Rajamäki, Robin; Koivunen, Visa

Comparison of sparse sensor array configurations with constrained aperture for passive sensing

Published in:
2017 IEEE Radar Conference, RadarConf 2017

DOI:
10.1109/RADAR.2017.7944312

Published: 07/06/2017

Document Version
Peer-reviewed accepted author manuscript, also known as Final accepted manuscript or Post-print

Please cite the original version:
Rajamäki, R., & Koivunen, V. (2017). Comparison of sparse sensor array configurations with constrained aperture for passive sensing. In *2017 IEEE Radar Conference, RadarConf 2017* (pp. 797-802). Article 7944312 (Proceedings of the IEEE Radar Conference). IEEE. https://doi.org/10.1109/RADAR.2017.7944312

Comparison of Sparse Sensor Array Configurations with Constrained Aperture for Passive Sensing

Robin Rajamäki and Visa Koivunen
School of Electrical Engineering
Aalto University, 02150 Espoo, Finland
Email: robin.rajamaki@aalto.fi, visa.koivunen@aalto.fi

***Abstract*—Sparse sensor arrays can match the performance of fully populated arrays in many tasks, such as direction-of-arrival estimation, using substantially fewer elements. However, finding the sparse array configuration that uses the smallest number of elements is generally a hard problem. Consequently, several closed-form, but sub-optimal solutions have been developed in the past. These designs are typically specified for a given number of elements, although when the area occupied by the array is the main limitation, it is more convenient to compare arrays of similar aperture instead. This paper outlines procedures for synthesizing three sparse linear array geometries for a specified aperture, namely the Wichmann, Nested and Super nested array. These configurations are compared to the optimal Minimum-redundancy array and their deviation from optimality is quantified in the limit of large apertures.**

I. Introduction

Phased sensor arrays have become a ubiquitous technology with wide-spread applications in e.g. radar, communications, radio astronomy, sonar, medical ultrasound and seismology [1]. Some of the central advantages of sensor arrays include spatial selectivity, adaptive interference rejection capability and high signal-to-noise ratio. The complexity and cost of conventional arrays with uniformly spaced elements may however become impractical when the number of sensors grows large. Fortunately, a *sparse* or *thinned* sensor array using only $\mathcal{O}(\sqrt{N})$ elements can achieve the same number of *degrees of freedom* (DOF) as a filled array with N elements [2]. The available DOF determine, for instance, the number of sources that can be resolved in *direction-of-arrival* estimation [3], [4], or the *beamforming* capabilities of the array [5], [6]. Consequently, sparse arrays may offer a significant reduction in hardware costs compared to conventional arrays, at little or no performance loss. Reduced *mutual coupling* due to the larger average spacing between elements is another benefit of sparse arrays [7], [8].

Finding the optimal sparse array geometry is often a computationally difficult combinatorial problem [9]. The lack of a closed-form optimal solution has led to the development of several sub-optimal but mathematically tractable array configurations [4], [8], [10]–[12]. These configurations are typically generated for a fixed number of elements, although the area available for the array (instead of the number of elements) is the dominating constraint in many applications. For example, a radar antenna system mounted on a platform, such as a vehicle, aircraft or vessel, is allowed to occupy only a certain limited space. Consequently, it is necessary to synthesize and compare sparse array geometries of similar aperture.

This paper analyzes three sparse linear array configurations used in passive sensing, i.e. receive only operation. The considered geometries are: the *Wichmann* [10], *Nested* [4] and *Super nested array* [8]. The contributions of the paper are i) extending the methods for generating sparse array configurations to the case when the array aperture is constrained; ii) deriving an expression for the maximum aperture, respectively minimum number of elements achievable by the array; iii) comparing the asymptotic performances of the sub-optimal arrays to that of the optimal *Minimum-redundancy array*; and iv) verifying the results in simulations.

The paper is organized as follows. Section II introduces the necessary definitions for sparse arrays. Section III provides procedures for generating the array geometries for a constrained aperture. An expression for the largest achievable aperture and lowest possible number of elements is formulated for each array configuration in section IV. Additionally, the asymptotic properties of the configurations are evaluated. The results of sections III and IV are verified by numerical examples in section V.

II. Sparse array definitions and configurations

A. Considered array configurations

This paper considers a sparse array to be a thinned *uniform linear array* (ULA). Consequently, the elements of the sparse array fall on a regular grid with unit distance $\tilde{d}$, i.e. the inter-element spacing of the ULA. This grid is usually normalized to yield a set of integers for convenience.

The sparse ruler is a convenient analogy for illustrating the problem of optimal sparse array design. Actually, the theoretical groundwork for this problem was already laid down in mid 20$^{\text{th}}$ century number theoretical studies on minimal *restricted difference bases* [13], [14]. Essentially, a sparse ruler can measure each integer distance up to its length L, with the least number of markers N possible. The *perfect sparse ruler* represents each distance exactly once and thus has a length [13]:

$$L = \binom{N}{2} = \frac{1}{2}N(N-1). \tag{1}$$

Equivalently, Eq. (1) may be interpreted as the maximum possible aperture achievable by any sparse *restricted array* [9], i.e. an array whose set of inter-element distances contains all non-negative integers up to L. The restricted array with no repetitions in this set achieves the aperture in Eq. (1) and is referred to as the *perfect array*. Unfortunately, these kind of non-redundant arrays do not exist for $N > 4$ [13], and array configurations with low redundancy have to be found instead. The sparse arrays considered in this paper are all of the low-redundancy type. Only restricted array configurations are considered, since they achieve the same number of DOF for a given aperture and are thus fair to compare with each other. This means that arrays whose set of inter-element distances contains gaps, such as the *Co-prime* [11] and *Minimum-hole array* [9], are excluded from this study.

B. Difference co-array

In passive array processing, one usually deals with auto-correlations of measurements. Assuming uncorrelated narrow-band signals impinging on the array from the far-field, it turns out that the spatial data covariance matrix of the array has support on the set of element position differences, referred to as *co-array* elements or *lags* [2]. Given the element positions $\mathbf{d}$, the support of the *difference co-array* is

$$\mathcal{C}_\Delta = \{\mathbf{d}_n - \mathbf{d}_{\tilde{n}}\},\ n, \tilde{n} = 1, 2\ ...\ N. \tag{2}$$

The virtual array defined by the difference co-array in Eq. (2) is symmetrical and has twice the aperture of the physical array. The co-array is *contiguous* if its support contains every integer between $-L$ and L. The multiplicity of the co-array at lag d_Δ is given by $\upsilon_\Delta(d_\Delta) \in \mathbb{N}$ (set of non-negative integers), $d_\Delta = -L, -(L-1), ..., L-1, L$. For example, if two different element pairs are spaced apart by 3 units, then $\upsilon_\Delta(3) = \upsilon_\Delta(-3) = 2$. Two arrays are *co-array equivalent* if their co-arrays have the same support $\mathcal{C}_\Delta$.

C. Figures of merit of a sparse array

1) Redundancy: The redundancy of a passive linear array is defined as the ratio of the aperture (or equivalently, DOF) of the perfect array defined in Eq. (1) to the number of contiguous elements $N_{\text{c, max}}$ in the difference co-array of the actual array [9]:

$$R = \frac{N(N-1)}{2N_{\text{c,max}}}. \tag{3}$$

If the co-array has no holes, then $N_{\text{c,max}} = L$. In this case, the number of elements may also be expressed as a function of redundancy and aperture:

$$N = \frac{1}{2}(\sqrt{8RL+1} + 1). \tag{4}$$

Alternatively, the ratio $A = N^2/L$ is sometimes used as a measure of redundancy [10], [13]–[15]. It follows from Eq. (3) that

$$R = \frac{1}{2}(\frac{N^2}{L} - \frac{N}{L}) = \frac{1}{2}(A - \frac{N}{L}), \tag{5}$$

and therefore $R \approx \frac{1}{2}A$ for large L.

2) Sparseness: While redundancy measures how efficiently the co-array is populated, it does not take into account the distribution of possible redundancies. For example, two co-array equivalent arrays can have the same redundancy, while one may have fewer closely spaced elements than the other. Since element proximity is a factor contributing to e.g. mutual coupling, it is reasonable to have a measure of the sparseness (or denseness) of the array. The number of unit spacings in the array, $\upsilon_\Delta(1)$, serves as a first-order approximation of denseness and has been used in e.g. [8] as a guideline for array design.

D. Sparse array configurations with contiguous co-arrays

1) Minimum-redundancy array (MRA): The MRA [9] achieves the smallest redundancy for a given number of elements, while having no holes in the co-array. Unfortunately, finding MRAs is a combinatorial problem and no efficient search algorithm is known. Consequently, several array configurations with closed-form solutions, but sub-optimal redundancies have been developed in e.g. [4], [8], [10]. Three of these are presented next.

2) Wichmann array (WA): The WA is based on a regular pattern in restricted difference bases originally observed by B. Wichmann in the 1960's [14]. The pattern was later rediscovered independently in the context of MRAs in [10], [15]. The inter-element spacings of the WA can be expressed as [10]:

$$d\mathcal{D}_{\text{WA}} = \{1^{(l)}, l+1, (2l+1)^{(l)}, (4l+3)^{(q)}, (2l+2)^{(l+1)}, 1^{(l)}\}, \tag{6}$$

where $l,\ q \in \mathbb{N}$ and the notation $x^{(y)}$ denotes " y repetitions of x". The number of elements $N \in \mathbb{N}_+$ (set of positive integers) and aperture $L \in \mathbb{N}$ may be derived from Eq. (6):

$$\begin{cases} N = 4l + q + 3 \\ L = 4l(l+q+2) + 3(q+1). \end{cases} \tag{7}$$

3) Nested array (NA): The NA is proposed in [4] as a concatenation of two ULAs with different inter-element spacings. The element positions of the 2-level NA are given by

$$\mathcal{D}_{\text{NA}} = \{0, 1,\ ...\ , N_1{-}1, N_1, 2(N_1{+}1){-}1, ...\ , N_2(N_1{+}1){-}1\}, \tag{8}$$

where $N_1, N_2 \in \mathbb{N}$ are the number of elements of the two subarrays. From Eq. (8), the aperture of the NA is simply:

$$L = N_2(N_1 + 1) - 1. \tag{9}$$

Since the total number of elements is given by $N = N_1 + N_2$, Eq. (9) is maximimized when [4]:

$$\begin{cases} N_1^\star = N_2^\star = \frac{N}{2}, & \text{when } N \text{ is even} \\ N_1^\star = \frac{N-1}{2};\ N_2^\star = \frac{N+1}{2}, & \text{when } N \text{ is odd.} \end{cases} \tag{10}$$

4) Super nested array (SNA): The SNA [8] improves upon the NA in terms of sparseness. Several elements of the NA are close to each other due to the ULA sub-array with N_1 unit inter-element spacings. The SNA relaxes this limitation, whilst retaining co-array equivalence with the NA. The element positions of the SNA are given by [8]:

$$\mathcal{D}_{\text{SNA}} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5 \cup \mathcal{D}_6, \tag{11}$$

where

$$\begin{cases} \mathcal{D}_1 = \{2l\}, & l = 0, 1, ..., A_1 \\ \mathcal{D}_2 = \{N_1 - (1+2l)\}, & l = 0, 1, ..., B_1 \\ \mathcal{D}_3 = \{N_1 + (2+2l)\}, & l = 0, 1, ..., A_2 \\ \mathcal{D}_4 = \{2(N_1+1) - (1+2l)\}, & l = 0, 1, ..., B_2 \\ \mathcal{D}_5 = \{l(N_1+1) - 1\}, & l = 2, 3, ..., N_2 \\ \mathcal{D}_6 = \{N_2(N_1+1) - 2\} \end{cases} \tag{12}$$

and

$$(A_1, B_1, A_2, B_2) = \begin{cases} (q, q-1, q-1, q-2), & \text{if } N_1 = 4q \\ (q, q-1, q-1, q-1), & \text{if } N_1 = 4q+1 \\ (q+1, q-1, q, q-2), & \text{if } N_1 = 4q+2 \\ (q, q, q, q-1), & \text{if } N_1 = 4q+3. \end{cases} \tag{13}$$

As can be seen from Eq. (12) and (13), the SNA is completely determined by the parameters N_1, N_2 used for generating the seed NA.

III. Fixed aperture array synthesis

In this section, procedures for generating the sparse array configurations of section II-D, for a constrained aperture, are presented. This allows for different array geometries of the same spatial extent to easily be compared with each other, provided that solutions for the given aperture exist.

A. Minimum-redundancy array

The MRA can be considered as the solution to the optimization problem [16]:

$$\begin{aligned} \text{minimize} \quad & N \\ \text{subject to} \quad & N_{\text{c,max}} = L. \end{aligned} \tag{14}$$

Eq. (14) may be solved through a search which involves: i) incrementally increasing N; ii) checking if any of the possible $\binom{L+1}{N}$ combinations fill the constraint of a difference co-array with no holes; and iii) stopping once a solution is found. Since there may be several equally redundant solutions for a given N, the configuration with the largest sparseness is selected. The worst case search space of this method consists of $\sum_{N=2}^{L+1} \binom{L+1}{N} = \mathcal{O}(2^L)$ combinations. This becomes prohibitively large even for moderate L. For example, for an array with $L = 40$ it is $\mathcal{O}(2^{40}) = \mathcal{O}(10^{12})$. The complexity can be somewhat reduced by e.g. fixing the position of the second or second-to-last element and sorting the possible array configurations into a search tree, which allows for pruning and parallelization of the problem [10], [16]. Note that a solution to Eq. (14) is guaranteed for each L, although the MRA is traditionally defined only for the maximum achievable aperture for a given number of elements [9].

B. Wichmann array

The WA parameter pair l, q that minimizes the number of elements for a given aperture L, is obtained by optimizing Eq. (7) assuming $l, q \in \mathbb{R}$ (a detailed derivation is provided in section IV-B):

$$l^\star = \left\lfloor \frac{\sqrt{12L+9} - 9}{12} \right\rceil. \tag{15}$$

In Eq. (15), the operator $\lfloor \cdot \rceil$ rounds to the closest integer. The optimal parameter q is obtained by evaluating Eq. (15), inserting the result into Eq. (7) and rounding:

$$q^\star = \left\lfloor \frac{-4(l^\star)^2 - 8l^\star + L - 3}{4l^\star + 3} \right\rceil. \tag{16}$$

The parameter pair $l^\star, q^\star \geq 0$, given by Eq. (15) and (16), produces a valid WA with the minimum number of elements for a given aperture. No optimal solution exists if $l^\star, q^\star < 0$.

C. Nested array

Substituting the optimal NA parameters from Eq. (10) into Eq. (9) yields the aperture for a fixed number of elements [4]:

$$\begin{cases} L = \frac{1}{2}(\frac{N^2}{2} + N - 2), & \text{when } N \text{ is even} \\ L = \frac{1}{2}(\frac{N^2}{2} + N - \frac{3}{2}), & \text{when } N \text{ is odd.} \end{cases} \tag{17}$$

Alternatively, Eq. (17) may be solved for N:

$$\begin{cases} N = \sqrt{4L+5} - 1, \text{when } N \text{ is even} \\ N = 2\sqrt{L+1} - 1, \text{when } N \text{ is odd.} \end{cases} \tag{18}$$

The integer-valued solutions to Eq. (18) yield the smallest number of elements of realizable NAs. Once N is computed for a given L, the optimal NA may be generated using Eq. (10) and (8).

D. Super nested array

The SNA is generated by first finding the NA of a given aperture, provided that $N_1 \geq 4$ and $N_2 \geq 3$ [8]. Knowing N_1, the parameter $q \in \mathbb{N}$ may be calculated using:

$$q = \frac{1}{4}(N_1 - N_1 \bmod 4), \tag{19}$$

which determines the rest of the parameters A_1, B_1, A_2, B_2 and ultimately yields a unique solution to Eq. (11). Eq. (19) follows directly from the case conditions listed in Eq. (13). The SNA has exactly the same aperture and number of elements as the NA that is used to generate it.

IV. Derivation of bounds

In the following section, bounds on the largest aperture for a fixed number of elements, or equivalently, the smallest number of elements for a given aperture, are derived for the array geometries presented in section III. Also, the corresponding asymptotic redundancy of each array is calculated. Denoting the optimal aperture and number of elements by $\tilde{L}$ and $\tilde{N}$, the respective L and N of any realizable array satisfy:

$$\begin{cases} L \leq \tilde{L} \\ N \geq \tilde{N}. \end{cases} \tag{20}$$

In case of the WA, NA and SNA, the parameters that maximize the aperture for a given number of elements are found in a straightforward fashion by allowing for non-integer valued optimization variables. The aperture maximization problem is concave in all three cases under this relaxation. Consequently, the integer-valued solutions of any realizable array must satisfy Eq. (20).

A. Minimum-redundancy array

In [17], a graph-theoretic approach was taken in an attempt to derive an expression for the aperture of the MRA. Unfortunately, the obtained expression $\tilde{L} = \lfloor N(3N-2)/8 + 1 \rfloor$ ($\lfloor \cdot \rfloor$ is the floor operator) was later shown not to hold for $N \geq 12$ in [15], [16]. Although the optimal aperture or number of elements of the MRA are unknown, asymptotic limits on the redundancy of restricted difference bases are known. Inserting the bounds on $A = N^2/L$ reported in [13], [14] into Eq. (5) yields:

$$1.217 \leq \lim_{N \to \infty} R_{\text{MRA}} = R_{\infty,\text{MRA}} = A_{\infty,\text{MRA}}/2 \leq 1.5. \tag{21}$$

B. Wichmann array

Assuming N fixed and solving Eq. (7) for L yields

$$L(l) = -12l^2 + (4N-19)l + 3N - 6, \tag{22}$$

which is clearly concave. Relaxing the assumption that $l \in \mathbb{N}$ to $l \in \mathbb{R}$ enables differentiation of Eq. (22) with respect to l. The value l that maximizes Eq. (22) is found to be

$$\tilde{l}^\star = \frac{N-4}{6}. \tag{23}$$

Naturally, l assumes the closest non-negative integer solution to Eq. (23) for any physically feasible array. Inserting Eq. (23) into (22) yields the maximum possible aperture that a WA can achieve for a given number of elements:

$$\tilde{L} = L(\tilde{l}^\star) = \frac{1}{3}(N^2 + N - 2). \tag{24}$$

Alternatively, Eq. (24) may be expressed as the minimum number of elements for a given aperture

$$\tilde{N} = N(\tilde{l}^\star) = \frac{1}{2}(\sqrt{12L+9} - 1). \tag{25}$$

The number of unit spacings in the optimal WA can be deduced from Eq. (6) directly:

$$\upsilon_\Delta(1) = \begin{cases} 1, & \text{when } l = 0 \\ 2l = \frac{N-4}{3}, & \text{otherwise.} \end{cases} \tag{26}$$

C. Nested array

By substituting $N_1 = N - N_2$ into Eq. (9), the aperture of the NA may be expressed as

$$L(N_2) = -N_2^2 + (N+1)N_2 - 1. \tag{27}$$

Because Eq. (27) is a convex function of N_2, it is maximized (under the relaxation that $N_1, N_2 \in \mathbb{R}$) when

$$\begin{cases} N_1 = \tilde{N}_1^\star = \frac{N-1}{2} \\ N_2 = \tilde{N}_2^\star = \frac{N+1}{2}, \end{cases} \tag{28}$$

which corresponds to the solution in Eq. (10) for odd N. Consequently, from Eq. (17) and Eq. (18), the maximum aperture of the NA is

$$\tilde{L} = \frac{1}{2}(\frac{N^2}{2} + N - \frac{3}{2}), \tag{29}$$

and the minimum number of elements is

$$\tilde{N} = 2\sqrt{L+1} - 1. \tag{30}$$

The number of unit inter-element spacings in the optimal NA is determined by the number of elements the sub-array with N_1 elements, which according to (10) is

$$\upsilon_\Delta(1) = N_1 = \begin{cases} \frac{N}{2}, & \text{when } N \text{ is even} \\ \frac{N-1}{2}, & \text{when } N \text{ is odd.} \end{cases} \tag{31}$$

D. Super nested array

The SNA follows the derivation of the optimal NA in the previous section, with the exception of the number of unit spacings, which is [8]

$$\upsilon_\Delta(1) = \begin{cases} 2, & \text{when } N_1 \text{ is even} \\ 1, & \text{when } N_1 \text{ is odd.} \end{cases} \tag{32}$$

E. Asymptotic results

The results of section IV are summarized in Table I. Expressions for the maximum aperture $\tilde{L}$ and minimum number of elements $\tilde{N}$ are listed for the considered sparse array configurations. These expressions provide a convenient tool for comparing the optimal solutions of each geometry for arbitrary N and L. Note that not all aperture/number-of-elements pairs yield a realizable array, because N and L have to be integer-valued in practice.

In order to convert the essential information in Table I into numerical values, two asymptotic bounds are calculated in Table II. Firstly, the *asymptotic redundancy*

$$R_\infty = \lim_{N\to\infty} R(N) \tag{33}$$

is evaluated by substituting Eq. (3) and the respective expression for optimal aperture from Table I into Eq. (33). Secondly, the *element redundancy*

$$\eta_\infty = \lim_{L\to\infty} \frac{\tilde{N}(L)}{\tilde{N}_{\text{MRA}}(L)} = \sqrt{\frac{R_\infty}{R_{\infty,\text{MRA}}}} \tag{34}$$

is evaluated by substituting Eq. (4) into Eq. (34). The ratio η_∞ quantifies the fraction of elements required by a given design in comparison to the minimum possible among all sparse restricted arrays in the limit of large apertures. Since the asymptotic redundancy of the MRA is not known exactly, η_∞ is given as an interval that is calculated using Eq. (21).

Table II shows the asymptotic redundancy and the element redundancy evaluated for the considered array configurations. A couple of remarks about the results are in place. First, the asymptotic redundancy of the WA is consistent with the value $A_\infty = 3 = 2R_\infty$ reported in [10], [14], [15]. The NA and SNA are approximately 33% more redundant than the WA. Second, the element redundancy of the WA suggests that it requires at most 11% more elements than the MRA for large apertures. However, all WAs checked against MRAs for $N \geq 14$ so far have been found to be minimally redundant [16]. It is therefore possible that the WA is actually optimal in terms of redundancy. In case of the NA and SNA, the increase in the number of elements is about $15-28\%$ compared to the MRA. It is interesting to note that the perfect array would require $9-18\%$ fewer elements than the MRA. However, as mentioned in section II-A, the perfect array cannot be constructed for $N > 4$, and it is included here only as a point of reference.

As a final observation, one may see from Table I that the SNA is clearly superior to the WA and NA in terms of sparseness. This would likely result in reduced mutual couping for the SNA. For both the WA and NA $\upsilon_\Delta(1) \propto N \propto \sqrt{L}$, whereas for the SNA $\upsilon_\Delta(1)$ is practically constant. The WA has $1-2/3 \approx 33\%$ less unit spacings than the NA as $N \to \infty$.

TABLE I. SUMMARY OF RESULTS IN SECTION IV. MRA = MINIMUM-REDUNDANCY ARRAY, SNA = SUPER NESTED ARRAY, NA = NESTED ARRAY, AND WA = WICHMANN ARRAY.

	$\tilde{L}$	$\tilde{N}$	$\upsilon_\Delta(1)$
Perfect array	$\frac{1}{2}(N^2 - N)$	$\frac{1}{2}(\sqrt{8L+1} + 1)$	1
WA	$\frac{1}{3}(N^2 + N - 2)$	$\frac{1}{2}(\sqrt{12L+9} - 1)$	$\frac{1}{3}(N-4)$
NA	$\frac{1}{4}(N^2 + 2N - 3)$	$2(\sqrt{L+1} - \frac{1}{2})$	$\frac{1}{2}(N-1)$
SNA	—"—	—"—	1 or 2 [8]

TABLE II. ASYMPTOTIC REDUNDANCY R_∞ AND ELEMENT REDUNDANCY η_∞ AS $N, L \to \infty$. THE WA REQUIRES BETWEEN 0 AND 11% MORE ELEMENTS THAN THE MRA FOR LARGE APERTURES. THE CORRESPONDING FIGURE FOR THE NA AND SNA IS $15 - 28\%$.

	R_∞	η_∞
Perfect array	1	$0.82 - 0.91$
MRA	$1.217 - 1.5$	1
WA	$\frac{3}{2} = 1.5$	$1 - 1.11$
NA / SNA	2	$1.15 - 1.28$

V. NUMERICAL RESULTS

In this section, the presented array configurations are generated as described in section III for various designer constrained apertures. The configurations are compared using the figures of merit of redundancy and sparseness introduced in section II-C. A practical array design example is also given in order to highlight the advantages and disadvantages of sparse arrays compared to uniformly spaced arrays.

A. *Comparison of sparse configurations*

Fig. 1 shows the geometry of the MRA, WA, NA and SNA for an aperture of $L = 29$ unit inter-element spacings. This is the smallest common aperture for which the NA, SNA and WA produce feasible solutions. Fig. 2 shows the respective difference co-arrays. Table III provides the figures of merit calculated for each configuration. As expected, the WA achieves a lower redundancy than the NA and SNA. Actually, the WA matches the redundancy of the MRA, although the two array geometries are clearly not equivalent (Fig. 1). The SNA and MRA obtain the lowest number of unit inter-element spacings. In case of the NA and WA, a close agreement can be seen between the true (Table III) and estimated values (Table I), i.e. $v_\Delta(1) = (10-1)/2 = 4.5 \approx 5$ and $(9-4)/3 = 1.666... \approx 2$.

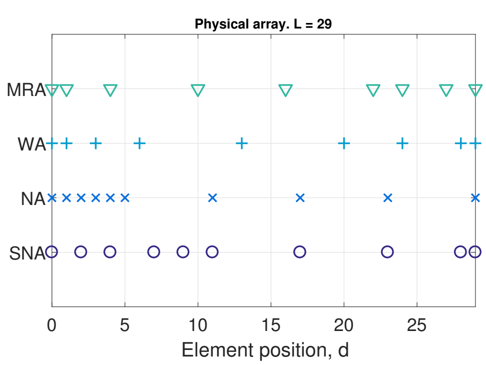

Fig. 1. Sparse array configurations for fixed aperture ($L = 29$). The MRA and WA need one element less than the NA and SNA, which both have $N = 10$ elements. A ULA of equivalent aperture would require $N_{\text{ULA}} = 30$ elements.

TABLE III. FIGURES OF MERIT OF THE SPARSE ARRAY CONFIGURATIONS IN FIG. 1 ($L = 29$). THE WA MATCHES THE REDUNDANCY, R, OF THE MRA, WHEREAS THE MRA AND SNA HAVE THE SMALLEST NUMBER OF UNIT SPACINGS, $v_\Delta(1)$.

	N	R	$v_\Delta(1)$
MRA	9	1.24	1
SNA	10	1.55	1
NA	10	1.55	5
WA	9	1.24	2

Fig. 3 shows the redundancy of the WA, NA and SNA calculated for $L = 0, 1, ..., 500$. The WA consistently yields a lower redundancy than the NA and SNA. The two latter are equally redundant since they have the same number of elements and aperture. The redundancies converge to their asymptotic values of 1.5 and 2 for large L, as expected (see Table II).

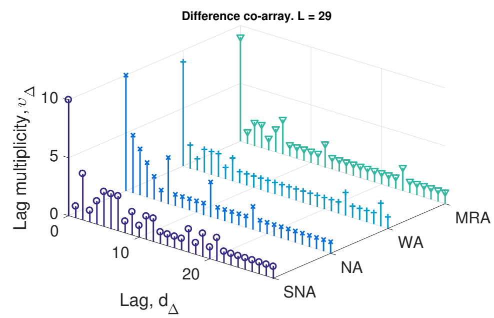

Fig. 2. Difference co-arrays of the sparse array configurations in Fig. 1 ($L = 29$). All arrays have contiguous co-arrays (only positive lags are displayed).

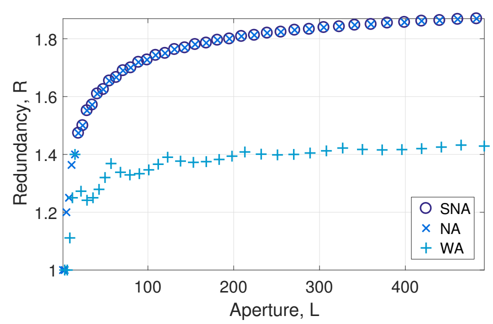

Fig. 3. Redundancy of the sparse array configurations. The WA generally achieves a lower redundancy than the NA and SNA. The redundancies can be seen to converge to their asymptotic values of 1.5 and 2.

In Fig. 4, the number of unit spacings in the arrays are plotted as a function of aperture. The SNA achieves $v_\Delta(1) = 1$ or 2 independent of aperture, whereas $v_\Delta(1) \propto \sqrt{L}$ in case of the NA and WA. The step-like structure of the WA is a result of quantizing the array parameters l and q. The rightmost point of each step yields the maximum aperture for a given number of elements and unit spacings.

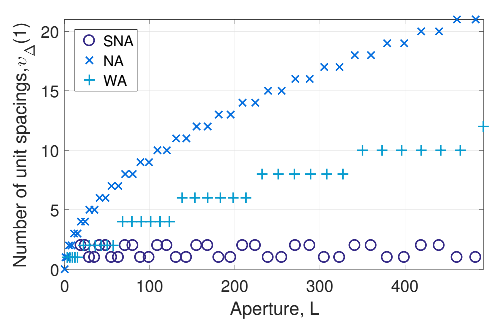

Fig. 4. Number of unit inter-element spacings in the sparse array configurations. The SNA always has only one or two unit spacings, whereas for the NA and WA this number increases proportionally to $\sqrt{L}$.

B. *Practical example*

Next, the advantages and disadvantages of the sparse and uniform arrays are illustrated through an example. Consider that the arrays in Fig. 1 are operating at a center frequency of $f = 15$ GHz. Assuming half wavelength inter-element spacing yields $\tilde{d} = \lambda/2 = c/(2f) \approx 3 \cdot 10^8/(2 \cdot 15 \cdot 10^9)\text{m} = 1$ cm between elements and a total physical aperture of $L = 29$ cm. The ULA of equivalent aperture has $N_{\text{ULA}} = 30$ elements spaced 1 cm apart, whereas the sparse arrays only require around a third of the elements (Table III), resulting in an average inter-element spacing of 3 cm. Consequently, mutual coupling can be expected to be less of a problem in the sparse arrays. Using delay-and-sum type beamforming with Chebyshev weighting, the ULA can achieve a -30 dB side lobe level with $4.2°$ -3 dB main lobe width at broadside, as is shown in Fig. 5. Due to their co-array equivalence with the ULA, all of the considered sparse configurations can match this performance using co-array based array processing methods, such as *covariance matrix augmentation* [18] or *intensity image addition* [2]. The sparse arrays can likewise resolve up to $N_{\text{ULA}} - 1 = 29$ incoherent sources using e.g. sub-space techniques like *spatial smoothing MUSIC* [4]. On the other hand, the fewer number of elements means that the array gain of the sparse array is roughly a third of the gain of the ULA, which translates to an SNR loss of approximately 5 dB.

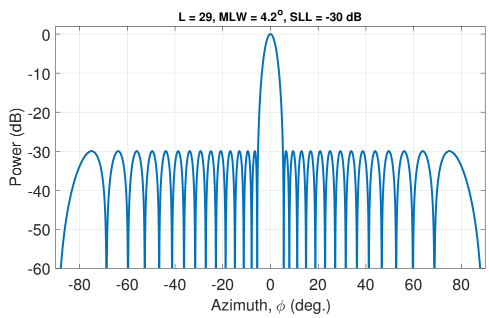

Fig. 5. Delay-and-sum beampattern of 30 element ULA with half wavelength inter-element spacing. Using Chebyshev weighting, the array achieves $4.2°$ main lobe width (MLW) for a -30 dB side lobe level (SLL) at broadside. The sparse arrays in Fig. 1 may achieve the same performance with only 9 or 10 elements using co-array based processing.

VI. Conclusion

This paper compared the geometrical properties of three sparse linear array designs: the Wichmann (WA), Nested (NA) and Super nested array (SNA). The configurations were chosen due to their co-array equivalence with the uniform linear array in far-field narrow-band passive sensing applications. Procedures were outlined for synthesizing the geometries given a target aperture. Additionally, expressions for the optimal aperture and number of elements of each array were given. The asymptotic redundancy and element redundancy of the geometries were also computed.

The WA has a low redundancy at large apertures, and uses at most 11% more elements than the Minimum-redundancy array (MRA). The NA and SNA require $15 - 28\%$ more elements than the MRA. On the other hand, the SNA has the least number of closely spaced elements, which makes it an attractive alternative when mutual coupling is a problem. The number of unit spacings grows proportionally to the square-root of aperture in both the NA and WA, whereas for the SNA this figure is constant. The WA has approximately 33% fewer unit spacings than the NA, making it less susceptible to mutual coupling effects.

Acknowledgment

The authors would like to thank Dr. Veli-Matti Kolmonen from Nokia Bell Labs for his insightful comments.

References

[1] H. L. Van Trees, *Optimum Array Processing, Part IV of Detection, Estimation, and Modulation Theory*. Wiley-Interscience, 2002.

[2] R. T. Hoctor and S. A. Kassam, "The unifying role of the coarray in aperture synthesis for coherent and incoherent imaging," *Proceedings of the IEEE*, vol. 78, no. 4, pp. 735–752, Apr 1990.

[3] ——, "High resolution coherent source location using transmit/receive arrays," *IEEE Transactions on Image Processing*, vol. 1, no. 1, pp. 88–100, Jan 1992.

[4] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Transactions on Signal Processing*, vol. 58, no. 8, pp. 4167–4181, Aug 2010.

[5] F. Ahmad, G. J. Frazer, S. A. Kassam, and M. G. Amin, "Design and implementation of near-field, wideband synthetic aperture beamformers," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 40, no. 1, pp. 206–220, Jan 2004.

[6] K. Adhikari, J. R. Buck, and K. E. Wage, "Extending coprime sensor arrays to achieve the peak side lobe height of a full uniform linear array," *EURASIP Journal on Advances in Signal Processing*, vol. 2014, no. 1, pp. 1–17, 2014.

[7] E. BouDaher, F. Ahmad, M. G. Amin, and A. Hoorfar, "Mutual coupling effect and compensation in non-uniform arrays for direction-of-arrival estimation," *Digital Signal Processing*, pp. –, 2016.

[8] C. L. Liu and P. P. Vaidyanathan, "Super nested arrays: Linear sparse arrays with reduced mutual coupling - part i: Fundamentals," *IEEE Transactions on Signal Processing*, vol. 64, no. 15, pp. 3997–4012, Aug 2016.

[9] A. Moffet, "Minimum-redundancy linear arrays," *IEEE Transactions on Antennas and Propagation*, vol. 16, no. 2, pp. 172–175, Mar 1968.

[10] D. A. Linebarger, I. H. Sudborough, and I. G. Tollis, "Difference bases and sparse sensor arrays," *IEEE Transactions on Information Theory*, vol. 39, no. 2, pp. 716–721, Mar 1993.

[11] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Transactions on Signal Processing*, vol. 59, no. 2, pp. 573–586, Feb 2011.

[12] R. J. Kozick and S. A. Kassam, "Linear imaging with sensor arrays on convex polygonal boundaries," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 21, no. 5, pp. 1155–1166, Sep 1991.

[13] J. Leech, "On the representation of 1, 2, , n by differences," *Journal of the London Mathematical Society*, vol. s1-31, no. 2, pp. 160–169, 1956.

[14] B. Wichmann, "A note on restricted difference bases," *Journal of the London Mathematical Society*, vol. s1-38, no. 1, pp. 465–466, 1963.

[15] D. Pearson, S. U. Pillai, and Y. Lee, "An algorithm for near-optimal placement of sensor elements," *IEEE Transactions on Information Theory*, vol. 36, no. 6, pp. 1280–1284, Nov 1990.

[16] K. A. Blanton and J. H. McClellan, "New search algorithm for minimum redundancy linear arrays," in *International Conference on Acoustics, Speech, and Signal Processing, ICASSP 1991*, Apr 1991, pp. 1361–1364 vol.2.

[17] S. D. Bedrosian, "Nonuniform linear arrays: Graph-theoretic approach to minimum redundancy," *Proceedings of the IEEE*, vol. 74, no. 7, pp. 1040–1043, July 1986.

[18] S. U. Pillai, Y. Bar-Ness, and F. Haber, "A new approach to array geometry for improved spatial spectrum estimation," *Proceedings of the IEEE*, vol. 73, no. 10, pp. 1522–1524, Oct 1985.