Niiranen, Jarkko; Khakalo, Sergei; Balobanov, Viacheslav

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Published in:
GACM Colloquium on Computational Mechanics for Young Scientists from Academia and Industry

Published: 11/10/2017

Proceedings of the 7th GACM Colloquium on Computational Mechanics
for Young Scientists from Academia and Industry
October 11-13, 2017 in Stuttgart, Germany

Strain gradient elasticity theories in lattice structure modelling

Jarkko Niiranen[1]*, Sergei Khakalo[1] and Viacheslav Balobanov[1]

Micro Abstract
The first and second strain gradient elasticity theories, resulting in higher-order governing equations, are studied in the framework of continualization, or homogenization, of lattice structures such as trusses in plane and space, with auxetic metamaterials as a special application. In particular, the role of length scale parameters and classical dimensions, such as the beam thickness, is addressed by parameter studies. Finite element and isogeometric methods are utilized for discretizations.

[1]Department of Civil Engineering, Aalto University, Espoo, Finland
***Corresponding author**: jarkko.niiranen@aalto.fi

Introduction

Generalized continuum theories have been developed in order to widen the range of applicability or to increase the accuracy of the theories of the classical Cauchy continua. Surprisingly, the history of generalized continuum theories dates back to the birth of the classical theories. In short, generalized (multi-scale) continuum theories aim at bringing in some length scale information lacking from the classical (single-scale) continuum theories. New information is brought into the continuum models by enriching the classical strain energy expressions essentially by either new independent variables of local nature (e.g. micro-rotations in micro-polar theories) or gradients of the classical variables of global nature (e.g. strain gradients in strain gradient theories). In both types of approaches, length scale parameters – related to the new variables or the gradients, respectively – are introduced, basically implying new constitutive relationships.

This contribution focuses on applying the first and second strain gradient elasticity theories for modeling the size-dependent mechanical response of lattices structures: bending, buckling and vibrations of triangular lattices are analyzed by beam and plane stress/strain models of strain gradient elasticity.

1 First Strain Gradient Elasticity Theory for Modeling Planar Trusses

This section first recalls the strain energy of the first strain gradient elasticity theory and its application to engineering beams via the classical dimension reduction assumptions of Euler and Bernoulli. Then the strain gradient beam model is applied for modeling planar trusses having a uniform triangular microarchitecture.

1.1 Strain Energy in the First Strain Gradient Elasticity Theory

Let us consider Mindlin's strain gradient elasticity theory of form II [5] with the strain energy density written in the form

$$\begin{aligned}\mathcal{W}_{II} = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + g_1\gamma_{iik}\gamma_{kjj} + g_2\gamma_{ijj}\gamma_{ikk} \\ + g_3\gamma_{iik}\gamma_{jjk} + g_4\gamma_{ijk}\gamma_{ijk} + g_5\gamma_{ijk}\gamma_{kji},\end{aligned} \tag{1}$$

where the (third-order) micro-deformation tensor is defined as the strain gradient

$$\gamma = \nabla \varepsilon, \tag{2}$$

with operator ∇ denoting the third-order tensor-valued gradient and with the classical linear strain tensor written as

$$\varepsilon = \varepsilon(\boldsymbol{u}) = \frac{1}{2}\big(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\big), \tag{3}$$

where the nabla operator now denotes the second-order tensor-valued gradient. The (third-order) double stress tensor is defined by a set of five additional material parameters $g_1, ..., g_5$ as $\tau_{ijk} = \partial \mathcal{W}_{II}/\partial \gamma_{ijk} = \tau_{jik}$ with indices i, j, k getting values x, y, z within a Cartesian coordinate system. The constitutive relation between the strain tensor and the classical Cauchy stress tensor $\boldsymbol{\sigma}$ follows the generalized Hooke's law as in the classical elasticity theory:

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}\, \boldsymbol{\varepsilon} \boldsymbol{I}, \tag{4}$$

with the Lamé material parameters μ and λ. The displacement field of body $\mathcal{B}$ is denoted by $\boldsymbol{u} : \mathcal{B} \to \mathbb{R}^3$.

Finally, the virtual internal work is written in the form [5]

$$\delta W_{\mathrm{int}} = \int_{\mathcal{B}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta \boldsymbol{u})\, \mathrm{d}\mathcal{B} + \int_{\mathcal{B}} \boldsymbol{\tau} \,\vdots\, \boldsymbol{\gamma}(\delta \boldsymbol{u})\, \mathrm{d}\mathcal{B}, \tag{5}$$

where $:$ and $\vdots$ denote the scalar products for second- and third-order tensors, respectively.

A one-parameter simplified strain gradient elasticity theory proposed originally by Altan and Aifantis [1] reduces the strain energy density (1) to the form

$$\mathcal{W}_{II} = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + g^2\big(\frac{1}{2}\lambda\varepsilon_{ii,k}\varepsilon_{jj,k} + \mu\varepsilon_{ij,k}\varepsilon_{ij,k}\big), \tag{6}$$

where the non-classical material parameter g describes the length scale of the micro-structure of the material. Double stresses

$$\tau_{ijk} = \frac{\partial \mathcal{W}_{II}}{\partial \varepsilon_{ij,k}} = g^2(\lambda\varepsilon_{ll,k}\delta_{ij} + 2\mu\varepsilon_{ij,k}) = \tau_{jik} \tag{7}$$

are related to the partial derivatives of the strain components by the Lamé parameters and the gradient parameter g. For constant Lamé parameters, the double stress tensor takes the form $\boldsymbol{\tau} = g^2\nabla\boldsymbol{\sigma}$, and the virtual work expression can be written as

$$\delta W^g_{\mathrm{int}} = \int_{\mathcal{B}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta \boldsymbol{u})\, \mathrm{d}\mathcal{B} + \int_{\mathcal{B}} g^2 \nabla\boldsymbol{\sigma} \,\vdots\, \nabla\boldsymbol{\varepsilon}(\delta \boldsymbol{u})\, \mathrm{d}\mathcal{B}. \tag{8}$$

1.2 Euler–Bernoulli Beams within the First Strain Gradient Elasticity Theory

Let us consider a three-dimensional beam structure occupying the domain

$$\mathcal{B} = A \times \Omega, \tag{9}$$

where $\Omega = (0, L)$ denotes the central axis of the structure with L standing for the length of the structure. The x-axis of a Cartesian coordinate system is assumed to follow the central axis of the beam. $A \subset \mathbb{R}^2$ denotes the cross section of the beam, which requires that $\mathrm{diam}(A) \ll L$.

Let us assume that the material properties and the cross section of the beam as well as surface and body loads, and both static and kinematic boundary conditions on the end point cross sections, are of such a form that one can focus on *uni-axial bending* in the xz-plane governed by displacement field $\boldsymbol{u} = (u_x, u_z)$. The dimension reduction hypotheses of Euler and Bernoulli then imply the displacement components of the form

$$u_x = -z\frac{\partial w(x)}{\partial x}, \quad u_z = w(x), \tag{10}$$

leaving the transverse deflection $w : \Omega \to \mathbb{R}$ as the only independent unknown of the problem.

Inserting the kinematical descriptions in the virtual work expression (8) and defining the force resultants, the classical Cauchy type bending moment and a generalized moment, respectively, as

$$M(x) = \int_A \sigma_x(x, y, z)z \,\mathrm{d}A, \quad R(x) = \int_A \frac{\partial \sigma_x(x, y, z)}{\partial z} \,\mathrm{d}A, \tag{11}$$

results in an energy expression over the central axis of the beam and finally gives, with the principle of virtual work, the governing equation of the problem in terms of bending moments in the form [7],

$$(M + g^2R - (g^2M')')'' = f \quad \text{in } \Omega. \tag{12}$$

In terms of deflection, the governing equation reads as

$$\big((EI + g^2EA)w'' - (g^2EIw''')'\big)'' = f \quad \text{in } \Omega. \tag{13}$$

With constant material parameters, this equation still reduces to the form

$$(EI + g^2EA)w^{(4)} - g^2EIw^{(6)} = f. \tag{14}$$

The boundary conditions of the problem are able to describe the three standard types: clamped, simply supported and free. The clamped and simply supported boundaries can be distinguished, however, into two different types according to the curvature $\kappa = -w''(x)$ of the beam axis. In this way, five different boundary condition types can be defined [7]: *doubly clamped* and *singly clamped* boundaries, respectively,

$$w = \overline{w} \;\text{ and }\; w' = \overline{\beta} \;\text{ and }\; -w'' = \overline{\kappa} \quad \text{on } \Gamma_{\mathrm{C_d}}, \tag{15}$$

$$w = \overline{w} \;\text{ and }\; w' = \overline{\beta} \;\text{ and }\; g^2M' = \overline{G}^g \;\text{ on } \Gamma_{\mathrm{C_s}}, \tag{16}$$

doubly simply supported and *singly simply supported* boundaries, respectively,

$$w = \overline{w} \;\text{ and }\; (M + g^2R - (g^2M')') = \overline{M}^g \;\text{ and }\; -w'' = \overline{\kappa} \;\text{ on } \Gamma_{\mathrm{S_d}}, \tag{17}$$

$$w = \overline{w} \;\text{ and }\; (M + g^2R - (g^2M')') = \overline{M}^g \;\text{ and }\; g^2M' = \overline{G}^g \;\text{ on } \Gamma_{\mathrm{S_s}}, \tag{18}$$

and *free* boundaries,

$$\begin{aligned} (M + g^2R - (g^2M')')' &= \overline{Q}^g \;\text{ and} \\ (M + g^2R - (g^2M')') &= \overline{M}^g \;\text{ and} \\ g^2M' &= \overline{G}^g \;\text{ on } \Gamma_{\mathrm{F}}, \end{aligned} \tag{19}$$

Finally, we note that setting $g = 0$ results in the classical boundary conditions of Euler–Bernoulli beams. For gradient-elastic Timoshenko beams, we refer to [2, 3], whereas higher-order beam models including material unisotropy have been studied in [8], for instance.

1.3 Numerical Tests for Trusses

Let us consider a uniform triangular lattice structure in plane formed by equilateral triangles made of bulk material with Young's modulus E as depicted in Figure 1 (left). A series of trusses of length $L = Nl$ and thickness $T = Nt$, with $N = 1, 2, 3, \ldots$, are extracted from the triangular lattice such that t denotes the thickness of one layer of triangles and l denotes a chosen fundamental length such that assumption $t \ll l$ is valid (implying that the fundamental truss with $N = 1$ can be considered as a thin beam and, accordingly, the subsequent trusses of the series as well). It should be noticed that the (thick) internal and (thin) surface bars of the lattice are of constant thickness, say, d and $d/2$, respectively (see Figure 1).

The lattice beams of the series are homogenized as straight beams following the strain gradient theory and it is shown by numerical results that the generalized beam model is able to take into account the size dependency caused by the lattice microstructure of the beams in bending and buckling by two material parameters: the effective Young's modulus and the length scale parameter g (one more parameter is needed for free vibrations) [3]. In auxetic metamaterials, in particular, the size dependency plays a key role due to the bending dominance of internal deformations of the material. Isogeometric Galerkin methods have been used for discretizing the generalized beam model, whereas for fine scale validation models standard finite element methods have been utilized.

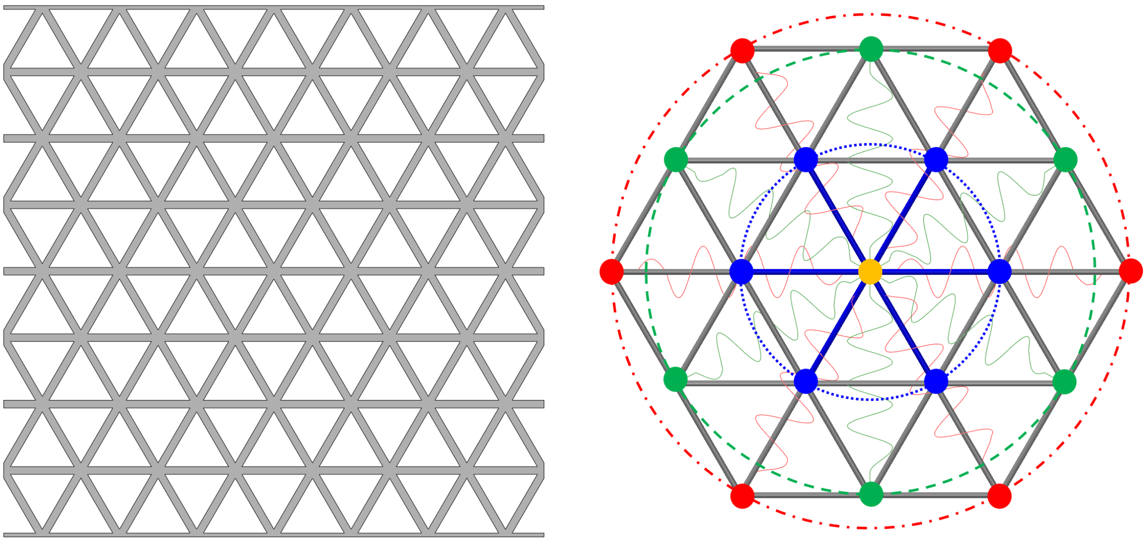

Figure 1. Left: A piece of a uniform triangular lattice structure formed by equilateral triangles made of straight bars of bulk material depicted by (grey) solid lines. Right: A piece of a triangular lattice structure with the basic bars depicted by (blue or grey thick) solid lines and (red and green thin) springs indicating two types of distant bars.

2 Second Strain Gradient Elasticity Theory for Modeling Spring Lattices

Models of the second strain gradient elasticity theory can be formulated by extending the basic principles of the first strain gradient elasticity theory; see [4,6] providing the general descriptions and a simplied model with a thorough analysis.

A triangular lattice structure of multiple bars or springs, depicted in Figure 1 (right), can be shown to behaves as a second strain gradient continuum [4]. In particular, it can be shown that initial stresses prescribed on boundaries can be associated to one of the higher-order parameters, the so-called modulus of cohesion, giving rise to surface tension.

Conclusions

This contribution has focused on applying the first and second strain gradient elasticity theories for modeling the size-dependent mechanical response of lattices structures. First, the strain energy of the first strain gradient elasticity theory has been recalled and then applied to Euler–Bernoulli beams. Then the strain gradient beam model has been applied for modeling planar trusses having a uniform triangular microarchitecture. In particular, the mechanical response of the lattice beams in bending, buckling and free vibrations have been reported to be microstructure-dependent, i.e., size-dependent. It has been finally reported as well that a more complex triangular lattice structure behaves as a second strain gradient continuum possessing size-dependent surface tension effects, in particular.

Acknowledgements

Authors have been supported by Academy of Finland through the project Adaptive isogeometric methods for thin-walled structures (decision numbers 270007, 273609, 304122). Access and licenses for the commercial FE software Abaqus have been provided by CSC – IT Center for Science, Finland.

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