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Detecting Non-Markovianity of Quantum Evolution via Spectra of Dynamical Maps

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We provide an analysis on non-Markovian quantum evolution based on the spectral properties of dynamical maps. We introduce the dynamical analog of entanglement witness to detect non-Markovianity and we illustrate its behavior with several instructive examples. It is shown that for several important classes of dynamical maps the corresponding evolution of singular values and/or eigenvalues of the map provides a simple non-Markovianity witness.

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Introduction.—Spectral theorem is one of the mathematical pillars of quantum theory [1]. This celebrated result of von Neumann states that for any normal operator (i.e., $AA^\dagger = A^\dagger A$) in the Hilbert space one has the corresponding spectral decomposition $A = \sum_k a_k |\phi_k\rangle\langle\phi_k|$ with complex a_k and $\langle\phi_k|\phi_l\rangle = \delta_{kl}$. In particular, if A is not only normal but also Hermitian then a_k are real. Many problems from quantum physics are directly related to finding the spectrum $\{a_k\}$ of some normal or Hermitian operator.

In this Letter we apply some tools from spectral analysis to study the evolution of an open quantum system. Such systems provide a fundamental tool to study the interaction between a quantum system and its environment, causing dissipation, decay, and decoherence [2–4]. It is, therefore, clear that open quantum systems are important for quantum-enhanced applications, as both entanglement and quantum coherence are basic resources in modern quantum technologies, such as quantum communication, cryptography, and computation [5].

Recently, much effort has been devoted to the description, analysis, and classification of non-Markovian quantum evolution (see, e.g., recent review papers [6–8]). In analogy to entanglement theory [9] several non-Markovianity measures were proposed that characterize various concepts of non-Markovianity. The two most influential approaches to non-Markovian evolution are based on divisibility of dynamical maps [10,11] and distinguishability of states [12] (for other approaches see also [13–19]). The results we present in this Letter allow us to introduce for the first time a witness of non-Markovianity in the same spirit of entanglement witnesses. An entanglement witness method applied to the Choi-Jamiolkovski state of a quantum channel was recently developed [20] in order to detect properties based on convexity features. The method was tested experimentally for entanglement breaking channels and for separable random unitary channels [21].

Besides the fundamental interest, our approach simplifies, in certain cases, the experimental detection of non-Markovianity of a dynamical map.

Let us recall that a dynamical map Λ_t is *CP* divisible if for any $t > s$ one has $\Lambda_t = V_{t,s}\Lambda_s$, with $V_{t,s}$ being completely positive. We call quantum evolution Markovian if the corresponding dynamical map is *CP* divisible. Recently, this notion was refined as follows [22]: Λ_t is k divisible if $V_{t,s}$ is k positive. In particular, 1-divisible maps are called P divisible ($V_{t,s}$ is positive). Maps that are even not P divisible are called essentially non-Markovian. These types of dynamical maps have been recently simulated and detected experimentally [23].

Note that *CP* divisibility is a mathematical property of the map. Another approach more operationally oriented is based on distinguishability of quantum states [12]: according to [12] the evolution is Markovian if

$$\frac{d}{dt}\|\Lambda_t[\rho_1 - \rho_2]\|_1 \leq 0, \tag{1}$$

for any pair of initial states ρ_1 and ρ_2. Actually, assuming that Λ_t is invertible one shows [22] that Λ_t is k divisible if $(d/dt)\|(\mathbb{1}_k \otimes \Lambda_t)[X]\|_1 \leq 0$, for all Hermitian $X \in M_k(\mathbb{C}) \otimes \mathcal{B}(\mathcal{H})$ [$M_k(\mathbb{C})$ denotes $k \times k$ complex matrices and $\mathcal{B}(\mathcal{H})$ bounded operators in $\mathcal{H}$]. Note that if $k = 1$ and $X = \rho_1 - \rho_2$ one recovers (1).

In the following we develop further the analysis of non-Markovian evolution based on the spectral properties of dynamical maps, and provide the dynamical analog of entanglement witness for detecting non-Markovianity. In particular, we analyze three classes of dynamical maps: (i) unital maps, $\Lambda_t[\mathbb{1}] = \mathbb{1}$, (ii) normal maps, $\Lambda_t\Lambda_t^* = \Lambda_t^*\Lambda_t$, and (iii) commutative maps, $\Lambda_t\Lambda_s = \Lambda_s\Lambda_t$, where Λ_t^* denotes dual map (Heisenberg picture). While the class of dynamical maps considered in this Letter is not the most general, it does comprise noise sources that are dominant in a vast number of experimental scenarios. Indeed, on the one hand the Pauli channel case considered in example 2 encompasses classical stochastic noise arising from random fluctuations in the experimental parameters. On the other hand, in many experiments where both dephasing and relaxation are present, the dephasing time scale is much shorter than relaxation, so in fact dephasing in these systems is the most relevant decoherence mechanism. Examples of experimental systems where the main decoherence source is of the commutative type described in this Letter are trapped ion systems [24], ultracold gases [25], nitrogen-vacancy (NV) centers in diamonds [26], nuclear magnetic resonance (NMR) systems [27], and solid-state systems such as, e.g., Cooper pair boxes in the pure-dephasing limit ($1/f$ noise) [28].

Volume and body of accessible states.—Let us denote by B the space of density operators. Clearly $B(t)=\Lambda_t[B]$ denotes the body of accessible states at time t. In a recent paper [18] an interesting geometric characterization is proposed, namely, if Λ_t is P divisible, then

$$\frac{d}{dt}\mathrm{Vol}(t)\leq 0, \tag{2}$$

where $\mathrm{Vol}(t)$ denotes the volume of accessible states at time t, i.e., the volume of the convex body $B(t)$. This result follows from the fact that $\mathrm{Vol}(t)=|\mathrm{Det}\Lambda_t|\mathrm{Vol}(0)$ and for the P-divisible map one has $(d/dt)|\mathrm{Det}\Lambda_t|\leq 0$ (cf., [29]).

Let us provide more geometrical insight passing to the matrix representation $\Lambda_t\to F_{\alpha\beta}(t)\coloneqq \mathrm{Tr}(G_\alpha\Lambda_t[G_\beta])$, where G_α is an orthonormal basis in $\mathcal{B}(\mathcal{H})$. A suitable choice of G_α is the set of generalized Gell-Mann matrices with $G_0=\mathbb{I}/\sqrt{d}$ and Hermitian G_α ($\alpha=1,\ldots,d^2-1$). In this case $F(t)$ has the following form,

$$F(t)=\left(\begin{array}{c|c}1 & 0\\ \hline \mathbf{q}_t & \Delta_t\end{array}\right), \tag{3}$$

with $\mathbf{q_t}\in\mathbb{R}^{d^2-1}$ and Δ_t being the $(d^2-1)\times(d^2-1)$ real matrix. It is clear that $F(t)$ encodes all properties of the original dynamical map Λ_t. In particular, Λ_t and $F(t)$ have exactly the same spectrum $\lambda_\alpha(t)$ ($\alpha=0,1,\ldots,d^2-1$, where $d=\dim\mathcal{H}$), and hence $\mathrm{Det}\Lambda_t=\mathrm{Det}F(t)=\mathrm{Det}\Delta_t$. This shows that the volume of the set of accessible states is fully controlled by the matrix Δ_t itself. Using singular value decomposition of the matrix $F(t)$,

$$F(t)=\mathcal{O}_1(t)\Sigma(t)\mathcal{O}_2^{-1}(t), \tag{4}$$

where $\mathcal{O}_k(t)$ ($k=1,2$) are orthogonal matrices and $\Sigma(t)$ is a diagonal matrix containing singular values of $F(t)$. Hence the action of $F(t)$ consists in a rotation $\mathcal{O}_2^{-1}(t)$, a contraction governed by $\Sigma(t)$ [all singular values $s_k(t)\leq 1$] followed by the rotation $\mathcal{O}_1(t)$. Since rotation does not change the volume the latter is fully controlled by $\Sigma(t)$,

$$|\mathrm{Det}F(t)|=|\mathrm{Det}\Delta_t|=\mathrm{Det}\Sigma(t)=\prod_{k=1}^{d^2-1}s_k(t).$$

Note that $s_0(t)=1$ and all singular values $s_k(t)$ are by definition non-negative.

Now, defining the generalized Bloch representation $\rho=(1/d)(\mathbb{I}+\sum_{\alpha=1}^{d^2-1}x_\alpha G_\alpha)$, the action of the channel Λ_t on ρ corresponds to the following affine transformation of the generalized Bloch vector $\mathbf{x}\to\mathbf{x}_t=\Delta_t\mathbf{x}+\mathbf{q}_t$. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are Bloch vectors corresponding to ρ_1 and ρ_2, then $[\rho_1-\rho_2]\to\Lambda_t[\rho_1-\rho_2]$ corresponds to the linear transformation $\Delta_t(\mathbf{x}_1-\mathbf{x}_2)$ and hence does not depend upon the vector $\mathbf{q}_t$. It clearly shows that Breuer-Laine-Piilo (BLP) Markovianity is controlled only by Δ_t whereas the full P divisibility is controlled by the entire map $F(t)$, i.e., both Δ_t and $\mathbf{q}_t$. Note that divisibility of $F(t)$, that is, $F(t)=F(t,s)F(s)$ implies the quite nontrivial relations $\Delta_t=\Delta_{t,s}\Delta_s$ and $\mathbf{q}_t=\mathbf{q}_{t,s}+\Delta_{t,s}\mathbf{q}_s$, where $\mathbf{q}_{t,s}$ and $\Delta_{t,s}$ parametrize $F(t,s)$. They considerably simplify if the dynamical map Λ_t is unital. In this case $\mathbf{q}_t=0$ and one is left with a simple divisibility condition $\Delta_t=\Delta_{t,s}\Delta_s$. In this case one proves the following.

Proposition 1 If Λ_t is P divisible and unital, then

$$\frac{d}{dt}\|\Lambda_t[X]\|_2\leq 0, \tag{5}$$

for all operators X.

For the proof see [30]. In particular, $(d/dt)\|\Delta_t\mathbf{x}\|_2\leq 0$, which shows that the Euclidean norm of the Bloch vector $\mathbf{x}$ decreases monotonically [30]. This observation immediately implies the following.

Corollary 1 If Λ_t is P divisible and unital, then $(d/dt)s_k(t)\leq 0$.

Hence for unital maps a rather weak witness—monotonicity of the volume of accessible states—is replaced by d^2-1 conditions for monotonicity of the singular values. Conditions $(d/dt)s_k(t)\leq 0$ mean that the body of states monotonically shrinks in time, that is, for any $t>s$ there exists an affine transformation $A(t,s)$ in $\mathbb{R}^{d^2-1}$ such that $A(t,s)[B(t)]\subset B(s)$. Obviously one has $\mathrm{Vol}(t)\leq\mathrm{Vol}(s)$.

Example 1 Consider the qubit evolution governed by the following time-local generator

$$\mathcal{L}_t[\rho]=\frac{1}{2}\sum_{k=1}^{3}\gamma_k(t)(\sigma_k\rho\sigma_k-\rho), \tag{6}$$

which leads to the following unital dynamical map (time-dependent Pauli channel): $\Lambda_t[\rho]=\sum_{\alpha=1}^{3}p_\alpha(t)\sigma_\alpha\rho\sigma_\alpha$. In this case condition (2) is equivalent to $\gamma_1(t)+\gamma_2(t)+\gamma_3(t)\geq 0$. It was shown [31] that monotonicity of singular values implies $\gamma_1(t)+\gamma_2(t)\geq 0$, $\gamma_2(t)+\gamma_3(t)\geq 0$, and $\gamma_3(t)+\gamma_1(t)\geq 0$. Note that $B(t)$ defines an ellipsoid

$$\frac{x_1^2}{s_1^2(t)}+\frac{x_2^2}{s_2^2(t)}+\frac{x_3^2}{s_3^2(t)}\leq 1$$

and this evolution is P divisible if $B(t)\subset B(s)$ for $t>s$. Clearly, one may have $\mathrm{Vol}(t)<\mathrm{Vol}(s)$ even if $B(t)$ is not a subset of $B(s)$.

Normal maps.—Consider now a class of normal dynamical maps, that is, maps satisfying $\Lambda_t\Lambda_t^* = \Lambda_t^*\Lambda_t$, for any $t \geq 0$. Note that in this case $F(t)$ has to be a normal matrix, which means that $\mathbf{q}_t = 0$ and Δ_t is a normal matrix. Hence, any normal dynamical map is necessarily unital. Moreover, for normal maps one has $s_k(t) = |\lambda_k(t)|$, where $\lambda_k(t)$ denotes eigenvalues of Λ_t, or, equivalently, of the matrix $F(t)$. Hence the following:

Corollary 2 If Λ_t is a normal dynamical map, then P divisibility implies $(d/dt)|\lambda_k(t)| \leq 0$.

The characteristic feature of the normal map is a spectral representation $\Lambda_t[\rho] = \sum_{k=0}^{d^2-1} \lambda_k(t)F_k(t)\mathrm{Tr}[F_k^\dagger(t)\rho]$, where $\lambda_0(t) = 1$ and $F_0(t) = \mathbb{I}/\sqrt{d}$. The dual map Λ_t^* has exactly the same representation with λ_k replaced by λ_k^*, that is, one has $\Lambda_t[F_k(t)] = \lambda_k(t)F_k(t)$ and $\Lambda_t^*[F_k(t)] = \lambda_k^*(t)F_k(t)$. If $\lambda_k^*(t) = \lambda_k(t)$, the map is Hermitian; i.e., it satisfies $\Lambda_t^* = \Lambda_t$ for all $t > 0$.

Corollary 3 If Λ_t is a Hermitian dynamical map, then P divisibility implies $(d/dt)\lambda_k(t) \leq 0$.

Indeed, $\lambda_k(t) \geq 0$; otherwise the map Λ_t is not divisible. Hence $s_k(t) = |\lambda_k(t)| = \lambda_k(t)$ and the result follows. Note that a time-local generator

$$\mathcal{L}_t[\rho] = -\sum_k \gamma_k(t)[A_k(t), [A_k(t), \rho]] \tag{7}$$

gives rise to a Hermitian dynamical map if $A_k^\dagger(t) = A_k(t)$. The simplest example is provided by $A(t) = \sigma_z$, which leads to the qubit dephasing $\mathcal{L}_t[\rho] = -\gamma(t)[\sigma_z[\sigma_z, \rho]]$.

Interestingly, in the case of Hermitian maps we may provide an extra tool for analyzing P divisibility. In entanglement theory one defines an entanglement witness, i.e., a Hermitian operator W in $\mathcal{H}\otimes\mathcal{H}$ such that (i) $\langle\psi_1\otimes\psi_1|W|\psi_1\otimes\psi_2\rangle \geq 0$, and (ii) $\mathrm{Tr}(W\rho) < 0$ for some entangled state ρ. Any such operator may be constructed as $W \coloneqq (\mathbb{1}\otimes\Phi)|\alpha\rangle\langle\alpha|$, where $\Phi\colon \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a positive but not completely positive map, and $|\alpha\rangle = (1/\sqrt{d})\sum_{i=1}^d |i\otimes i\rangle$ denotes the maximally entangled state in $\mathcal{H}\otimes\mathcal{H}$. Consider now an arbitrary diagonalizable linear map $\Phi\colon\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ and define

$$f_\Phi = \langle\alpha|(\mathbb{1}\otimes\Phi)[P^+]|\alpha\rangle, \tag{8}$$

with $P^+ = |\alpha\rangle\langle\alpha|$. Interestingly, f_Φ is fully characterized by the spectral properties of the map Φ. One has the following.

Proposition 2 Function f_Φ is fully determined by the spectrum of Φ, that is, $f_\Phi = d^{-2}\sum_{\alpha=0}^{d^2-1}\lambda_\alpha$, where λ_α are eigenvalues of Φ.

Indeed, consider the spectral representation $\Phi[\rho] = \sum_\alpha \lambda_\alpha F_\alpha \mathrm{Tr}(G_\alpha^\dagger\rho)$, where $\{F_\alpha, G_\alpha\}$ provide a damping basis [32] for the map Φ, that is, $\Phi[F_\alpha] = \lambda_\alpha F_\alpha$ and $\Phi^*[G_\alpha] = \lambda_\alpha^* G_\alpha$ such that $\mathrm{Tr}(F_\alpha G_\beta^\dagger) = \delta_{\alpha\beta}$. One has

$$\begin{aligned} d^2 f_\Phi &= \sum_{i,j}\sum_{k,l}\mathrm{Tr}((|i\rangle\langle j|\otimes|i\rangle\langle j|)\cdot(|k\rangle\langle l|\otimes\Phi[|k\rangle\langle l|])) \\ &= \sum_\alpha\sum_{i,j}\lambda_\alpha\langle i|F_\alpha|j\rangle\langle j|G_\alpha^\dagger|i\rangle = \sum_\alpha\lambda_\alpha, \end{aligned}$$

due to $\mathrm{Tr}(F_\alpha G_\alpha^\dagger) = 1$. Therefore, one arrives at the following.

Proposition 3 If Λ_t is a P divisible Hermitian map, then

$$\frac{d}{dt}\langle\alpha|(\mathbb{1}\otimes\Lambda_t)[P^+]|\alpha\rangle \leq 0, \tag{9}$$

for all $t \geq 0$.

We note here that condition (9) can be easily detected in an experimental scenario without performing all the measurements required for quantum process tomography. It may be considered as a dynamical analog of an entanglement witness. Actually, $f(t) = \langle\alpha|(\mathbb{1}\otimes\Lambda_t)[P^+]|\alpha\rangle$ is the probability of projecting the global state of the system and the ancilla onto state $|\alpha\rangle$. Let us consider for simplicity the case of two-dimensional systems. We can write $|\alpha\rangle$ in terms of local Pauli operators as $|\alpha\rangle\langle\alpha| = 1/4(\mathbb{I}\otimes\mathbb{I} + \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z)$. This means that $f(t)$ can be measured from the expectation value of the local observables $\sigma_x\otimes\sigma_x$, $\sigma_y\otimes\sigma_y$, $\sigma_z\otimes\sigma_z$ without requiring a complete set of two-qubit operators that would be needed for entanglement-assisted quantum process tomography. Moreover, the detection scheme considered here would be particularly suited in a linear optical scenario. Actually, the projection onto the maximally entangled state $|\alpha\rangle\langle\alpha|$ could be performed in a single measurement because it corresponds to a single projection onto a Bell state while there is no need to distinguish between the four Bell states, which is usually considered a drawback of linear optical implementations.

Recall that the Hermitian map allows for the Kraus representation with Hermitian Kraus operators $K_\alpha(t)$, that is, $\Lambda_t[\rho] = \sum_\alpha K_\alpha(t)\rho K_\alpha(t)$. In this case proposition 3 implies the following.

Corollary 4 If Λ_t is a P divisible Hermitian map, then

$$\frac{d}{dt}\sum_\alpha[\mathrm{Tr}K_\alpha(t)]^2 \leq 0, \tag{10}$$

for all $t \geq 0$.

Example 2 The time-dependent Pauli channel $\Lambda_t[\rho] = \sum_\alpha p_\alpha(t)\sigma_\alpha\rho\sigma_\alpha$ defines a Hermitian dynamical map with $K_\alpha(t) = \sqrt{p_\alpha(t)}\sigma_\alpha$ (with $\sigma_0 = \mathbb{I}$). One finds from (10) that P divisibility implies $(d/dt)p_0(t) \leq 0$.

Example 3 (Weyl channel) The Pauli channel may be easily generalized for $d > 2$ as follows,

$$\mathcal{L}_t[\rho] = \sum_{k+l>0}^{d-1}\gamma_{kl}(t)[U_{kl}\rho U_{kl}^\dagger - \rho], \tag{11}$$

where U_{kl} are Weyl operators $U_{kl} = \sum_{m=0}^{d-1}\omega^{mk}|m\rangle\langle m+l|$, with $\omega = e^{2\pi i/d}$. Because of the well-known properties

$$U_{kl}U_{rs} = \omega^{ks}U_{k+r,l+s}, \qquad U_{kl}^\dagger = \omega^{kl}U_{-k,-l},$$

this generator gives rise to the normal dynamical map $\Lambda_t[\rho] = \sum_{k,l=0}^{d-1}p_{kl}(t)U_{kl}\rho U_{kl}^\dagger$, and hence $(d/dt)|\lambda_\alpha(t)| \leq 0$ defines a necessary condition for P divisibility.

Example 4 (generalized Pauli channel) The generalized Pauli channel [33,34] is a special example of the Weyl channel defined as follows: let $\{|\psi_0^{(\alpha)}\rangle, ..., |\psi_{d-1}^{(\alpha)}\rangle\}$ denote $d+1$ mutually unbiased bases in $\mathbb{C}^d$. Define the quantum channels $\mathcal{P}_\alpha[\rho] = \sum_{l=0}^{d-1} |\psi_l^{(\alpha)}\rangle\langle\psi_l^{(\alpha)}|\rho|\psi_l^{(\alpha)}\rangle\langle\psi_l^{(\alpha)}|$ and let

$$\mathcal{L}_t[\rho] = \sum_{\alpha=1}^{d+1} \gamma_\alpha(t)(\mathcal{P}_\alpha[\rho] - \rho). \qquad (12)$$

This map is Hermitian and P divisibility implies [34] $\gamma(t) - \gamma_\alpha(t) \geq 0$, where $\gamma(t) = \sum_\alpha \gamma_\alpha(t)$.

Example 5 (perfect decoherence) Consider the following time-independent Hamiltonian in $\mathcal{H}_A \otimes \mathcal{H}_B$,

$$H = H_A \otimes \mathbb{I}_B + \mathbb{I}_A \otimes H_B + \sum_k P_k \otimes B_k, \qquad (13)$$

where $P_k = |k\rangle\langle k|$ are projectors into the computational basis vectors $|k\rangle$ in $\mathcal{H}_A$ and B_k are Hermitian operators in $\mathcal{H}_B$. Assuming that $H_A = \sum_k \epsilon_k P_k$ one finds $H = \sum_k P_k \otimes Z_k$, where $Z_k = \epsilon_k \mathbb{I}_B + H_B + B_k$. Such a Hamiltonian leads to a pure decoherence of the density operator ρ_A of subsystem A,

$$\rho_A(t) = \mathrm{tr}_B(e^{-iHt}\rho_A \otimes \rho_B e^{iHt}) = \sum_{k,l} c_{kl}(t) P_k \rho_A P_l,$$

with $c_{kl}(t) = \mathrm{tr}(e^{-iZ_k t}\rho_B e^{iZ_l t})$. It turns out that $c_{kl}(t)$ define eigenvalues of the map $\Lambda_t[\rho_A] = \sum_{k,l} c_{kl}(t) P_k \rho_A P_l$, which is normal. Hence, P divisibility implies $(d/dt)|c_{kl}(t)| \leq 0$. The map Λ_t becomes Hermitian if $c_{kl}(t)$ are real. In this case (9) gives $(d/dt)\sum_{k,l} c_{kl}(t) \leq 0$.

Commutative maps.—Finally consider a class of quantum evolutions satisfying the following commutativity condition $\Lambda_t\Lambda_s = \Lambda_s\Lambda_t$, for any $t, s > 0$. Equivalently, the time-local generator satisfies $\mathcal{L}_t\mathcal{L}_s = \mathcal{L}_s\mathcal{L}_t$. The commutativity condition implies that Λ_t and its dual (Heisenberg picture) possess time-independent eigenvectors

$$\Lambda_t[F_\alpha] = \lambda_\alpha(t) F_\alpha, \qquad \Lambda_t^*[G_\alpha] = \lambda_\alpha^*(t) G_\alpha, \qquad (14)$$

for $\alpha = 0, 1, ..., d^2 - 1$. This condition is indeed very restrictive. However, in practice many examples considered in the literature belong to the commutative class. The reason is very simple: assuming that Λ_t satisfies the time-local master equation $(d/dt)\Lambda_t = \mathcal{L}_t\Lambda_t$, with suitable time-local generator $\mathcal{L}_t$, one has $\Lambda_t = \mathcal{T} e^{\int_0^t \mathcal{L}_\tau d\tau}$, where $\mathcal{T}$ denotes the chronological operator. In general the above formula has only a formal meaning and it is defined by the Dyson expansion $\Lambda_t = \mathbb{1} + \int_0^t dt_1 \mathcal{L}_{t_1} + \int_0^t dt_1 \int_0^{t_1} dt_2 \mathcal{L}_{t_1}\mathcal{L}_{t_2} + \cdots$ Now, in the commutative case the chronological product drops out and the solution is represented by the simple exponential formula $\Lambda_t = e^{\int_0^t \mathcal{L}_\tau d\tau}$. Moreover, the eigenvalues $\lambda_\alpha(t)$ of the dynamical map are related to the corresponding eigenvalues $\mu_\alpha(t)$ of the time-local generator $\mathcal{L}_t$ via $\lambda_\alpha(t) = e^{\int_0^t \mu_\alpha(\tau) d\tau}$. Actually, examples 1–5 belong to the commutative class. One has, therefore, the following obvious property:

Proposition 4 If Λ_t defines commutative P-divisible map, then $(d/dt)|\lambda_\alpha(t)| \leq 0$ or equivalently $\mathrm{Re}\mu_\alpha(t) \leq 0$ for $\alpha = 1, ..., d^2 - 1$.

Indeed, one has $(d/dt)\|\Lambda_t[F_\alpha]\|_1 = (d/dt)|\lambda_\alpha(t)|\|F_\alpha\|_1$ and hence $(d/dt)\|\Lambda_t[F_\alpha]\|_1 \leq 0$ implies $(d/dt)|\lambda_\alpha(t)| \leq 0$.

It should be stressed that normal maps and commutative maps define two different classes with nontrivial intersection. In particular, a map that is not unital cannot be normal but can be commutative (cf., example 6). Consider a time-local generator

$$\begin{aligned}\mathcal{L}_t[\rho] &= \mathcal{L}_t^{(1)}[\rho] + \mathcal{L}_t^{(2)}[\rho] \\ &= -i[H(t), \rho] - \sum_k \gamma_k(t)[A_k(t), [A_k(t), \rho]],\end{aligned}$$

with $A_k^\dagger(t) = A_k(t)$ and $H^\dagger(t) = H(t)$ [cf., (7)]. This map is unital but it need not be normal. It is normal if $\mathcal{L}_t^{(1)}$ and $\mathcal{L}_t^{(2)}$ commute. However, being normal it still might be noncommutative.

Let us observe that for commutative maps condition (2) may be easily translated to the condition upon the time-local generator $\mathcal{L}_t$. Using the well-known property of matrices $\mathrm{Det}e^A = e^{\mathrm{Tr}A}$ one finds that (2) is equivalent to $\mathrm{Tr}\mathcal{L}_t \leq 0$, and hence the following:

Corollary 5 If Λ_t is a commutative divisible map, then its time-local generator $\mathcal{L}_t$ satisfies

$$\langle\alpha|(\mathbb{1} \otimes \mathcal{L}_t)[P^+]|\alpha\rangle \leq 0. \qquad (15)$$

Example 6 (amplitude damping channel) The dynamics of a single amplitude-damped qubit is governed by a single function $G(t)$,

$$\Lambda_t[\rho] = \begin{pmatrix} \rho_{11} + (1 - |G(t)|^2)\rho_{22} & G(t)\rho_{12} \\ G^*(t)\rho_{21} & |G(t)|^2\rho_{22} \end{pmatrix}, \qquad (16)$$

where the function $G(t)$ depends on the form of the reservoir spectral density $J(\omega)$ [2]. The dynamical map Λ_t is commutative but not normal. The corresponding eigenvalues read as follows: $\lambda_0(t) = 1$, $\lambda_1(t) = G(t)$, $\lambda_2(t) = G^*(t)$, and $\lambda_3(t) = |G(t)|^2$. This evolution is generated by the following time-local generator,

$$\mathcal{L}_t[\rho] = -\frac{is(t)}{2}[\sigma_+\sigma_-, \rho] + \gamma(t)\left(\sigma_-\rho_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right),$$

where $\sigma_\pm$ are the spin lowering and rising operators together with $s(t) = -2\mathrm{Im}(\dot{G}(t)/G(t))$, and $\gamma(t) = -2\mathrm{Re}(\dot{G}(t)/G(t))$. It is clear that $(d/dt)|\lambda_k(t)| \leq 0$ implies $\gamma(t) \geq 0$. Again in this case this condition is necessary and sufficient for Markovianity. Note that now eigenvalues are in general complex. Interestingly, for Lorentzian spectral density $J(\omega) = (\gamma_M\lambda^2/2\pi[(\omega - \omega_c)^2 + \lambda^2])$ the function $G(t)$ becomes real and hence $f(t) = \frac{1}{4}[1 + G(t)]^2$ and condition (9) implies $\gamma(t) \geq 0$. This example may be considered as an analog of the non-Hermitian Hamiltonian with real spectra analyzed by Bender [35].

Conclusions.—In this Letter we provided further characterization of non-Markovian evolution for three important classes of dynamical maps: unital, normal, and commutative. It is shown that P divisibility implies simple conditions for the spectra of the dynamical maps—singular values in the case of unital maps, and eigenvalues in the case of normal and commutative maps. These conditions provide much stronger non-Markovianity witness than the volume of accessible states [18]. Finally, it is argued that the quantity $\langle\alpha|(\mathbb{1}\otimes\Lambda_t)[P^+]|\alpha\rangle$ may be considered as a dynamical analog of entanglement witness that can be accessed in the experimental scenario. Our analysis is illustrated by several paradigmatic examples. It should be stressed, however, that the notion of non-Markovianity, or memory effects, based solely on a reduced dynamics described by the dynamical map satisfying the corresponding master equation has to be taken with care. It turns out [36] that the random unitary qubit evolution Λ_t satisfying the non-Markovian master equation—corresponding to negative decoherence rates—may be realized as stochastic averaging of the purely unitary evolution governed by dephasing dynamics in random directions, or equivalently, as a classical perfectly Markov process. This only proves that the notion of non-Markovian quantum evolution still deserves a thorough analysis.

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