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Comment on "Self-organized criticality and absorbing states: Lessons from the Ising model"

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According to Pruessner and Peters [G. Pruessner and O. Peters, Phys. Rev. E **73**, 025106(R) (2006)], the finite-size scaling exponents of the order parameter in sandpile models depend on the tuning of driving and dissipation rates with system size. We point out that the same is not true for *avalanches* in the slow driving limit.

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In a recent paper, Pruessner and Peters investigated the relation between self-organized criticality (SOC) in sandpile models and absorbing state phase transitions, on the basis of an analogy with standard equilibrium critical phenomena, in particular, the two-dimensional Ising model [1]. According to Ref. [1], only a careful choice of the system size dependence of the driving and dissipation parameters would yield the scaling results of the underlying phase transition. Here we point out that this reasoning does not apply, in particular, when one studies "SOC variables" such as various measures of avalanches in the slow driving limit, as is traditionally done in this context. This is confirmed by numerical simulations presented below.

In SOC sandpile models, the steady state is maintained by a balance of dissipation ϵ and driving h. In particular, the control parameter of the absorbing phase transition, i.e., the average height of the pile ζ, evolves on the average as

$$\dot{\zeta} = h - \epsilon\rho, \tag{1}$$

where ρ is the density of active sites, the order parameter of the absorbing phase transition. In the limit $h \to 0^+$, $\epsilon \to 0^+$, the control parameter flows to the critical value ζ_c and the model shows scale invariance [2]. The question raised by the authors of Ref. [1] is, how to apply finite-size scaling (FSS) to the order parameter ρ if $\epsilon(L) \sim L^{-\kappa}$ and $h(L) \sim L^{-\omega}$ are taken to be functions of the system size L. To investigate this issue, Pruessner and Peters employ a similar "driving" to the Ising model, using a fluctuating inverse temperature θ evolving as

$$\dot{\theta} = h - \epsilon|m|, \tag{2}$$

where $|m|$ is the absolute value of the magnetization. Using this driving, the system size dependence of the order parameter $\langle|m|\rangle \sim L^{\kappa-\omega}$ coincides with that expected from standard FSS only when $\omega - \kappa = \beta/\nu$, where β and ν are the equilibrium exponents for the order parameter and the correlation length, respectively. We notice that the case $\omega - \kappa > \beta/\nu$ corresponds precisely to the slow driving limit of SOC, where there is complete time-scale separation between driving and dissipation (i.e., for most purposes one may even take $\omega \to \infty$ and wait an infinite amount of time after each driving event). In this limit, the effective temperature defined in Ref. [1] also would diverge. The authors, in analogy with the Ising model, conclude that the slow driving SOC state should *not* correspond to the critical point of an absorbing phase transition, in contradiction with the evidence from numerical simulations of sandpile models [3].

The apparent contradiction disappears when we notice that avalanche statistics, as typically studied in the case of SOC, and the average order parameter studied in Ref. [1] are not equivalent measures of the criticality of the underlying absorbing state phase transition. As pointed out by Pruessner and Peters, it is indeed possible to tune the system size dependence of the order parameter, or average activity, by choosing the size scaling of driving and dissipation rates appropriately. In the slow driving limit relevant for SOC, however, this is just a trivial consequence of the drive rate dependence of the quiescent periods between avalanches. The avalanches themselves are not affected by the drive rate in any way as long as it is slow enough such that no new grains are added while the system is active.

In SOC sandpiles, one usually implements open boundary conditions and infinitely slow drive, corresponding to $\kappa=2$ and $\omega \to \infty$. The mappings to absorbing phase transitions and to depinning transitions allows one to obtain the scaling behavior for any value of κ, provided that we remain in the time-scale separation regime [2]. In general, sandpile models exhibit FSS forms for the avalanche size s, of the type

$$P(s,\xi) = s^{-\tau_s} P(s/\xi_s^D). \tag{3}$$

τ_s and D are critical exponents related to the underlying depinning transition and $\xi(\omega,\kappa)$ is the cutoff scale that is determined by the condition of balance between dissipation and drive (which also results in the steady state condition $\rho = h/\epsilon$). One transparent argument to compute ξ is to look at the dynamics following the addition of a single grain, which gives rise to an avalanche of average size $\langle s\rangle$ dissipating on the average one grain [4]. Thus one has the condition $\epsilon(L)\langle s\rangle(\xi) = 1$ and obtains

$$\xi \sim L^{\kappa/[D(2-\tau_s)]}. \tag{4}$$

Notice, in particular, that ξ is not dependent on the drive rate or ω.

To confirm this, we consider the Manna sandpile model with periodic boundary conditions, slow driving (i.e., $\omega = \infty$), and bulk dissipation. The model is studied here on a 2D

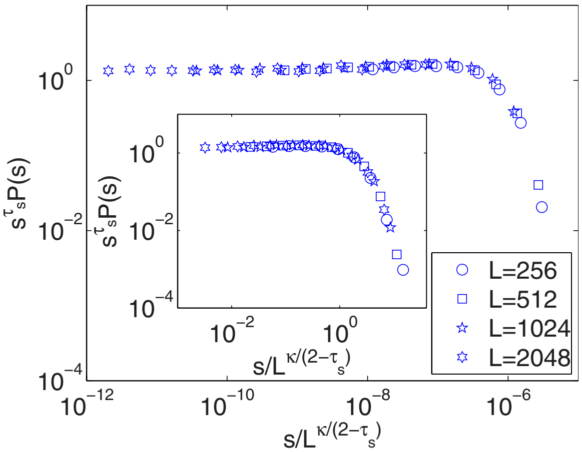

FIG. 1. (Color online) Scaling plots of the avalanche-size distributions from the two-dimensional Manna model with bulk dissipation. The inset presents the collapse in the case $\kappa=1$, while in the main figure the case $\kappa=3$ is shown. Both collapses correspond to the value $\tau_s=1.28$.

lattice, where for each lattice site i one assigns an integer variable z_i (the number of "grains"). If $z_i > z_c = 1$, a "toppling" occurs and the grains are redistributed according to $z_i \to z_i - 2$ and $z_{nn} = z_{nn} + 1$, where z_{nn} are two randomly chosen nearest neighbors of site i. The dissipation is implemented by removing a toppling grain from the system with probability $\epsilon \propto L^{-\kappa}$. Here we consider the two cases with $\kappa = 1$ and $\kappa = 3$, respectively. In Fig. 1 we show that the scaling follows the predictions of absorbing phase transitions: the cutoff scale of the avalanche-size distribution scales according to Eq. (4), regardless of the L dependence of ϵ.

To summarize, we would like to point out that the conclusions of Pruessner and Peters are misleading in the sense that avalanche statistics in sandpile models indeed follow from the underlying absorbing state transition whenever one studies the slow driving limit $\omega - \kappa > \beta/\nu$. This is the condition of complete time-scale separation between driving and avalanche propagation that has been recognized already in the early literature as the crucial ingredient for SOC in sandpile models [5].

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