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Isogeometric finite element analysis of mode I cracks within strain gradient elasticity

Jarkko Niiranen[1], Sergei Khakalo and Viacheslav Balobanov

Summary. A variational formulation within an H^2 Sobolev space setting is formulated for fourth-order plane strain/stress boundary value problems following a widely-used one parameter variant of Mindlin's strain gradient elasticity theory. A corresponding planar mode I crack problem is solved by isogeometric C^{p-1}-continuous discretizations with NURBS basis functions of order $p \geq 2$. Stress field singularities of the classical elasticity are shown to be removed by the strain gradient formulation.

Key words: strain gradient elasticity, fracture, mode I crack, isogeometric analysis

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Introduction

Generalized continuum theories have been developed in order to include length scale information lacking from the classical continuum theories by enriching classical strain energy expressions essentially by either new independent variables of local nature (e.g. micro-rotations in micro-polar theories) or by gradients of the classical variables of global nature (e.g. strain gradients in strain gradient theories). In particular, length scale parameters are introduced by both types of approaches.

Regarding fracture mechanics, unphysical singularities at crack tips realized in the classical elasticity theory have been shown to be removed, or better regularized, within the strain gradient elasticity theory (see, e.g., [1, 13, 6, 7, 4, 3, 12, 9]). Most of the related results in literature have been, however, obtained by analytical or semi-analytical methods and surprisingly few studies with numerical methods exist [4, 3, 12, 5].

In this contribution, isogeometric finite element methods, shown to be appropriate for solving higher-order boundary value problems in the context of strain gradient theories [10, 11], are applied for analyzing a plane mode I crack problem following a one parameter variant of Mindlin's strain gradient elasticity theory [8]. First, a variational formulation for the adopted strain gradient model is recalled [10], and then some numerical results for the crack problem are presented.

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Simplified strain gradient elasticity in plane

Let us first consider Mindlin's strain gradient elasticity theory of Form II [8] giving the virtual work expression over a body $\mathcal{B} \subset \mathbb{R}^3$ in the form

$$\delta W_{\text{int}} = \int_{\mathcal{B}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta \boldsymbol{u}) \, \mathrm{d}\mathcal{B} + \int_{\mathcal{B}} \boldsymbol{\tau} \,\vdots\, \boldsymbol{\gamma}(\delta \boldsymbol{u}) \, \mathrm{d}\mathcal{B}, \tag{1}$$

where $:$ and $\vdots$ denote scalar products for second- and third-order tensors, respectively.

The classical (second-order) Cauchy-like stress tensor $\boldsymbol{\sigma} : \mathcal{B} \to \mathbb{R}^{3\times3}$ is related to its work conjugate, linear strain tensor $\boldsymbol{\varepsilon} : \mathcal{B} \to \mathbb{R}^{3\times3}$, defined as the symmetric (second-order) tensor-valued gradient of the displacement field $\boldsymbol{u} : \mathcal{B} \to \mathbb{R}^3$, through the generalized Hooke's law $\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \operatorname{tr} \boldsymbol{\varepsilon} \boldsymbol{I}$, with Lame material parameters $\mu = \mu(x, y, z)$ and $\lambda = \lambda(x, y, z)$, and $\boldsymbol{I}$ denoting an identity tensor. The (third-order) micro-deformation tensor $\boldsymbol{\gamma} : \mathcal{B} \to \mathbb{R}^{3\times3\times3}$ is defined by the strain gradient as $\boldsymbol{\gamma} = \boldsymbol{\nabla}\boldsymbol{\varepsilon}$, where operator $\boldsymbol{\nabla}$ denotes the (third-order) tensor-valued gradient. The (third-rank) double stress tensor $\boldsymbol{\tau} : \mathcal{B} \to \mathbb{R}^{3\times3\times3}$, in turn, is related to its work conjugate by a (sixth-order) constitutive tensor involving, for centrosymmetric isotropic materials, a set of five material parameters $g_1 = g_1(x, y, z), ..., g_5 = g_5(x, y, z)$ giving the strain energy density in the form ((11.3) in [8])

$$\mathcal{W} = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + g_1\gamma_{iik}\gamma_{kjj} + g_2\gamma_{ijj}\gamma_{ikk} + g_3\gamma_{iik}\gamma_{jjk} + g_4\gamma_{ijk}\gamma_{ijk} + g_5\gamma_{ijk}\gamma_{kji}. \tag{2}$$

A one-parameter simplified strain gradient elasticity theory, originally proposed by Altan and Aifantis [2], reduces the strain energy density (2) to the form

$$\mathcal{W} = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + g^2\Big(\frac{1}{2}\lambda\varepsilon_{ii,k}\varepsilon_{jj,k} + \mu\varepsilon_{ij,k}\varepsilon_{ij,k}\Big), \tag{3}$$

where the non-classical material parameter g describes the length scale of the microstructure of the material ($g_1 = 0, g_2 = 0, g_3 = g^2\lambda/2, g_4 = g^2\mu, g_5 = 0$). With constant Lamé parameters, the double stress tensor then takes the form $\boldsymbol{\tau} = g^2\boldsymbol{\nabla}\boldsymbol{\sigma}$. In general, the gradient parameter can be assumed to be non-constant, i.e., $g = g(x, y, z)$. In what follows, however, g is assumed to be constant as usual.

For plane problems, with $\boldsymbol{u} = (u_x(x, y), u_y(x, y))$ denoting now the in-plane displacement vector, and $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ standing for the corresponding restrictions of the strain and stress tensors, respectively, and $\boldsymbol{\nabla}$ now including partial derivatives with respect to x and y only, the virtual work expression (1) takes the form

$$\delta W_{\text{int}} = \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta \boldsymbol{u}) \, \mathrm{d}\Omega + \int_{\Omega} g^2 \boldsymbol{\nabla}\boldsymbol{\sigma} \,\vdots\, \boldsymbol{\nabla}\boldsymbol{\varepsilon}(\delta \boldsymbol{u}) \, \mathrm{d}\Omega, \tag{4}$$

where the constitutive relation $\boldsymbol{\sigma} = \boldsymbol{E}\,\boldsymbol{\varepsilon}$ is now defined by the symmetric and positive definite (fourth-order) in-plane elasticity tensor $\boldsymbol{E} : \Omega \to \mathbb{R}^{2\times2\times2\times2}$ following the generalized Hooke's law of the chosen plane elasticity model.

A variational formulation of the plane gradient elasticity problem corresponding to (4) and the corresponding external energy reads as follows: for $\boldsymbol{f} \in [L^2(\Omega)]^2$, find $\boldsymbol{u} \in \boldsymbol{U} \subset [H^2(\Omega)]^2$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = l(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V} \subset [H^2(\Omega)]^2, \tag{5}$$

where the bilinear form $a : \boldsymbol{U} \times \boldsymbol{V} \to \mathbb{R}$, $a(\boldsymbol{u}, \boldsymbol{v}) = a^{\mathrm{c}}(\boldsymbol{u}, \boldsymbol{v}) + a^{\nabla}(\boldsymbol{u}, \boldsymbol{v})$, and the load functional $l : \boldsymbol{V} \to \mathbb{R}$ are, respectively, defined as

$$a^{\mathrm{c}}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \boldsymbol{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}\Omega, \tag{6}$$

$$a^{\nabla}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} g^2 \boldsymbol{\nabla}(\boldsymbol{E}\boldsymbol{\varepsilon}(\boldsymbol{u})) \,\vdots\, \boldsymbol{\nabla}\boldsymbol{\varepsilon}(\boldsymbol{v})\, \mathrm{d}\Omega, \tag{7}$$

$$l(\boldsymbol{v}) = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}\, \mathrm{d}\Omega. \tag{8}$$

The trial function set $\boldsymbol{U} = \{\boldsymbol{v} \in [H^2(\Omega)]^2 \mid \boldsymbol{v}_{|\Gamma_{\mathrm{C_s}} \cup \Gamma_{\mathrm{C_d}}} = \overline{\boldsymbol{u}},\ (\boldsymbol{\nabla}\boldsymbol{v})\boldsymbol{n}_{|\Gamma_{\mathrm{C_d}} \cup \Gamma_{\mathrm{F_s}}} = \overline{\boldsymbol{w}}\}$ consists of functions satisfying the essential boundary conditions, with the given Dirichlet data $\overline{\boldsymbol{u}}$ and $\overline{\boldsymbol{w}}$, whereas the test function space $\boldsymbol{V}$ consists of $[H^2]^2$ functions satisfying the corresponding homogeneous Dirichlet boundary conditions.

As proved in [10], the energy norm of the problem induced by the bilinear form is equivalent to the H^2-norm whenever $\boldsymbol{U} = \boldsymbol{V}$, whereas symmetry, continuity and coercivity of the bilinear form (for $g > 0$) guarantee the solvability of the problem. Furthermore, for conforming Galerkin methods these results imply optimal error estimates [10].

Numerical results via isogeometric analysis

Let us analyze a mode I crack problem in plane with a NURBS discretization of order $p = 5$ (see [10] for details) depicted in Fig. 1 (left). The stress distributions and crack openings compared for the classical and strain gradient models demonstrate, in particular, (1) the qualitative differences in the shapes of the openings and (2) the removal of the stress singularity as illustrated in Figs. 1 and 2 (for two different parameter values). As a conclusion, it should be noticed that the stress level in the strain gradient model is finally determined by the value of the (experimentally validated) length scale parameter g.

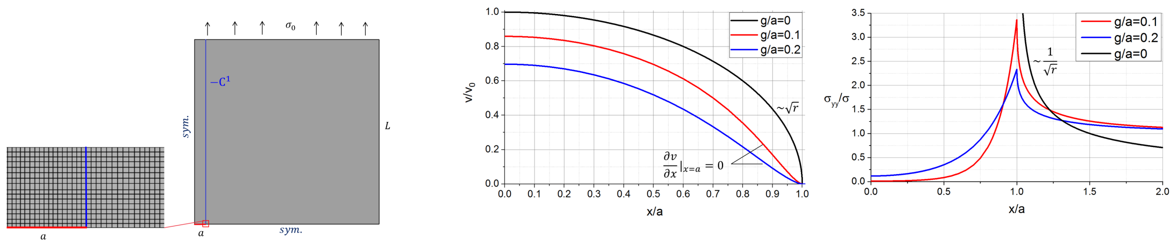
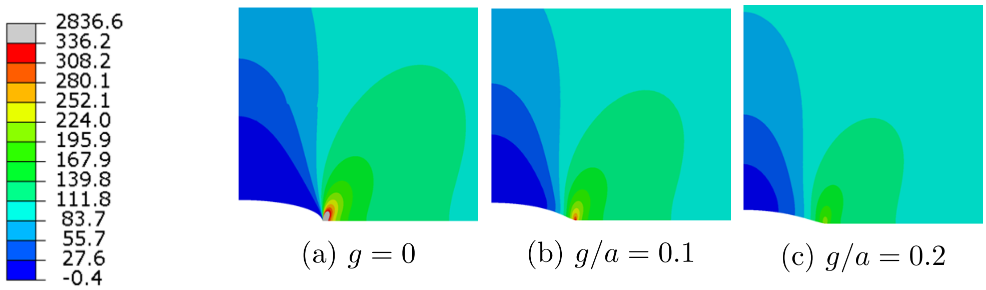


Figure 1: (left) Problem setting and computational domain with a uniform mesh; (middle) Shape of the crack opening; (right) Stress σ_{yy} along the crack line.



(a) $g = 0$ (b) $g/a = 0.1$ (c) $g/a = 0.2$

Figure 2: Stress σ_{yy} with the classical (a) and strain gradient (b,c) models.

References

[1] B. S. Altan and E. C. Aifantis. On the structure of the mode III crack-tip in gradient elasticity. *Scripta Mellurgica et Materialia*, 26:319–324, 1992. URL https://doi.org/10.1016/0956-716X(92)90194-J.

[2] B. S. Altan and E. C. Aifantis. On some aspects in the special theory of gradient elasticity. *Journal of the Mechanical Behavior of Materials*, 8:231–282, 1997. URL https://doi.org/10.1515/JMBM.1997.8.3.231.

[3] N. Aravas and A.E. Giannakopoulos. Plane asymptotic crack-tip solutions in gradient elasticity. *International Journal of Solids and Structures*, 46(25):4478 – 4503, 2009. URL https://doi.org/10.1016/j.ijsolstr.2009.09.009.

[4] H. Askes, I. Morata, and E. C. Aifantis. Finite element analysis with staggered gradient elasticity. *Comp. Struct.*, 86:1266–1279, 2008.

[5] S.H. Chen and T.C. Wang. Interface crack problems with strain gradient effects. *International Journal of Fracture*, 117(1):25–37, 2002. URL https://doi.org/10.1023/A:1020904510702.

[6] G. Exadaktylos. Gradient elasticity with surface energy: Mode-I crack problem. *International Journal of Solids and Structures*, 35:421–456, 1998. URL https://doi.org/10.1016/S0020-7683(97)00036-X.

[7] M. Lazar and G. A. Maugin. Nonsingular stress and strain fields of dislocations and disclinations in first strain gradient elasticity. *International Journal of Engineering Science*, 43: 1157–1184, 2005. URL https://doi.org/10.1016/j.ijengsci.2005.01.006.

[8] R. D. Mindlin. Micro-structure in linear elasticity. *Archive for Rational Mechanics and Analysis*, 16:51–78, 1964.

[9] S. Mahmoud Mousavi and Markus Lazar. Distributed dislocation technique for cracks based on non-singular dislocations in nonlocal elasticity of helmholtz type. *Engineering Fracture Mechanics*, 136:79 – 95, 2015. URL https://doi.org/10.1016/j.engfracmech.2015.01.015.

[10] J. Niiranen, S. Khakalo, V. Balobanov, and A. H. Niemi. Variational formulation and isogeometric analysis for fourth-order boundary value problems of gradient-elastic bar and plane strain/stress problems. *Computer Methods in Applied Mechanics and Engineering*, 308:182–211, 2016.

[11] J. Niiranen, J. Kiendl, A. H. Niemi, and A. Reali. Isogeometric analysis for sixth-order boundary value problems of gradient-elastic Kirchhoff plates. *Computer Methods in Applied Mechanics and Engineering*, 316:328–348, 2017. URL https://doi.org/10.1016/j.cma.2016.05.008.

[12] S.-A. Papanicolopulos and A. Zervos. Numerical solution of crack problems in gradient elasticity. *Proceedings of the ICE - Engineering and Computational Mechanics*, 163(2): 73–82, 6 2010.

[13] I. Vardoulakis, G. Exadaktylos, and E. Aifantis. Gradient elasticity with surface energy: mode-iii crack problem. *International Journal of Solids and Structures*, 30:4531–4559, 1996. URL https://doi.org/10.1016/0020-7683(95)00277-4.

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