpha \mathrm{t}_\alpha^2$ is assumed to be much smaller than the typical energy scales in the scanned superconductor. Here, ν_α corresponds to the density of states in the sample. Then, the first term in the Eq. (8) is negligible and the current can be approximated as

$$I \approx -\frac{2e}{\hbar}\sum_{\alpha\beta} \mathrm{t}_\alpha \mathrm{t}_\beta\, \mathsf{P}\int \frac{d\omega}{2\pi}\frac{1}{\omega}\mathrm{Re}\left[F^{(0)K}_{\alpha\beta}(\omega)\right], \tag{9}$$

where the symbol P denotes the principal value of the integral. Interestingly, this latter equation illustrates a straight connection between the supercurrent through the Majorana STM and the symmetry of the anomalous Green's function of the superconducting substrate with respect to ω. We now illustrate our general result on the basis of a concrete example where the amount of odd-frequency SC can be nicely tuned.

IV. OTE SUPERCONDUCTING QUANTUM DOT

Let us consider a single-level quantum dot (QD) with Coulomb energy U subject to an external magnetic field $\mathbf{B} = (B_\perp\cos\theta, B_\perp\sin\theta, B_z)$ pointing in an arbitrary direction with respect to the spin quantization axis that is effectively defined by the MBS at the tip of the STM. The magnetic field acts only on the spin degree of freedom $\hat{\mathbf{S}} = \frac{1}{2}\sum_{\sigma\sigma'} c^\dagger_\sigma \boldsymbol{\sigma}_{\sigma\sigma'} c_{\sigma'}$ of the QD, where $\boldsymbol{\sigma}$ is the vector of Pauli matrices. The QD level with energy ε is coupled via the tunnel coupling Γ_Δ to a conventional s-wave superconductor with order parameter $\Delta e^{i\phi}$. In the following, we will focus on subgap transport where quasiparticle contributions are exponentially suppressed in Δ/T. This allows us to integrate out the superconducting degrees of freedom and leave aside sophisticated Kondo physics [31]. Hence, we obtain an effective dot Hamiltonian [32–34]

$$H_{\mathrm{dot}} = \sum_\sigma \varepsilon c^\dagger_\sigma c_\sigma + \mathbf{B}\cdot\hat{\mathbf{S}} + U n_\uparrow n_\downarrow - \frac{\Gamma_\Delta e^{i\phi}}{2} c^\dagger_\uparrow c^\dagger_\downarrow + \mathrm{H.c.} \tag{10}$$

The eigenstates of the isolated QD superconductor system are given by states of a single occupied QD with spin parallel $|\uparrow_\mathbf{B}\rangle$ and antiparallel $|\downarrow_\mathbf{B}\rangle$ to the magnetic field with energies $E_{\uparrow/\downarrow} = \varepsilon \pm B/2$. Furthermore, there exist the mixtures of empty $|0\rangle$ and fully occupied $|\uparrow\downarrow\rangle$ dot states

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\phi/2}\sqrt{1\mp\frac{\delta}{2\varepsilon_\mathrm{A}}}|0\rangle \mp e^{i\phi/2}\sqrt{1\pm\frac{\delta}{2\varepsilon_\mathrm{A}}}|\uparrow\downarrow\rangle\right)$$

with energies $E_\pm = \frac{\delta}{2} \pm \varepsilon_\mathrm{A}$, where $\varepsilon_\mathrm{A} = \frac{1}{2}\sqrt{\delta^2 + \Gamma_\Delta^2}$ and $\delta = 2\varepsilon + U$.

In order to characterize the superconducting correlations induced on the QD, we consider the time-ordered anomalous Green's functions $F^c_{\sigma\sigma'}(t) = \langle \mathsf{T} c_{\sigma'}(t) c_\sigma(0)\rangle$, which can be written in terms of the density matrix elements of the QD $\langle\alpha|\rho|\beta\rangle = P_\alpha\delta_{\alpha\beta}$, where $|\alpha\rangle$ are the eigenstates of the

Hamiltonian (10) given above, $P_\alpha = Z^{-1}e^{-E_\alpha/T}$, and $Z = \sum_\alpha e^{-E_\alpha/T}$. We find

$$F^c_{\sigma'\sigma}(t) = \sum_{\alpha\beta}\int \frac{d\omega}{2\pi}e^{i\omega t}P_\alpha\left(\frac{\langle\alpha|c_\sigma|\beta\rangle\langle\beta|c_{\sigma'}|\alpha\rangle}{\omega - E_\beta + E_\alpha + i0^+} + \frac{\langle\alpha|c_{\sigma'}|\beta\rangle\langle\beta|c_\sigma|\alpha\rangle}{\omega + E_\beta - E_\alpha - i0^+}\right). \quad (11)$$

Parametrizing the Green's functions as $F^c_{\sigma\sigma'}(t) = \{i[F^c_s(t) + \mathbf{F}^c_t(t)\cdot\boldsymbol{\sigma}]\sigma_y\}_{\sigma\sigma'}$, we can define an effective order parameter for the singlet part, which corresponds to the even-frequency component and is equal to

$$F^c_s(0) = \frac{i\pi\Gamma_\Delta}{2\varepsilon_A}(P_+ - P_-). \quad (12)$$

To characterize the triplet part, we employ the time derivative of the Green's function as an effective order parameter of the odd-frequency component [35–38]

$$\partial_t\mathbf{F}^c_t(0) = \pi\Gamma_\Delta\mathbf{S} - \frac{i}{2}\mathbf{B}F^c_s(0). \quad (13)$$

Evidently, these order parameters depend on the expectation value of the spin operator $\mathbf{S} = \sum_\alpha P_\alpha\langle\alpha|\hat{\mathbf{S}}|\alpha\rangle$ of the QD and on the magnetic field $\mathbf{B}$.

The coupling between the dot level and the MBS on the tip is given by the general tunneling Hamiltonian (3) (with $\psi_\alpha = c_\sigma$ and $\mathrm{t}_\alpha = \mathrm{t}_\sigma$). In the following, we represent the MBS γ by a conventional spinless (nonlocal) fermion f as $\gamma = f + f^\dagger$. (This representation implies that there is a second MBS on the STM far away from the tunneling tip, which naturally happens in any realization of a 1D topological superconductor.) Then, the full Hamiltonian decomposes into two blocks corresponding to even and odd parity sectors. As both blocks are equivalent to each other, we now focus on the odd parity sector. It is spanned by the states $|\uparrow\downarrow,1\rangle$, $|\uparrow,0\rangle$, $|\downarrow,0\rangle$, and $|0,1\rangle$, where the first ket entry denotes the dot occupation, while the second ket entry is the occupation of the nonlocal fermion described by the operator $f^\dagger f$. Choosing the spin quantization axis such that $\mathrm{t}_\uparrow = \mathrm{t}$ is real and $\mathrm{t}_\downarrow = 0$, the Hamiltonian takes the form (all gauge transformations related to the tunneling Hamiltonian are addressed in the Appendix B)

$$H = \begin{pmatrix} \delta & 0 & \mathrm{t} & -\frac{\Gamma_\Delta}{2}e^{i\phi - i\theta} \\ 0 & \varepsilon + \frac{B_z}{2} & \frac{B_\perp}{2} & -\mathrm{t} \\ \mathrm{t} & \frac{B_\perp}{2} & \varepsilon - \frac{B_z}{2} & 0 \\ -\frac{\Gamma_\Delta}{2}e^{-i\phi + i\theta} & -\mathrm{t} & 0 & 0 \end{pmatrix}. \quad (14)$$

The eigenvalues of this Hamiltonian are the energies $E_\alpha(\phi - \theta)$ corresponding to many-body eigenstates $|\alpha\rangle$, which depend on the superconducting phase. Then, the supercurrent can be calculated via the derivative of the free energy with respect to the phase,

$$I(\phi) = \frac{2e}{\hbar}\partial_\phi(-T\ln Z) = \frac{2e}{\hbar}\sum_\alpha P_\alpha\partial_\phi E_\alpha(\phi - \theta). \quad (15)$$

We find that a supercurrent can only flow if the direction of the external magnetic field is not collinear with the spin quantization axis of the MBS, i.e., $B_\perp \neq 0$. In Appendix C, we demonstrate how Eqs. (5) and (15) can be transformed into each other. Furthermore, in Sec. V, we calculate the supercurrent of a MBS-QD-superconductor setup directly on the basis of the Green's function formalism to explicitly demonstrate how the symmetry of the Green's functions affects the supercurrent.

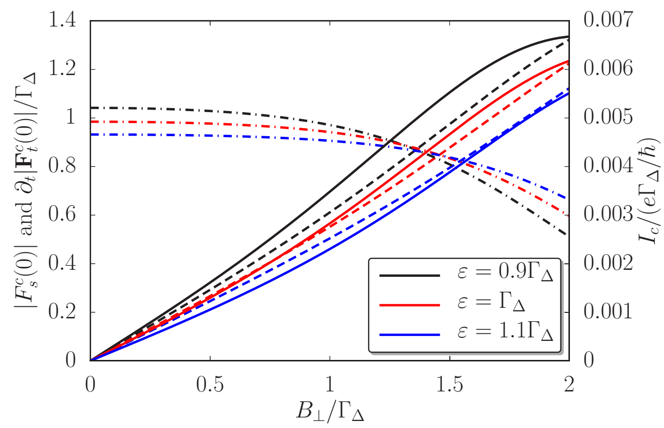

FIG. 2. The critical current I_c (solid lines) in comparison with the odd-frequency order parameter $\partial_t|\mathbf{F}^c_t(0)|$ (dashed lines) and the even-frequency order parameter $|F^c_s(0)|$ (dashed-dotted lines) for $B_z = 0$, $U = \Gamma_\Delta$, weak coupling $\mathrm{t} = 0.1\Gamma_\Delta$, and temperature $T = 0.2\Gamma_\Delta$.

The current is 2π periodic (with respect to ϕ) with a dominating first harmonics, so one can approximate $I(\phi) \approx I_c\sin(\phi - \theta)$. Interestingly, the influence of the angle θ on the current allows for a so-called ϕ_0 junction [39]. The dependence of I_c on the system parameters is shown in Fig. 2. Evidently, the supercurrent correlates nicely with the odd-frequency pairing (dashed line) defined in Eq. (13) instead of the even-frequency component (dash-dotted line) given in Eq. (12). The difference to the case of a conventional superconducting STM tip (which would follow the even-frequency component) is the sign of the gradient of the $B_\perp$ dependence.

V. CURRENT IN SC-QD-MBS SETUP FROM GREEN'S FUNCTIONS FORMALISM

To connect better between the more general part in the Sec. III and the example of the SC-QD-MBS setup in Sec. IV, we now apply the Green's function formalism, equivalent to Eqs. (5)–(8), to calculate the current flowing through a single-level quantum dot of energy ϵ coupled to a Majorana bound state and a conventional s-wave superconductor. We consider the regime where Kondo interactions can be neglected and the effect of the Coulomb repulsion is to renormalize the level position and tunneling amplitudes [40]. Further, we allow the quantum level to couple with an external magnetic field $\mathbf{B} = (B_\perp\cos\theta, B_\perp\sin\theta, B_z)$ and assume that the level separation is the largest energy scale. We describe the system in the combined Nambu-spin space with spinor fields $\Psi = (d_\uparrow, d_\downarrow, d^\dagger_\uparrow, d^\dagger_\downarrow)^T$, where $d^\dagger_\sigma$ (d_σ) annihilates (creates) an electron with spin $\sigma = \uparrow,\downarrow$ on the dot. The Hamiltonian of the isolated quantum dot is given by

$$\check{H}_{\mathrm{qd}} = \epsilon\hat{\sigma}_0\hat{\tau}_3 + B_\perp\cos\theta\hat{\sigma}_1\hat{\tau}_3 + B_\perp\sin\theta\hat{\sigma}_2\hat{\tau}_0 + B_z\hat{\sigma}_3\hat{\tau}_3,$$

where $\hat{\sigma}_\nu$ $(\hat{\tau}_\nu)$, with $\nu = 0,1,2,3$, are Pauli matrices in spin (Nambu) space with identity matrix $\hat{\sigma}_0$ $(\hat{\tau}_0)$. The transport properties of the system are described by the Green's function

$$\check{G}_{qd}(\omega) = [\omega\hat{\sigma}_0\hat{\tau}_0 - \check{H}_{\text{qd}} - \check{\Sigma}_S(\omega) - \check{\Sigma}_M(\omega)]^{-1}, \qquad (16)$$

where ω stands for $\omega \pm i0^+$ for retarded/advanced Green's functions or $i\omega_n = 2\pi i(n + 1/2)k_B T$, with temperature T, for Matsubara Green's functions. The superconducting substrate and the Majorana state are included as self-energies

$$\check{\Sigma}_S(\omega) = t^2[g_s(\omega)\hat{\sigma}_0\hat{\tau}_0 + f_s(\omega)(i\hat{\sigma}_2)\hat{\tau}_1],$$

$$\check{\Sigma}_M(\omega) = t_\uparrow^2(g_m(\omega)\hat{\tau}_0 - f_m(\omega)e^{i\phi\hat{\tau}_3}\hat{\tau}_1)\frac{\hat{\sigma}_0 + \hat{\sigma}_3}{2}.$$

By an appropriate charge $U(1)$ gauge choice, the static phase ϕ is included in the Green's function of the Majorana state. Due to the $SU(2)$ spin symmetry of the conventional superconductor, its tunnel coupling to the QD t is spin independent. Choosing the appropriate spin gauge we let the spinless Majorana state couple only to the fermions in the superconductor with spin $\uparrow$ with the amplitude $t_\uparrow$. For the singlet s-wave superconductor, the normal and anomalous dimensionless Green's functions are $f_s(\omega) = -(\Delta/\omega)g_s(\omega) = \Delta/\sqrt{\Delta^2 - \omega^2}$. The dimensionless Green's functions for the Majorana state are obtained from the edge Green's function of a topological superconductor as $f_m(\omega) = (\Delta/\sqrt{\Delta^2 - \omega^2})g_m(\omega) = \Delta/\omega$ (see Refs. [10,41–43]). For simplicity, we have assumed that both superconductors have the same gap Δ. As in the main text, we are only considering the equilibrium case where the junction is phase biased, i.e., the applied voltage is zero. As a result, the current adopts the simple form

$$I(V{=}0,T) = \frac{e}{2\hbar}\int \frac{\mathrm{d}\omega}{2\pi}\tanh\left(\frac{\omega}{2T}\right) \times \mathrm{Tr}\{\hat{\tau}_3[\check{\Sigma}_M^r(\omega)\check{G}_{qd}^r(\omega) - \check{\Sigma}_M^a(\omega)\check{G}_{qd}^a(\omega)]\}. \qquad (17)$$

Let us now consider the Green's function of the isolated system formed by the quantum dot and the superconducting substrate. Using the Dyson equation, we can recast Eq. (16) as

$$\check{G}_{qd} = [(\check{g}_{qd})^{-1} - \check{\Sigma}_M]^{-1},$$

where $\check{g}_{qd} = (\omega\hat{\sigma}_0\hat{\tau}_0 - \check{H}_{\text{qd}} - \check{\Sigma}_S)^{-1}$ is the Green's function of the dot-superconductor system. To further analyze the proximity-induced superconducting pair amplitude at the quantum dot, we focus on the anomalous part of $\check{g}_{qd}$, which in spin space adopts the form

$$(\check{g}_{qd})_{eh} = -\frac{t^2 f_s(\omega)}{\det[\check{g}_{qd}]}\begin{pmatrix} -2\omega_s B_\perp e^{-i\theta} & F_0(\omega) + 2\omega_s B_z \\ -F_0(\omega) + 2\omega_s B_z & 2\omega_s B_\perp e^{i\theta}\end{pmatrix}$$

$$= \frac{t^2 f_s(\omega)}{\det[\check{g}_{qd}]}(F_0\hat{\sigma}_0 + 2\omega_s \mathbf{B}\cdot\hat{\sigma})(-i\hat{\sigma}_2) \qquad (18)$$

with $\omega_s = \omega - t^2 g_s$ an odd function of the frequency and $F_0(\omega) = F_0(-\omega) = \omega_s^2 - t^4 f_s(\omega) - \epsilon^2 + B^2$, where $B^2 = B_\perp^2 + B_z^2$ is the magnitude of the magnetic field. It is thus clear from Eq. (18) that the induced pairing in the dot-superconductor system is a superposition of spin-singlet and triplet states, where the spin-triplet state is parallel to the external magnetic field. Additionally, since both $f_s(\omega)$ and $\det[\check{g}_{qd}]$ are even functions of frequency, the induced singlet state accounts for the even-frequency component of the pair amplitude while the triplet is odd in frequency. To characterize the superconducting correlations induced in the quantum dot by proximity to the superconductor, we define the quantities

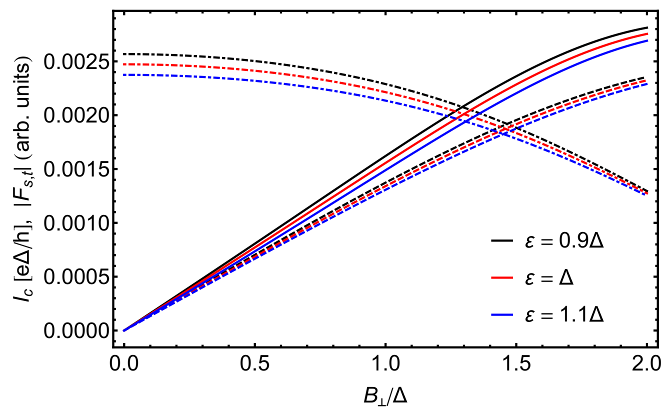

FIG. 3. Critical current of the SC-QD-MBS system (solid lines) as a function of the noncollinear component of the external magnetic field $B_\perp$ compared to the even-frequency singlet $|F_0|$ (dashed-dotted lines) and the magnitude of the odd-frequency triplet $|\mathbf{F}_t|$ (dashed lines) of the induced pairing in the dot. The current is calculated using Eq. (20) with $\theta = 0$, $B_z = 0$, $t = 3\Delta/2$, $t_\uparrow = 0.1\Delta$, $\beta = 1/(k_B T) = 5\Delta$, and $\Delta = 1$. The pairing amplitudes are normalized by the scaling factor $(t_\uparrow^2/t^2)$. This figure nicely confirms the results displayed in Fig. 2 on the basis of Green's functions calculation.

$$F_s = \sum_{\omega_n}\frac{t^2 f_s(\omega_n)}{\det[\check{g}_{qd}(\omega_n)]}F_0(\omega_n), \qquad (19a)$$

$$\mathbf{F}_t = \sum_{\omega_n}\omega_n\frac{2t^2 f_s(\omega_n)\omega_s}{\det[\check{g}_{qd}(\omega_n)]}\mathbf{B}, \qquad (19b)$$

which play the same role as the corresponding expressions in Eqs. (12) and (13), though they should not be compared directly, as the models are complementary to each other (here the QD has finite Δ and no double occupancy).

We can now find a more compact expression for the Josephson current in terms of the Green's functions of the Majorana state and the dot-superconductor system. We thus rewrite Eq. (17) as

$$I(V=0,T) = \frac{2e}{\beta h}t^2 t_\uparrow^2 \sin(\phi - \theta)\sum_{\omega_n}\frac{2\omega_s B_\perp f_s}{\det[\check{G}_{qd}]}f_m, \qquad (20)$$

with $\beta = 1/(k_B T)$. The Josephson current is finite only if $B_\perp \neq 0$, as it is shown in Fig. 3. According to Eq. (19), this is equivalent to inducing an odd-frequency triplet (OTE) component in the superconducting substrate. Therefore, as it is demonstrated in Sec. III on general grounds, a finite supercurrent is only possible in the presence of odd-frequency pairing in the superconducting substrate.

Additionally, in the limit where the tunneling to the Majorana state is much smaller than the energy scales of the studied superconductor, we can neglect the self-energy of the Majorana bound state Σ_M in the determinant to find $\det[\check{G}_{qd}] \approx \det[\check{g}_{qd}]$. In such a case, the current expression in

Eq. (20) adopts a simpler form equivalent to Eq. (8), namely

$$I(t_\uparrow \ll \Delta) \simeq \frac{2e}{\beta h} t_\uparrow^2 \sin(\phi-\theta) \sum_{\omega_n} f_m[(\mathbf{F}_t)_\perp(\theta=0)].$$

VI. CONCLUSIONS

We suggest a new device corresponding to a Majorana bound state at the tip of a scanning tunneling microscope, which we dub Majorana STM. It is shown that a single Majorana bound state exhibits a pair amplitude that is an odd function of time. This feature is decisive that the Majorana STM serves as an ideal detector for odd-frequency superconductivity. If a supercurrent builds up between the Majorana STM and an unknown superconducting sample then the latter superconductor has to experience odd-frequency pairing itself. We illustrate this general result on the basis of a simple quantum dot model coupled to the Majorana STM.

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APPENDIX A: CORRELATORS, GREEN'S FUNCTIONS, AND THEIR SYMMETRIES

In this section, we study the symmetries of the general correlators, apply the obtained results to the fermionic Green's functions, and examine the manifestation of the Fermion anticommutativity for the different types of Green's functions.

Let us consider two different objects described by the operators A and B in Heisenberg representation. We can define three correlators that account for the different order of the operators with respect to time coordinates, or for their Hermitian conjugated versions. Expressing the resulting Green's functions on the Keldysh contour [29,44], we can write

$$\begin{aligned} \mathcal{G}_\mathsf{K}(t_i,t_j') &= -i\langle \mathrm{T}_\mathsf{K} A(t_i)B(t_j')\rangle, \\ \widetilde{\mathcal{G}}_\mathsf{K}(t_i,t_j') &= -i\langle \mathrm{T}_\mathsf{K} B(t_i)A(t_j')\rangle, \\ \overline{\mathcal{G}}_\mathsf{K}(t_i,t_j') &= -i\langle \mathrm{T}_\mathsf{K} B^+(t_i)A^+(t_j')\rangle, \end{aligned} \tag{A1}$$

where T_K is the time-ordering operator on the Keldysh contour. Here, we follow a notation similar to Ref. [44], where for the time coordinate t_i, the index $i=+(-)$ corresponds to the c_1 (c_2) contour that lies above (below) the time axis and is the first (second) part of the full Keldysh contour. We thus express the Green's functions in matrix form as

$$\begin{aligned} \mathcal{G}_\mathsf{K}(t_i,t_j') = [\mathcal{G}_\mathsf{K}(t-t')]_{ij} &\equiv \begin{pmatrix} \mathcal{G}^c & \mathcal{G}^< \\ \mathcal{G}^> & \mathcal{G}^{ac} \end{pmatrix}(t-t') \\ &= -i\begin{pmatrix} \langle \mathrm{T}A(t)B(t')\rangle & \mp\langle B(t')A(t)\rangle \\ \langle A(t)B(t')\rangle & \langle \widetilde{\mathrm{T}}A(t)B(t')\rangle \end{pmatrix}, \end{aligned} \tag{A2}$$

where T is the time-ordering operator, $\widetilde{\mathrm{T}}$ is the reverse time-ordering one, and the sign $-(+)$ corresponds to fermion (boson) operators. $\mathcal{G}^c$, $\mathcal{G}^{ac}$, $\mathcal{G}^<$, and $\mathcal{G}^>$ are causal, anticausal (with reverse time ordering), greater, and lesser Green's functions. Note that

$$\begin{aligned} \langle \mathrm{T}A(t)B(t')\rangle &= \mp\langle \mathrm{T}B(t')A(t)\rangle, \\ \langle \widetilde{\mathrm{T}}A(t)B(t')\rangle &= \mp\langle \widetilde{\mathrm{T}}B(t')A(t)\rangle, \\ \langle \mathrm{T}A(t)B(t')\rangle^* &= \langle \widetilde{\mathrm{T}}B^+(t')A^+(t)\rangle, \\ \langle \widetilde{\mathrm{T}}A(t)B(t')\rangle^* &= \langle \mathrm{T}B^+(t')A^+(t)\rangle. \end{aligned}$$

These relations allow us to establish the connection between the different correlators defined in Eq. (A1). Assuming that the Hamiltonian is time independent, the correlators depend on the time difference only, and are transformed into each other as

$$\begin{aligned} \widetilde{\mathcal{G}}_\mathsf{K}(t-t') &= \mp[\mathcal{G}_\mathsf{K}(t'-t)]^\mathsf{T} \\ \overline{\mathcal{G}}_\mathsf{K}(t-t') &= -\tau^1[\mathcal{G}_\mathsf{K}(t'-t)]^+\tau^1, \end{aligned} \tag{A3}$$

where the $^+$ superscript denotes Hermitian conjugation in Keldysh matrix space and τ^i are the Pauli matrices acting on the same space. The four Green's functions defined in Eq. (A2) are, however, linearly dependent. We can eliminate one component, if we rotate the basis of the Keldysh space [44] as follows

$$\hat{\mathcal{G}}(t-t') = L\tau^3\mathcal{G}_\mathsf{K}(t-t')L^\mathsf{T} = \begin{pmatrix} \mathcal{G}^R & \mathcal{G}^K \\ 0 & \mathcal{G}^A \end{pmatrix}(t-t'), \tag{A4}$$

where $L=(1-i\tau^2)/\sqrt{2}$ and $\mathcal{G}^{R,A,K}$ are retarded, advanced, and Keldysh Green's functions, respectively. Applying the same rotation to the other Green's functions defined in Eq. (A1), we obtain

$$\hat{\widetilde{\mathcal{G}}}(t) = \mp\tau^1\hat{\mathcal{G}}^\mathsf{T}(-t)\tau^1, \quad \hat{\overline{\mathcal{G}}}(t) = \tau^2\hat{\mathcal{G}}^+(-t)\tau^2. \tag{A5}$$

Performing the Fourier transform over the time variable, we get the relation between the different correlators in frequency representation, namely,

$$\widetilde{\mathcal{G}}^{R/A}(\omega) = \mp\mathcal{G}^{A/R}(-\omega), \quad \widetilde{\mathcal{G}}^K(\omega) = \mp\mathcal{G}^K(-\omega), \tag{A6}$$

$$\overline{\mathcal{G}}^{R/A}(\omega) = [\mathcal{G}^{A/R}(\omega)]^*, \quad \overline{\mathcal{G}}^K(\omega) = -[\mathcal{G}^K(\omega)]^*. \tag{A7}$$

Let us illustrate these relations at the example of the Majorana bound states and the electron normal/anomalous Green's functions defined as

$$\begin{aligned} D^{R/A}(t) &= \mp i\langle\{\gamma(t),\gamma(0)\}_+\rangle\theta(\pm t), \\ D^K(t) &= -i\langle[\gamma(t),\gamma(0)]_-\rangle, \\ G^{R/A}_{\alpha\beta}(t) &= \mp i\langle\{\psi_\alpha(t),\psi^\dagger_\beta(0)\}_+\rangle\theta(\pm t), \\ G^K_{\alpha\beta}(t) &= -i\langle[\psi_\alpha(t),\psi^\dagger_\beta(0)]_-\rangle, \\ F^{R/A}_{\alpha\beta}(t) &= \mp i\langle\{\psi_\alpha(t),\psi_\beta(0)\}_+\rangle\theta(\pm t), \\ F^K_{\alpha\beta}(t) &= -i\langle[\psi_\alpha(t),\psi_\beta(0)]_-\rangle. \end{aligned} \tag{A8}$$

The normal electron Green's function can be obtained from the definitions in Eq. (A1) by substitution of the generic operators $A=\psi_\alpha$ and $B=\psi^+_\beta$. It is thus defined as

$$G_{\mathsf{K}\alpha\beta}(t_i,t_j') = -i\langle \mathrm{T}_\mathsf{K}\psi_\alpha(t_i)\psi^+_\beta(t_j')\rangle, \tag{A9}$$

where the indexes α and β denote the full set of the electron quantum numbers, such as spin, momentum, etc.

TABLE II. The transformation of the different correlators with respect to the exchange of the ladder operators. The sign and the position of the cell denotes the connection between the Green's functions. The nonstandard correlators ac (anticausal) and s (spectral) are defined in the Eqs. (A2) and (A18), respectively.

	$F^R_{\alpha\beta}(\omega)$	$F^A_{\alpha\beta}(\omega)$	$F^K_{\alpha\beta}(\omega)$	$F^c_{\alpha\beta}(\omega)$	$F^{ac}_{\alpha\beta}(\omega)$	$F^>_{\alpha\beta}(\omega)$	$F^<_{\alpha\beta}(\omega)$	$F^s_{\alpha\beta}(\omega)$
$F^R_{\beta\alpha}(-\omega)$		$-$						
$F^A_{\beta\alpha}(-\omega)$	$-$							
$F^K_{\beta\alpha}(-\omega)$			$-$					
$F^c_{\beta\alpha}(-\omega)$				$-$				
$F^{ac}_{\beta\alpha}(-\omega)$					$-$			
$F^>_{\beta\alpha}(-\omega)$							$-$	
$F^<_{\beta\alpha}(-\omega)$						$-$		
$F^s_{\beta\alpha}(-\omega)$								$+$

In that case, two of the correlators listed in Eq. (A1) are equivalent up to the exchange of the quantum numbers, namely, $\overline{G}_{\mathsf{K}\alpha\beta}(t_i,t'_j) = G_{\mathsf{K}\beta\alpha}(t_i,t'_j)$. Together with Eqs. (A7), we obtain the symmetry of the normal electron Green's function in frequency representation

$$G^{R/A}_{\alpha\beta}(\omega) = \left[G^{A/R}_{\beta\alpha}(\omega)\right]^*,\ \ G^K_{\alpha\beta}(\omega) = -\left[G^K_{\beta\alpha}(\omega)\right]^*. \quad \text{(A10)}$$

The anomalous electron Green's function is defined via the substitution $A = \psi_\alpha$ and $B = \psi_\beta$, resulting in

$$F_{\mathsf{K}\alpha\beta}(t_i,t'_j) = -i\langle \mathrm{T}_{\mathsf{K}}\psi_\alpha(t_i)\psi_\beta(t'_j)\rangle. \quad \text{(A11)}$$

Since the two generic operators are of the same type, we immediately find that $\widetilde{F}_{\mathsf{K}\alpha\beta}(t_i,t'_j) = F_{\mathsf{K}\beta\alpha}(t_i,t'_j)$. As a result, the symmetry dependence with respect to frequency of the anomalous Green's function is

$$F^{R/A}_{\alpha\beta}(\omega) = -F^{A/R}_{\beta\alpha}(-\omega),\ \ F^K_{\alpha\beta}(\omega) = -F^K_{\beta\alpha}(-\omega). \quad \text{(A12)}$$

For the Majorana fermion, due to its fundamental Hermiticity, $\gamma^+ = \gamma$, only one correlator can be defined. By setting $A = B = \gamma$, we find

$$D_{\mathsf{K}}(t_i,t'_j) = -i\langle \mathrm{T}_{\mathsf{K}}\gamma(t_i)\gamma(t'_j)\rangle, \quad \text{(A13)}$$

which combines the properties of both normal and anomalous Green's functions, $\widetilde{D}_{\mathsf{K}}(t_i,t'_j) = \overline{D}_{\mathsf{K}}(t_i,t'_j) = D_{\mathsf{K}}(t_i,t'_j)$. Therefore, it must fulfill the same symmetry properties with respect to frequency as the normal and the anomalous Green's functions,

$$D^{R/A}(\omega) = -D^{A/R}(-\omega), \quad \text{(A14)}$$

$$D^K(\omega) = -D^K(-\omega), \quad \text{(A15)}$$

$$D^{R/A}(\omega) = [D^{A/R}(\omega)]^*, \quad \text{(A16)}$$

$$D^K(\omega) = -[D^K(\omega)]^*. \quad \text{(A17)}$$

Next we discuss the manifestation of the fermionic anticommutation of the anomalous Green's function. For clarity, we summarize the transformations under the exchange of the particles of various anomalous correlators in Table II. The usual choice of Keldysh, retarded and advanced Green's functions does not fully reflect the fermionic nature of the particles. While the Keldysh component indeed changes the sign under the exchange of particles, the retarded and advanced components transform into each other. However, we can construct two independent correlators from the symmetric and antisymmetric superposition of the retarded and advanced Green's functions. From the sum of F^R and F^A, we obtain an independent correlator, which is odd with respect to particle exchange. Further, using the initial Keldysh Green's function in Eq. (A2), we notice that $F^R + F^A = F^c - F^{ac}$.

The other independent correlator is given by $F^R - F^A = F^> - F^< \equiv F^s$, and it is even with respect to particle exchange, as can be seen from

$$F^s_{\alpha\beta}(t-t') = -i\langle \psi_\alpha(t)\psi_\beta(t') + \psi_\beta(t')\psi_\alpha(t)\rangle. \quad \text{(A18)}$$

The correlation function F^s, which is expressed in terms of the greater and lesser Green's functions $F^{<,>}$, describes the spectral properties of the system. We would like to stress that, even in the case of pure odd-frequency superconductivity, this correlator is still even in ω. This is not surprising when we consider that, for thermal equilibrium, the Keldysh Green's function adopts the form $F^K = F^s \tanh(\omega/2T)$. As long as the Keldysh component is odd in frequency, the spectral one is bound to be even.

APPENDIX B: GAUGE IN THE TUNNELING HAMILTONIAN OF THE SC-QD-MBS SETUP

The full Hamiltonian of the quantum dot on the superconducting substrate coupled to the Majorana state is $H = H_{\mathrm{dot}} + H_t$, where H_{dot} is given in Eq. (10) of the main text. The tunneling term in the most general gauge of the dot ladder operators can be written in the form

$$H_t = \sum_\sigma (\mathrm{t}^*_\sigma \gamma c_\sigma + \mathrm{t}_\sigma c^\dagger_\sigma \gamma). \quad \text{(B1)}$$

The general gauge transformations of the operators c_σ belong to the $U(2) = U(1) \otimes SU(2)$ Lee group, which can be split into the $U(1)$ charge gauge and the $SU(2)$ spin gauge. The Hamiltonian H_{dot} is invariant under $SU(2)$ transformations, but $U(1) = e^{i\varphi}$ changes the superconducting phase by $\phi \to \phi + 2\varphi$. The tunneling Hamiltonian H_t, due to the Hermicity of the Majorana operator γ, is not invariant under either $U(1)$ or $SU(2)$ transformations. As a result, the tunneling coefficients t_σ change if the spin gauge changes. In an experimental realization of a MBS, this $SU(2)$ symmetry is usually broken, not by the tunneling amplitude, but by the magnetic order and the spin-orbit interaction in the STM tip, in a typical setup for the creation of the MBS [16,17]. Thus, in the effective model of Eq. (B1), the gauge change $c_\sigma \to \sum_{\sigma'} U_{\sigma\sigma'} c_{\sigma'}$ by a $SU(2)$ matrix U changes the tunneling coefficients $\mathrm{t}_\uparrow$ and $\mathrm{t}_\downarrow$ in the same way: $t_\sigma \to \sum_{\sigma'} U_{\sigma\sigma'} t_{\sigma'}$. Choosing the matrix $\tilde{U}$ such that

$$\sum_{\sigma'} \tilde{U}_{\sigma\sigma'} \mathrm{t}_{\sigma'} = \begin{pmatrix} \mathrm{t} \\ 0 \end{pmatrix}, \quad \tilde{U} = \frac{i}{\mathrm{t}} \begin{pmatrix} \mathrm{t}_\uparrow^* & \mathrm{t}_\downarrow^* \\ \mathrm{t}_\downarrow & -\mathrm{t}_\uparrow \end{pmatrix} \in SU(2), \quad \text{(B2)}$$

where $\mathrm{t} = \sqrt{|\mathrm{t}_\uparrow|^2 + |\mathrm{t}_\downarrow|^2} \in \mathrm{Re} > 0$, we can eliminate the tunneling in one channel and make the tunneling amplitude real in the other one.

The closest analogy to this gauge freedom can be found in the tunnel coupling for two normal subsystems 1 and 2. Then, the tunneling Hamiltonian looks like $\mathrm{t}\psi_1^+\psi_2 + \mathrm{H.c.}$ where $\psi_{1(2)}$ are the annihilation operators of the subsystems 1(2). If $\mathrm{t} = |\mathrm{t}|e^{i\theta}$ one can perform a $U(1)$ charge gauge transformation $\psi_1 \to \psi_1 e^{i\theta_0 + i\theta}$ and $\psi_2 \to \psi_2 e^{i\theta_0}$, which will make the tunneling amplitude real.

Let us now explicitly show how the gauge transformation works in the Fock subspace with odd fermion parity defined in the main text, and how one obtains the Hamiltonian in the form of Eq. (14). The total Hamiltonian, in the initial (arbitrary) gauge, is

$$H = \begin{array}{r} \langle\uparrow\downarrow,1| \\ \langle\uparrow,0| \\ \langle\downarrow,0| \\ \langle 0,1| \end{array} \overset{\begin{array}{cccc} |\uparrow\downarrow,1\rangle & |\uparrow,0\rangle & |\downarrow,0\rangle & |0,1\rangle \end{array}}{\begin{pmatrix} \delta & -\mathrm{t}_\downarrow & \mathrm{t}_\uparrow & -\frac{\Gamma_\Delta}{2}e^{i\phi} \\ -\mathrm{t}_\downarrow^* & \varepsilon + \frac{B_z'}{2} & \frac{B_\perp'}{2}e^{-i\theta'} & -\mathrm{t}_\uparrow \\ \mathrm{t}_\uparrow^* & \frac{B_\perp'}{2}e^{i\theta'} & \varepsilon - \frac{B_z'}{2} & -\mathrm{t}_\downarrow \\ -\frac{\Gamma_\Delta}{2}e^{-i\phi} & -\mathrm{t}_\uparrow^* & -\mathrm{t}_\downarrow^* & 0 \end{pmatrix}}, \quad \text{(B3)}$$

where we displayed the corresponding basis, for clarity, next to the matrix. Using the block-diagonal matrix $V_U = \mathrm{diag}(1,U,1)$ corresponding to the $SU(2)$ rotation in second quantization, we can eliminate $\mathrm{t}_\downarrow$ and set $\mathrm{t}_\uparrow$ to t, which is real and positive, resulting in

$$V_U^+ H V_U = \begin{pmatrix} \delta & 0 & \mathrm{t} & -\frac{\Gamma_\Delta}{2}e^{i\phi} \\ 0 & \varepsilon + \frac{B_z}{2} & \frac{B_\perp}{2}e^{-i\theta} & -\mathrm{t} \\ \mathrm{t} & \frac{B_\perp}{2}e^{i\theta} & \varepsilon - \frac{B_z}{2} & 0 \\ -\frac{\Gamma_\Delta}{2}e^{-i\phi} & -\mathrm{t} & 0 & 0 \end{pmatrix}$$

$$= V_\theta \begin{pmatrix} \delta & 0 & \mathrm{t} & -\frac{\Gamma_\Delta}{2}e^{i\phi - i\theta} \\ 0 & \varepsilon + \frac{B_z}{2} & \frac{B_\perp}{2} & -\mathrm{t} \\ \mathrm{t} & \frac{B_\perp}{2} & \varepsilon - \frac{B_z}{2} & 0 \\ -\frac{\Gamma_\Delta}{2}e^{-i\phi + i\theta} & -\mathrm{t} & 0 & 0 \end{pmatrix} V_\theta^+.$$

In the second step, we have used the matrix $V_\theta = \mathrm{diag}(e^{i\theta},1,e^{i\theta},1)$, which is a combination of the rotation by an angle θ around the z axis [$SU(2)$ by $\mathrm{diag}(1,e^{-i\theta/2},e^{i\theta/2},1)$], and a charge gauge change by $\theta/2$ [$U(1)$ by $\mathrm{diag}(e^{i\theta},e^{i\theta/2},e^{i\theta/2},1)$].

APPENDIX C: FORMULA FOR THE SUPERCURRENT IN THE SC-QD-MBS SETUP

This subsection is devoted to the derivation of the expression for the current in Eq. (15) of the manuscript. We will show that this formula is equivalent to Eq. (5). Starting from the definition of the current operator, given in the Eq. (4), and writing down the ladder operators $\gamma c_\sigma^{(\dagger)}$ in the Fock space, specified in Eq. (B3), we obtain for the current operator

$$\hat{I} = i\frac{e}{\hbar}\sum_\sigma \left(\mathrm{t}_\sigma \gamma c_\sigma - \mathrm{t}_\sigma^* c_\sigma^\dagger \gamma\right), \quad \text{(C1)}$$

which in the matrix representation is

$$\hat{I} = i\frac{e}{\hbar}\begin{pmatrix} 0 & \mathrm{t}_\downarrow & -\mathrm{t}_\uparrow & 0 \\ -\mathrm{t}_\downarrow^* & 0 & 0 & \mathrm{t}_\uparrow \\ \mathrm{t}_\uparrow^* & 0 & 0 & \mathrm{t}_\downarrow \\ 0 & -\mathrm{t}_\uparrow^* & -\mathrm{t}_\downarrow^* & 0 \end{pmatrix}. \quad \text{(C2)}$$

The superconducting phase dependence in the Hamiltonian (B3) can be moved from the order parameter to the tunneling coefficients using the $U(1)$ charge gauge transformation $V_\phi = \mathrm{diag}(e^{i\phi},e^{i\phi/2},e^{i\phi/2},1)$. As a result, we find

$$V_\phi^+ H V_\phi = \begin{pmatrix} \delta & -\mathrm{t}_\downarrow e^{-i\phi/2} & \mathrm{t}_\uparrow e^{-i\phi/2} & -\frac{\Gamma_\Delta}{2} \\ -\mathrm{t}_\downarrow^* e^{i\phi/2} & \varepsilon + \frac{B_z}{2} & \frac{B_\perp}{2}e^{-i\theta} & -\mathrm{t}_\uparrow e^{-i\phi/2} \\ \mathrm{t}_\uparrow^* e^{i\phi/2} & \frac{B_\perp}{2}e^{i\theta} & \varepsilon - \frac{B_z}{2} & -\mathrm{t}_\downarrow e^{-i\phi/2} \\ -\frac{\Gamma_\Delta}{2} & -\mathrm{t}_\uparrow^* e^{i\phi/2} & -\mathrm{t}_\downarrow^* e^{i\phi/2} & 0 \end{pmatrix},$$

which provides the relation

$$2\frac{e}{\hbar}\partial_\phi(V_\phi^+ H V_\phi) = V_\phi^+ \hat{I} V_\phi. \quad \text{(C3)}$$

This is not a coincidence, but a relation reflecting the fact that the electromagnetic gauge field $\mathbf{A}$ changes the amplitude t_{12} of the tunneling matrix element from 1 to 2 as $\mathrm{t}_{12} \to \mathrm{t}_{12}e^{i\frac{e}{c}\int_1^2 \mathbf{A}d\mathbf{r}}$. Since the current is $\mathbf{j} = \delta H/\delta\mathbf{A}$, we then obtain Eq. (C3). Starting from Eq. (15) for the current defined in the manuscript, we find

$$\begin{aligned} I &= -2\frac{e}{\hbar}\beta^{-1}\partial_\phi \ln Z = -2\frac{e}{\hbar}\beta^{-1}Z^{-1}\partial_\phi \mathrm{Tr}\, e^{-\beta H} \\ &= -2\frac{e}{\hbar}\beta^{-1}Z^{-1}\mathrm{Tr}\, \partial_\phi e^{-\beta V_\phi^+ H V_\phi}, \end{aligned} \quad \text{(C4)}$$

where $Z = \mathrm{Tr}\, e^{-\beta H} = \sum_\alpha e^{-E_\alpha/T}$, and E_α are the energies of the Andreev levels defined in the main text after Eq. (10). Using the standard formula for the derivative of the exponential map [45], $\partial e^X = \int_0^1 e^{sX}(\partial X)e^{(1-s)X}ds$, and the relation between current and Hamiltonian in Eq. (C3), we get

$$\begin{aligned} I &= Z^{-1}\mathrm{Tr}\int_0^1 e^{-s\beta V_\phi^+ H V_\phi} V_\phi^+ \hat{I} V_\phi e^{-(1-s)\beta V_\phi^+ H V_\phi} ds \\ &= Z^{-1}\mathrm{Tr}\, V_\phi^+ \hat{I} V_\phi e^{-\beta V_\phi^+ H V_\phi} = Z^{-1}\mathrm{Tr}\, \hat{I} e^{-\beta H}. \end{aligned} \quad \text{(C5)}$$

This result corresponds to the expectation value of the current $I = \langle \hat{I} \rangle$ where the angle brackets denote the averaging over the thermodynamic equilibrium state of the system. This latter equation is precisely Eq. (5).

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