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# On hp-Adaptive Solution of Thin Shells of Revolution

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**Abstract.** We present an a posteriori hp-adaptive algorithm for shells. In an hp-solver there are two ways to increase accuracy: one can either refine the mesh (h-step) or increase the degree of the polynomial (p-step). In the adaptive setting it is also necessary to have the capability to reverse already made decisions.

Our *p*-approach is influenced by work of Houston and Süli on the estimation of Sobolev regularity and analycity. Due to the possibility of numerical locking we initialize our algorithm only after initial probing of the energy distribution of the solution.

#### 1. Introduction

Thin shell problems are known to be challenging due to their dependence on the dimensionless thickness and effects of the shell geometry. The solution of a given shell problem can be viewed as a linear combination of features each of which has its own characteristic length scale: the smooth component with scale equal to the diameter of the shell and the layers which can occur at the boundaries or inside the domain.

Here we consider the a posteriori adaptive hp-FEM solution of thin shells of revolution using standard finite elements. The choice of high order finite elements is due to numerical locking in shells, see e.g. [4].

In the context of shells with the problem of numerical locking present, the question of choosing the correct initial setup is central. Our approach is: Probe for locking by solving the problem with minimal mesh but various polynomial degrees. The idea is not to minimize the energy but to get a picture of the relative sizes of the energy components. In the numerical experiments we show this is to be a very effective strategy.

#### 2. Shell Model

2.1. Shell geometry

By shell we mean a three-dimensional domain

$$\Omega = \big\{(x,y,z) \in \mathbb{R}^3 | (x,y) \in \omega, -\frac{d}{2} < z < \frac{d}{2}\big\},$$

where d is the thickness of the shell, and  $\omega$  is the mid-surface of the shell. Here we are interested in shells of revolution for which the mid-surface can be defined as

$$\omega = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2^2 + x_3^2 = \Phi(x_1)^2, -1 < x_1 < 1, 0 < \Phi \in C^1(-1, 1) \right\}$$

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#### 2.2. Naghdi Model

Our two-dimensional shell model is the so-called Naghdi model, with a five-component displacement vector field  $\underline{u} = (u, v, w, \theta, \psi)$ , where the first three components are the axial displacements and the remaining two rotations in longitudinal and latitudinal directions, respectively.

The total energy is given by the quadratic functional

$$\mathcal{F}(\underline{u}) = \frac{1}{2}\mathcal{A}(\underline{u},\underline{u}) - \mathcal{Q}(\underline{u}), \tag{1}$$

where  $\mathcal{A}$  represents deformation energy and  $\mathcal{Q}$  is the load-induced potential energy. The deformation energy consists of three parts: bending  $(\mathcal{A}_B)$ , membrane $(\mathcal{A}_M)$ , and shear $(\mathcal{A}_S)$ :

$$\mathcal{A}(\underline{u},\underline{u}) = d^2 \mathcal{A}_B(\underline{u},\underline{u}) + \mathcal{A}_M(\underline{u},\underline{u}) + \mathcal{A}_S(\underline{u},\underline{u}), \tag{2}$$

with definitions

$$d^{2}\mathcal{A}_{B}(\underline{u},\underline{u}) = d^{2} \int_{\omega} [\nu(\kappa_{11}(\underline{u}) + \kappa_{22}(\underline{u}))^{2} + (1-\nu) \sum_{i,j=1}^{2} \kappa_{ij}(\underline{u})^{2}] A_{1}A_{2} dxdy, \qquad (3)$$

$$\mathcal{A}_{M}(\underline{u},\underline{u}) = 12 \int_{\omega} [\nu(\beta_{11}(\underline{u}) + \beta_{22}(\underline{u}))^{2} + (1-\nu) \sum_{i,j=1}^{2} \beta_{ij}(\underline{u})^{2}] A_{1}A_{2} dxdy, \qquad (4)$$

$$\mathcal{A}_{S}(\underline{u},\underline{u}) = 6(1-\nu) \int_{\omega} \left[ (\rho_{1}(\underline{u})^{2} + \rho_{2}(\underline{u}))^{2} \right] A_{1}A_{2} \, dxdy, \tag{5}$$

where  $\nu$  is the Poisson number and  $A_1$ ,  $A_2$  are Lamé parameters, defined for shells of revolution in terms of the geometry function  $\Phi$ :

$$A_1(x) = \sqrt{1 + [\Phi'(x)]^2}, \quad A_2 = \Phi(x).$$

Further we need the principal curvature radii:

$$R_1(x) = -\frac{A_1(x)^3}{\Phi''(x)}, \quad R_2(x) = A_1(x)A_2(x).$$

Thus, we can write the expressions for bending, membrane, and shear strains,  $\kappa_{ij}$ ,  $\beta_{ij}$ ,  $\rho_i$ ,

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respectively:[1]

$$\begin{split} \kappa_{11} &= \frac{1}{A_1} \frac{\partial \theta}{\partial x} + \frac{\psi}{A_1 A_2} \frac{\partial A_2}{\partial y}, \\ \kappa_{22} &= \frac{1}{A_2} \frac{\partial \psi}{\partial y} + \frac{\theta}{A_1 A_2} \frac{\partial A_2}{\partial x}, \\ \kappa_{12} &= \kappa_{21} = \frac{1}{2} \left[ \frac{1}{A_1} \frac{\partial \psi}{\partial x} + \frac{1}{A_2} \frac{\partial \theta}{\partial y} - \frac{\theta}{A_1 A_2} \frac{\partial A_2}{\partial y} - \frac{\psi}{A_1 A_2} \frac{\partial A_2}{\partial x} \right] \\ &\quad - \frac{1}{R_1} \left( \frac{1}{A_2} \frac{\partial u}{\partial y} - \frac{v}{A_1 A_2} \frac{\partial A_2}{\partial x} \right) \\ &\quad - \frac{1}{R_2} \left( \frac{1}{A_1} \frac{\partial v}{\partial x} - \frac{u}{A_1 A_2} \frac{\partial A_1}{\partial y} \right) \right], \\ \beta_{11} &= \frac{1}{A_1} \frac{\partial u}{\partial x} + \frac{v}{A_1 A_2} \frac{\partial A_2}{\partial y} + \frac{w}{R_1}, \\ \beta_{22} &= \frac{1}{A_2} \frac{\partial v}{\partial y} + \frac{u}{A_1 A_2} \frac{\partial A_2}{\partial x} + \frac{w}{R_2}, \\ \beta_{12} &= \beta_{21} = \frac{1}{2} \left( \frac{1}{A_1} \frac{\partial v}{\partial x} + \frac{1}{A_2} \frac{\partial u}{\partial y} - \frac{u}{A_1 A_2} \frac{\partial A_1}{\partial y} - \frac{v}{A_1 A_2} \frac{\partial A_2}{\partial x} \right), \\ \rho_1 &= \frac{1}{A_1} \frac{\partial w}{\partial x} - \frac{u}{R_1} - \theta, \\ \rho_2 &= \frac{1}{A_2} \frac{\partial w}{\partial y} - \frac{v}{R_2} - \psi, \end{split}$$

The energy norm  $||| \cdot |||$  is defined in terms of the deformation energy (2):

$$\mathcal{E}(\underline{u}) := |||\underline{u}|||^2 := \mathcal{A}(\underline{u}, \underline{u}) \tag{6}$$

Let us also introduce notation for different energy components:

$$\mathcal{B}(\underline{u}) := d^2 \mathcal{A}_B(\underline{u}, \underline{u}) \tag{7}$$

$$\mathcal{M}(\underline{u}) := \mathcal{A}_M(\underline{u}, \underline{u}) \tag{8}$$

$$\mathcal{S}(\underline{u}) := \mathcal{A}_S(\underline{u}, \underline{u}) \tag{9}$$

#### 2.3. Variational Formulation

We solve by minimizing the total energy (1), which in turn leads to a following variational problem: Find  $\underline{u} \in \mathcal{U} \subset [H^1(\omega)]^5$  such that

$$\mathcal{A}(\underline{u},\underline{v}) = \mathcal{Q}(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}.$$
(10)

#### 3. Adaptive Algorithm

#### 3.1. Detecting Locking

Our basic tenet is that detection of possible bending locking is central to success of any a posteriori scheme for shells. Here we rely on the *p*-method: We simply use a minimal mesh to probe for any possible changes in the energy distributions.

Consider a cylindrical shell defined by  $\Phi = 1$  with d = 1/100 and load  $f(x, y) = \cos(2y)$ . The computational domain  $(0, 1) \times (0, \pi/4)$  is discretized using two triangles  $K_1 = \{(0,0), (1,0), (0,\pi/4)\}$  and  $K_2 = \{(1,\pi/4), (1,0), (0,\pi/4)\}$ . Due to the special form of the loading we can use symmetries at boundaries x = 0, y = 0, and  $y = \pi/4$ , and at x = 1

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$\frac{\ \underline{\boldsymbol{u}} - \underline{\boldsymbol{u}}_h}{\ \underline{\boldsymbol{u}}\ }$	≞ (%)	$ \mid \frac{\mathcal{B}(\underline{u}_h)}{\mathcal{E}(\underline{u}_h)} $	$\frac{\mathcal{M}(\underline{u}_h)}{\mathcal{E}(\underline{u}_h)}$	$\frac{\mathcal{S}(\underline{u}_h)}{\mathcal{E}(\underline{u}_h)} \ (\%)$
p = 1	93,1	0,0113	32,0	68,0
p = 2 $p = 3$	$^{62,3}_{20,2}$	0,127 0,339	$43,2 \\94,1$	56,6 5,59
p = 4	7,8	0,675	98,1	1,27
p = 5 $p = 6$	$\begin{array}{c} 2,9\\ 0,96\end{array}$	$\left  \begin{array}{c} 0,941\\ 1,32 \end{array} \right $	98,7 98,6	$\substack{0,34\\0,13}$

Table 1. Cylindrical Shell: Clamped Boundary

 Table 2. Cylindrical Shell: Free Boundary

$\frac{\ \underline{u} - \underline{u}_{\mu}\ }{\ \underline{u}\ }$	≝∭ (%)	$\frac{\mathcal{B}(\underline{u}_h)}{\mathcal{E}(\underline{u}_h)}$	$\frac{\mathcal{M}(\underline{u}_h)}{\mathcal{E}(\underline{u}_h)}$	$\frac{\mathcal{S}(\underline{u}_h)}{\mathcal{E}(\underline{u}_h)} \ (\%)$
p = 1	99,99	0,023	40,8	59,1
p = 2	$98,\!92$	$1,\!36$	92,2	$6,\!44$
p = 3	$23,\!8$	94,2	$^{5,5}$	0,318
p = 4	$1,\!11$	99,4	$0,\!56$	0,034
p = 5	$0,\!19$	$99,\!6$	$0,\!33$	0,0204
p = 6	$0,\!034$	99,7	$0,\!25$	0,0134

either free or clamped boundary conditions. In Tables 1 and 2 we give a break-up of energy distributions in terms of p. Note how in the bending-dominating case (free boundary) we observe a dramatic change in bending energy at p = 3.

#### 3.2. Error Indicators

Our error indicators are bubble-mode based. Let us denote the solution space (without bubbles) with  $\mathcal{U}_h$  and the additional bubble modes with  $\mathcal{U}_h^+$ . Let  $\underline{u}_h$  be the discrete solution: Find  $\underline{u}_h \in \mathcal{U}_h$  such that

$$\mathcal{A}(\underline{u}_h, \underline{v}) = \mathcal{Q}(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_h.$$

Taking  $\underline{u}_h$  as known, we add bubbles  $\underline{u}_h^+ \in \mathcal{U}_h^+$  to the solution vector. Thus, the problem becomes: Find  $\underline{u}_h^+ \in \mathcal{U}_h^+$  such that

$$\mathcal{A}(\underline{u}_h + \underline{u}_h^+, \underline{v}) = \mathcal{Q}(\underline{v}) \quad \forall \underline{v} \in \mathcal{U}_h^+.$$
(11)

Since every bubble is supported by exactly one element, the problem (11) can be solved elementby-element:

$$\mathcal{A}(\underline{u}_h^+, \underline{v})_e = \mathcal{Q}(\underline{v})_e - \mathcal{A}(\underline{u}_h, \underline{v})_e \quad \forall \underline{v} \in \mathcal{U}_h^+,$$
(12)

 $e = 1, \ldots, e_{max}$ . Since the solution lies in a subspace of  $\mathcal{U}$  we can transform (12) with (10) so that we end up with

$$\mathcal{A}(\underline{u}_h^+, \underline{v})_e = \mathcal{A}(u - \underline{u}_h, \underline{v})_e \quad \forall \underline{v} \in \mathcal{U}_h^+$$
(13)

The problem (13) can be interpreted so that the error  $\underline{u}_{err} = \underline{u} - \underline{u}_h$  is approximated in subspace  $\mathcal{U}_h^+ \subset \mathcal{U}$ .

Error is measured in the energy norm, so the elemental error indicator is

$$\eta_e^+ := |||\underline{u}_h^+|||_{K_e} \tag{14}$$

(21)

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and the corresponding global indicator

$$\eta^{+} := \sqrt{\sum_{e} (\eta_{e}^{+})^{2}}.$$
(15)

#### 3.3. Estimation of Sobolev Regularity

Let us first consider the reference interval (-1, 1) and a function  $\hat{u} \in L^2(-1, 1)$  with Legendre series

$$\hat{u}(\xi) = \sum_{i=0}^{\infty} \hat{a}_i \hat{L}_i(\xi),$$
(16)

where  $\hat{L}_i$  is a Legendre polynomial of degree *i*. Legendre polynomials are orthogonal

$$\int_{-1}^{1} \hat{L}_i(\xi) \hat{L}_j(\xi) d\xi = \delta_{ij} \frac{2}{2i+1},$$

so the coefficients  $\hat{a}_i$  can be written as

$$\hat{a}_i = \frac{2i+1}{2} \int_{-1}^1 \hat{u}(\xi) \hat{L}_i(\xi) d\xi.$$
(17)

Let us define a sequence  $\{l_i\}_{i\geq 2}$  using  $\hat{a}_i$ :

$$l_i = \frac{\log\left(\frac{2i+1}{2|a_i|^2}\right)}{2\log i}.$$
(18)

If  $l = \lim_{i \to \infty} l_i$  exists and l > 1/2, then

$$u \in H_{loc}^{l-1/2-\epsilon}(-1,1), \qquad 0 < \epsilon < l-1/2.$$

In 2D proceed as above. Let K be a triangle and  $u \in L^2(K)$  a function with Legendre series

$$u(x,y) = \sum_{i,j=0}^{\infty} a_{ij} L_{ij}(x,y).$$
(19)

Here we assume that the shape functions  $L_{ij}$  are also orthogonal. Let F denote the mapping from the reference quadrilateral  $\hat{Q} = (-1, 1)^2$  to K. Then

$$L_{ij}(x,y) := (\hat{L}_{ij} |\det J_F|^{-1/2}) \circ F^{-1}(x,y),$$
(20)

where  $\hat{L}_{ij}(\xi,\eta) = \hat{L}_i(\xi)\hat{L}_j(\eta)$  and  $J_F$  is the Jacobian of F. The coefficients  $a_{ij}$  are

 $\begin{aligned} a_{ij} &= c_{ij} \int_{K} u(x,y) L_{ij}(x,y) \, dx dy \\ &= c_{ij} \int_{\hat{Q}} u \circ F(\xi,\eta) \, L_{ij} \circ F(\xi,\eta) |\det J_F| \, d\xi d\eta \\ &= c_{ij} \int_{\hat{Q}} u \circ F(\xi,\eta) \, \hat{L}_{ij}(\xi,\eta) |\det J_F|^{1/2} \, d\xi d\eta, \end{aligned}$ 

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where  $c_{ij} = \frac{2i+1}{2} \frac{2j+1}{2}$ . We want to examine the convergence of the coefficient in  $\xi$ - and  $\eta$ -directions, so we define

$$|\alpha_i|^2 = \sum_{j=0}^{\infty} |a_{ij}|^2 \frac{2}{2j+1} \quad \text{and} \quad |\beta_j|^2 = \sum_{i=0}^{\infty} |a_{ij}|^2 \frac{2}{2i+1}.$$
 (22)

and the sequences  $\{l_{\xi,i}\}_{i\geq 2}$ :  $\{l_{\eta,i}\}_{i\geq 2}$ :

$$l_{\xi,i} = \frac{\log\left(\frac{2i+1}{2|\alpha_i|^2}\right)}{2\log i} \quad \text{and} \quad l_{\eta,j} = \frac{\log\left(\frac{2j+1}{2|\beta_j|^2}\right)}{2\log j}.$$
 (23)

If the limits  $l_{\xi} = \lim_{i \to \infty} l_{\eta,i}$  and  $l_{\eta} = \lim_{i \to \infty} l_{\eta,i}$  exist and  $l_{\xi}, l_{\eta} > 1/2$ , function u belongs to a locally anisotropic Sobolev-space

$$u \in H^{k_{\xi},k_{\eta}}(K)$$

where

$$k_{\xi} = l_{\xi} - 1/2 - \epsilon, \quad k_{\eta} = l_{\eta} - 1/2 - \epsilon.$$

In this paper we only consider the isotropic case, so  $k := \min\{k_{\xi}, k_{\eta}\}$ , and

$$u \in H^k_{loc}(K).$$

#### 3.4. Estimation of Regularity of Solution $\underline{u}_h$

Let us examine an element e and assume that there are equal number (m + 1) of Legendre coefficients in both directions;  $a_{ij}$ , i, j = 0, ..., m. Let us define  $l_{\xi,i}$  and  $l_{\eta,i}$  as in (23). We approximate the limits using the last coefficients of the sequences:

$$l_{\xi} := \min\{l_{\xi,m-1}, l_{\xi,m}\}, \quad l_{\eta} := \min\{l_{\eta,m-1}, l_{\eta,m}\}.$$
(24)

 $l_{\xi}$  and  $l_{\eta}$  are used to compute the highest suitable order of the polynomials on e:

$$\hat{p}_e := \lfloor \min\{l_{\xi}, l_{\eta}\} - 1/2 \rfloor.$$
(25)

Note that scaling  $c_u u_h$  ( $c_u \in \mathbb{R}$ ) does not affect the regularity in the general case, since

$$\tilde{l}_i := \frac{\log\left(\frac{2i+1}{2|c_u a_i|^2}\right)}{2\log i} = \frac{-\log|c_u|^2}{2\log i} + l_i,$$

that is,  $\lim_{i\to\infty} \tilde{l}_i = \lim_{i\to\infty} l_i$ . However, it is reasonable to require that  $u_h$  and  $c_u u_h$  have the properties in this sense. Let  $u_h$  be the numerical solution on the discretization of domain  $\Omega$ . Let us define domain  $\tilde{\Omega}$  and the corresponding discretization using mapping  $G: (x, y) \to (c_x x, c_y y)$ :

$$\hat{\Omega} := \{ (\tilde{x}, \tilde{y}) \mid (\tilde{x}, \tilde{y}) = G(x, y), \ (x, y) \in \Omega \}.$$

We get the function  $\tilde{u}_h$  by scaling  $u_h$ :

$$\tilde{u}_h = c_u \, u_h \circ G^{-1}(\tilde{x}, \tilde{y}).$$

We get the Legendre coefficients of  $\tilde{u}_h$  in one of the elements  $\tilde{K}$  in  $\tilde{\Omega}$ :

$$\begin{split} \tilde{a}_{ij} &= c_{ij} \int_{\hat{Q}} \tilde{u}_h \circ \tilde{F}(\xi, \eta) \hat{L}_{ij} |\det J_{\tilde{F}}|^{1/2} d\xi d\eta \\ &= c_{ij} \int_{\hat{Q}} c_u \, u_h \circ F(\xi, \eta) \hat{L}_{ij} \Big(\frac{|\tilde{\Omega}|}{|\Omega|}\Big)^{1/2} |\det J_F|^{1/2} d\xi d\eta \\ &= c_u \Big(\frac{|\tilde{\Omega}|}{|\Omega|}\Big)^{1/2} a_{ij}, \end{split}$$

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where  $a_{ij}$  are Legendre coefficients of the solution  $u_h$ . On the other hand for  $L^2$ -norm of  $\tilde{u}_h$  we get

$$\begin{split} \|\tilde{u}_h\|_{0,\tilde{\Omega}}^2 &= \int_{\tilde{\Omega}} \tilde{u}_h^2 d\tilde{x} d\tilde{y} = \int_{\Omega} c_u^2 u_h^2 \Big(\frac{|\Omega|}{|\Omega|}\Big) dx dy = c_u^2 \frac{|\Omega|}{|\Omega|} \|u_h\|_{0,\Omega}^2,\\ \frac{\tilde{a}_{ij}}{\|\tilde{u}_h\|_{0,\tilde{\Omega}}} &= \frac{a_{ij}}{\|u_h\|_{0,\Omega}}. \end{split}$$

 $\mathbf{SO}$ 

Here we scale the solution so that their  $L^2$ -norms are equal to 1. Thus, the Legendre coefficients are

$$a_{ij} := \frac{|K|^{1/2}}{2||u_h||_{0,\Omega}} c_{ij} \int_{\hat{Q}} u_h \circ F(\xi,\eta) \hat{L}_{ij} d\xi d\eta.$$
(26)

The term  $|K|^{1/2}/2$  comes from  $|\det J_F|^{1/2}$ , which in fact depends on the coordinate  $\eta$  if K is a triangle. In practise we omit this dependence and precompute the term.

#### 3.5. The ith Step of the hp-Algorithm

Let us assume that the solution of the step i-1,  $\underline{u}_h^{i-1}$ , has been computed using the mesh  $\mathcal{T}_h^{i-1}$ and *p*-distribution  $\underline{p}^{i-1}$ . Our goal is to find a solution  $\underline{u}_h^i$  by refining the mesh and/or altering the *p*-distribution depending on the error indicators computed from the solution  $\underline{u}_h^{i-1}$ . At each step a set of elements will be subjected to splitting,  $S_0^i$ , increasing of degree,  $U_0^i$ , or decreasing of degree,  $D_0^i$ . We drop the subscript 0 to indicate modifications at step *i*. Further, the set  $\tilde{U}^i$ includes the elements subject to smoothing of the *p*-distribution.

- (i) Compute the elemental error indicators  $\eta_e$ , e = 1, ..., E. Stop if the global error estimate  $\eta^2 = \sum_{e=1}^{E} \eta_e^2$  is below the chosen tolerance.
- (ii) Collect into the list of modified elements M those elements e, for which it holds that  $\eta_e \ge \alpha \max_{1 \le i \le E} \eta_i$ . We choose  $\alpha = 1/2$ .
- (iii) Estimate the highest possible degree per element;  $\hat{p}_e$ ,  $e = 1, \ldots, E$ .
- (iv) Divide the elements  $e \in M$  into  $S_0^i, U_0^i$ , and  $D_0^i$ :
  - (i) If  $p_e < \hat{p}_e$  and  $p_e < p_{max}$ , then  $e \to U_0^i$ .
  - (ii) If  $p_e < \hat{p}_e$  and  $p_e = p_{max}$ , then  $e \to S_0^i$ .
  - (iii) If  $\hat{p}_e \leq p_e \leq \hat{p}_e + 1$ , then  $e \to S_0^i$ .
  - (iv) If  $p_e > \hat{p}_e + 1$ , then  $e \to S_0^i$  and  $e \to L_0^i$ .
- (v) Check and correct choices made at i-1. Initialize  $\Delta \underline{p} := 0$ , the list indicating the changes in degrees.
  - (i) For  $e \in U^{i-1}$ : If  $p_e > \hat{p}_e$ , then  $\Delta p_e := -1$  and  $e \to S_0^i$ .
  - (ii) For  $e \in \tilde{U}^{i-1}$ : If  $p_e > \hat{p}_e$ , then  $\Delta p_e := -1$ .
  - (iii) For  $e \in D^{i-1}$ : If  $p_e < \hat{p}_e$ , then  $\Delta p_e := +1$ .
  - (iv) Update *p*-distribution:  $p := p + \Delta p$ .
  - (v) Smoothen *p*-distribution and update  $\Delta p$  accordingly.
- (vi) Update the mesh and *p*-distribution following the error indicators:
  - (i) Lower the degree  $(p_e := p_e 1)$  of elements in  $D_0^i$  and set  $L^i := L_0^i$ .
    - (ii) Refine the mesh at  $S_0^i$ . Add newly created elements to  $J^i$ .
  - (iii) Let  $U^i := U_0^i \setminus (U_0^i \cap S^i)$  and increase the degree  $(p_e = p_e + 1)$  of elements in  $U^i$ . We do not increase the degree of elements created in refinement step.
  - (iv) Smoothen *p*-distribution and collect the modified elements to  $\tilde{U}^i$ . Remove those from the of lowered elements:  $D^i := D^i \setminus (D^i \cap \tilde{U}^i)$ .
- (vii) Let  $\underline{p}^i := \underline{p}$  and compute the new solution  $\underline{u}_h^i$  using the new discretization  $(\mathcal{T}_h^i, \underline{p}^i)$ .

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Local changes in p can lead to highly uneven p-distribution. We smoothen by increasing the degree in some additional elements: If the degree of at least two neighbours of e,  $e_1$  and  $e_2$ , is greater than that of e,  $p_e < p_{e_1}, p_{e_2}$ , we set  $p_e = \min\{p_{e_1}, p_{e_2}\}$ .

#### 4. Numerical Experiments

In this section we consider three configurations.

(i) Free Cylindrical Shell: As above, our cylinder is defined by  $\Phi(x) = 1$  over  $-1 \le x \le 1$ , and d = 1/100. Loading is  $f_w(x, y) = \cos 2y$  so that all symmetries can be taken into account. The expected boundary layers are of types  $\sqrt{t}$  and t.

In this example numerical locking plays a significant role. In Figure 1(a) we see that in terms of the error indicator, there is no convergence at p = 1. In reality the convergence is very slow, but the error indicator fails due to locking. On the other hand, for higher values of p we get the expected results, with hp-adaptive algorithm being the most efficient one when compared against fixed p variable h -variants. Yet, Figure 1(b) shows that the high condition number starts to affect the quality of the solution and convergence stalls at very large systems.

In Figure 2 we show various meshes with p-distributions. The most notable one is Figure 2(a), where we see that for fixed p = 2 the algorithm refines everywhere which is appropriate for the longest characteristic length scale. Yet, there is only very slow convergence.

(ii) Cut Cylindrical Shell: The domain is set so that for  $\pi/4 \le y \le 3\pi/4$  or  $5\pi/4 \le y \le 7\pi/4$  we have the previous shell but otherwise  $x \in [-3/4, 3/4]$ . Here we consider a pressure load  $f_w = 1$  so that the final computational domain is bounded by  $(0, 1) \times (0, \pi/2)$ . Boundary at x = 3/4 is free, otherwise boundaries at x = 1 and  $y = \pi/4$  are clamped.

There are three dominant features of the solution: Singularity at  $(3/4, \pi/4)$ ,  $\sqrt{t}$ -layer at x = 1, and  $\sqrt[4]{t}$ -layer along the characteristic line at  $y = \pi/4$ . Since we want to demonstrate also the  $\sqrt[4]{t}$ -layer, we choose d = 1/10000.

In Figure 3 we have two stages of the algorithm. The effect of the internal layer is evident along the characteristic line at  $y = \pi/4$ .

(iii) Hyperbolic Shell Under Concentrated Load: Consider  $\Phi(x) = 1 + 1/2x^2$  over  $-1 \le x \le 1$ . Shell is clamped at  $x = \pm 1$ , and the concentrated load at (0,0) is  $f_w(x,y) = \exp(-100r^2)$ , where  $r^2 = x^2 + y^2$ . Due to symmetries the computational domain is  $(0,1) \times (0,\pi/2)$ .

There are three dominant features of the solution: Large effect at (0,0),  $\sqrt{t}$ -layer at x = 1, and  $\sqrt[3]{t}$ -layer traveling along the characteristic line starting at (0,0). Since we want to demonstrate also the  $\sqrt[3]{t}$ -layer, we choose d = 1/10000 for contrast with the cut cylinder above.

In Figure 4 we have two stages of the algorithm. Note how the algorithm resolves the components of the solution in order: first the area close to or under the load, then the internal layer, and eventually (not really clear here) the boundary layer. Note also that the algorithm doesn't touch elements outside the layers (white regions).

#### 5. Conclusions

Our algorithm has the advantage that we do not assume any characteristics of the solution. However, the next challenge is to combine our adaptive scheme with a priori knowledge on the solution either given by an engineer or detected by some automatic means. IOP Conf. Series: Materials Science and Engineering 10 (2010) 012163

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Figure 2. Cylindrical Shell, Free Boundary, d = 1/100, (d.o.f.; element count)



Figure 4. Hyperbolic Shell, d = 1/10000,  $f_w = \exp(-100r^2)$ , (d.o.f.; element count)

#### References

- M. Malinen, On the classical shell model underlying bilinear degenerated shell finite elements: general shell geometry, *International Journal for Numerical Methods in Engineering*, Volume 55, Issue 6, Date: 30 October 2002, Pages: 629–652
- [2] J. Pitkranta, A.-M. Matache, C. Schwab, Fourier mode analysis of layers in shallow shell deformations, Computer Methods in Applied Mechanics and Engineering 190 (2001) 2943–2975
- [3] P. Houston, B. Senior, E. Süli, Sobolev regularity estimation for hp-adaptive finite element methods, in: Proceedings of ENUMATH 2001 the 4th European Conference on Numerical Mathematics and Advanced Applications, Ischia, July 2001, pp. 631–656
- [4] E. Rank, A. Duester, V. Nuebel, K. Preusch und O. T. Bruhns. High order finite elements for shells, Computer Methods in Applied Mechanics and Engineering (2005) 2494–2512.