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A Universal Sequence of Tensors for the Asymptotic Rank Conjecture

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Abstract

The *exponent* $\sigma(T)$ of a tensor $T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$ over a field $\mathbb{F}$ captures the base of the exponential growth rate of the tensor rank of T under Kronecker powers. Tensor exponents are fundamental from the standpoint of algorithms and computational complexity theory; for example, the exponent ω of square matrix multiplication can be characterized as $\omega = 2\sigma(\mathrm{MM}_2)$, where $\mathrm{MM}_2 \in \mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$ is the tensor that represents 2×2 matrix multiplication.

Strassen [FOCS 1986] initiated a duality theory for spaces of tensors that enables one to characterize the exponent of a tensor via objects in a dual space, called the *asymptotic spectrum* of the primal (tensor) space. While Strassen's theory has considerable generality beyond the setting of tensors – Wigderson and Zuiddam [*Asymptotic Spectra: Theory, Applications, and Extensions*, preprint, 2023] give a recent exposition – progress in characterizing the dual space in the tensor setting has been slow, with the first universal points in the dual identified by Christandl, Vrana, and Zuiddam [*J. Amer. Math. Soc.* 36 (2023)]. In parallel to Strassen's theory, the algebraic geometry community has developed a geometric theory of tensors aimed at characterizing the structure of the primal space and tensor exponents therein; the latter study was motivated in particular by an observation of Strassen (implicit in [*J. Reine Angew. Math.* 384 (1988)]) that matrix-multiplication tensors have limited universality in the sense that $\sigma(\mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d) \leq \frac{2\omega}{3} = \frac{4}{3}\sigma(\mathrm{MM}_2)$ holds for all $d \geq 1$. In particular, this limited universality of the tensor MM_2 puts forth the question whether one could construct explicit universal tensors that exactly characterize the worst-case tensor exponent in the primal space. Such explicit universal objects would, among others, give means towards a proof or a disproof of Strassen's asymptotic rank conjecture [*Progr. Math.* 120 (1994)]; the former would immediately imply $\omega = 2$ and, among others, refute the Set Cover Conjecture (cf. Björklund and Kaski [STOC 2024] and Pratt [STOC 2024]).

Our main result is an explicit construction of a *sequence* $\mathcal{U}_d$ of zero-one-valued tensors that is universal for the worst-case tensor exponent; more precisely, we show that $\sigma(\mathcal{U}_d) = \sigma(d)$ where $\sigma(d) = \sup_{T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d} \sigma(T)$. We also supply an explicit universal sequence $\mathcal{U}_\Delta$ localised to capture the worst-case exponent $\sigma(\Delta)$ of tensors with support contained in $\Delta \subseteq [d] \times [d] \times [d]$; by combining such sequences, we obtain a universal sequence $\mathcal{T}_d$ such that $\sigma(\mathcal{T}_d) = 1$ holds if and only if Strassen's asymptotic rank conjecture holds for d. Finally, we show that the limit $\lim_{d\to\infty} \sigma(d)$ exists and can be captured as $\lim_{d\to\infty} \sigma(D_d)$ for an explicit sequence $(D_d)_{d=1}^{\infty}$ of tensors obtained by diagonalisation of the sequences $\mathcal{U}_d$.

As our second result we relate the *absence* of polynomials of fixed degree vanishing on tensors of low rank, or more generally asymptotic rank, with upper bounds on the exponent $\sigma(d)$. Using this technique, one may bound asymptotic rank for all tensors of a given format, knowing enough specific tensors of low asymptotic rank.

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1 Introduction

1.1 Exponents of Tensors and the Quest for Universality

For an infinite field $\mathbb{F}$ and a positive integer constant d, the *exponent* of a nonzero tensor $T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$ is the least nonnegative real $\sigma(T)$ such that the sequence of tensor (Kronecker) powers

$$T^{\boxtimes \mathbb{N}} = (T^{\boxtimes p} : p = 1, 2, \ldots) \tag{1}$$

has its sequence of tensor ranks bounded by $\mathrm{R}(T^{\boxtimes p}) \leq d^{\sigma(T)p+o(p)}$. Exponents of specific *constant-size* tensors T are of fundamental significance in the study of algorithms and algebraic complexity theory [16, 76]. For example, Strassen showed that the exponent ω of square matrix multiplication satisfies $\omega = 2\sigma(\mathrm{MM}_2)$ [71, 67], where MM_2 is the $4 \times 4 \times 4$ tensor representing the multiplication map for two 2×2 matrices. Recently, it was shown that the Set Cover Conjecture [29, 30, 40] fails if the exponent of a specific $7 \times 7 \times 7$ tensor Q is sufficiently close to one [7, 59].

The study of exponents of tensors – or, what is the same up to exponentiation, the study of *asymptotic rank*[1] [34] of tensors – is difficult. The exponent ω of square matrix multiplication is perhaps the best studied nontrivial exponent, and even in its case the best current lower bound $\omega \geq 2$ remains the trivial one (but see [8, 43, 52, 47, 24]), and the best current upper bound $\omega \leq 2.371866$ [31] is a result of extensive work spanning decades (e.g. [70, 58, 6, 62, 61, 25, 71, 26, 65, 75, 53, 2]) and relying on increasingly sophisticated techniques. In parallel to the study of exponents of individual tensors, research effort has been invested into developing a structural theory for spaces of tensors and their exponents, a theory to which the present paper also contributes. Two rough lines of research most relevant to our present work are as follows.

Strassen's duality theory – the asymptotic spectrum of tensors. The first line of research, announced by Strassen [71] in his 1986 FOCS paper and developed in a sequence of papers [66, 67, 68] and PhD theses [15, 74, 54] (cf. [72]), builds a duality theory for asymptotic rank of tensors based on the theory of preordered commutative semirings, with the direct sum and tensor (Kronecker) product of tensors as the pertinent semiring operations, and the preorder defined by a rank-capturing tensor restriction relation. Wigderson and Zuiddam [76] give a recent comprehensive exposition of Strassen's theory and the preorder-theoretic topological dual spaces – the *asymptotic spectrum* of a space of tensors – which enable a tight characterization of the asymptotic rank of a tensor via the preorder-monotone homomorphisms in the dual. Yet, progress in terms of understanding the structure and identifying explicit points in the asymptotic spectrum of tensors has been slow. Beyond Strassen's original construction of support functionals, which are restricted to the subspace of oblique tensors, only recently Christandl, Vrana, and Zuiddam [18] constructed a family

[1] The *asymptotic rank* of $T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$ is $\tilde{\mathrm{R}}(T) = \lim_{p\to\infty} \mathrm{R}(T^{\boxtimes p})^{1/p}$ and we have $\tilde{\mathrm{R}}(T) = d^{\sigma(T)}$.

of explicit *universal* spectral points – called quantum functionals – using theory of quantum entropy and covariants; however, also this dual family is far from yielding broadly tight lower bounds on asymptotic rank.

Viewed from the standpoint of Strassen's duality theory, rather than working in a dual space, in this paper we seek and obtain as our main result an explicit and universal *primal* characterization of tensor exponents and thus asymptotic rank in $\mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$ – before proceeding to our results, let us review a second pertinent line of prior research and motivation.

Geometry of tensors and the asymptotic rank conjecture. The second line of research seeks to study spaces of tensors to identify the *worst-case* behaviour in the space using tools from algebraic geometry, in particular building on the seminal concept of *border rank* of a tensor due to Bini, Capovani, Romani, and Lotti [6] (see also Schönhage [62]) and its geometric characterization via secant varieties of Segre and Veronese varieties (e.g. [13, 14, 50, 42, 45, 78]). For a space $\mathcal{F}$ of tensors, let $\sigma(\mathcal{F}) = \sup_{T\in\mathcal{F}} \sigma(T)$; as a special case, for $d = 1, 2, \ldots$ let $\sigma(d) = \sigma(\mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d)$. It is immediate that $1 \leq \sigma(d) \leq 2$ and that $\sigma(1) = 1$. Over the complex numbers, it is a nontrivial consequence of the geometry of tensors that $\sigma(2) = 1$. Already the value $\sigma(3)$ is open and would be of substantial interest to determine. For example, it is known that $\sigma(3) = 1$ implies $\omega = 2$ by the application of the Coppersmith–Winograd method [26] to a specific $3 \times 3 \times 3$ tensor. It appears very difficult to prove lower bounds on $\sigma(d)$ apart from the trivial $\sigma(d) \geq 1$. Indeed, any tensor $T \in (\mathbb{F}^d)^{\otimes 3}$ with $\sigma(T) > 1$ would yield an explicit sequence of tensors, namely its Kronecker powers, such that for any constant $C > 0$ for $k \gg 0$ we have rank and border rank of $T^{\boxtimes k}$ greater than Cd^k. Currently, we do not know explicit examples of such sequences for $C = 3$ with the current world record lower bound on rank in [1]. One of the main obstacles is lack of methods to prove that a given tensor has high rank or border rank. In case of rank, the state of the art is the substitution method, based on linear algebra. In case of border rank, one looks for polynomial witnesses that vanish on tensors of given bounded border rank. However in this case, most known equations, not found via explicit computations, also vanish on so-called cactus varieties, which fill the ambient space quickly and thus known methods cannot provide super-linear lower bounds on tensor rank [4, 5, 12, 32, 38]. The state of the art for border rank is based on mixture of different methods, barely breaking the bound for $C = 2$ [48]. As the rank of the generic tensor grows quadratically with d the problem is often referred to as an instance of the *hay in a haystack* problem (phrase due to H. Karloff): find an explicit object that behaves generically. There is hope in the new introduced method of border apolarity [13], still there is currently no clear path of getting past $C = 3$.

The difficulty of lower bounds and progressively improving upper bounds for specific exponents, the exponent ω in particular, has prompted bold conjectures on worst-case exponents for broad families of tensors. Most notably, writing $\mathcal{T}_d$ for the space of all tight tensors in $\mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$, *Strassen's asymptotic rank conjecture* (cf. [69, Conjecture 5.3]) states that the worst-case exponent for this space is the least possible:

▶ **Conjecture 1** (Strassen's asymptotic rank conjecture). *For all $d \geq 1$ it holds that $\sigma(\mathcal{T}_d) = 1$.*

Strassen's asymptotic rank conjecture, if true, immediately implies the algorithmically serendipitous corollary $\omega = 2$ in particular; cf. also [7, 59] for further consequences to algorithms. A yet stronger conjecture (cf. Bürgisser, Clausen, and Shokrollahi [16, Problem 15.5]; also e.g. Conner, Gesmundo, Landsberg, Ventura, and Wang [23, Conjecture 1.4] as well as Wigderson and Zuiddam [76, Section 13, p. 122]) states that the least possible exponent is shared by all concise tensors in $\mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$:

▶ **Conjecture 2** (Extended asymptotic rank conjecture). *For all $d \geq 1$ it holds that $\sigma(d) = 1$.*

A key result supporting Conjecture 1 and Conjecture 2, implicit in Strassen [67, Proposition 3.6] and highlighted by Christandl, Vrana, and Zuiddam [17, Proposition 2.12] as well as Conner, Gesmundo, Landsberg, and Ventura [22, Remark 2.1], is that tensor rank is known to be nontrivially submultiplicative under Kronecker products; stated in terms of worst-case tensor exponents, Strassen proved that for all $d \geq 1$ it holds that $\sigma(d) \leq \frac{2\omega}{3}$.

Viewed in terms of exponents of tensors and universality, we can rephrase Strassen's result as stating that the exponent of the matrix multiplication tensor MM_2 controls from above the exponent of all other tensors, namely we have $\sigma(d) \leq \frac{4}{3}\sigma(\mathrm{MM}_2) = \frac{2\omega}{3}$. This rephrasing suggests that one should seek explicit constructions of tensors $U \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$ that are worst-case *universal* for the class with $\sigma(d) = \sigma(U)$. In rough analogy with computational complexity theory, such explicit tensors capture the "hardest" tensors in a class of tensors and, for example, would provide an explicit object of study towards resolving Conjectures 1 and 2.

1.2 Our Results – Explicit Universal Sequences of Zero-One-Valued Tensors

While we do not present explicit individual tensors U that are universal, as our main result we present an explicit *sequence* $\mathcal{U}$ of tensors that is universal and consists of zero-one-valued tensors in coordinates. Towards this end, let us extend the definition of the exponent of a tensor T to a sequence

$$\mathcal{T} = (T_j \in \mathbb{F}^{s_j} \otimes \mathbb{F}^{s_j} \otimes \mathbb{F}^{s_j} : j = 1, 2, \ldots) \tag{2}$$

of nonzero tensors. The *exponent* of the sequence $\mathcal{T}$ is the least nonnegative real $\sigma(\mathcal{T})$ such that $\mathrm{R}(T_j) \leq s_j^{\sigma(\mathcal{T})+o_j(1)}$. From (1) and (2) we immediately have $\sigma(T^{\boxtimes\mathbb{N}}) = \sigma(T)$ for all tensors $T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$. Our main result is that there is an explicit sequence of tensors that characterizes the exponent of $\mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$ and thus enables an approach towards resolving Strassen's conjecture.

▶ **Theorem 3** (Main; A universal sequence of tensors for d fixed). *For all $d \geq 1$ there is an explicit sequence $\mathcal{U}_d$ of zero-one-valued tensors with $\sigma(\mathcal{U}_d) = \sigma(d)$.*

In coordinates, the q^{th} tensor in the sequence $\mathcal{U}_d$ admits explicit combinatorial expression as a union of orbit-indicator tensors under a particular action of the symmetric group $\mathfrak{S}_q$. This combinatorial structure enables us to study the exponent $\sigma(\Delta)$ of the space of tensors spanned by $\{e_i \otimes e_j \otimes e_k : (i,j,k) \in \Delta\} \subseteq \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$ for a nonempty $\Delta \subseteq [d] \times [d] \times [d]$; when equality holds, we have $\sigma(\Delta) = \sigma(d)$.

▶ **Theorem 4** (Support-localized universal sequences of tensors). *For all nonempty $\Delta \subseteq [d] \times [d] \times [d]$ there is an explicit sequence $\mathcal{U}_\Delta$ of zero-one-valued tensors with $\sigma(\mathcal{U}_\Delta) = \sigma(\Delta)$.*

From Theorem 4 we obtain the following corollary for tight tensors and Strassen's conjecture (Conjecture 1). We need short preliminaries. We say that the set Δ is *tight* if there exist injective functions $\alpha, \beta, \gamma : [d] \to \mathbb{Z}$ such that $\alpha(i) + \beta(j) + \gamma(k) = 0$ for all $(i,j,k) \in \Delta$. A tensor $T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$ is *tight* if it admits a coordinate-representation whose support is contained in a tight set.

▶ **Theorem 5** (A universal sequence of tensors for Strassen's conjecture). *For all $d \geq 1$ there is an explicit sequence $\mathcal{T}_d$ of tight zero-one-valued tensors with $\sigma(\mathcal{T}_d) = \sigma(\mathfrak{T}_d)$.*

Further, by diagonalising the sequences $\mathcal{U}_d$ for increasing d we obtain a universal sequence providing worst possible exponent irrespective of d; that is, a universal sequence for the extended asymptotic rank conjecture (Conjecture 2):

▶ **Theorem 6** (A universal sequence of tensors for the extended asymptotic rank conjecture). *There is an explicit sequence $\mathcal{D} = (D_d : d = 1, 2, \ldots)$ of zero-one-valued tensors with $\lim_{d\to\infty} \sigma(D_d) = \lim_{d\to\infty} \sigma(d)$.*

We note that the limit in the theorem above exists and is the supremum of all $\sigma(d)$; cf. Lemma 19. The above theorem may be regarded as a solution to the aforementioned hay in the haystack problem for the exponent of tensors.

1.3 Overview of Techniques

Before proceeding to review our further results, it will be convenient to give an overview of our main techniques and concepts underlying Theorem 3. In particular, a basis induced from integer compositions for the linear span of the image of the Kronecker power map $S \mapsto S^{\boxtimes q}$ will be our key tool.

The Kronecker power map $\mathrm{K}_{d,q}$ in coordinates. For a field $\mathbb{F}$ and positive integers d and q, our key object of study is the *Kronecker power map*

$$\mathrm{K}_{d,q} : \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d \to \mathbb{F}^{d^q} \otimes \mathbb{F}^{d^q} \otimes \mathbb{F}^{d^q}$$

that takes a tensor $S \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$ to its q^{th} tensor (Kronecker) power $\mathrm{K}_{d,q}(S) = S^{\boxtimes q}$.

It will be convenient to study the Kronecker power map $\mathrm{K}_{d,q}$ by working with tensors in coordinates, so let us set up conventions accordingly. Let us write $[d] = \{1, 2, \ldots, d\}$ and identify the spaces $\mathbb{F}^{d\times d\times d} = \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$. For a tensor $S \in \mathbb{F}^{d\times d\times d}$ in the domain of $\mathrm{K}_{d,q}$, we write $S_{i,j,k} \in \mathbb{F}$ for the entry of S at position $i, j, k \in [d]$. For a tensor $T \in \mathbb{F}^{d^q\times d^q\times d^q}$ in the codomain of $\mathrm{K}_{d,q}$, it is convenient to index the entries $T_{I,J,K} \in \mathbb{F}$ of T with q-tuples $I = (i_1, i_2, \ldots, i_q) \in [d]^q$, $J = (j_1, j_2, \ldots, j_q) \in [d]^q$, $K = (k_1, k_2, \ldots, k_q) \in [d]^q$. Indeed, with this convention, the image $\mathrm{K}_{d,q}(S) = S^{\boxtimes q}$ of a tensor $S \in \mathbb{F}^{d\times d\times d}$ can be defined entrywise for all $I, J, K \in [d]^q$ by

$$S^{\boxtimes q}_{I,J,K} = S_{i_1,j_1,k_1} S_{i_2,j_2,k_2} \cdots S_{i_q,j_q,k_q} \,. \tag{3}$$

An explicit basis for the linear span of the image of $\mathrm{K}_{d,q}$. Next we embed the image $\mathrm{K}_{d,q}(\mathbb{F}^{d\times d\times d})$ in the least-dimensional subspace of $\mathbb{F}^{d^q\times d^q\times d^q}$ possible, so-called linear span, and provide an explicit basis for this subspace in the assumed coordinates. Towards this end, working with integer compositions that capture the distinct right-hand sides of (3) for a generic S with support in a nonempty set $\Delta \subseteq [d] \times [d] \times [d]$ will be convenient. In precise terms, a function $g : \Delta \to \mathbb{N}_{\geq 0}$ with $\sum_{\delta\in\Delta} g(\delta) = q$ is a *composition* of the positive integer q with domain Δ. Let us write $\mathcal{C}^{\Delta}_q$ for the set of all compositions of q with domain Δ. The distinct right-hand sides of (3) for a generic S are now enumerated by the compositions $g \in \mathcal{C}^{\Delta}_q$; explicitly, associate with g the product

$$S^g = \prod_{i,j,k\in[d]} S^{g(i,j,k)}_{i,j,k} \in \mathbb{F} \,. \tag{4}$$

Writing $\mathbb{F}^{\Delta}$ for the subspace of all tensors in $\mathbb{F}^{d\times d\times d}$ with support in Δ, we next provide a basis $(T^{(g)} \in \mathbb{F}^{d^q\times d^q\times d^q} : g \in \mathcal{C}^{\Delta}_q)$ that enables us to express the Kronecker power $S^{\boxtimes q}$ of an arbitrary tensor $S \in \mathbb{F}^{\Delta}$ as the linear combination

$$S^{\boxtimes q} = \sum_{g \in \mathcal{C}_q^{\Delta}} S^g T^{(g)} \,. \tag{5}$$

We need short preliminaries to define the tensors $T^{(g)}$ in the assumed coordinates. For $I, J, K \in [d]^q$, define the triple-counting composition $\Phi_{I,J,K} \in \mathcal{C}_q^{[d]\times[d]\times[d]}$ for all $u, v, w \in [d]$ by

$$\Phi_{I,J,K}(u, v, w) = |\{\ell \in [q] : i_\ell = u,\ j_\ell = v,\ k_\ell = w\}| \,. \tag{6}$$

We use *Iverson's bracket notation*; for a logical proposition P, we write $[\![P]\!]$ to indicate a 0 if P is false and a 1 if P is true.

▶ **Definition 7** (Composition basis for the linear span of the image $\mathrm{K}_{d,q}(\mathbb{F}^{\Delta})$)**.** *For $g \in \mathcal{C}_q^{\Delta}$, define the tensor $T^{(g)} \in \mathbb{F}^{d^q \times d^q \times d^q}$ in coordinates for all $I, J, K \in [d]^q$ by*

$$\left(T^{(g)}\right)_{I,J,K} = [\![\Phi_{I,J,K} = g]\!] \,. \tag{7}$$

The tensors $(T^{(g)} : g \in \mathcal{C}_q^{\Delta})$ are the composition basis *for the linear span of* $\mathrm{K}_{d,q}(\mathbb{F}^{\Delta})$.

We provide various equivalent characterisations of these tensors $T^{(g)}$ in Lemma 9 (on page 10). The characterisation via integer compositions in Definition 7 in particular enables an immediate verification via (7), (4), and (3) that (5) holds. In particular, the linear span of the image $\mathrm{K}_{d,q}(\mathbb{F}^{\Delta})$ is contained in the linear span of the composition basis. What is less immediate is that the reverse containment holds under mild assumptions on the field $\mathbb{F}$ by identifying the Kronecker power map with the Veronese map on homogeneous Δ-variate polynomials of degree q. In coordinates, by relying on homogeneous polynomial interpolation we show in Corollary 13 and Proposition 15 that for every $g \in \mathcal{C}_q^{\Delta}$ there exist tensors $S_f \in \mathbb{F}^{\Delta}$ and scalars $\lambda_{f,g} \in \mathbb{F}$ indexed by $f \in \mathcal{C}_q^{\Delta}$ such that

$$T^{(g)} = \sum_{f \in \mathcal{C}_q^{\Delta}} \lambda_{f,g} S_f^{\boxtimes q} \,. \tag{8}$$

Now (5) and (8) imply the composition basis spans exactly the linear span of $\mathrm{K}_{d,q}(\mathbb{F}^{\Delta})$.

Sublinearity and serendipity of polynomial dimensionality of the span. By subadditivity of tensor rank, from the linear combination (5) we have immediately for an arbitrary tensor $S \in \mathbb{F}^{\Delta}$ that

$$\mathrm{R}\big(S^{\boxtimes q}\big) \le \sum_{g \in \mathcal{C}_q^{\Delta}} \mathrm{R}\big(T^{(g)}\big) \,.$$

Conversely, for an arbitrary $g \in \mathcal{C}_q^{\Delta}$, we have from (8) that

$$\mathrm{R}\big(T^{(g)}\big) \le \sum_{f \in \mathcal{C}_q^{\Delta}} \mathrm{R}\big(S_f^{\boxtimes q}\big) \,.$$

Recalling from combinatorics of integer compositions that $|\mathcal{C}_q^{\Delta}| = \binom{|\Delta|-1+q}{|\Delta|-1} \le (|\Delta| - 1 + q)^{|\Delta|-1}$, we observe that dimension of the linear span of $\mathrm{K}_{d,q}(\mathbb{F}^{\Delta})$ grows only polynomially in q when Δ is fixed. Theorem 3 and Theorem 4 follow essentially immediately by taking $\mathcal{U}_{\Delta} = (U_{\Delta,q} : q = 1, 2, \ldots)$ with $U_{\Delta,q} = \bigoplus_{g \in \mathcal{C}_q^{\Delta}} T^{(g)}$ as the universal sequence.

Structure and description of the tensors $T^{(g)}$. The tensors $T^{(g)}$ admit several descriptions, e.g. a group-theoretic description can be given using orbit-indicators of the group $\mathfrak{S}_q$, see Lemma 9, and an alternative description can be given via type decompositions [76, Section 9.3] (see also [28]). There are several important applications of group theory in the study or tensor rank and asymptotic rank. We note that when G is an Abelian group then the structure tensor S_G of the group algebra is an orbit-indicator of the group G^2 identified with $\{(g_1, g_2, g_3) \in G^3 : g_1 + g_2 = g_3\}$; in this case, the tensor S_G has also minimal rank, equal to $|G|$, via the Discrete Fourier Transform. When G is not Abelian, the representation theory of G allows for nontrivial bounds on the rank of S_G; similar observations have been leveraged in the study of fast matrix multiplication; e.g. Cohn and Umans [20], Cohn, Kleinberg, Szegedy, and Umans [19], Cohn and Umans [21], Blasiak, Cohn, Grochow, Pratt, and Umans [10]. We expect this structure to enable further work towards an eventual proof or disproof of the (extended) asymptotic rank conjecture.

A win-win dichotomy. In addition to enabling worst-case characterisation of tensor exponents for families of tensors, the tensors $T^{(g)}$ enable the following "win-win" dichotomy (cf. Corollary 21 for a precise statement) that motivates their further study:

Either the extended asymptotic rank conjecture (Conjecture 2) holds, implying $\omega = 2$ and a disproof of the Set Cover Conjecture;

or the tensors $T^{(g)}$ form an explicit sequence of tensors with superlinear border rank growth, providing substantial progress for the "hay in the haystack" problem for rank and border rank.

A remark on explicitness. We stress that we here view explicitness as the property of not only a single tensor but rather a sequence of tensors; cf. e.g. [48, Section 3]. Both the tensors $T^{(g)}$ and the tensors in our universal sequences have zero-one entries in coordinates, and these entries may be computed fast. Indeed, from (7) it is immediate that we have linear-time algorithms that output $T^{(g)}_{I,J,K}$ given I, J, K, g as input. Similarly, listing (or ranking/unranking) integer compositions $g \in \mathcal{C}^{\Delta}_q$ for given Δ, q admit fast algorithms.

Invariant Specht tensors. Our techniques reduce the question about asymptotic ranks to questions about ranks of specific tensors, where $T^{(g)}$ is only one possible choice. Another one is to apply representation theory of the symmetric group $\mathfrak{S}_p$. It turns out that ranks of very special tensors, precisely invariants in the tensor product of three Specht modules $(S^\alpha \otimes S^\beta \otimes S^\gamma)^{\mathfrak{S}_p}$, where α, β, γ are partitions of p with at most d parts, govern the exponent $\sigma(d)$. Precise results are given in Section 4.

1.4 Further Results

Our second result relates extended Strassen's conjecture to equations of varieties, in particular secant varieties. We tacitly assume sufficient background in algebraic geometry and geometry of tensors (e.g. [27, 42, 55]). Also, unless otherwise mentioned, we assume that the field $\mathbb{F}$ is the field of complex numbers.

To set the context, it is a well-established method to prove that tensors have high border rank by exhibiting polynomials vanishing on the k^{th} secant variety of the Segre variety $X = (\mathbb{P}^{d-1})^{\times 3}$ and evaluating it on a given tensor T. Among the state of the art theoretical equations in this context are the Koszul flattenings and the Young flattenings, which can provide border rank bounds up $(2 - \epsilon)d$ for any $\epsilon > 0$ for $d \gg 0$. The degree of those equations grows as a polynomial in d for fixed ϵ, however the degree of this polynomial also grows as $\epsilon \to 0$. The second method to obtain equations of secant varieties is computational, based on representation theory and linear algebra. This is an exhaustive method, finding all equations in the given degree and partitioning them into so-called isotypic components [11, 36]. Still of course its scope is limited due to computational obstacles. For border rank $k < d$, the smallest degree of a polynomial vanishing on the k^{th} secant variety is exactly $k + 1$ [46]. However, for border rank above d, one observes fast grow of the minimal degree in which equations exist; for example, the smallest degree of an equation vanishing on the 6^{th} secant variety of $(\mathbb{P}^3)^{\times 3}$ is 19 and on the 18^{th} secant variety of $(\mathbb{P}^6)^{\times 3}$ is at least 187000 [36]. The lack of low-degree (or otherwise easy) equations has been perceived so far as an obstacle in proving that tensors have high border rank, in particular in disproving Strassen's conjecture or its extensions. In this paper, we proceed in the *opposite* direction, namely that *absence* of low-degree equations of secant varieties implies upper bound on $\sigma(d)$.

▶ **Theorem 8** (Absence of low-degree equations implies low asymptotic rank). *Let $X \subseteq \mathbb{P}((\mathbb{F}^d)^{\otimes 3})$ be a variety contained in the locus of tensors of asymptotic rank at most r. Suppose that no polynomial of degree p vanishes on X. Then every tensor in $(\mathbb{F}^d)^{\otimes 3}$ has asymptotic rank at most*

$$r\binom{d^3 - 1 + p}{d^3 - 1}^{\frac{1}{p}}.$$

We note that the theorem above implies that bounds on asymptotic rank of special tensors may imply bounds for all tensors. As the variety X may be always assumed to be $GL(d)^{\times 3}$ invariant one may combine the computational method of finding isotypic decomposition of homogeneous polynomials with obtaining good bounds on p, in order to obtain new bounds on $\sigma(d)$. We leave this line of research for the future.

1.5 Related Work

It is known that computing the tensor rank of a given tensor is NP-hard [35]; see also [37, 60, 63] for pertinent hardness results. It is also difficult in practice to determine the rank and border rank of small tensors; for example, the rank, border rank and border support rank of MM_2 are known to be seven [9, 36, 41, 70].

There has been extensive interest in constructing explicit tensors of high rank or border rank [1, 44, 48, 51]. This study is motivated by the need for new methods to provide lower complexity bounds. Currently, we do not know how to construct sequences of explicit tensors $T \in \mathbb{F}^d \otimes \mathbb{F}^d \otimes \mathbb{F}^d$ of rank or border rank above $3d$. In fact, there are no known examples of tensors with entries 0 or 1 and rank or border rank greater than $3d$. The only method to construct such tensors, is by making the entries incomparable in size, i.e. each entry is of different order of magnitude then other entries, or making them algebraically independent, which makes the tensors very far from explicit. However, it is easy to prove that tensors of super-linear border rank, with respect to their size exist. Precisely, for any constant $C > 0$ and d sufficiently large there exists a tensor $T \in (\mathbb{F}^d)^{\otimes 3}$ of border rank greater than Cd. Even more: general tensors will have greater border rank than Cd for d sufficiently large and the growth of maximal border rank is quadratic in d. We simply do not know how to provide explicit examples of such tensors, as the methods we have do not allow to prove that particular tensors have large rank. These obstructions are related to the fact that the cactus variety, that contains the secant variety, fills the whole ambient space, while most of the equations we know for secant varieties, also vanish on cactus varieties [12, 33, 32, 38, 49, 73]. We even do not know how to provide a sequence of explicit tensors so that infinitely many elements of the sequence would have high rank, say above $3d$.

The families of tensors with fixed support are also studied. An important concept is that of support rank [21], which has given rise to other support-based algorithm-design techniques (e.g. [3, 39]). Further, bounding asymptotic rank of special tensors was recently tied to NP-hard problems; in particular, Strassen's conjecture (Conjecture 1) would imply unexpected (but not polynomial) upper bounds on complexity of (randomized) algorithms for NP-hard problems [7, 59].

Beyond the present invariant Specht tensors, representation theory of the symmetric and general linear groups has extensive connections to algebraic complexity theory via Strassen's theory of asymptotic spectra (e.g. [18]) as well as the geometric complexity theory program (e.g. [56, 57]).

1.6 Organization of This Paper

Section 2 reviews notational preliminaries and definitions. Section 3 studies the composition basis and proves our main theorems on explicit universal sequences of tensors (Theorems 3 to 6). Section 4 introduces invariant Specht tensors and shows their universality for the exponent $\sigma(d)$ (Corollary 38). Section 5 relates the equations of varieties to bounds on asymptotic rank and proves Theorem 8.

2 Preliminaries

Recall that we write $[d] = \{1, 2, \ldots, d\}$. The set $[d]^q$ consists of sequences of length q of integers in $[d]$. We fix the canonical basis $e_1, e_2, \ldots, e_m \in \mathbb{F}^m$.

In this article, we exclusively work with tensors of format $a \times b \times c$, where in most cases $a = b = c$. These are elements of the vector space $\mathbb{F}^a \otimes \mathbb{F}^b \otimes \mathbb{F}^c \simeq \mathbb{F}^{a\times b\times c}$. In analogy to the case of matrices, the reader may freely think about tensors as three dimensional arrays filled with elements of $\mathbb{F}$. For a vector $v \in \mathbb{F}^a$ we write $v_i := e_i^*(v) \in \mathbb{F}$ for $1 \le i \le a$. We write $S_{i,j,k} \in \mathbb{F}$ for the entry of $S \in \mathbb{F}^a \otimes \mathbb{F}^b \otimes \mathbb{F}^c$ at position $(i, j, k) \in [a] \times [b] \times [c]$.

For three vectors $v_1 \in \mathbb{F}^a, v_2 \in \mathbb{F}^b$ and $v_3 \in \mathbb{F}^c$ we define the tensor $v_1 \otimes v_2 \otimes v_3 \in \mathbb{F}^a \otimes \mathbb{F}^b \otimes \mathbb{F}^c$ where the (i, j, k) coordinate equals $(v_1)_i(v_2)_j(v_3)_k$. Tensors of this form are called *rank one* tensors. The *rank* of a tensor T is the smallest r such that T is sum of r rank one tensors.

In case $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ we define the *border rank* of a tensor T as the smallest r such that in any neighbourhood of T there exists a tensor of rank r. For more details about rank and border rank we refer to [16, 42, 55]. The Kronecker power of a tensor is defined as in formula (3). *Asymptotic rank* [34] of a tensor T is defined as $\lim_{n\to\infty} \mathrm{R}(T^{\boxtimes n})^{\frac{1}{n}}$. We write $\langle s\rangle$ to be the unit tensor $\langle s\rangle = \sum_{i=1}^{s} e_i \otimes e_i \otimes e_i$.

3 Universal Sequences of Tensors for the Asymptotic Rank Conjecture

In this section we prove that the tensors $T^{(g)}$ in the composition basis (Definition 7) form a universal family for the asymptotic rank conjecture. We start with a more detailed analysis of the structure of the composition basis for fixed d (Section 3.1), and follow with an analysis for increasing d (Section 3.2); in particular, we establish the existence of the limit exponent $\lim_{d\to\infty} \sigma(d)$. Theorem 3 and Theorem 6 are restated and proved next (Section 3.3). Using techniques of Strassen, we then show that the universal sequences have nontrivially low tensor rank (Section 3.4). We end this section by setting up our techniques for support-localization (Section 3.5) as well as restate and prove Theorem 5, our main result for tight tensors (Section 3.6).

3.1 Properties of the Composition Basis

We start with the following easy lemma that provides various alternative characterisations of tensors $T^{(g)}$ in Definition 7.

We note that the symmetric group $\mathfrak{S}_q$ acts by permutations on $\mathbb{F}^{[d]^q}$ and diagonally on $(\mathbb{F}^{[d]^q})^{\otimes 3}$.

▶ **Lemma 9** (Equivalent definitions of the composition basis). *We have the following equivalent definitions of tensors $T^{(g)}$ for any $g \in \mathcal{C}_q^{[d]^3}$.*

1. *The tensors $T^{(g)}$ are precisely sums of $\mathfrak{S}_q$ orbits of canonical basis tensors in $(\mathbb{F}^{[d]^q})^{\otimes 3}$.*
2. *The tensors $T^{(g)}$ are coefficients of monomials for the Kronecker map $K_{d,q}$, explicitly:*

$$K_{d,q}(S) = \sum_{g \in \mathcal{C}_q^{[d]^3}} S^g T^{(g)} .$$

In particular, the linear span of the image is contained in the linear span of tensors $T^{(g)}$.

3. *If $|\mathbb{F}| > q$, then the tensors $T^{(g)}$, up to scaling by a constant, are precisely tensors with inclusion minimal supports in the linear span of the image of $K_{d,q}$.*

Proof. We note that the condition $\Phi_{I,J,K} = g$ from Definition 7 determines the triple (I, J, K) up to simultaneous action by an element of $\mathfrak{S}_q$. This proves equivalence of point (1) with the original definition.

The coordinates of $K_{d,p}(S)$ in the canonical basis are monomials of degree q in the coordinates of S. A monomial S^g appears exactly on the entries indexed by such (I, J, K)'s that $\Phi_{I,J,K} = g$. This proves equivalence of point (2) with Definition 7.

In Proposition 15, we show that the linear span of the image of $K_{d,q}$ coincides with the linear span of $T^{(g)}$'s for $g \in \mathcal{C}_q^{[d]^3}$ and $|\mathbb{F}| > q$. As $T^{(g)}$'s have disjoint supports, we obtain point (3). ◀

▶ Remark 10 (Invariant subspaces defined by marginals of g). It is immediate that $T^{(g)} \in (\mathbb{F}^{[d]^q})^{\otimes 3}$, however the tensor $T^{(g)}$ also belongs to smaller invariant subspace defined by g. Namely, for $g \in \mathcal{C}_q^{[d]^3}$ let $g_1 : [d] \to \mathbb{N}$ be the first marginal of g; that is, let $g_1(j) = \sum_{a,b \in [d]} g(j, a, b)$ for all $j \in [d]$. In the same way, define the second marginal g_2 and third marginal g_3. For $i = 1, 2, 3$, let $U_i \subseteq [d]^q$ be the set of all q-tuples $I \in [d]^q$ such that the value j appears in I exactly $g_i(j)$ times for all $j \in [d]$. We have $|U_i| = \binom{q}{g_i(1), g_i(2), \ldots, g_i(d)}$ as well as $T^{(g)} \in \mathbb{F}^{U_1} \otimes \mathbb{F}^{U_2} \otimes \mathbb{F}^{U_3}$. Clearly, the ambient space is $\mathfrak{S}_q$ invariant.

▶ **Example 11** (The small Coppersmith–Winograd tensor). Let $d = 2$, $q = 3$ and g assign value 1 on $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$. The tensor $T^{(g)}$ is the small Coppersmith–Winograd tensor in $(\mathbb{F}^3)^{\otimes 3}$, that is:

$$e_0 \otimes e_1 \otimes e_2 + e_0 \otimes e_2 \otimes e_1 + e_1 \otimes e_0 \otimes e_2 + e_1 \otimes e_2 \otimes e_0 + e_2 \otimes e_0 \otimes e_1 + e_2 \otimes e_1 \otimes e_0 .$$

▶ **Definition 12** (Linear span of the composition basis). *Let $L_{d,q}$ be the linear span of the tensors $T^{(g)}$ for $g \in \mathcal{C}_q^{[d]^3}$. As the tensors $T^{(g)}$ have disjoint supports they are linearly independent and hence form a basis of $L_{d,q}$. We note that* $\dim L_{d,q} = \binom{d^3+q-1}{q}$ *which is the cardinality of $\mathcal{C}_q^{[d]^3}$.*

▶ **Corollary 13** (Span coincides with the space of invariants). *The tensors $T^{(g)}$ for $g \in \mathcal{C}_q^{[d]^3}$ form a basis of the invariants space $((\mathbb{F}^{[d]^q})^{\otimes 3})^{\mathfrak{S}_q}$, which coincides with $L_{d,q}$. Up to rescaling, it is the unique basis of that space made of tensors with disjoint supports.*

From now on we will assume that the cardinality of the field is greater than q.

▶ **Lemma 14** (Dual space of homogeneous polynomials)**.** *There is an isomorphism of vector spaces: $L_{d,q}^*$ of linear forms and functions on $(\mathbb{F}^d)^{\otimes 3}$ given by homogeneous polynomials of degree q. The isomorphism sends a linear form l to the polynomial function $l \circ K_{d,q}$.*

Proof. Let $T^{(g)*}$ be the basis of $L_{d,q}^*$ dual to $T^{(g)}$. The image of the linear function $\sum \lambda_g T^{(g)*}$ is $\sum \lambda_g S^g$. Surjectivity is obvious.

By induction on the number n of variables, one proves that no polynomial of degree q may vanish identically on $\mathbb{F}^n$, when $|\mathbb{F}| > q$. Hence, no linear form l is mapped to the zero function and the linear map $l \mapsto l \circ K_{d,p}$ is injective. This finishes the proof. ◀

▶ **Proposition 15** (Span of the image of the Kronecker power map)**.** *The space $L_{d,q}$ is the linear span of the image of $K_{d,q}$.*

Proof. Clearly $L_{d,q}$ contains the image of $K_{d,q}$. If the containment was strict, there would exist a nonzero linear function $l \in L_{d,q}$ vanishing on the image. Then $l \circ K_{d,q} = 0$. This is not possible by Lemma 14. ◀

▶ Remark 16 (The Veronese map of degree q). Up to isomorphism, $K_{d,q}$ may be identified with the q^{th} Veronese map, that is a map defined by all degree q monomials. However, in coordinates each monomial may appear more than once, as each monomial represented by $g \in \mathcal{C}_q^{[d]^3}$ appears on the support of $T^{(g)}$.

▶ **Lemma 17** (Maximum rank in the composition basis controls rank in $L_{d,q}$)**.** *Let r be the maximum rank (respectively, asymptotic rank) of $T^{(g)}$ over $g \in \mathcal{C}_q^{[d]^3}$. Every tensor in $L_{d,q}$ has rank (respectively, asymptotic rank) at most*

$$r|\mathcal{C}_q^{[d]^3}| = r\binom{d^3 - 1 + q}{d^3 - 1}.$$

In particular, every tensor in $(\mathbb{F}^d)^{\otimes 3}$ has asymptotic rank at most

$$\left(r\binom{d^3 - 1 + q}{d^3 - 1}\right)^{\frac{1}{q}}.$$

Proof. Fix $T \in (\mathbb{F}^d)^{\otimes 3}$. By Lemma 9 the tensor $K_{d,q}(T)$ is a linear combination of $T^{(g)}$'s for $g \in \mathcal{C}_q^{[d]^3}$. As $|\mathcal{C}_q^{[d]^3}| = \binom{d^3-1+q}{d^3-1}$ we obtain:

$$\mathrm{R}(K_{d,q}(T)) \leq r\binom{d^3 - 1 + q}{d^3 - 1}.$$

The statement under the assumption of asymptotic rank r for $T^{(g)}$'s is proved in the same way, noting that asymptotic rank is subadditive and submultiplicative (e.g. [76]). ◀

3.2 The Extended Asymptotic Rank Conjecture and the Limit Exponent

We are now ready to connect the composition-basis tensors $T^{(g)}$ to the extended asymptotic rank conjecture (Conjecture 2).

▶ **Corollary 18** (The composition basis suffices for the extended asymptotic rank conjecture)**.** *If there exists an infinite set $S = \{d_1, d_2, \ldots\} \subseteq \mathbb{Z}_{\geq 1}$ such that for any d_i there exist infinitely many $q_{i,j} \in \mathbb{Z}_{\geq 1}$ such that Conjecture 2 holds for the tensor $T^{(g)}$ for all $g \in \mathcal{C}_{q_{i,j}}^{[d_i]^3}$, then Conjecture 2 holds for all tensors.*

Proof. First fix d_i. By Lemma 17 we see that any $T \in ((\mathbb{F}^{d_i})^{\otimes 3})$ has asymptotic rank at most:

$$(d_i^q)^{\frac{1}{q}} \binom{d_i^3 - 1 + q}{d_i^3 - 1}^{\frac{1}{q}} .$$

Note that for fixed d_i we have

$$\lim_{q \to \infty} \binom{d_i^3 - 1 + q}{d_i^3 - 1}^{\frac{1}{q}} = 1 .$$

This confirms Conjecture 2 for any $T \in (\mathbb{F}^{d_i})^{\otimes 3}$. We conclude that it must hold for each d from the next Lemma 19. ◀

▶ **Lemma 19** (Existence of the limit exponent). *The limit* $\lim_{d\to\infty} \sigma(d)$ *exists and equals the supremum of the set* $\{\sigma(d) : d \in \mathbb{Z}_{\geq 1}\}$*.*

Proof. As the sequence $\sigma(d)$ is bounded it is enough to prove:

$$\forall_{d_0 \in \mathbb{Z}_{\geq 1}} \forall_{\epsilon > 0} \exists_{D \in \mathbb{Z}_{\geq 1}} \forall_{d > D} \quad \sigma(d) > \sigma(d_0) - \epsilon .$$

Fix $\epsilon > 0$ and $d_0 \in \mathbb{Z}_{\geq 1}$. For contradiction assume there are infinitely many d_i such that $\sigma(d_i) \leq \sigma(d_0) - \epsilon$. Hence, we also obtain infinitely many k_i such that $d_0^{k_i} \leq d_i < d_0^{k_i+1}$.

We note that we have $\sigma(d_0^{k_i}) \geq \sigma(d_0)$ as taking Kronecker power of a tensor does not change the exponent. Hence, for any $\delta > 0$ there exists $T_\delta \in (\mathbb{F}^{d_0^{k_i}})^{\otimes 3}$ of asymptotic rank at least $d_0^{k_i(\sigma(d_0)-\delta)}$. As we may embed T_δ in larger space, we see that its asymptotic rank is at most $d_i^{\sigma(d_i)} \leq d_0^{(k_i+1)(\sigma(d_0)-\epsilon)}$. We obtain:

$$k_i(\sigma(d_0) - \delta) \leq (k_i + 1)(\sigma(d_0) - \epsilon).$$

As this holds for any $\delta > 0$ we obtain:

$$k_i\sigma(d_0) \leq (k_i + 1)(\sigma(d_0) - \epsilon),$$

which is a contradiction for k_i sufficiently large. ◀

The following more precise result shows that the exponent governing the asymptotic rank is completely determined via asymptotic ranks of the tensors $T^{(g)}$.

▶ **Corollary 20** (The composition bases govern tensor exponents). *For all* $\tau \geq 0$ *it holds that each tensor* $T^{(g)} \in (\mathbb{F}^m)^{\otimes 3}$ *(where* m *is uniquely determined by* g*) has asymptotic rank at most* m^τ *if and only if any tensor* $T \in (\mathbb{F}^d)^{\otimes 3}$ *(where* d *is arbitrary) has asymptotic rank at most* d^τ*. In particular, for any tensor* T *we have* $\sigma(T) \leq \sup_g \sigma(T^{(g)})$*.*

Proof. The implication "⇐" is immediate. For the implication "⇒", assume that a tensor T has asymptotic rank greater than $d^{\tau+\epsilon}$ for fixed $\epsilon > 0$. By applying Lemma 17 for $q \gg 0$ such that $\binom{d^3-1+q}{d^3-1}^{\frac{1}{q}} < d^\epsilon$ we obtain a contradiction. ◀

We note that there are countably many tensors $T^{(g)}$ and they form a sequence of explicit tensors. This sequence gives rise to the following dichotomy.

▶ **Corollary 21** (A dichotomy on improved algorithms or explicit rank lower bounds)**.** *We have the following dichotomy. Either*

(i) *the extended asymptotic rank conjecture (Conjecture 2) holds; that is, for all tensors T we have $\sigma(T) \leq 1$, which implies $\omega = 2$ and disproves the Set Cover Conjecture; or*

(ii) *the tensors $T^{(g)}$ form a sequence of explicit tensors such that for any constant $C \in \mathbb{Z}_{\geq 1}$ there exist infinitely many $T^{(g)} \in (\mathbb{F}^m)^{\otimes 3}$, such that rank, border rank, and asymptotic rank is greater than Cm.*

It is clear that for special g the rank of $T^{(g)}$ can be small. For example if g is supported at one element, then $T^{(g)}$ has support of cardinality one and thus is of rank one.

3.3 Explicit Universal Sequences and Main Results

We first prove our main theorem for fixed d, and define the universal sequence accordingly.

▶ **Definition 22** (Universal sequence for a fixed d)**.** *For a fixed positive integer d, we define the sequence $\mathcal{U}_d = (U_{d,q} : q = 1, 2, \ldots)$ of tensors, where we set*

$$U_{d,q} = \bigoplus_{g \in \mathcal{C}_q^{[d]^3}} T^{(g)} .$$

We are now ready to prove our main theorem; Theorem 3 is restated below for convenience.

▶ **Theorem 23** (Main; Explicit universal sequences of tensors)**.** *For all $d \geq 1$ we have $\sigma(\mathcal{U}_d) = \sigma(d)$.*

Proof. We prove the equality by proving both inequalities.

First, suppose that $\sigma(\mathcal{U}_d) < \sigma$ for some constant σ. Fix an $\epsilon > 0$ so that for q sufficiently large $\mathrm{R}(U_{d,q}) < (|\mathcal{C}_q^{[d]^3}|d^q)^{\sigma-\epsilon}$. Then, for q sufficiently large and all $g \in \mathcal{C}_q^{[d]^3}$ we have $\mathrm{R}(T^{(g)}) \leq \mathrm{R}(U_{d,q}) < (|\mathcal{C}_q^{[d]^3}|d^q)^{\sigma-\epsilon}$. As $\lim_{q\to\infty} |\mathcal{C}_q^{[d]^3}|^{1/q} = 1$ we see that for q sufficiently large and all $g \in \mathcal{C}_q^{[d]^3}$ we have $\sigma(T^{(g)}) \leq \sigma$. Applying Lemma 17 and taking limit, as $q \to \infty$ we see that $\sigma(T) \leq \sigma$ for all $T \in (\mathbb{F}^d)^{\otimes 3}$.

Suppose now that for all $T \in (\mathbb{F}^d)^{\otimes 3}$ we have $\sigma(T) < \sigma$. Fix $\epsilon > 0$, such that $\sigma(T) < \sigma - \epsilon$ for all $T \in (\mathbb{F}^d)^{\otimes 3}$. By Proposition 15 for any $g \in \mathcal{C}_q^{[d]^3}$ the tensor $T^{(g)}$ is a sum of at most $|\mathcal{C}_q^{[d]^3}|$-many tensors of type $K_{d,q}(T)$. Hence, for q sufficiently large we have:

$$\mathrm{R}(T^{(g)}) \leq |\mathcal{C}_q^{[d]^3}|d^{q(\sigma-\epsilon)} .$$

It follows that:

$$\mathrm{R}(U_{d,q}) \leq |\mathcal{C}_q^{[d]^3}|^2 d^{q(\sigma-\epsilon)} .$$

As $\lim_{q\to\infty} |\mathcal{C}_q^{[d]^3}|/d^{\epsilon q} = 0$ we see that $\mathrm{R}(U_{d,q}) \leq (|\mathcal{C}_q^{[d]^3}|d^q)^{\sigma}$ for $q \gg 0$. Thus $\sigma(\mathcal{U}_d) \leq \sigma$, which finishes the proof. ◀

Next, we construct one universal sequence realizing the worst possible exponent, irrespective of d via a diagonal argument on the sequences $\mathcal{U}_d$. Accordingly, recalling Definition 22, define the diagonal sequence $\mathcal{D} = (D_d : d = 1, 2, \ldots)$ of tensors for all positive integers d by setting $D_d = U_{d,d^4}$. It is immediate that the sequence is explicit. Theorem 6 is restated below for convenience.

▶ **Theorem 24** (An explicit universal sequence for the extended asymptotic rank conjecture)**.** *The sequence $\mathcal{D}$ satisfies $\lim_{d\to\infty} \sigma(D_d) = \lim_{d\to\infty} \sigma(d)$.*

Proof. By Lemma 19 it is enough to prove that for every $\epsilon > 0$, for all d sufficiently large we have $\sigma(U_{d,d^4}) > \sigma(d) - \epsilon$. For any (concise) $T \in (\mathbb{F}^d)^{\otimes 3}$ we have:

$$d^{d^4 \sigma(T)} = \tilde{\mathrm{R}}(T)^{d^4} \le \sum_{g \in \mathcal{C}_{d^4}^{[d]^3}} \tilde{\mathrm{R}}(T^{(g)}) \le |\mathcal{C}_{d^4}^{[d]^3}| \tilde{\mathrm{R}}(U_{d,d^4}) \le |\mathcal{C}_{d^4}^{[d]^3}| (|\mathcal{C}_{d^4}^{[d]^3}| d^{d^4})^{\sigma(U_{d,d^4})} .$$

After taking root of order d^4 we only need to prove that $\lim_{d\to\infty} |\mathcal{C}_{d^4}^{[d]^3}|^{1/d^4} = 1$. We have:

$$\lim_{d\to\infty} |\mathcal{C}_{d^4}^{[d]^3}|^{1/d^4} = \lim_{d\to\infty} \binom{d^4 + d^3 - 1}{d^3 - 1}^{1/d^4} \le \lim_{d\to\infty} (2d^4)^{d^3/d^4} = 1 ,$$

which finishes the proof. ◀

3.4 Upper Bounds on Tensor Rank in the Universal Sequences

A natural question arises: what are ranks of the tensors $T^{(g)}$. In order to provide upper bounds, we recall the following lemma due to Strassen (implicit in [67, Proposition 3.6]).

▶ **Lemma 25** (Limited universality of matrix multiplication tensors)**.** *The matrix multiplication tensor* MM_{d^2} *degenerates to* $T^{\boxtimes 3}$ *for any tensor* $T \in (\mathbb{F}^d)^{\otimes 3}$*. In particular,* $\sigma(T) \le 2\omega/3$*.*

Proof. Let us write in coordinates $T = (\lambda_{p,q,r})$ and

$$\mathrm{MM}_{d^2} = \sum_{i_1,i_2,j_1,j_2,k_1,k_2 \in [d]} e_{j_1,j_2}^{i_1,i_2} \otimes e_{k_1,k_2}^{j_1,j_2} \otimes e_{i_1,i_2}^{k_1,k_2} .$$

For $i = 1, 2, 3$, define $l_i : \mathbb{F}^{d^4} \to \mathbb{F}^{d^3}$ as follows:

$$l_1(e_{j_1,j_2}^{i_1,i_2}) = \sum_{a\in[d]} \lambda_{a,j_1,i_1} e_{a,j_2,i_2} ,$$

$$l_2(e_{k_1,k_2}^{j_1,j_2}) = \sum_{b\in[d]} \lambda_{j_2,b,k_1} e_{j_1,b,k_2} ,$$

$$l_3(e_{i_1,i_2}^{k_1,k_2}) = \sum_{c\in[d]} \lambda_{i_2,k_2,c} e_{i_1,k_1,c} .$$

We have $l_1 \otimes l_2 \otimes l_3(\mathrm{MM}_{d^2}) = T^{\boxtimes 3}$. ◀

In a similar way we obtain:

▶ **Lemma 26** (Limited universality of matrix multiplication under powers)**.** *The matrix multiplication tensor* $\mathrm{MM}_{d^{2k}}$ *degenerates to* $T^{\boxtimes 3k}$ *for any tensor* $T \in (\mathbb{F}^d)^{\otimes 3}$*.*

We obtain the following corollary, which currently is asymptotically in $k \gg 0$ the best one we know, that works for arbitrary $g \in \mathcal{C}_{3k}^{[d]^3}$.

▶ **Corollary 27** (Upper bound on rank via matrix multiplication)**.** *For any* $g \in \mathcal{C}_{3k}^{[d]^3}$ *the tensor* $T^{(g)}$ *has rank at most* $\mathrm{R}(\mathrm{MM}_{d^{2k}}) \cdot \binom{d^3-1+3k}{d^3-1}$*.*

▶ Remark 28 (Note on special g). We note that Corollary 27 may be improved for many special g. For any $g \in \mathcal{C}_{3k}^{[d]^3}$ we have three marginal distributions on $[d]$. If such a distribution is close to uniform, then we obtain about d^k compatible sequences of length k; that is, asymptotically every sequence is compatible. However, if the distribution is far from uniform, we obtain less sequences, say d^{ck} for some $c < 1$. In such a case $\mathrm{MM}_{d^{2k}}$ in Lemma 26 may be changed into smaller matrix multiplication.

3.5 Support-Localized Universality

Many of our results remain true if we consider tensors $T \in (\mathbb{F}^d)^{\otimes 3}$ with support contained in a nonempty $\Delta \subseteq [d]^3$. Let us write $\mathbb{F}^\Delta$ be the linear space of tensors with such a support. As before $K_{d,q}$ restricted to $\mathbb{F}^\Delta$ is still a Veronese map, simply in variables corresponding to elements from Δ. The image of $K_{d,q}|_{\mathbb{F}^\Delta}$ has a basis made of tensors $T^{(g)}$ for $g \in \mathcal{C}_q^\Delta$. Below we present a variant of Lemma 17, which proof is analogous.

▶ **Lemma 29** (Maximum rank in the support-localized composition basis controls rank in $\mathbb{F}^\Delta$)**.** *Let r be the maximum rank (respectively, asymptotic rank) of $T^{(g)}$ over $g \in \mathcal{C}_q^\Delta$. The q^{th} Kronecker power of every tensor in $\mathbb{F}^\Delta$ has rank (respectively, asymptotic rank) at most*

$$r|\mathcal{C}_q^\Delta| = r\binom{|\Delta|-1+q}{|\Delta|-1}.$$

In particular, every tensor in $(\mathbb{F}^d)^{\otimes 3}$ with support contained in Δ has asymptotic rank at most

$$\left(r\binom{|\Delta|-1+q}{|\Delta|-1}\right)^{\frac{1}{q}}.$$

Next, we present a variant of Strassen's result for tensors with fixed support. For $\Delta \subseteq [d]^3$, we will use the following notation: $\Delta_{1,2} = \{(i,j) : \exists_k (i,j,k) \in \Delta\}$, $\Delta_{1,3} = \{(i,k) : \exists_j (i,j,k) \in \Delta\}$, and $\Delta_{2,3} = \{(j,k) : \exists_i (i,j,k) \in \Delta\}$.

Consider the following support-localized restriction of the matrix multiplication tensor MM_{d^2}:

$$\mathrm{MM}_\Delta := \sum_{\substack{(j_1,i_1)\in\Delta_{2,3}\\(j_2,k_1)\in\Delta_{1,3}\\(i_2,k_2)\in\Delta_{1,2}}} e_{j_1,j_2}^{i_1,i_2} \otimes e_{k_1,k_2}^{j_1,j_2} \otimes e_{i_1,i_2}^{k_1,k_2}.$$

We obtain the following version of Lemma 25.

▶ **Lemma 30** (Limited universality of support-localized matrix multiplication)**.** *The tensor MM_Δ degenerates to $T^{\boxtimes 3}$ for any tensor $T \in (\mathbb{F}^d)^{\otimes 3}$ with support contained in Δ.*

▶ **Corollary 31** (Upper bound on rank via support-localized matrix multiplication)**.** *For any $g \in \mathcal{C}_{3k}^\Delta$ the tensor $T^{(g)}$ has rank at most $\mathrm{R}(\mathrm{MM}_\Delta) \cdot \binom{|\Delta|-1+3k}{|\Delta|-1}$.*

Next we present the proof of Theorem 4.

▶ **Theorem 32** (Support-localized universal sequences of tensors)**.** *For all nonempty $\Delta \subseteq [d] \times [d] \times [d]$ there is an explicit sequence $\mathcal{U}_\Delta$ of zero-one-valued tensors with $\sigma(\mathcal{U}_\Delta) = \sigma(\Delta)$.*

Proof. Let $\mathcal{U}_\Delta = (U_{\Delta,q} : q = 1, 2, \ldots)$ for

$$U_{\Delta,q} = \bigoplus_{g \in \mathcal{C}_q^\Delta} T^{(g)}.$$

As the image of $K_{d,q}|_{\mathbb{F}^\Delta}$ has a basis made of tensors $T^{(g)}$ for $g \in \mathcal{C}_q^\Delta$ the proof is analogous to the proof of Theorem 23 where we replace reference to Lemma 17 by reference to Lemma 29. ◀

3.6 Tight Tensors and Strassen's Conjecture

We now proceed to our main theorem for tight tensors and fixed d; we start by defining a corresponding universal sequence.

▶ **Definition 33** (Universal sequence for tight tensors and fixed d). *For a fixed positive integer d, we define the sequence $\mathcal{T}_d = (T_{d,q} : q = 1, 2, \ldots)$ of tensors, where we set*

$$T_{d,q} = \bigoplus_{\substack{\Delta \subseteq [d]^3 \\ \Delta \text{ tight}}} U_{\Delta,q} = \bigoplus_{\substack{\Delta \subseteq [d]^3 \\ \Delta \text{ tight}}} \bigoplus_{g \in \mathcal{C}_q^{\Delta}} T^{(g)} .$$

Next we prove Theorem 5, which we restate below for convenience.

▶ **Theorem 34** (A universal sequence of tensors for Strassen's conjecture). *We have $\sigma(\mathcal{T}_d) = \sigma(\mathfrak{T}_d)$ and each $T_{d,q}$ is tight.*

Proof. First, we prove that for $g \in \mathcal{C}_q^{\Delta}$, where Δ is tight, the tensor $T^{(g)}$ is tight. Indeed, if we take any tensor T with support Δ, it is tight, thus so is $T^{\boxtimes q}$. As $T^{(g)}$ has smaller support it is tight. As direct sum of tight tensors is tight, we see that each $T_{d,q}$ is tight.

Next we note that for fixed d each $T_{d,q}$ is a sum of a fixed number of tensors $U_{\Delta,q}$. Thus, there exists a constant C such that:

$$\max_{\substack{\Delta \subseteq [d]^3 \\ \Delta \text{ tight}}} \mathrm{R}(U_{\Delta,q}) \leq \mathrm{R}(T_{d,q}) \leq C \max_{\substack{\Delta \subseteq [d]^3 \\ \Delta \text{ tight}}} \mathrm{R}(U_{\Delta,q})$$

and the same inequalities hold for the size of the tensors. For every tight T, we may find tight Δ such that $\sigma(T) \leq \sigma(\Delta)$. By Theorem 4 this equals $\sigma(\mathcal{U}_\Delta)$ and by the inequality above this is upper bounded by $\sigma(\mathcal{T}_d)$.

For the other inequality, we see that $\sigma(\mathcal{T}_d)$ is upper bounded by $\sigma(\mathcal{U}_\Delta)$ for some tight Δ. Clearly, $\sigma(\mathcal{U}_\Delta) \leq \sigma(\mathfrak{T}_d)$ which finishes the proof. ◀

4 Representation Theory and Universality of Specht Tensors

Let us now take a representation-theoretic view to the invariant subspace $(\mathbb{C}^{[d]^p \times [d]^p \times [d]^p})^{\mathfrak{S}_p}$ and start by observing that the $\mathfrak{S}_p$-module $\mathbb{C}^{[d]^p \times [d]^p \times [d]^p}$ is isomorphic to the $\mathfrak{S}_p$-module $\mathbb{C}^{d^p} \otimes \mathbb{C}^{d^p} \otimes \mathbb{C}^{d^p}$. A k-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ is an *(integer) partition* of the integer $p = \lambda_1 + \lambda_2 + \ldots + \lambda_k$ into k *parts*. We write $\mathcal{I}(p)$ for the set of all integer partitions of p and $\mathcal{I}(p, d)$ for the set of all integer partitions with at most d parts. For $\lambda \in \mathcal{I}(p)$ we write $\lambda! = (\lambda_1!)(\lambda_2!) \cdots (\lambda_k!)$. For $\lambda \in \mathcal{I}(p, d)$ we write $D(\lambda) = (d-k)! \prod_{\ell=1}^{p} (|\{j \in [k] : \lambda_j = \ell\}|!)$. For $\lambda \in \mathcal{I}(p)$, let us write M^λ for the $\mathfrak{S}_p$-permutation module on Young tabloids of shape λ. Recall that the dimension of M^λ is $p!/\lambda!$. We have the $\mathfrak{S}_p$-module isomorphism

$$\mathbb{C}^{d^p} \cong \bigoplus_{\lambda \in \mathcal{I}(p,d)} \frac{d!}{D(\lambda)} M^\lambda$$

and thus

$$\mathbb{C}^{d^p} \otimes \mathbb{C}^{d^p} \otimes \mathbb{C}^{d^p} \cong \bigoplus_{\mu,\nu,\lambda \in \mathcal{I}(p,d)} \frac{d!}{D(\mu)} \frac{d!}{D(\nu)} \frac{d!}{D(\lambda)} M^\mu \otimes M^\nu \otimes M^\lambda . \tag{9}$$

For $\alpha \in \mathcal{I}(p)$, let us write S^α for the Specht module associated with α. For $\alpha, \mu \in \mathcal{I}(p)$, let us write $m_{\alpha,\mu}$ for the Kostka numbers so that the $\mathfrak{S}_p$-module isomorphism

$$M^\mu \cong \bigoplus_{\alpha \in \mathcal{I}(p)} m_{\alpha,\mu} S^\alpha \tag{10}$$

holds. We have the following elementary lemma that puts a strong restriction on the Specht modules that need to be considered for small values of d in (9) and (10). Recall that a partition $(\alpha_1, \alpha_2, \dots)$ is greater or equal to a partition $(\mu_1, \mu_2, \dots)$ in the *dominance order* if and only if for any i, we have $\sum_{j=0}^{i} \alpha_j \geq \sum_{j=0}^{i} \mu_j$. Here, if one of partitions, say α, has k parts, we let $\alpha_j = 0$ for $j > k$.

▶ **Lemma 35** (Restriction to at most d parts)**.** *For $\alpha, \mu \in \mathcal{I}(p)$ we have $m_{\alpha,\mu} \neq 0$ if and only if α is larger than μ in the dominance order. In particular, for all $\mu \in \mathcal{I}(p, d)$ we have $m_{\alpha,\mu} = 0$ unless $\alpha \in \mathcal{I}(p, d)$.*

Proof. See e.g. [64, p. 315]. ◀

From (9), (10), and Lemma 35 we thus have

$$\mathbb{C}^{d^p} \otimes \mathbb{C}^{d^p} \otimes \mathbb{C}^{d^p} \cong \bigoplus_{\mu,\nu,\lambda \in \mathcal{I}(p,d)} \frac{d!}{D(\mu)} \frac{d!}{D(\nu)} \frac{d!}{D(\lambda)} \bigoplus_{\alpha,\beta,\gamma \in \mathcal{I}(p,d)} m_{\alpha,\mu} m_{\beta,\nu} m_{\gamma,\lambda} S^\alpha \otimes S^\beta \otimes S^\gamma \,. \tag{11}$$

Writing $K(\alpha, \beta, \gamma)$ for the Kronecker coefficient of $\alpha, \beta, \gamma \in \mathcal{I}(p)$, we have that the invariant space $(S^\alpha \otimes S^\beta \otimes S^\gamma)^{\mathfrak{S}_p}$ has dimension $K(\alpha, \beta, \gamma)$. Since $(\mathbb{C}^{[d]^p \times [d]^p \times [d]^p})^{\mathfrak{S}_p}$ has dimension $\binom{d^3-1+p}{d^3-1}$, from (11) we immediately have

$$\binom{d^3 - 1 + p}{d^3 - 1} = \sum_{\mu,\nu,\lambda \in \mathcal{I}(p,d)} \frac{d!}{D(\mu)} \frac{d!}{D(\nu)} \frac{d!}{D(\lambda)} \sum_{\alpha,\beta,\gamma \in \mathcal{I}(p,d)} m_{\alpha,\mu} m_{\beta,\nu} m_{\gamma,\lambda} K(\alpha, \beta, \gamma) \,. \tag{12}$$

Assuming that d is fixed and p grows, we observe that the left-hand side of (12) grows at most polynomially in p, and all summands on the right-hand side are nonnegative, implying that every summand on the right-hand grows at most polynomially in p. The above observations motivate the following definition.

▶ **Definition 36.** *For any three Specht modules $S^\alpha, S^\beta, S^\gamma$, where α, β, γ are partitions of p, we call any element of $(S^\alpha \otimes S^\beta \otimes S^\gamma)^{\mathfrak{S}_p}$ an* invariant Specht tensor *with* margin *α, β, γ and* degree *p.*

▶ Remark 37. For fixed α, β, γ the dimension of the space of invariant Specht tensors equals the Kronecker coefficient $K(\alpha, \beta, \gamma)$.

By Lemma 35 and Equation (11) only invariant Specht tensors with margins that have at most d parts appear in the decomposition of $(\mathbb{C}^{[d]^p \times [d]^p \times [d]^p})^{\mathfrak{S}_p}$ and each such tensor does appear. Further, for fixed d the dimension of all invariant Specht tensors with arbitrary margins $\alpha, \beta, \gamma \in \mathcal{I}(p, d)$ is polynomial in p. Thus, we can equivalently define $\sigma(d)$ as the infimum of all positive real numbers such that for all $\alpha, \beta, \gamma \in \mathcal{I}(p, d)$ the maximum tensor rank in $(S^\alpha \otimes S^\beta \otimes S^\gamma)^{\mathfrak{S}_p}$ is at most $d^{\sigma(d)p+o(p)}$. We obtain the following corollary.

▶ **Corollary 38.** *Fix an integer $d \geq 2$. Then $\sigma(d)$ is the infimum of all positive real numbers such that for all $\alpha, \beta, \gamma \in \mathcal{I}(p, d)$ the maximum tensor rank of an invariant Specht tensor is at most $d^{\sigma(d)p+o(p)}$.*

The above corollary motivates the following questions:

- What is the maximum tensor rank for an invariant Specht tensor of degree p with margins in $\mathcal{I}(p,d)$ as p grows?
- Which margins witness this maximum?

Specht modules and tensor products of Specht modules have considerable structure; e.g. [77]. As highlighted above, already the case $d=3$ is interesting.

▶ Remark 39. Our results allow to translate between asymptotic rank and rank of invariant Specht tensors. As $\sigma(d) \leq \frac{2\omega}{3} < 2$, we see that in the range $\alpha, \beta, \gamma \in \mathcal{I}(p,d)$ one indeed obtains nontrivial upper bounds on ranks of invaiant Specht tensors; that is, such tensors in general do not have generic ranks in their ambient spaces $S^\alpha \otimes S^\beta \otimes S^\gamma$.

5 Equations of Secant Varieties and Asymptotic Rank

In this section we relate existence of polynomials vanishing on special varieties, like secant varieties, and asymptotic rank. In particular, we obtain upper-bound control on $\sigma(d)$ via the *absence* of low-degree equations of secants.

5.1 Equations and Bounds on $\sigma(d)$

We start with a generalization of Proposition 15.

▶ **Proposition 40** (Span of the image of a subset). *For any subset $S \subseteq (\mathbb{F}^d)^{\otimes 3}$, the following are equivalent:*

1. *no homogeneous polynomial of degree p vanishes on S,*
2. *$K_{d,p}(S)$ linearly spans $L_{d,p}$.*

Proof. The second condition is equivalent to the fact that no linear form in $L^*_{d,p}$ vanishes on $K_{d,p}(S)$. By Lemma 14 this is also equivalent to the first condition. ◀

▶ **Corollary 41** (Absence of low-degree equations implies low asymptotic rank). *Let X be the Segre variety of rank one tensors in $(\mathbb{F}^d)^{\otimes 3}$. Suppose that for fixed n no polynomial of degree p vanishes on the n^{th} secant variety $\sigma_n(X)$. Then every tensor in $L_{d,p}$ has rank at most*

$$\binom{d^3-1+p}{d^3-1} n^p .$$

Every tensor in $(\mathbb{F}^d)^{\otimes 3}$ has asymptotic rank at most

$$n\binom{d^3-1+p}{d^3-1}^{\frac{1}{p}} .$$

Proof. Note that $\dim L_{d,p} = \binom{d^3-1+p}{d^3-1}$ and by Proposition 40 we have a basis of $L_{d,p}$ made of tensors of rank at most n^p. The last statement follows. ◀

A more general, but less explicit version is given by Theorem 8, which we derive as the following corollary.

▶ **Corollary 42** (Absence of low-degree equations implies low asymptotic rank; implicit version). *Let $Y \subseteq (\mathbb{F}^d)^{\otimes 3}$ be a subset with the property that for all $T \in Y$ the asymptotic rank of T is at most n. Suppose that no homogeneous polynomial of degree p vanishes on Y. Then, every tensor in $(\mathbb{F}^d)^{\otimes 3}$ has asymptotic rank at most*

$$n\binom{d^3-1+p}{d^3-1}^{\frac{1}{p}} .$$

▶ Remark 43. The dimension component $\binom{d^3-1+p}{d^3-1}$ may be improved (by going to border rank) by the smallest s, such that $\sigma_s(K_{d,p}(\sigma_n(X))) = L_{d,p}$. The latter number is in most cases much smaller then $\dim L_{d,p}$. Each particular case may be exactly studied by the Teraccini Lemma.

Alternatively we could greedily, starting from rank one and going up, take tensors T_i such that $K_{d,p}(T_i)$ are linearly independent, building a basis of $L_{d,p}$.

For tensors belonging to a fixed subspace $W \subseteq (\mathbb{F}^d)^{\otimes 3}$ we also have the following variant.

▶ **Theorem 44.** *Let $Y \subseteq W \subseteq (\mathbb{F}^d)^{\otimes 3}$ be a subset with the property that for all $T \in Y$ the asymptotic rank of T is at most n. Suppose that no homogeneous polynomial on W of degree p vanishes on Y. Then every tensor in W has asymptotic rank at most*

$$n\binom{\dim W - 1 + p}{\dim W - 1}^{\frac{1}{p}}.$$

Proof. We recall that the restriction of the Veronese map of degree p to any linear subspace is still a Veronese map of degree p; that is, a map defined by polynomials spanning the space of degree p polynomials, simply in smaller number of variables.

We restrict $K_{d,p}$ to W. The linear span of $K_{d,p}(W)$ is spanned by $K_{d,p}(Y)$ and has dimension $\binom{\dim W-1+p}{\dim W-1}$. Hence for any tensor $T \in W$ the asymptotic rank of $K_{d,p}(T)$ is at most:

$$n^p\binom{\dim W - 1 + p}{\dim W - 1}.$$

As $K_{d,p}(T)$ is the p^{th} Kronecker power of T, the statement follows. ◀

The previous result could be particularly useful if we can provide many examples of low rank tensors inside a linear subspace.

5.2 Equations of Secant Varieties and Applications

We note that deciding if there exists a nonzero polynomial of degree p on $(\mathbb{F}^d)^{\otimes 3}$ vanishing on tensors of rank r is a question in linear algebra. The naïve approach is a as follows.

Consider $3r$ vectors $v_{i,j} \in \mathbb{F}^d$ where $1 \le i \le 3$ and $1 \le j \le r$ with coordinates $(v_{i,j})_k$ treated as variables. The tensor $T := \sum_{j=1}^{r} v_{1,j} \otimes v_{2,j} \otimes v_{3,j}$ is a general tensor of rank r. We have:

$$T_{a,b,c} = \sum_{j=1}^{r}(v_{1,j})_a(v_{2,j})_b(v_{3,j})_c\,.$$

If we assign the weight $(1,0,0) \in \mathbb{Z}^3$ to each $(v_{1,j})_a$, the weight $(0,1,0)$ to each $(v_{2,j})_b$, and the weight $(0,0,1)$ to each $(v_{3,j})_c$, we see that an evaluation of a degree p polynomial P on T is a degree (p,p,p) polynomial. Furthermore, P vanishes on all tensors of rank at most r if and only if $P(T) = 0$ as a polynomial. To check the last condition we may build a matrix $N_{d,r,p}$ with:

1. $\binom{p+d^3-1}{p}$ columns indexed by $\mathcal{C}_p^{[d]^3}$, equivalently monomials of degree p,
2. $\binom{p+dr-1}{p}^3$ rows indexed by monomials of degree (p,p,p) in variables $(v_{i,j})_k$,
3. the entry in a column indexed by a monomial $g \in \mathcal{C}_p^{[d]^3}$ and a row indexed by a monomial m is the coefficient of m in the polynomial $g(T)$.

The kernel of $N_{d,r,p}$ is naturally identified with the space of homogeneous degree p polyomials vanishing on rank r tensors. In particular, the kernel is trivial if and only if there are no such polynomials. While this approach is very general, it is also not applicable in most cases due to the large sizes of the matrices $N_{d,r,p}$.

A better approach to understand the equations of secant varieties is to exploit group actions. For this, one decomposes the space $S^p(\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d)$ of homogeneous degree p polynomials on the tensor space as a direct sum of irreducible $G := GL(n) \times GL(n) \times GL(n)$ representations. The irreducible polynomial representations of G are indexed by triples of Young diagrams, in our case each one in the triple will have p elements:

$$S^p(\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d) = \bigoplus V_{\lambda,\mu,\nu}^{\bigoplus g_{\lambda,\mu,\nu}},$$

where $V_{\lambda,\mu,\nu}$ is the triple corresponding to three Young diagrams λ, μ, ν with p boxes each and $V_{\lambda,\mu,\nu}$ is the Kronecker coefficient. The vector space of homogeneous polynomials of degree p that vanish on the n^{th} secant variety is a subrepresentation and hence also may be decomposed as:

$$\bigoplus V_{\lambda,\mu,\nu}^{\bigoplus a_{\lambda,\mu,\nu}},$$

where now the coefficients $a_{\lambda,\mu,\nu} \leq g_{\lambda,\mu,\nu}$ are unknown. One way to determine them is to consider highest weigh space $HWS_{\lambda,\mu,\nu}$, that is a vector space of dimension $g_{\lambda,\mu,\nu}$ inside $V_{\lambda,\mu,\nu}^{\bigoplus g_{\lambda,\mu,\nu}}$. It turns out that $a_{\lambda,\mu,\nu}$ equals the dimension of the subspace of $HWS_{\lambda,\mu,\nu}$ consisting on those polynomials that vanish on the n^{th} secant variety. Finding the basis of $HWS_{\lambda,\mu,\nu}$ and efficiently evaluating it on points of the secant variety is an art on its own. However, in many examples, the procedure above was successfully carried out in practice. We refer to [11, 36] for details.

▶ **Example 45.** The generic rank in $\mathbb{C}^7 \otimes \mathbb{C}^7 \otimes \mathbb{C}^7$ is 19. Tensors of border rank 18 form a hypersurface with the defining equation of degree at least 187000 [36, Section 3.2]. Applying Corollary 41 we see that every tensor in that space has asymptotic rank smaller than 18.25.

We believe that many further results on asymptotic rank may be obtained in a similar way, through Corollary 41 and 42, especially in combination with Remark 43. We leave this for future work.

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