Sharma, R.; Sihvola, Ari

Studies on polarizabilities and scattering behavior of small spherical particles

Published in:
Radio Science

DOI:
10.1029/1999RS900101

Published: 01/01/2000

Document Version
Publisher's PDF, also known as Version of record

Please cite the original version:
Sharma, R., & Sihvola, A. (2000). Studies on polarizabilities and scattering behavior of small spherical particles. *Radio Science*, *35*(1), 83-96. https://doi.org/10.1029/1999RS900101

Radio Science, Volume 35, Number 1, Pages 83–96, January–February 2000

Studies on polarizabilities and scattering behavior of small spherical particles

Reena Sharma and Ari Sihvola

Electromagnetics Laboratory, Helsinki University of Technology, Espoo, Finland

Abstract. It is possible to relate the response of a sphere to an applied uniform static electric field with the scattering from a small spherical particle due to plane wave incidence. The limits up to which these relations between the polarizability and the extinction, scattering, and radar cross sections of a sphere are valid in the dynamic range are presented here. From the observations it can be concluded that radar cross section is a very good measure in predicting the polarizability. A related aspect studied here is the evaluation of the effective permittivity of a sparse mixture of spherical inclusions using a generalized Maxwell Garnett mixing rule. This is compared with extinction of a plane wave by a slab of n spherical inclusions sparsely located. The extinction by such a slab is calculated using the quasistatic approximation to Mie theory, and also using the full Mie theory, as the size of the inclusions is increased. The studies have been carried out for both lossless and lossy inclusions. The generalized mixing rule was found to be quite accurate in predicting the value of effective permittivity up to size parameters of 0.5 at least for small ε_r of the inclusion.

1. Introduction

The scattering from dielectric spherical inclusions has been analyzed by *Mie* [1908] and *Debye* [1909], popularly known as the Mie scattering theory, using the vector spherical harmonic functions to represent the fields. It is also well known that this theory is not so elegant when one is dealing with scattering from electrically small particles. By using the quasistatic analysis, it is possible to get simple expressions for the Mie coefficients without worrying about the behavior of spherical Bessel functions. Under the quasistatic limit the fundamental mode given by the first Mie coefficient of the scattered field is similar to the field scattered by an oscillating dipole with a dipole moment the same as that of a sphere under the influence of uniform static field. Therefore one can relate the polarizability of an electrically small dielectric sphere to its scattering and radar cross section (RCS) in the dynamic range.

Since there are practical situations where it is much more convenient to calculate the RCS of the particle, it is sometimes easier to evaluate its polarizability from the definition of RCS. One of the aspects studied in this paper is to see the limit of validity of relating the RCS of the particle to its polarizability when the scatterer size is very small compared with the operating wavelength. Relations have been derived between the static polarizability and the scattering, extinction, and radar cross section for both lossless and lossy dielectrics.

A related aspect discussed here is to measure or estimate the effective dielectric constant of a composite material made up of small spherical inclusions embedded in a homogeneous matrix. The imaginary part of the effective value of permittivity $-\mathrm{Im}\{\varepsilon_{\mathrm{eff}}/\varepsilon_0\}$ is calculated using the definition of the polarizability of a dielectric sphere and the generalized Maxwell Garnett (MG) mixing formula which was derived by *Sihvola and Sharma* [1999]. *Sihvola and Sharma* [1999] derived the generalized MG mixing formula by adding a few correction terms to the polarizability (explained in section 5). The effective permittivity is generalized to include the first-order scattering effects of the inclusions, which contribute to the imaginary part of the effective permittivity and are discussed in section 2. The problem can be analyzed by treating the mixture as a slab of particles that are electrically small and sparse, as shown in Figure 1. The incident field, which is assumed to be a plane wave, and the transmitted field are represented as $\mathbf{E}_\mathrm{i}$ and $\mathbf{E}_\mathrm{t}$. The imaginary part of the effective permittivity is compared with the extinction due to scattering by small inclusions and is seen to be consistent up to a certain upper limit of the frequency.

Paper number 1999RS900101.
0048-6604/00/1999RS900101$11.00

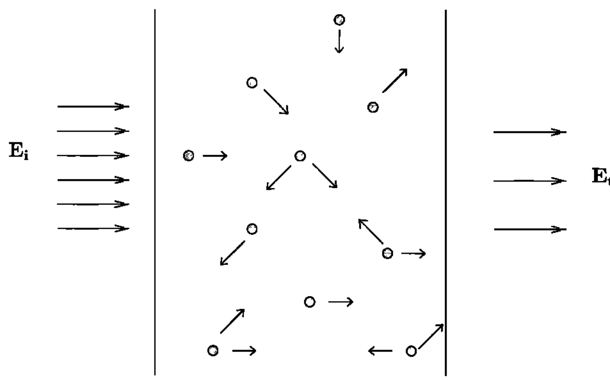

Figure 1. Extinction from a slab of small spherical particles.

Since the formula takes into account the size of the scatterer at least to the lowest-order terms, the $-\mathrm{Im}\{\varepsilon_{\mathrm{eff}}/\varepsilon_0\}$ is plotted as a function of the size parameter $k_o a = x$ using different definitions, explained in section 4. The radius of the spherical inclusion is a, and k_o is the free-space wave number. The limit up to which the generalized MG formula is accurate is shown clearly by considering inclusions of different sizes and permittivities satisfying the sparse mixture conditions.

Table 1 illustrates the approach of the study carried out in this paper. The interaction of the electromagnetic wave with a single spherical particle and with a dilute mixture of spherical inclusions is considered for both the static and dynamic cases. The aim here is to see to what extent the static mixing rules can be extended into the higher frequency ranges.

One of the motivations for this study is the conception of new materials with particular electromagnetic properties (like absorbers). These materials utilize heterogeneous structures made up of inclusions of different shapes and properties dispersed in a dielectric matrix. Another possible application of this study is analyzing the atmospheric aerosols. The interaction of electromagnetic waves with particles such as clouds or rain is one such example.

2. Single Scattering Extinction

The problem of analyzing the scattering from a sparse mixture can be treated as scattering from a slab of n particles embedded in a host medium (as shown in Figure 1), which is considered to be free space in this paper. The attenuation due to a plane wave incident on a slab of n such scatterers is calculated. This is then compared with scattering from a single spherical scatterer whose scattering characteristics can be evaluated using the standard Mie theory [*Mie*, 1908], the scattering code of which is available on the World Wide Web (http://imperator.cip-iw1.uni-bremen.de/ fg01/codes2.html) and also given by *Sharma and Sihvola* [1998]. In the present paper all the results are presented using the code used by *Sharma and Sihvola* [1998].

The Mie theory is well established and used to solve scattering from cylindrical- and spherical-shaped conducting and dielectric scatterers. It has been very well discussed by *de Hulst* [1957], *Kerker* [1969], and *Bohren and Huffman* [1983]. Mie theory involves expansion of the electric and magnetic fields inside and outside the sphere in terms of vector spherical harmonic functions with unknown coefficients, also called Mie coefficients. Application of the boundary conditions at the interface results in solving for the unknown Mie coefficients. The Mie coefficients are used to compute various cross sections, namely, the scattering (C_{scat}), extinction (C_{ext}), absorption (C_{abs}), and radar cross sections (σ_{rcs}). From the optical extinction theorem one can write the absorption cross section as $C_{\mathrm{abs}} = C_{\mathrm{ext}} - C_{\mathrm{scat}}$. The expressions for these cross sections in terms of the Mie coefficients are given by *de Hulst* [1957], *Kerker* [1969], and *Bohren and Huffman* [1983].

The scattering efficiencies, namely, the scattering (Q_{scat}), extinction (Q_{ext}), and absorption (Q_{abs}) efficiencies, are defined as the respective cross sections normalized with respect to the geometric cross section of the sphere (πa^2). Thus we can relate the scattering efficiencies to the cross sections as

$$C_{\mathrm{ext}} = \pi a^2 Q_{\mathrm{ext}} \qquad C_{\mathrm{scat}} = \pi a^2 Q_{\mathrm{scat}} \tag{1}$$

Now if such spherical scatterers are sparsely distributed and occupy a number density n in the mixture, a plane wave that travels through it suffers the extinction in power density (the Poynting vector) $S(z) \sim |E(z)|^2 \sim \exp(-2k''_{\mathrm{eff}} z)$, with

$$2k''_{\mathrm{eff}} = nC_{\mathrm{ext}}, \tag{2}$$

Table 1. Scattering Problem Classification

	Single Particle	Mixture
$\omega = 0$	polarizability α_n	Maxwell Garnett
$\omega \neq 0$	Mie theory	?

where $C_{ext} = C_{scat} + C_{abs}$. For lossless inclusions the extinct power is due to scattering alone, as there is no absorption, and therefore in the above equation we can replace C_{ext} by C_{scat}. The connection of the extinction coefficient k''_{eff} with the imaginary part of the effective permittivity is obviously

$$k''_{eff} = -k_0 \,\mathrm{Im}\, \sqrt{\varepsilon_{eff}/\varepsilon_0} = \frac{nC_{ext}}{2}, \tag{3}$$

where the minus sign is due to the engineering notation $k_{eff} = k'_{eff} - jk''_{eff}$. If the mixture is sparse, then we can accurately determine the value of the imaginary part of the effective permittivity using the computed values of C_{scat} and C_{ext} for lossless and lossy inclusions, respectively, provided multiple scattering is ignored.

3. Quasistatic Approximations

The Mie scattering from dielectric spherical particles of dimensions small compared with wavelength can be approximated using the asymptotic series expansions for the spherical Bessel functions [*Abramowitz and Stegun*, 1970] for the unknown scattered field coefficients [*Bohren and Huffman*, 1983]. The normalized extinction, scattering, and absorption cross sections or efficiencies for a small dielectric sphere with radius a and relative permittivity ε_r under the quasistatic approximation can be written in terms of series expansion of the size parameter $x = k_o a$. Below, we define the quasistatic efficiencies [*Bohren and Huffman*, 1983] (by taking the series expansion terms up to order x^4):

$$Q_{\text{ext-qs}} = -4x \cdot \mathrm{Im}\left\{\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\left[1 + \frac{x^2}{15}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)\frac{\varepsilon_r^2 + 27\varepsilon_r + 38}{2\varepsilon_r + 3}\right]\right\} + \frac{8}{3}x^4 \,\mathrm{Re}\left\{\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)^2\right\} \tag{4}$$

$$Q_{\text{scat-qs}} = \frac{8}{3}x^4\left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \tag{5}$$

$$Q_{\text{abs-qs}} = Q_{\text{ext-qs}} - Q_{\text{scat-qs}}, \tag{6}$$

where, Re and Im represent the real and imaginary parts of the argument, respectively. The subscript qs has been attached to the scattering, extinction, and absorption efficiencies to represent them for quasistatic approximations only. These will be compared later on in this paper, while discussing the results, with the full Mie theory definitions of scattering and extinction cross sections. The expressions given above are for the relative permittivity $\varepsilon_r = \varepsilon'_r - j\varepsilon''_r$ corresponding to the $e^{j\omega t}$ notation. The value of ε''_r is assumed to be greater than zero to represent passive dielectric materials.

It can be seen from (5) that the scattered power is quadratic to the field. Substituting the expression for $C_{\text{scat-qs}}$ (for a lossless inclusion) from (5) into (3), we can derive the expression for $-\mathrm{Im}\{\varepsilon_{eff}/\varepsilon_0\}$. The result is

$$-\mathrm{Im}\,\{\varepsilon_{eff}\} = \varepsilon''_{eff} = 2f\varepsilon_0(k_0 a)^3\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)^2, \tag{7}$$

where the assumptions $\varepsilon'_{eff}/\varepsilon_0 \approx 1$ and $\varepsilon''_{eff}/\varepsilon_0 \ll 1$ have been made for a sparse mixture, allowing the relation $-\mathrm{Im}\,\sqrt{\varepsilon_{eff}/\varepsilon_0} \approx \varepsilon''_{eff}/(2\varepsilon_0)$.

4. Polarizability: Relation to Extinction, Scattering, and Radar Cross Sections

4.1. Scattering and Extinction Cross Sections

The response of an isotropic sphere to an applied uniform static electric field which induces a dipole moment is well known [*Jackson*, 1975], and its polarizability is defined as

$$\alpha = 4\pi a^3 \varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r + 2}.$$

Let us denote the static polarizability given above as $\alpha = \alpha(0)$, representing the static case $\omega = 0$ and differentiating it from the quasistatic polarizability $\alpha(\omega)$. Normalizing the above equation with respect to the volume and ε_0 to make it dimensionless gives

$$\alpha_n(0) = \frac{\alpha}{\varepsilon_0 V} = 3\,\frac{\varepsilon_r - 1}{\varepsilon_r + 2}, \tag{8}$$

where $V = \frac{4}{3}\pi a^3$ is the volume of the spherical inclusion. The dimensionless normalized polarizability for the static fields is denoted as $\alpha_n(0)$.

In order to determine the polarizability as a function of frequency $\alpha(\omega)$, one needs to calculate the scattering and absorption from a spherical particle. Under the quasistatic limits $\omega \to 0$ the definition for normalized polarizability for a sphere should reduce to that given in (8). Therefore we proceed to relate the normalized polarizability to the extinction and scattering cross sections.

Under the quasistatic approximations the cross sections for the extinction and scattering can be obtained for small size parameters ($x \ll 1$) and for $|\sqrt{\varepsilon_r}|x \ll 1$ and thus can be written in terms of the static polarizability ($\omega \to 0$), denoted here as $\alpha(0)$ as [*Bohren and Huffman*, 1983, p. 140]

$$C_{\text{ext-qs}} = -k_o \operatorname{Im}\left\{\frac{\alpha(0)}{\varepsilon_0}\right\} \tag{9}$$

and

$$C_{\text{scat-qs}} = \frac{k_o^4}{6\pi}\left|\frac{\alpha(0)}{\varepsilon_0}\right|^2. \tag{10}$$

The coupled dipole method was first applied by *Purcell and Pennypacker* [1973] for scattering from nonspherical particles using the Clausius-Mossotti (CM) [*Clausius*, 1879; *Mossotti*, 1850] relations to evaluate the complex polarizability. There is a shortcoming, however, in using the CM relations in the coupled dipole method, as pointed out by *Dungey and Bohren* [1991] and also described below.

From (9) we can see that the extinction cross section $C_{\text{ext-qs}}$ for small spheres is directly proportional to the imaginary part of its polarizability. When the relative permittivity is real, the polarizability $\alpha(0)$ is also real. Since $C_{\text{ext-qs}}$ cannot be zero because the incident wave gets attenuated by scattering, $\alpha(0)$ must be complex. In the literature, *Draine* [1988], *Goedecke and O'Brien* [1988], and *Dungey and Bohren* [1991] have provided methods to satisfy this criterion. In this paper we have derived the equation for the polarizability, which is complex and satisfies the above criterion, as will be seen in section 5.

Here we would like to extend this definition of polarizability to include the extinction effects. This results in polarizability being a function of frequency, denoted here as $\alpha(\omega)$. To derive a relation between the scattering cross section and the polarizability of a sphere, (10) is used. Thus the magnitude of the normalized polarizability $|\alpha_n(\omega)|$ for lossless or lossy inclusions can be written in terms of the scattering cross section as

$$\left|\frac{\alpha(\omega)}{\varepsilon_0 V}\right| = \frac{\sqrt{6\pi C_{\text{scat-qs}}}}{k_o^2 V}. \tag{11}$$

The magnitude of the normalized polarizability makes sense only for lossy inclusions, since it is a complex number. The imaginary part of $\alpha_n(\omega)$ for lossy inclusions can be obtained from the extinction cross section and by using (9). The value of $\alpha_n(\omega)$ is also calculated by replacing $C_{\text{scat-qs}}$ with C_{scat}, obtained from the full Mie theory solution, in the above equation.

4.2. Radar Cross Section

Next we would like to relate the radar cross section to the polarizability $\alpha(\omega)$. For this we consider a plane wave varying in time and space to be incident on an infinitesimal dipole. When a plane wave $E_0\hat{u}_x$ ($\hat{u}_x$ is an x-directed unit vector, and $e^{j\omega t}$ time-dependence is assumed and omitted everywhere) is incident on a dipole located at $z = 0$, it oscillates with the frequency of the applied field; the dipole radiates or scatters an electric field as given by *Stratton* [1941]:

$$\mathbf{E}_s = \frac{e^{-jk_o r}}{jk_o r}\frac{(-jk_0^3)}{4\pi\varepsilon_0}\hat{u}_r \times (\hat{u}_r \times \mathbf{p}), \qquad (k_0 r \gg 1), \tag{12}$$

where $\mathbf{p} = \alpha(\omega)E_0\hat{u}_x$ is the dipole moment of an ideal dipole located at $z = 0$ and r is the distance to the point from the origin at which the scattered field is evaluated ($\hat{u}_r$ is the radial unit vector). The scattered electric field intensity after some manipulations can be written as

$$\mathbf{E}_s = \frac{e^{-jk_o r}}{jk_o r}\mathbf{X}E_0 \tag{13}$$

$$\mathbf{X} = \frac{-jk_o^3}{4\pi}\alpha(\omega)\hat{u}_r \times (\hat{u}_r \times \hat{u}_x). \tag{14}$$

$\mathbf{X}$ is called the vector scattering amplitude.

For a unit power incident on a scatterer we define the monostatic RCS in terms of the scattered amplitude in the backward direction as

$$\sigma_{\text{rcs}} = \frac{4\pi}{k_o^2}|\mathbf{X}|^2. \tag{15}$$

Using the above definition for the RCS, we can write the normalized polarizability $\alpha_n(\omega)$, as a function of the radar cross section, as

$$\left|\frac{\alpha(\omega)}{\varepsilon_0 V}\right| = \frac{\sqrt{4\pi\sigma_{\text{rcs}}}}{k_o^2 V}. \tag{16}$$

It can be seen that (16) gives a relation between the magnitude of $\alpha_n(\omega)$ and RCS for lossy inclusions. The plots for the variation of the normalized polarizability as a function of size parameter are shown in section 6 using the above definitions.

5. Corrections to the Maxwell Garnett Mixing Formula

A heterogeneous medium made up of spherical inclusions in a host medium can be treated as an effective homogeneous medium, with its constitutive parameters being the effective permittivity and the permeability. Assuming the medium to be nonmagnetic and its relative permeability to be the same as free space ($\mu_r = 1$), the effective permittivity of such an heterogeneous mixture can be calculated as a function of the parameters of the host and the inclusion materials and their fractional volume. The effective permittivity could be complex in general. The well-known Maxwell Garnett (MG) [*Garnett*, 1904] mixing rule

$$\varepsilon_{\text{eff}} = \varepsilon_0 + 3f\varepsilon_0 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0 - f(\varepsilon - \varepsilon_0)} \tag{17}$$

gives the effective permittivity ε_{eff} of a mixture where spherical inclusions of permittivity $\varepsilon = \varepsilon_r\varepsilon_0$ occupy a volume fraction f in the background medium with permittivity ε_0, here assumed vacuum. As can be seen from the simple appearance of the formula, the effective permittivity does not depend on the size of the scatterers or on the wavelength of the operating field. Indeed, it is approximatively valid for wavelengths that are much larger than the size of the spheres. Because of this low-frequency character of the mixing rule, it is often characterized as quasistatic.

Although there have been size-dependent corrections to the Maxwell Garnett mixing rule earlier [*Fikioris*, 1965; *Wang*, 1982], we have used the *Peltoniemi* [1996] approach for evaluating the polarizability of a finite sphere asymptotically. (The time-dependence behavior is given by $e^{j\omega t}$ notation.) Using the quasistatic assumption in calculating the dipole moment, the size-dependent polarizability denoted here as $\alpha(\omega)$ can be calculated as [*Sihvola and Sharma*, 1999]

$$\alpha(\omega) = 3V\varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \cdot \left(1 - 3\frac{\varepsilon_r - 1}{\varepsilon_r + 2}[G_1(x) + \varepsilon_r G_2(x)]\right)^{-1}, \tag{18}$$

where

$$G_1(x) = \tfrac{2}{3}[(1 + jx)e^{-jx} - 1], \tag{19}$$

$$G_2(x) = (1 + jx - \tfrac{7}{15}x^2 - j\tfrac{2}{15}x^3)e^{-jx} - 1, \tag{20}$$

and $x = k_o a$ is the size parameter of the scatterer. The Taylor series expansion of the polarizability gives

$$\alpha(\omega) \approx 3V\varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \cdot \left(1 + \frac{\varepsilon_r - 1}{\varepsilon_r + 2}\frac{\varepsilon_r + 10}{10}x^2 - j\frac{2}{3}\frac{\varepsilon_r - 1}{\varepsilon_r + 2}x^3 + \cdots\right). \tag{21}$$

The effective permittivity of a sparse mixture with $f \ll 1$ can be written as $\varepsilon_{\text{eff}} \approx \varepsilon_0 + n\alpha(\omega)$ and is thus given as

$$\varepsilon_{\text{eff}} \approx \varepsilon_0 + 3f\varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \cdot \left(1 + \frac{\varepsilon_r - 1}{\varepsilon_r + 2}\frac{\varepsilon_r + 10}{10}x^2 - j\frac{2}{3}\frac{\varepsilon_r - 1}{\varepsilon_r + 2}x^3 + \cdots\right). \tag{22}$$

The comparison of (7) with the imaginary part of the size-dependent Maxwell Garnett mixing rule (22) results in perfect agreement.

The numerical results presented here for the effective permittivity are obtained not by terminating the series for $\alpha(\omega)$ but by using the exact expression given in (18). The value of $\alpha(\omega)$, a function of $G_1(x)$ and $G_2(x)$, is evaluated by using the complete expressions of $G_1(x)$ and $G_2(x)$ as given in (19) and (20).

For a lossless scatterer the relation between the imaginary part of the effective permittivity of the mixture and the scattering cross section $C_{\text{scat-qs}}$ (since there is no attenuation due to absorption) can be found from (7) and written as

$$-\text{Im}\{\varepsilon_{\text{eff}}/\varepsilon_0\} = \frac{nC_{\text{scat-qs}}}{k_o}. \tag{23}$$

The above equation is also used by replacing the $C_{\text{scat-qs}}$ with C_{scat}, obtained by using the full Mie theory results. For a lossy scatterer, $C_{\text{ext-qs}}$ (and C_{ext}) has to be used in the above equation since it now takes into account the absorption due to the imaginary part of the dielectric constant of the inclusion.

Since for an arbitrarily shaped scatterer it is easier to evaluate the backscattered RCS, we can also relate the imaginary part of the effective relative permittivity ε_{eff} of the mixture to the radar cross section of the spherical scatterer using (16) and (7) to get

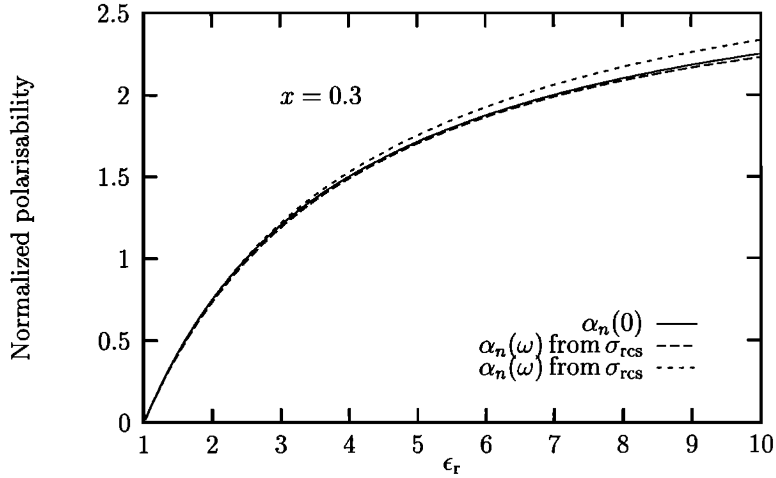

Figure 2. The variation of $\alpha_n(\omega)$ as a function of the relative permittivity of the inclusion.

$$-\mathrm{Im}\,\{\varepsilon_{\mathrm{eff}}/\varepsilon_0\} = \frac{2n\sigma_{\mathrm{rcs}}}{3k_o}. \tag{24}$$

This definition is correct only if the inclusions are lossless. For lossy inclusions, however, the attenuation of the plane wave as it passes through a slab of lossy particles cannot be related to RCS alone. The absorption phenomena also has to be taken into account. Thus evaluating the imaginary part of the effective permittivity of the mixture using the definition of RCS is not proper for lossy inclusions.

6. Results

The heterogenous mixture is analyzed when a plane wave is incident on it. Two types of inclusions are considered in order to study the behavior of the complex polarizability and the imaginary part of the effective permittivity of a sparse mixture. These are lossless inclusions and lossy inclusions.

The normalized polarizability values are computed using the definitions given in section 4 for different ε_r and loss tangents. Their deviation from the static value is also plotted as a function of the size parameter.

The generalized Maxwell Garnett mixing formula is used to calculate the effective permittivity of the mixture. The imaginary part of the complex effective permittivity is evaluated using the corrections made in the polarizability of the mixture. These results are then compared with the results obtained from scattering from a slab of particles using the Mie theory under the quasistatic approximation (equations (4) and (5)). The full Mie theory code given by *Sharma and Sihvola* [1998], by using which the scattering cross section and the radar cross section can be calculated, is also used to calculate the imaginary part of the effective permittivity of the mixture using (23) and (24).

The frequency has been fixed to 1 GHz, and the number density of the spheres is chosen to be 20 m^{-3}. The variation of size parameter results in variation of the radius of the dielectric inclusions. The variation in the fractional volume for a change in the size parameter x from 0 to 1 is 0–9.12 $\times$ 10^{-3}. The change in the fractional volume for a constant number density of 20 m^{-3} results in the variation of the radius of the dielectric inclusion from 0 to 4.78 $\times$ 10^{-2} m. These values satisfy the condition of a sparse mixture on which (22) is also based.

6.1. Lossless Dielectric Inclusions

For a lossless dielectric inclusion the value of the dielectric constant is real, and thus the extinct power is due to scattering alone, and hence we can replace C_{ext} in (2) by C_{scat}. The normalized polarizability was calculated for different ε_r and for different size parameters. Next the $-\mathrm{Im}\{\varepsilon_{\mathrm{eff}}/\varepsilon_0\}$ was calculated using the generalized MG formula. The definitions of C_{scat} and σ_{rcs} were used for calculating $\alpha_n(\omega)$ and $-\mathrm{Im}\{\varepsilon_{\mathrm{eff}}/\varepsilon_0\}$ using the complete Mie coefficients. A comparative study is made between the various methods used in finding the values of $\alpha_n(\omega)$ and $-\mathrm{Im}\{\varepsilon_{\mathrm{eff}}/\varepsilon_0\}$. First let us discuss the deviation of $\alpha_n(\omega)$ from (8) using different approaches.

6.1.1. Comparison of normalized static polarizability $\alpha_n(0)$ with $\alpha_n(\omega)$. The value of the normalized polarizability is calculated using the definitions

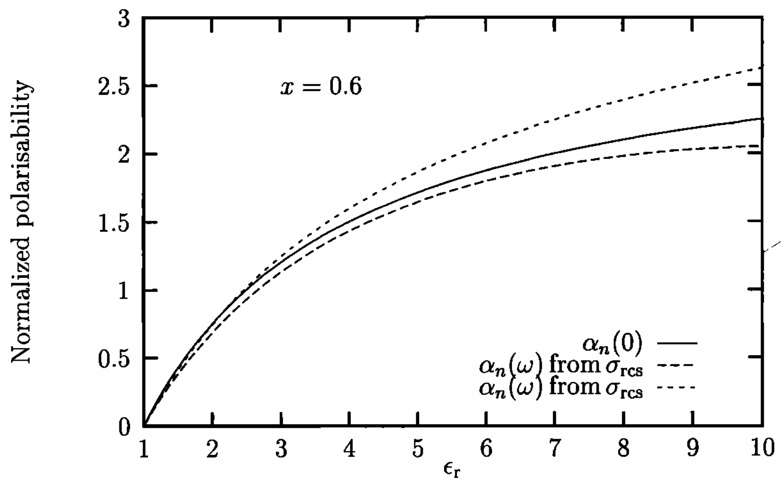

Figure 3. The variation of $\alpha_n(\omega)$ as a function of the relative permittivity of the inclusion.

of RCS and scattering cross section. Figures 2 and 3 show the behavior of $\alpha_n(\omega)$ with the variation of ε_r using the two definitions and for two different size parameters, namely, $x = 0.3$ and 0.6, respectively. The value of $\alpha_n(0)$ is also plotted in the same graphs for the static case. It is seen from the plots that the value of $\alpha_n(\omega)$ obtained using the definition of RCS can be successfully used to determine the static polarizability, as compared with that obtained from C_{scat}.

The normalized values of polarizability and the percentage deviation in its value from the static normalized polarizability

$$\left(\alpha_n(0) = 3\,\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)$$

calculated using different approaches are given in Table 2 for $\varepsilon_r = 1.3$. The percentage deviation in the values of normalized polarizability is higher when σ_{rcs} is used to calculate $\alpha(\omega)/\varepsilon_0 V$ as compared with that obtained using C_{scat}. It is also seen that the percentage deviation increases as the size parameter increases in both cases.

6.1.2. Comparison of the values of $-\text{Im}\{\varepsilon_{\text{eff}}/\varepsilon_0\}$ with full Mie theory calculations. The $-\text{Im}\ \{\varepsilon_{\text{eff}}/\varepsilon_0\}$ is calculated by using the generalized MG mixing formula (equation (22)), $C_{\text{scat-qs}}$, C_{scat} (equation (23)), and σ_{rcs} (equation (24)). These are then compared with the value of C_{scat} obtained from the full Mie theory, which is considered to be exact. The percentage errors in calculation of $-\text{Im}\ \{\varepsilon_{\text{eff}}/\varepsilon_0\}$ using different methods are given in Table 3 for a dielectric inclusion of $\varepsilon_r = 1.3$. The results show that

Table 2. Percentage Deviations in the Values of Normalized Polarizability $\alpha_n(\omega)$ From the Static Polarizability ($\alpha_n(0) = 0.27273$) Using Different Methods for a Lossless Inclusion of $\varepsilon_r = 1.3$

x	$\alpha_n(\omega)$ From C_{scat}	$\alpha_n(\omega)$ From σ_{rcs}	Percent Deviation in $\alpha_n(\omega)$ From C_{scat}	Percent Deviation in $\alpha_n(\omega)$ From σ_{rcs}
0.2	0.27133	0.26907	−0.51	−1.34
0.3	0.26958	0.26453	−1.15	−3.01
0.4	0.26711	0.25819	−2.06	−5.33
0.5	0.26389	0.25011	−3.24	−8.29
0.6	0.25992	0.24033	−4.70	−11.88
0.7	0.25517	0.22890	−6.44	−16.07

Table 3. Percentage Errors in the Values of $-\text{Im}\ \{\varepsilon_{\text{eff}}/\varepsilon_0\}$ Using Different Methods for Dielectric Inclusions of $\varepsilon_r = 1.3$

x	Percent Error From MG	Percent Error From $C_{\text{scat-qs}}$	Percent Error From σ_{rcs}
0.2	0.92	1.03	−1.66
0.3	2.07	2.35	−3.72
0.4	3.69	4.25	−6.56
0.5	5.77	6.81	−10.17
0.6	8.30	10.09	−14.51
0.7	11.29	14.23	−19.53

MG, Maxwell Garnett.

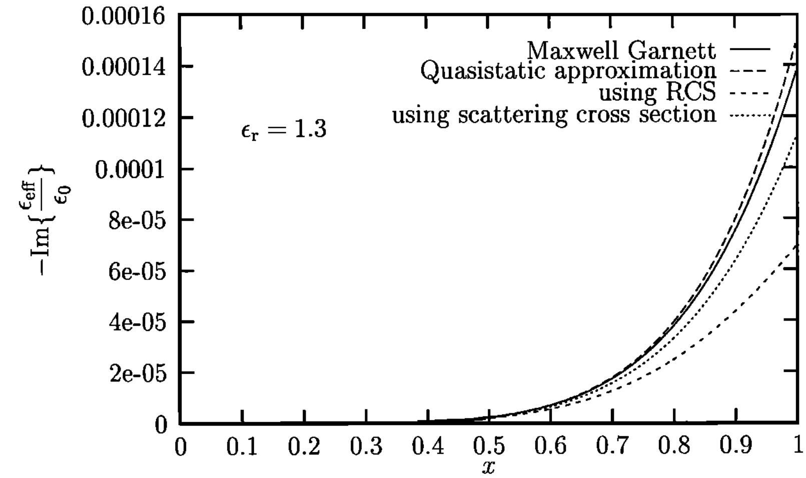

Figure 4. The variation of $-\mathrm{Im}\,\{\varepsilon_{\mathrm{eff}}/\varepsilon_0\}$ of a sparse mixture at a frequency of 1 GHz as a function of x.

the percentage error increases with the increase in the size parameter for all the methods adopted for calculations.

Figures 4–6 show the variation of $-\mathrm{Im}\,\{\varepsilon_{\mathrm{eff}}/\varepsilon_0\}$ as a function of the size parameter using different methods for dielectric inclusions of $\varepsilon_r = 1.3$, 2, and 10, respectively. Figure 6 has an expanded x range, with x varying from 0 to 0.5. It can be seen from these plots that the generalized Maxwell Garnett formula is a good approximation for calculating the imaginary part of the effective dielectric constant of the mixture only for small values of x. For larger size parameters, however, it becomes an overestimate as compared with the values obtained using the full Mie theory. This should indeed be the case, as it is valid only for quasistatic limits, and we are trying to see the extent up to which we can apply it. The quasistatic approximations are valid only when the condition $|\sqrt{\varepsilon_r}|x \ll 1$ is satisfied. Thus it can be seen that as we increase both ε_r and x the error in computing the imaginary part of the permittivity of the mixture also increases, but it is reasonably accurate for small x.

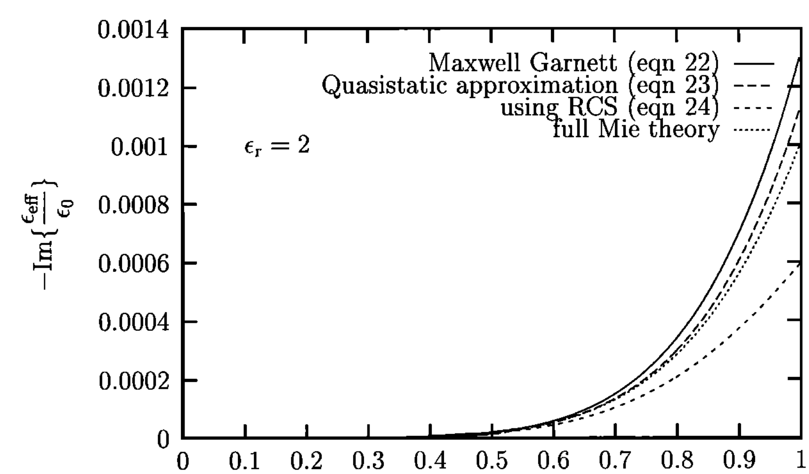

Figure 5. The variation of $-\mathrm{Im}\,\{\varepsilon_{\mathrm{eff}}/\varepsilon_0\}$ of a sparse mixture at a frequency of 1 GHz as a function of x.

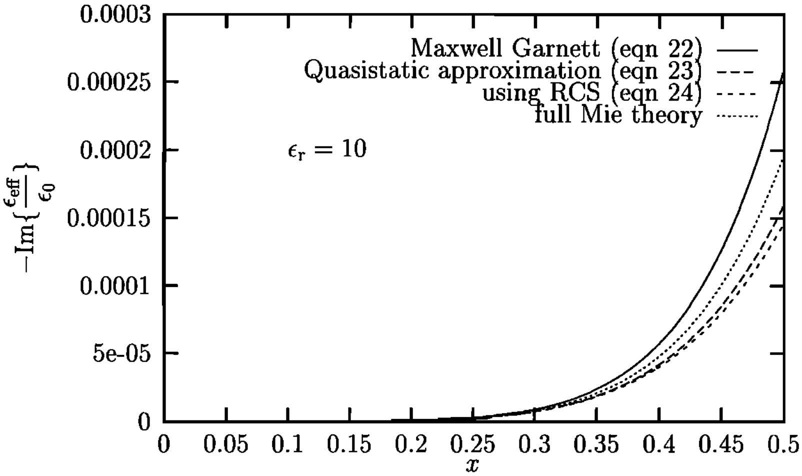

Figure 6. The variation of $-\mathrm{Im}\ \{\varepsilon_{\mathrm{eff}}/\varepsilon_0\}$ of a sparse mixture at a frequency of 1 GHz as a function of x, with x ranging from 0 to 0.5 only.

6.2. Lossy Dielectric Inclusions

In the case of lossy dielectric inclusions the extinct power is due to both scattering and absorption of the particles. Thus we use the definitions of C_{scat} and C_{ext} obtained from the complete Mie formulation to calculate the magnitude and the imaginary part of the normalized polarizability, respectively. The imaginary part of the effective permittivity of the mixture is also calculated by using the same definitions. As in the case of lossless inclusions, we will first compare the values of normalized polarizability obtained by different methods and then the values of $-\mathrm{Im}\ \{\varepsilon_{\mathrm{eff}}/\varepsilon_0\}$. The frequency has been fixed to 1 GHz and the number density n was fixed to 20 m^{-3}, as was done in the lossless case.

Table 4. Percentage Deviations in the Magnitude of Normalized Polarizability $|\alpha_n(\omega)|$ From the Magnitude of Static Normalized Polarizability ($\alpha_n(0) = 0.27523 - \mathrm{j}0.08257$ and $|\alpha_n(0)| = 0.28735$) Using Different Methods for a Lossy Inclusion of $\varepsilon_r = 1.3 - \mathrm{j}0.1$

x	$\lvert\alpha_n(\omega)\rvert$ From C_{scat}	$\lvert\alpha_n(\omega)\rvert$ From σ_{rcs}	Percent Deviation in $\lvert\alpha_n(\omega)\rvert$ From C_{scat}	Percent Deviation in $\lvert\alpha_n(\omega)\rvert$ From σ_{rcs}
0.2	0.285846	0.283464	−0.52	−1.35
0.3	0.283910	0.278583	−1.20	−3.05
0.4	0.281127	0.271735	−2.17	−5.43
0.5	0.277453	0.262932	−3.44	−8.50
0.6	0.272852	0.252208	−5.04	−12.2
0.7	0.267300	0.239618	−6.98	−16.61

6.2.1. Comparison of normalized static polarizability $\alpha_n(0)$ with $\alpha_n(\omega)$.

First, a lossy dielectric inclusion of $\varepsilon_r = 1.3 - \mathrm{j}0.1$ was considered. Since the normalized polarizability $\alpha_n(0)$ (see equation (8)) is complex for lossy inclusions, its magnitude was calculated and compared with the magnitude of normalized polarizability $\alpha_n(\omega)$ obtained using the scattering cross section and the radar cross section.

The values of the magnitudes of normalized polarizabilities ($|\alpha_n(\omega)|$) calculated using different approaches and their deviation from the static value are given in Table 4 for $\varepsilon_r = 1.3 - \mathrm{j}0.1$. It can be seen from Table 4 that the deviation has increased slightly for lossy inclusions for the same size parameter x and for the same real value of the relative permittivity, as compared to the lossless case, for both methods (using scattering cross section and the radar cross section).

The magnitude of $\alpha_n(\omega)$ is also plotted for different values of the real part of the relative permittivity, for two different size parameters, using the definitions of the radar cross section and the scattering cross section. The results are shown in Figures 7 and 8 for $x = 0.3$ and $x = 0.6$, respectively, for a $\tan\delta = 0.5$ (loss tangent $\tan\delta$ is defined as a ratio of ε_r'' to ε_r'). For a given $\tan\delta$ the variation in the real part of the relative permittivity results in the variation of its imaginary part also.

6.2.2. Comparison of the values of $-\mathrm{Im}\ \{\varepsilon_{\mathrm{eff}}/\varepsilon_0\}$ with full Mie theory calculations.

The value of $-\mathrm{Im}\ \{\varepsilon_{\mathrm{eff}}/\varepsilon_0\}$ was calculated using the generalized MG,

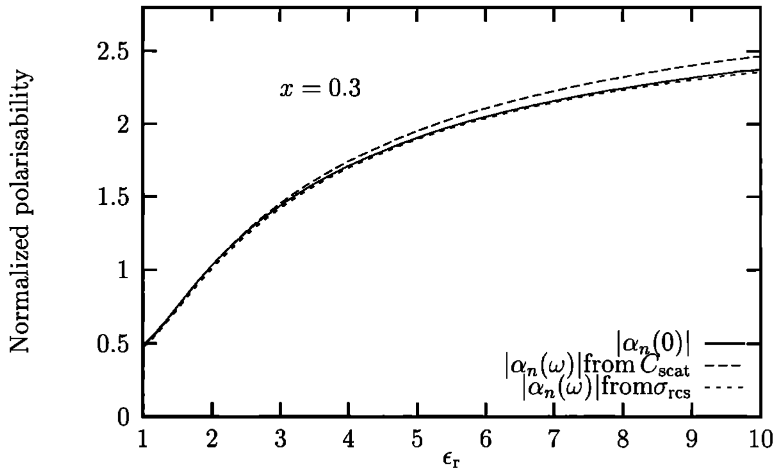

Figure 7. The variation of $|\alpha_n(\omega)|$ as a function of the real part of the relative permittivity of the inclusion ($\tan\delta = 0.5$).

using $C_{\text{ext-qs}}$, C_{ext}, and σ_{rcs} for a dielectric inclusion of relative permittivity of $\varepsilon_\mathrm{r} = 1.3 - \mathrm{j}0.1$. The errors in the values of $-\mathrm{Im}\,\{\varepsilon_{\text{eff}}/\varepsilon_0\}$ evaluated using different methods are shown in Table 5. As seen from the table, the percentage errors are very small for x ranging from 0 to 1 as compared with the lossless case. The error increases, however, with the increase with the size parameter, as is expected. The percentage error using σ_{rcs} was very high for small values of ε_r' and is not given in Table 5, although it has been plotted in the figures given.

The variation of $-\mathrm{Im}\,\{\varepsilon_{\text{eff}}/\varepsilon_0\}$ with the size parameter is shown in Figure 9 using different definitions to get the imaginary part of the effective permittivity of the mixture. The results of the generalized MG match very well with that obtained using the extinction cross section from Mie code, even for reasonably large values of the size parameters. The results computed using the radar cross section are quite inaccurate, however.

Next a slightly higher dielectric inclusion of $\varepsilon_\mathrm{r} = 2 - \mathrm{j}1.9$ was chosen as an inclusion. The dielectric inclusion was deliberately chosen to have a high loss tangent. The results are shown in Figure 10. The results of the generalized Maxwell Garnett formula and using (5) matched very well with that of using the

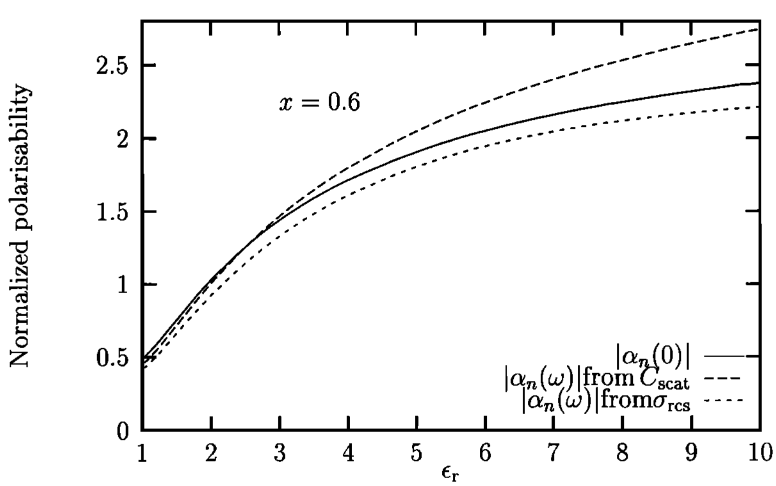

Figure 8. The variation of $|\alpha_n(\omega)|$ as a function of the real part of the relative permittivity of the inclusion ($\tan\delta = 0.5$).

Table 5. Percentage Errors in the Values of $-\text{Im}\{\varepsilon_{\text{eff}}/\varepsilon_0\}$ Using Different Methods for a Lossy Inclusion Having $\varepsilon_r = 1.3 - j0.1$

x	Percent Error From MG	Percent Error From Equation (4)
0.2	0.188	0.012
0.3	0.437	0.064
0.4	0.815	0.210
0.5	1.351	0.532
0.6	2.077	1.130
0.7	3.020	2.127

extinction cross section from the Mie theory up to size parameters of 0.6.

When the relative permittivity of the inclusions was chosen to be as high as $\varepsilon_r = 10 - j0.5$, it was seen that the generalized MG formula was quite an overestimate. The results are as shown in Figure 11 for an expanded x range (x varying from 0 to 0.5). The quasistatic approximation does not match at all beyond $x = 0.1$, since (5) is approximate only for low dielectric constant scatterers. The MG formula is accurate only up to $x = 0.2$, beyond which the error slowly increases.

7. Discussion and Conclusions

The two main aspects studied here are the following: First, the idea of complex polarizability is explained and related to various cross sections, namely, extinction, scattering, and radar cross sections. These definitions are used in a related aspect, which is the evaluation of the imaginary part of the effective permittivity of a sparse heterogeneous mixture. A generalized Maxwell Garnett formula (equation (22)), which is dependent on the electrical (optical) size of the spheres, in addition to the permittivity and volume fractions of the components in the mixture, is derived and used to calculate the effective properties of the heterogeneous mixture. The imaginary part of the complex permittivity obtained from this new generalized MG mixing formula is compared with the value obtained using the quasistatic Mie theory, radar cross section, and the full Mie scattering analysis. The range of applicability of these rules is then examined.

The results for both the polarizability and the imaginary part of the effective permittivity are given for two types of dielectric inclusions: lossless and lossy. For the lossless case the attenuation in a slab of n particles is described by scattering alone, but in the case of lossy inclusions the absorption, in addition to scattering, is taken into account to satisfy the optical extinction theorem. Thus the magnitude of the normalized polarizability is calculated using scattering and radar cross sections for both lossless and lossy inclusions, and its imaginary part can be obtained from the extinction cross section (using the full Mie theory code).

It was observed that for both lossless and lossy dielectric inclusions the deviation in calculating the magnitude of polarizability increases with the increase in the size parameter from its static value. The deviations in the values of magnitudes of normalized

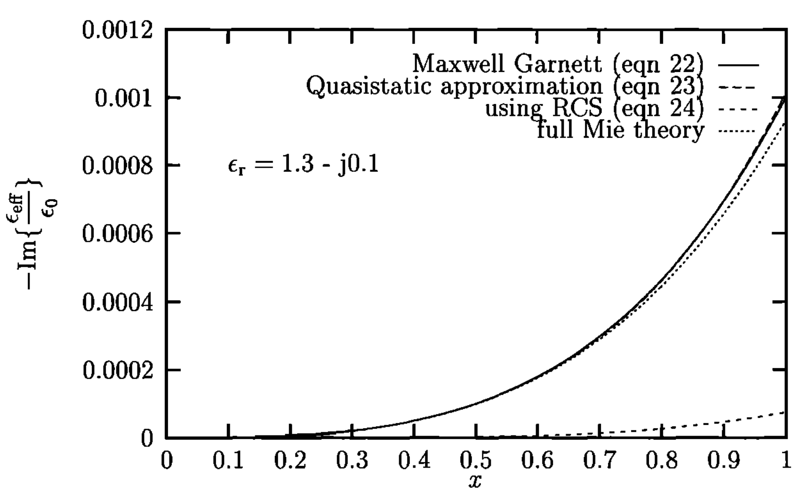

Figure 9. The variation of $-\text{Im}\{\varepsilon_{\text{eff}}/\varepsilon_0\}$ of a sparse mixture at a frequency of 1 GHz as a function of x.

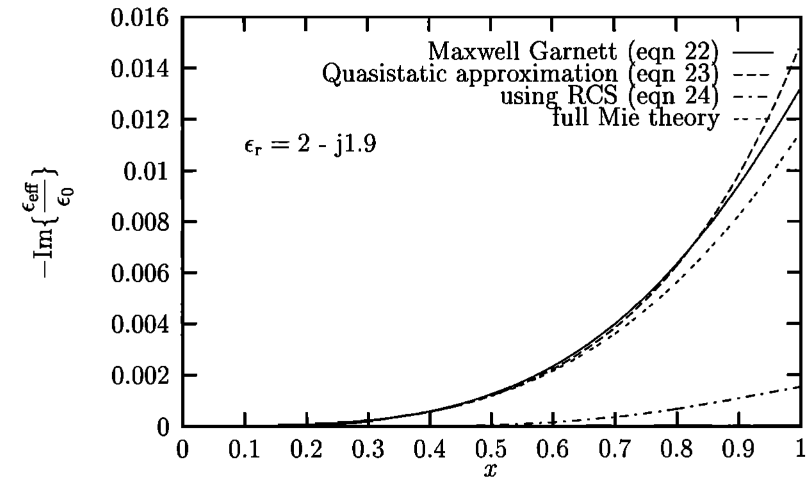

Figure 10. The variation of $-\mathrm{Im}\,\{\varepsilon_{eff}/\varepsilon_0\}$ of a sparse mixture at a frequency of 1 GHz as a function of x.

polarizability were slightly higher for lossy inclusions as compared with the lossless case. The definition of radar cross section gave smaller deviation in computing the value of $|\alpha_n(\omega)|$ from its static value when ε_r was high as compared with that obtained from the definition of C_{scat}. This is because the relation between the scattering cross section and polarizability becomes inaccurate much faster when the size parameter or the dielectric constant is increased, as the higher-order terms also start contributing.

The $-\mathrm{Im}\,\{\varepsilon_{eff}/\varepsilon_0\}$ is calculated using the generalized Mie theory and compared with the results obtained from the extinction cross section and radar cross section from the full Mie theory. For lossless inclusions the extinction cross section and the scattering cross sections are the same.

For lossless inclusions it was seen that the error in $-\mathrm{Im}\,\{\varepsilon_{eff}/\varepsilon_0\}$ increases with the increase in the size parameter and also as the dielectric constant was increased. For lossy inclusions, however, the generalized MG and the quasistatic approximation gave smaller errors in the values of $-\mathrm{Im}\,\{\varepsilon_{eff}/\varepsilon_0\}$ when compared with the full Mie theory calculations.

We may stop here to take a closer look at the first few Mie coefficients when expanded as a series and compare them with the coefficients obtained in the

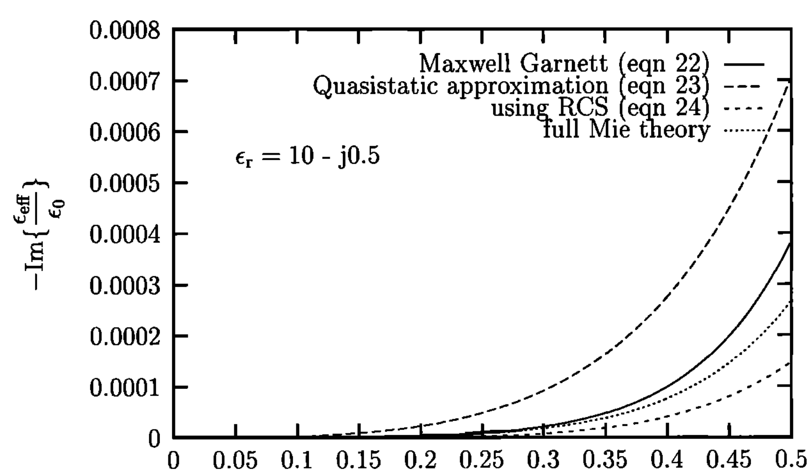

Figure 11. The variation of $-\mathrm{Im}\,\{\varepsilon_{eff}/\varepsilon_0\}$ of a sparse mixture at a frequency of 1 GHz as a function of x.

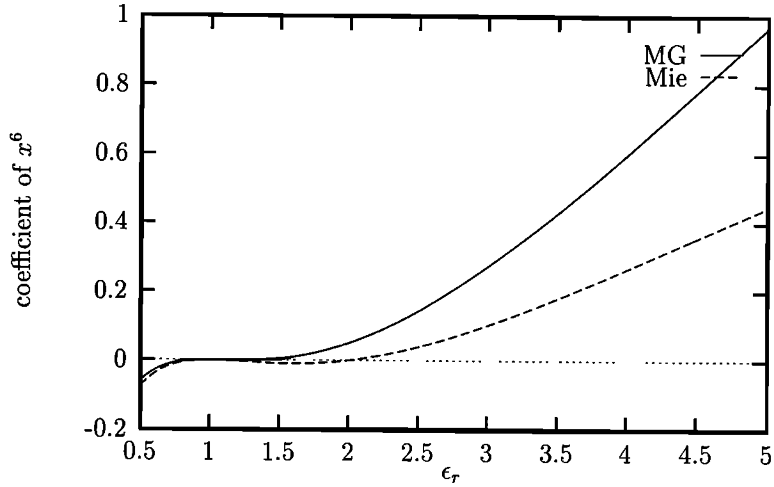

Figure 12. Variation of the x^6 term as a function of ε_r for Maxwell Garnett and Mie expansions.

generalized Maxwell Garnett mixing rule. At low frequencies, mainly the first few Mie coefficients are contributing to the scattered fields. In section 3 the Mie coefficients a_1, a_2, b_1, and b_2 were expanded using the Bessel function series up to the $O(x^5)$. Using these Mie coefficients, the scattering, extinction, and absorption efficiencies were derived up to $O(x^4)$ or the cross sections up to $O(x^6)$. It is worth looking at the behavior of the Mie scattering using the higher-order expansion of the Bessel functions in the Mie coefficients, especially around the value $\varepsilon_r = 2$. Below are given the order x^6 and x^8 contributions to the scattering cross section C_{scat}:

Order x^6

$$\frac{16}{5}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)^2 \frac{\varepsilon_r - 2}{\varepsilon_r + 2}$$

Order x^8

$$\frac{24}{25}\frac{(\varepsilon_r - 2)^2}{(\varepsilon_r + 2)^4} + \frac{8}{4725}[(2862\varepsilon_r^4 - 13813\varepsilon_r^3 - 36548\varepsilon_r^2 + 17880\varepsilon_r + 49272 + 7\varepsilon_r^6 + 185\varepsilon_r^5) /(\varepsilon_r + 2)^4(2\varepsilon_r + 3)^2].$$

Similar expansions that were obtained from the generalized MG formula (equation (22)) are

Order x^6

$$\frac{4}{15}\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)^2\left(\frac{\varepsilon_r^2 + 15\varepsilon_r - 22}{\varepsilon_r + 2}\right)$$

Order x^8

$$\frac{1}{525}\left(\frac{\varepsilon_r-1}{\varepsilon_r+2}\right)^2[(14\varepsilon_r^4 + 228\varepsilon_r^3 + 1197\varepsilon_r^2 - 7412\varepsilon_r + 6180) /(\varepsilon_r + 2)^2].$$

We would like to compare the first correction of the MG with the quasistatic term. The coefficients of the x^6 terms of the Mie and the MG are plotted in Figure 12 as a function of ε_r. It is seen that this term goes to zero at $\varepsilon_r = 2$ and at $\varepsilon_r = 1.34$ for Mie and MG, respectively. This term is negative when $\varepsilon_r < 1.34$ for MG and is negative for Mie when $\varepsilon_r < 2$. The contribution of this term is higher for MG as compared with Mie. These observations give support to the results in the previous section (section 6), where the MG prediction of the effective permittivity of the mixture was good, for small values of ε_r.

Although it is easy to compute the RCS of various objects, the definition of RCS when used to calculate the value of $-\text{Im}\,\{\varepsilon_{\text{eff}}/\varepsilon_0\}$ is highly inaccurate. This is because although the polarizability magnitude is related to the scattering cross section, the attenuation of the electromagnetic wave as it passes through a slab of lossy particles of spherical shapes cannot be explained by RCS alone. The absorption phenomena also has to be taken into account.

Thus it can be concluded that the polarizability of a spherical particle can be obtained from its RCS when it is small compared with the wavelength. In addition, it can be said that the generalized Maxwell Garnett formula is quite accurate for smaller values of $k_o a\sqrt{\varepsilon_r}$, and for larger values it is an overestimate for both lossless and lossy inclusions.

References

Abramowitz, M., and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, Mineola, N. Y., 1970.

Bohren, C. F., and D. R. Huffman, *Absorption and Scattering of Light by Small Particles*, John Wiley, New York, 1983.

Clausius, R. J. E., *Die Mechanische Behandlung der Electricität*, p. 94, F. Vieweg, Braunschweig, Germany, 1879.

Debye, P., Der Lichtdruck auf Kugeln von beliebigem Material, *Ann. Phys.*, *30*, 57–136, 1909.

de Hulst, H. C. V., *Light Scattering by Small Particles*, John Wiley, New York, 1957.

Draine, B. T., The discrete dipole approximation and its application to interstellar graphite grains, *Astrophys. J.*, *333*, 848–872, 1988.

Dungey, C. E., and C. F. Bohren, Light scattering by nonspherical particles: A refinement to the coupled dipole method, *J. Opt. Soc. Am.*, *8*(1), 81–87, 1991.

Fikioris, J. G., Electromagnetic field inside a current carrying region, *J. Math. Phys.*, *11*, 1617–1620, 1965.

Garnett, J. C. M., Colours in metal glasses and metal films, *Trans. R. Soc. London*, *CCIII*, 385–420, 1904.

Goedecke, G. H., and A. G. O'Brien, Scattering by irregular inhomogeneous particles via the digitized Green's function algorithm, *Appl. Opt.*, *27*(12), 2431–2438, 1988.

Jackson, J. D., *Classical Electrodynamics*, John Wiley, New York, 1975.

Kerker, M., *The Scattering of Light and Other Electromagnetic Radiation*, Academic, San Diego, Calif., 1969.

Mie, G., Beiträge zur Optik trüber Medien, speziell kolloidaler Metallösungen, *Ann. Phys.*, *25*, 377–445, 1908.

Mossotti, O. F., Discussione analitica sull'influenza che l'azione di un mezzo dielettrico ha sulla distribuzione dell'elettricità alla superficie di più corpi elettrici disseminati in esso, *Mem. Mat. Fis. Soc. Ital. Sci.*, *24*, 49–74, 1850.

Peltoniemi, J. I., Variational volume integral equation method for electromagnetic scattering by irregular grains, *J. Quant. Spectrosc. Radiat. Transfer*, *55*(5), 637–647, 1996.

Purcell, E. M., and C. R. Pennypacker, Scattering and absorption of light by nonspherical dielectric grains, *Astrophys. J.*, *186*, 705–714, 1973.

Sharma, R., and A. H. Sihvola, Mie scattering code for dielectric sphere, *Tech. Rep. 281*, Electromagn. Lab., Helsinki Univ. of Technol., Espoo, Finland, 1998.

Sihvola, A. H., and R. Sharma, Scattering corrections for the Maxwell Garnett mixing rule, *Microwave Opt. Technol. Lett.*, *22*(4), 229–231, 1999.

Stratton, J. A., *Electromagnetic Theory*, McGraw-Hill, New York, 1941.

Wang, J. J. H., A unified and consistent view on the singularity of the electric dyadic Green's function in the source region, *IEEE Trans. Antennas Propag.*, *30*(3), 463–468, 1982.

R. Sharma and A. Sihvola, Electromagnetics Laboratory, Helsinki University of Technology, P.O. Box 3000, 02015 HUT, Espoo, Finland. (ari.sihvola@hut.fi)

(Received May 18, 1999; revised September 7, 1999; accepted September 13, 1999.)