Analytical Modeling of a Six-Phase Bearingless Synchronous Reluctance Machine Using Winding Function Theory

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Abstract—This paper presents a modeling framework for bearingless synchronous reluctance machines that are equipped with a six-phase stator winding. The six-phase combined winding can be used to produce both electromagnetic torque and radial forces. Vector-space decomposition (or generalized space-vector transformation) and winding function theory are applied in the presented framework. To exemplify these concepts, a conceptual machine is studied by means of analytical expressions and by finite-element analysis. The presented framework can be used to develop dynamic models for other multiphase bearingless machines as well. Resulting dynamic models can be used in time-domain simulations and in development of real-time control algorithms.

Index Terms—Bearlingless, integrated force actuator, multiphase machines, radial forces, self-bearing, synchronous reluctance motors.

I. INTRODUCTION

Conventional bearingless machines use a separated suspension winding to produce radial forces [1]. A disadvantage of the conventional concept is that the force-producing winding occupies additional slot space, which compromises the torque production and the power density. Furthermore, the two winding sets require two different power converters. In recent years, a six-phase combined-winding configuration has become popular in bearingless machines [2]–[11]. It consists of two identical and symmetrically placed three-phase winding sets, which are fed from two identical power converters.

For regular multiphase machines, established modeling and analysis techniques are available, e.g., [12]–[17]. Vector-space decomposition is commonly used to transform the phase variables into orthogonal subspaces with a physical interpretation [12]. The components in the primary plane are related to the flux linkage and the torque, while the components in the secondary planes affect specific harmonics but do not contribute to the average torque. Vector-space decomposition is closely related to generalized space-vector transformation [13], [14]. Various extensions of vector-space decomposition have also been developed, e.g., [15], [16]. Furthermore, winding function theory can be used to determine the inductances in a systematical manner [17].

Interestingly, the secondary plane components resulting from vector-space decomposition can be used to generate radial force in symmetrical multiphase machines. However, the links between multiphase machines and bearingless machines are not very well known, despite their significant similarities in the hardware. One distinctly similar application is the electronic pole changing of multiphase machines in order to reach a wide speed range [18]. In the area of the machine design, the links between the multiphase and bearingless machines have recently been explored [8], [11], showing that the family of multiphase machines capable to produce controlled radial forces is wide. On the other hand, multiphase bearingless salient-pole machines seem not to have an established framework for dynamic models, although significant steps in that direction have recently been taken for nonsalient machines [9], [10].

In this paper, we aim to show that methodologies commonly used in the analysis and modeling of multiphase machines (such as vector-space decomposition and winding functions) can be effectively applied to multiphase combined-winding bearingless machines, including salient-pole machines. The presented framework can be used in the development of dynamic models for time-domain simulations and for development of real-time control algorithms. Using a conceptual machine as an example, the development of an analytical dynamic model is described and validated by means of finite-element analysis. The theoretical framework developed in this paper can be extended beyond this example machine.

II. PHASE-VECTOR MODEL

Fig. 1 shows a six-phase machine having two three-phase windings. The figure shows the conceptual winding, while the real windings typically are distributed in more slots, thus decreasing spatial harmonics of the magnetomotive force in the air gap.

The phase currents are collected to the column vector

\[ i_p = [i_1, i_2, \cdots, i_6]^T \]

where the indices match with Fig. 1 and transpose is marked with the superscript T. The phase-variable vectors for the voltage and flux linkage are defined similarly. Since the three-phase windings have separate neutral...
The six-phase stator winding consists of two sets of three phase windings. Dots and crosses mark the reference directions of the currents. The phase indices $k = 1 \ldots 6$ used in the equations are also marked.

Equation: $i_1 + i_3 + i_5 = 0 \quad i_2 + i_4 + i_6 = 0 \quad (1)$

The six-phase winding can produce two-pole and four-pole magnetomotive force in the air-gap.

The phase-variable model of the stator winding is

$$\frac{d\psi_p}{dt} = u_p - R_s i_p \quad (2)$$

where $R_s$ is the stator resistance. If the magnetic saturation is omitted, the phase flux linkage vector can be expressed as

$$\psi_p = L_p i_p \quad (3)$$

where the inductance matrix is

$$L_p = \begin{bmatrix} L_{11} & L_{12} & \cdots & L_{16} \\ L_{21} & L_{22} & \cdots & L_{26} \\ \vdots & \vdots & \ddots & \vdots \\ L_{61} & L_{62} & \cdots & L_{66} \end{bmatrix} \quad (4)$$

The phase inductances depend on the rotor angle and radial displacement of the rotor.

The magnetomotive force distribution in the air gap can be expressed as $F_k(\phi) = n_k(\phi)i_k$, produced by the $k$th phase.

The winding function for the $k$th phase can be presented as a Fourier series [17]

$$n_k(\phi) = \sum_{m=1}^{\infty} N_m \cos \left[ m\phi - \frac{m(k-1)}{3}\right] \quad (5)$$

where $N_m$ are its coefficients and $\phi$ is the circumferential angle along the air gap, as shown in Fig. 1(a). To enable radial force production, the winding functions should include the first and second harmonics, i.e., $m = 1, 2$. This condition is fulfilled for symmetrical six-phase machines. As an example, Fig. 2 shows the winding function $n_1$ for the geometry shown in Fig. 1(a). The Fourier series approximation of the winding function with the first two harmonics is also shown.

If the inverse air-gap function is also known, the phase inductances in (4) could be determined using winding function theory [17]. Alternatively, the winding functions can be determined in rotor coordinates, after applying vector-space decomposition to phase winding functions, as presented in Section IV-C.

III. VECTOR-SPACE DECOMPOSITION

The phase quantities can be transformed into the two-pole and four-pole vector spaces using vector-space decomposition [12]. The same result is obtained using the generalized space-vector transformation [9], [14]. For peak-valued complex space vectors, this transformation can be expressed as

$$i_{sh}^s = i_{\alpha h} + j i_{\beta h} = \frac{1}{3} \sum_{k=1}^{6} i_k e^{j(k-1)} \quad (6)$$

where $\alpha = e^{j\pi/3}$ is the complex factor, $k = 1 \ldots 6$ is the phase index, and the superscript $s$ marks that the vector is in stator coordinates. The indices $h = 1, 2$ correspond to vector spaces: $h = 1$ can be interpreted as the two-pole plane and $h = 2$ as the four-pole plane.
For salient-pole machines, the real column vectors can be more convenient choice than the complex space vectors. The transformation in (6) can be converted to matrix form, i.e.,

\[
\overline{i}^\text{sh} = \frac{1}{3} \begin{bmatrix} \text{Re}\{v_h\} \\ \text{Im}\{v_h\} \end{bmatrix} \overline{i}_p = T_h \overline{i}_p
\]  

(7a)

where the row vector is given by

\[
v_h = [1 \quad \alpha^h \quad \alpha^{2h} \quad \ldots \quad \alpha^{5h}]
\]  

(7b)

The resulting space vectors can be transformed to rotor coordinates,

\[
\overline{i}_h = e^{-h\theta_m} \overline{i}^\text{sh}
\]  

(8)

where \( \theta_m \) is the mechanical angle of the rotor, \( J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) is the skew-symmetric matrix, and space-vector quantities in rotor coordinates are marked without superscript.\(^1\) In addition to two space vectors, two zero-sequence components \( i_{s0} \) and \( i_{s3} \) can be defined,

\[
i_{s0} = \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \end{bmatrix} \overline{i}_p = t_0 \overline{i}_p
\]

(9a)

\[
i_{s3} = \frac{1}{6} \begin{bmatrix} 1 & -1 & 1 & \ldots & -1 \end{bmatrix} \overline{i}_p = t_3 \overline{i}_p
\]

(9b)

If the star points of three-phase windings are not connected, \( i_{s0} = i_{s3} = 0 \) holds. In this case, the zero-sequence components are not of interest. The combination of the space-vector transformation (7) and the coordinate transformation (8) is

\[
\overline{i}_s = \begin{bmatrix} i_{s1} \\ i_{s2} \end{bmatrix} = \begin{bmatrix} e^{-\theta_m J T_1} \\ e^{-2\theta_m J T_2} \end{bmatrix} \overline{i}_p = T_{sp}(\theta_m) \overline{i}_p
\]

(10)

where \( T_{sp} \) is of dimension 4 × 6. The inverse transformation from the space vectors in rotor coordinates to the phase quantities is

\[
\overline{i}_p = 3T_{sp}^T(\theta_m) \overline{i}_s = T_{ps}(\theta_m) \overline{i}_s
\]

(11)

where \( T_{ps} \) is of dimension 6 × 4. If needed, setting \( \theta_m = 0 \) or omitting the coordinate transformations yields the generalized space-vector transformation such that the resulting space vectors are in stator coordinates.

Even though not needed in the following, the zero-sequence components can be included in the transformations,

\[
\begin{bmatrix} i_{s0} \\ i_{s1} \\ i_{s2} \\ i_{s3} \end{bmatrix} = \begin{bmatrix} t_0 \\ e^{-\theta_m J T_1} \\ e^{-2\theta_m J T_2} \\ t_3 \end{bmatrix} \overline{i}_p
\]

(12)

As mentioned, this transformation can be seen to result either from vector-space decomposition or generalized space-vector transformation.

\(^1\)It is worth noticing that the matrix exponential can be written as \( e^{\alpha J} = \cos \alpha I + \sin \alpha J \), where \( I = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \).

IV. SPACE-VECTOR MODEL

A. State Equations

Using (10), the phase-variable model (2) can be converted into two space-vector state equations in rotor coordinates,

\[
\frac{d\psi_h}{dt} = u_{sh} - R_h \psi_h - h\omega_m J \psi_{sh} \quad h = 1, 2
\]

(13)

where \( \omega_m = \frac{d\theta_m}{dt} \) is the mechanical angular speed of the rotor. Using the presented transformations, the zero-sequence equations could be easily obtained, but they are not needed since the zero-sequence currents are zero.

If the magnetic saturation is omitted, the flux linkage space vectors can be presented using the inductance matrices as

\[
\psi_s = \begin{bmatrix} \psi_{s1} \\ \psi_{s2} \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} i_{s1} \\ i_{s2} \end{bmatrix} = \mathbf{L}_s \overline{i}_s
\]

(14)

where \( \mathbf{L}_s = \mathbf{L}_s(\theta_m, x, y) \). The inductance matrix can be obtained from the phase inductance as

\[
\mathbf{L}_s = T_{sp}(\theta_m) \mathbf{L}_{ps}(\theta_m)
\]

(15)

If spatial harmonics are omitted and non-eccentric rotor is assumed, the inductance matrix \( \mathbf{L}_s \) is constant in rotor coordinates. Instead of applying (15), the elements of \( \mathbf{L}_s \) can be computed directly in rotor coordinates by means of winding functions, as presented subsequently.

Typically, it is preferable to use the flux linkages as state variables according to (13), in order to avoid derivatives of the inductance matrix. In this case, the stator currents can be presented as function of the flux linkages, \( \overline{i}_s = \mathbf{L}_s^{-1} \overline{\psi}_s \), which completes the state equations in (13).

B. Electromagnetic Torque and Radial Forces

Omitting the magnetic saturation, the magnetic field energy can be expressed as

\[
W = \frac{3}{2} \overline{\psi}_s^T \mathbf{L}_s^{-1}(\theta_m, x, y) \overline{\psi}_s
\]

(16)

The electrical power fed to the magnetic field is obtained from (13). The rate of change of the field energy is the difference of the electrical power input and the mechanical power output,

\[
\frac{dW}{dt} = 3\overline{i}_{s1}^T \left( \frac{d\psi_{s1}}{dt} + \omega_m J \psi_{s1} \right)
\]

\[
+ 3\overline{i}_{s2}^T \left( \frac{d\psi_{s2}}{dt} + 2\omega_m J \psi_{s2} \right) - \tau_m \frac{d\theta_m}{dt} - F_x \frac{dx}{dt} - F_y \frac{dy}{dt}
\]

(17)

where \( \tau_m \) is the electromagnetic torque and \( F_x \) and \( F_y \) are the radial forces in rotor coordinates. Consequently, the electromagnetic torque is

\[
\tau_m = 3 \left( \overline{i}_{s1}^T J \psi_{s1} + 2\overline{i}_{s2}^T J \psi_{s2} \right) - \frac{\partial W}{\partial \theta_m}
\]

(18)
The last term is zero if spatial harmonics are omitted, i.e., if the inductance $L_s$ does not depend on the rotor angle. The force expressions are

\[
F_x = -\frac{\partial W}{\partial x} = \frac{3}{2}\psi_s^T \frac{\partial L_s^{-1}}{\partial x} \psi_s \quad (19a)
\]

\[
F_y = -\frac{\partial W}{\partial y} = \frac{3}{2}\psi_s^T \frac{\partial L_s^{-1}}{\partial y} \psi_s \quad (19b)
\]

Alternatively, the field coenergy can be considered,

\[
W_c = \frac{3}{2} i_s^T L_s (\vartheta_m, x, y) i_s \quad (20)
\]

based on which the force expressions can be written as

\[
F_x = \frac{\partial W_c}{\partial x} = \frac{3}{2} i_s^T \frac{\partial L_s}{\partial x} i_s \quad (21a)
\]

\[
F_y = \frac{\partial W_c}{\partial y} = \frac{3}{2} i_s^T \frac{\partial L_s}{\partial y} i_s \quad (21b)
\]

Both (19) and (21) result in the same force expressions.

As mentioned, the forces $F_x$ and $F_y$ are in rotor coordinates. They can transformed to stationary coordinates as

\[
\begin{bmatrix}
F_\alpha \\
F_\beta
\end{bmatrix} = e^{\vartheta_m J} \begin{bmatrix}
F_x \\
F_y
\end{bmatrix} \quad (22)
\]

where the subscripts $\alpha$ and $\beta$ refer to the coordinate axes.

C. Inductances Based on Winding Functions

The inductances can be computed by means of the winding function theory [17]. To simplify the following equations, it is convenient to introduce a new angular coordinate $\varphi_m = \phi - \vartheta_m$, cf. Fig. 1(a). The phase winding functions (5) can be collected to a vector $n_p = [n_1, n_2, \cdots, n_6]^T$ and transformed to rotor coordinates as

\[
n_s(\varphi_m, \vartheta_m) = T_{sp}(\vartheta_m) n_p(\varphi_m + \vartheta_m) \quad (23)
\]

where the elements of the transformed vector are labeled according to $n_s = [n_{1d}, n_{1q}, n_{2d}, n_{2q}]^T$. Using this labeling convention, the elements of the inductance matrix $L_{12}$ appearing in (14) are

\[
L_{12} = \begin{bmatrix}
L_{1d2d} & L_{1d2q} \\
L_{1q2d} & L_{1q2q}
\end{bmatrix} \quad (24)
\]

The same convention is used for other inductance matrices.

The air-gap distribution along the air gap also affects the phase inductances. Radial displacements of the rotor are first omitted. In salient-pole machines, the air-gap length is a function of $\varphi_m$. The inverse of the air-gap distribution is needed in order to determine the inductances. For even number of symmetrically shaped rotor poles, the inverse air-gap function consists of a constant term plus even harmonics [17]

\[
\frac{1}{g_{sal}(\varphi_m)} = \sum_{n=0}^{\infty} \frac{\cos(2n\varphi_m)}{g_{2n}} \quad (25)
\]

As an example, Fig. 3(a) shows a piecewise constant inverse air-gap distribution and its Fourier series approximation ($n = 0, 1, 2, 3$). The parameters are selected to approximately match with the example machine.

Fig. 3. Inverse air-gap function when the rotor is positioned at: (a) $x = y = 0$; (b) $x = 0.05g_a$ and $y = 0.1g_a$. The rotor pole span of $\pi/2$ and $g_0 = 5g_a$ are assumed. In (a), the dashed line shows a conceptual piecewise constant function. In (b), the eccentricity of the rotor is taken into account using (26) in the case of the dashed line. In (a) and (b), the solid lines show the approximation (31), which roughly matches with the geometry shown in Fig. 1(a).

To analyze the effects of an eccentric rotor, a cylindrical rotor surface is first assumed. Consider the rotor to be displaced along the positive $x$ and $y$ directions from its centered position, cf. Fig. 1(a). The air-gap distribution is

\[
g_{ecc}(\varphi_m, x, y) = 1 - \frac{x \cos(\varphi_m)}{g_a} - \frac{y \sin(\varphi_m)}{g_a} \quad (26)
\]

which is scaled by the nominal air gap $g_a$ for convenience. In the case of small displacements, the inverse of (26) can be approximated by means of the Taylor series expansion [19]

\[
g_a = \frac{1}{g_{ecc}(\varphi_m, x, y)} = 1 + \frac{x \cos(\varphi_m)}{g_a} + \frac{y \sin(\varphi_m)}{g_a} \quad (27)
\]

For salient-pole machines, (25) and (27) can be combined, resulting in an approximation

\[
\frac{1}{g(\varphi_m, x, y)} = g_a \frac{1}{g_{ecc}(\varphi_m, x, y)} g_{sal}(\varphi_m) \quad (28)
\]

in which case $g_a$ appearing in (27) can be interpreted as a scaling factor or an effective nominal air gap.
The inductances in (14) can now be expressed in terms of their winding functions and the inverse air-gap function. As an example, the inductance \( L_{1d2q}(\vartheta_m, x, y) \) is obtained from

\[
L_{1d2q} = \mu_0 r \ell \int_0^{2\pi} \frac{n_{1d}(\vartheta_m, \vartheta_m) n_{2q}(\vartheta_m, \vartheta_m)}{g(\vartheta_m, x, y)} d\vartheta_m
\]

where \( r \) is outer radius of the rotor, \( \ell \) is the effective length of the rotor, and \( \mu_0 \) is the air-gap permeability. The other inductances are obtained similarly. It is worth noticing that (29) assumes small radial displacements. An extended winding function theory [20], [21] might improve the accuracy of the inductances in the case of larger displacements.

V. ANALYTICAL EXAMPLE

The two-pole example machine shown in Fig. 1(a) is analyzed by means of simple analytical functions. The two-pole field (\( h = 1 \)) is used to magnetize the machine and to produce the electromagnetic torque. The four-pole field (\( h = 2 \)) allows to produce the radial force.

The winding function (5) with the first two terms is used,

\[
n_k(\varphi) = N_1 \cos \left[ \varphi - \frac{(k - 1)\pi}{3} \right] + N_2 \cos \left[ 2\varphi - \frac{2(k - 1)\pi}{3} \right]
\]

(30)

Fig. 2 shows this approximation for phase 1.

The inverse air-gap function (28) is defined by taking the components up to the third term. The rotor pole span of \( \pi/2 \) is assumed corresponding to Fig. 3. Under these assumptions, the inverse air-gap function becomes

\[
\frac{1}{g(\vartheta_m, x, y)} = \left[ 1 + \frac{x \cos(\vartheta_m)}{g_a} + \frac{y \sin(\vartheta_m)}{g_b} \right]: \left[ a - b \cos(2\varphi_m) + \frac{b}{3} \cos(6\varphi_m) \right]
\]

(31a)

The factors are given by [22]

\[
a = \frac{1}{2} \left( \frac{1}{g_a} + \frac{1}{g_b} \right), \quad b = \frac{2}{\pi} \left( \frac{1}{g_b} - \frac{1}{g_a} \right)
\]

(31b)

where \( g_a \) and \( g_b \) are the minimum and maximum air-gap lengths, respectively. Fig. 3 illustrates the function (31).

Applying (29), the following inductance matrices are obtained

\[
L_{1d} = \begin{bmatrix} L_{1d} & 0 \\ 0 & L_{1q} \end{bmatrix}, \quad L_{22} = L_2 I
\]

(32a)

where the self-inductances are

\[
L_{1d} = \frac{3N_1^2 \mu_0 r \ell}{2g_a g_b} \left[ \pi(g_a + g_b) + 2(g_a - g_b) \right]
\]

(32b)

\[
L_{1q} = \frac{3N_1^2 \mu_0 r \ell}{2g_a g_b} \left[ \pi(g_a + g_b) - 2(g_a - g_b) \right]
\]

(32c)

\[
L_2 = \frac{3N_2^2 \mu_0 r \ell}{2g_a g_b} (g_a + g_b)
\]

(32d)

Furthermore, the mutual coupling between the vector spaces is governed by the following inductance matrices

\[
L_{12} = \begin{bmatrix} m_d x & -m_d y \\ m_q y & m_q x \end{bmatrix}, \quad L_{21} = L_{12}^T
\]

(32e)

where the radial-force constants are

\[
m_d = 3N_1 N_2 \mu_0 r \ell \left[ g_b (\pi - 4) + g_a (\pi + 4) \right]
\]

(32f)

\[
m_q = 3N_1 N_2 \mu_0 r \ell \left[ g_b (\pi + 4) + g_a (\pi - 4) \right]
\]

(32g)

The torque expression given in (18) reduces to

\[
\tau_m = 3 \left( i_{s1}^T J_{\psi s1} + 2i_{s2}^T J_{\psi s2} \right)
\]

(33)

where the effect of the last term is insignificant due to the properties of \( L_{22} \) and \( L_{12} \). Using (21), the radial force expressions become

\[
F_x = 3i_{s1}^T \frac{\partial L_{12}}{\partial x} i_{s2} = 3(m_d i_{1d} i_{2d} + m_q i_{1q} i_{2q})
\]

(34a)

\[
F_y = 3i_{s1}^T \frac{\partial L_{12}}{\partial y} i_{s2} = 3(m_q i_{1q} i_{2d} - m_d i_{1d} i_{2q})
\]

(34b)

The radial forces depend on both two-pole and four-pole fields. As expected, the results of this analytical example match with the model for the separated winding in [1] and the model for the combined winding in [6].

VI. RESULTS OF FINITE-ELEMENT ANALYSIS

The model is validated by means of finite-element analysis. Fig. 4 shows the studied two-pole six-phase synchronous reluctance machine, and Table I gives its parameters.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values (mm)</th>
</tr>
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<tr>
<td>Stator outer diameter</td>
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<tr>
<td>Rotor outer diameter</td>
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</tr>
<tr>
<td>Air-gap length</td>
<td>1.0</td>
</tr>
<tr>
<td>Stack length</td>
<td>135.0</td>
</tr>
</tbody>
</table>

Table I: Machine Parameters Used in the Finite-Element Model
The machine has a double-layer winding distributed into 30 slots. The two-dimensional finite-element model was implemented using the Comsol software. The outer layer of the stator slots consists of the conductors which go into the slots and return from the inner layer of the slots. The combination of the coils for each phase is marked by the colour codes in Fig. 4. The phase windings are placed at $\pi/3$ angles with respect to each other.

Fig. 5 shows the flux density distribution with a two-pole excitation while the four-pole field is kept at zero, i.e., $i_{s2} = 0$. Similarly, Fig. 6 shows the four-pole excitation while $i_{s1} = 0$. Fig. 7 shows a combination of both two-pole and four-pole excitations.
excitations.

The analytical force expressions are compared with the results from the finite-element model. The inductance matrix is calculated, and $F_x$ and $F_y$ are computed using (34). Fig. 8 shows a comparison of the radial forces as a function of the rotor angle. It can be observed that the results match well.

VII. CONCLUSIONS

A model of a bearingless multiphase synchronous reluctance machine was developed in a systematical manner, based on the winding function theory and vector-space decomposition. The presented modeling framework can be used to develop dynamic models for time-domain simulations and for development of real-time control algorithms. Even though not considered in this paper, the effect of permanent magnets and different pole numbers could be straightforwardly taken into account. Furthermore, the presented framework can be a feasible starting point for future development of more advanced dynamic models, in which assumptions related to the eccentricity, spatial harmonics, and magnetic saturation are to be lifted.

REFERENCES