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Finite-size effects in thermodynamics: Negative compressibility and global instability in two-phase systems

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We have measured the equilibrium melting pressure of helium-4 as a function of the crystal size. Negative compressibility of a liquid with an inclusion of solid seed is predicted theoretically and verified experimentally with helium-4 crystal-superfluid system at 0.15 K. This two-phase system is shown to be stable if the crystal size is large enough, which is proven by the experiment. Crystal seeds that are too small spontaneously either melt completely or grow to a large enough size.

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I. INTRODUCTION

Positivity of the compressibility, i.e., an increase of the internal pressure of a body as a result of a decrease of its volume, is intuitively obvious and is proved in the first chapters of textbooks on thermodynamics [1]. The classical proof of the positivity of the compressibility, however, does not account for energy of interfaces which are present if the system consists of more than one phase. The importance of the surface energy becomes nonetheless obvious if one considers a phase of smaller and smaller size R in which the energy density (that is the internal pressure) α/R diverges (α is the surface tension). We will show below that the presence of a piece of one phase, e.g., solid, merged in the bulk of another phase, e.g., liquid, results in several unusual effects such as negative compressibility of the whole system and global instability of such two-phase system.

The experimental observation of the finite-size effects in usual systems is quite problematic because of an extremely long relaxation time needed for thermodynamic equilibrium to settle down. To exemplify, the relaxation time of a cubic centimeter size water-ice system is about 10^{11} s \sim 3000 years [2]. The reasons for such a long relaxation rate are the large latent heat of crystallization/melting and weakness of the capillary/gravity forces, which drive the system to equilibrium. The latent heat can be reduced by decreasing temperature, and helium is the only substance which exists in a liquid phase down to absolute zero: the equilibrium melting pressure P_0 amounts 25.3 bar for helium-4 and 34.4 bar for helium-3. At the same time, at low enough temperatures liquid phase of helium is superfluid, which provides extremely fast transport of heat and mass. Thus, helium presents a unique system where the conditions for phase equilibrium can be reached and tested experimentally.

In this work, we present a full theoretical description of the two-phase system including mechanical and thermodynamic equilibrium with finite surface energy and deduce the conditions for negative compressibility and its stability regime. We also describe an experimental observation of negative compressibility in the system consisting of liquid and solid helium-4 at 0.15 K.

II. COMPRESSIBILITY OF TWO-PHASE SYSTEM

Without loss of generality, we may consider a small seed of solid merged into liquid. For simplicity, we neglect the anisotropy of solid as well as the gravity (the latter assumption is always valid if the size R of the seed is smaller than the capillary length which is about 1 mm for the solid-liquid interface of helium). In equilibrium, the solid forms a sphere with radius R surrounded by the liquid. For the finite curvature of the solid surface, the pressure in the liquid, P_L , and in the solid, P_S , are shifted up with respect to the equilibrium melting pressure for a flat surface, P_0 , to satisfy the conditions of force balance at the interface, $P_S = P_L + 2\alpha/R$, and the thermodynamic equilibrium, $\mu_S(P_S) = \mu_S(P_0) + (P_S - P_0)/\rho_S = \mu_L(P_L) = \mu_L(P_0) + (P_L - P_0)/\rho_L$ (μ is the chemical potential per unit mass) [3]. We assume that the contributions from the entropy and temperature variations are negligible, which is the case for low enough temperatures. From the above equations we find the relation between the equilibrium pressure in the liquid and the size of the solid seed, $P_L(R) = P_0 + 2\alpha\rho_L/[\rho_S(\rho_S - \rho_L)]$.

Suppose that the total volume V of the considered two-phase system increases by an amount δV at fixed mass. This will result in the change δR of the radius of the solid and in the corresponding change δP_L of the pressure in the liquid,

$$\delta P_L = -2 \frac{\alpha\rho_L}{R^2(\rho_S - \rho_L)} \delta R + 2 \frac{\alpha}{R} \frac{\rho_S \delta\rho_L - \rho_L \delta\rho_S}{(\rho_S - \rho_L)^2}. \quad (1)$$

Change of the pressure leads to the change of the densities of both phases according to compressibilities β_L, β_S ,

$$\frac{\delta\rho_L}{\rho_L} = \beta_L \delta P_L, \quad (2)$$

$$\frac{\delta\rho_S}{\rho_S} = \beta_S \left(\delta P_L - 2 \frac{\alpha}{R^2} \delta R \right). \quad (3)$$

The pressures and densities change due to the total volume change and due to partial melting of the solid,

$$\frac{\delta\rho_L}{\rho_L} = \frac{\delta M_L}{M_L} - \frac{\delta V_L}{V_L}, \quad (4)$$

$$\frac{\delta\rho_S}{\rho_S} = -\frac{\delta M_L}{M_S} - \frac{\delta V}{V_S} + \frac{\delta V_L}{V_S}, \quad (5)$$

TABLE I. Comparison of the product of the surface tension and the compressibility, $\alpha\beta$, in different systems with the theoretical estimation, $a/6$ (see text). β is given for denser phase.

System	α [erg/cm ²]	β [cm ³ /erg]	$\alpha \times \beta$ [Å]	$a/6$ [Å]
Water liquid-vapour, NC	70	4.4×10^{-11}	0.31	0.52
Methanol liquid-vapour, NC	20	1.0×10^{-10}	0.20	0.68
Mercury liquid-vapour, NC	500	3.2×10^{-12}	0.16	0.48
Helium-4 solid-liquid $T < 2$ K	0.2	5.5×10^{-9}	0.11	0.52

where we have employed $\delta M_L = -\delta M_S$ and $\delta V_S = \delta V - \delta V_L$.

The geometrical relation

$$\frac{\delta R}{R} = \frac{1}{3} \frac{\delta V_S}{V_S} = \frac{1}{3} \frac{\delta V}{V_S} - \frac{1}{3} \frac{\delta V_L}{V_S} \quad (6)$$

closes the system of Eqs. (1)–(6) with respect to the six unknown increments δR , δP_L , $\delta \rho_L$, $\delta \rho_S$, δM_L , and δV_L . Finally, we obtain the sought-for compressibility $\beta \equiv -(1/V)(\partial V/\partial P)$ of the system

$$\beta = -\frac{1}{\alpha \rho_L^2 V (2\Delta\rho_{SL} - 4\alpha\rho_S\beta_S/R)} \times R [V_S(3\Delta\rho_{SL} - 2\alpha\beta_S\rho_S/R)(\Delta\rho_{SL}^2 - 2\alpha\rho_S\rho_L\Delta\beta_{LS}/R) - 2\alpha\rho_L(\Delta\rho_{SL} - 2\alpha\beta_S\rho_S/R)\{V_S\beta_S\rho_S + V_L\beta_L\rho_L\}], \quad (7)$$

where $\Delta\rho_{SL} = \rho_S - \rho_L$ and $\Delta\beta_{LS} = \beta_L - \beta_S$.

Each round bracket contains two competing terms corresponding to two effects of the volume increase: (i) the increase of the pressure due to the increase of curvature of the solid seed, and (ii) the decrease of the pressure due to compressibilities of both phases. The compressibility $-(1/V)(\partial V/\partial P)$ is negative if the first effect is stronger than the latter, which occurs if the radius R of the solid is large. The crossover radius $R_{cr} \sim (\rho/\Delta\rho)\alpha\beta$ can be estimated as follows. The internal pressure of a body is of the order of $3\varepsilon/a^3$, assuming 6 nearest neighbors, where a is the interatomic distance and ε is the characteristic interaction between neighboring particles. The compressibility β is thus of the order of $-(1/V)\partial V/\partial P = -(3/a)/(\partial P/\partial a) = (1/3)a^3/\varepsilon$. The surface tension α is about $(1/2)\varepsilon/a^2$, and the product $\alpha \times \beta$ is therefore of the order of $a/6$. To illustrate the validity of this estimation we list in Table I the corresponding measured values for several substances. As can be seen, the product $\alpha \times \beta$ indeed by the order of magnitude smaller than the interatomic distance a , which ensures that $R \gg \alpha\beta$, and that the compressibility of the two-phase system is *always negative*:

$$\beta \equiv -\frac{1}{V} \frac{\partial V}{\partial P} = -\frac{3R}{2\alpha} \left(\frac{\Delta\rho_{SL}}{\rho_L} \right)^2 \frac{V_S}{V}. \quad (8)$$

This conclusion remains valid also for system where the lighter phase (gas bubble) is surrounded by the heavier one (liquid). In this case the shift of the equilibrium pressure due to finite R is negative [see Eq. (1)], but increase of the volume leads to evaporation, i.e., to the increase of the radius, and the change in pressure is again positive.

Note also that the compressibility Eq. (8) goes to zero if $R \rightarrow 0$, meaning that the bulk modulus diverges. However, we will see later that too small crystals are unstable.

III. STABILITY OF TWO-PHASE SYSTEM

In the following we consider a stability of the two-phase system, i.e., sustainability with respect to a small change of the radius R of the solid. The change δR of the radius results in the change of pressures δP_S in the solid and δP_L in the liquid which are connected by the mechanical force balance, $\delta P_S = \delta P_L - (2\alpha/R^2)\delta R$. The condition of the conservation of mass $\Delta\rho_{SL}\delta V_L = V_S\delta\rho_S + (V - V_S)\delta\rho_L$ gives

$$\delta P_L = -\frac{4\pi R[R\Delta\rho_{SL} - (2/3)\alpha\beta_S\rho_S]}{[V\beta_L + V_S(\beta_S\rho_S - \beta_L\rho_L)]} \delta R, \quad (9)$$

and the corresponding change of chemical potential difference between the phases is

$$\begin{aligned} \delta\mu_L - \delta\mu_S &= \frac{\delta P_L}{\rho_L} - \frac{\delta P_S}{\rho_S} = \frac{\Delta\rho_{SL}}{\rho_S\rho_L} + \frac{2\alpha}{\rho_S R^2} \delta R \\ &= \left(-\frac{4\pi R[R\Delta\rho_{SL} - (2/3)\alpha\beta_S\rho_S]}{[V\beta_L + V_S(\beta_S\rho_S - \beta_L\rho_L)]} + \frac{2\alpha}{\rho_S R^2} \right) \delta R. \end{aligned} \quad (10)$$

As shown in Sec. I, the ratio $\alpha\beta/R$ is always much smaller than unity, and, with the assumption $V_S \ll V$, the exact Eq. (10) simplifies to

$$R^2 \frac{\partial \Delta\mu_{LS}}{\partial R} = \frac{2\alpha}{\rho_S} - \frac{3\Delta\rho_{SL} R V_S}{\beta_L \rho_L V}. \quad (11)$$

The derivative $\partial \Delta\mu_{LS}/\partial R$ is positive if the volume V_S of the solid is relatively small. This means global instability of the solid seed: a random decrease of its size results in the undersaturation $\Delta\mu_{LS} < 0$, and the crystal will continue melting and disappears completely. In the opposite case of small increase of the size, $\Delta\mu_{LS} > 0$, and the crystal will continue growing until its size reaches the critical value $R_c = (\rho_L/\Delta\rho_{SL})^{1/2}(\alpha\beta V/2\pi)^{1/4}$ at which the derivative Eq. (11) changes sign to negative, and the system becomes stable. For the solid-liquid helium-4 sample of 1 cm size, the critical radius R_c is about 0.1 mm.

Finally, let us consider the conditions for homogeneous nucleation of a solid in the case of a finite-size sample. In the infinite liquid the critical radius for 3D-nucleation is determined by the competition between the surface energy $4\pi R^2\alpha$ and the oversaturation $(4/3)\pi R^3(\Delta\mu_L - \Delta\mu_S)$: $R_{3D} = (2\alpha/\Delta P_L)(\rho_L/\Delta\rho_{SL})$ [1]. If the volume V of the liquid is finite, one should account for the compressibility of the liquid. After the nucleation of a solid seed with radius R , the chemical potential difference between liquid and solid phases becomes

$$\mu_L - \mu_S = \frac{\Delta\rho_{SL}}{\rho_S\rho_L} \Delta P_L + \frac{\Delta\rho_{SL}}{\rho_S\rho_L} \delta P_L - \frac{2\alpha}{\rho_S R}, \quad (12)$$

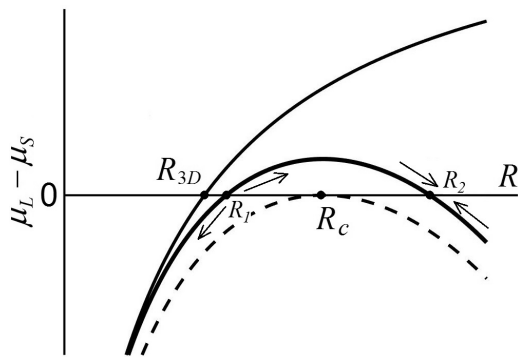


FIG. 1. Oversaturation $\mu_L - \mu_S$ after the nucleation of a crystal of size R , Eq. (14). If the initial oversaturation $(\Delta\rho_{SL}/\rho_S\rho_L)\Delta P_L$ is positive and the volume is infinite then the crystal grows to an infinite size (upper, thin solid curve). Accounting for the finite total volume V shifts the critical radius R_{3D} of the homogeneous 3D nucleation to a larger value R_1 and sets the maximum size of the crystal R_2 (thick solid curve). Decreasing the initial oversaturation brings R_1 and R_2 closer together until they coincide at the critical radius $R = R_c(V)$, which is the smallest possible stable size of the solid seed (lowest, dashed curve). See text for more details.

where $\Delta P_L = P_L^0 - P_0$ is the initial oversaturation of the liquid, and $\delta P_L = P_L(R) - P_L^0$ is the change of the pressure in the liquid due to the formation of the solid seed. The change $\delta P_L = P_L(R) - P_L^0$ is found from the mass conservation relation, $\rho_L\delta V_L + V_L\delta\rho_L + \rho_S\delta V_S = 0$:

$$\delta P_L = -\frac{4\pi R^3 \Delta\rho_{SL}}{V\beta_L\rho_L}. \quad (13)$$

By substituting Eq. (13) into Eq. (12) we find the chemical potential difference between phases after the nucleation,

$$\mu_L - \mu_S = \frac{\Delta\rho_{SL}}{\rho_S\rho_L}\Delta P_L - \frac{4\pi R^3 \Delta\rho_{SL}^2}{3V\beta_L\rho_S\rho_L^2} - \frac{2\alpha}{\rho_S R}. \quad (14)$$

Figure 1 displays a sketch of the dependence in Eq. (14). The finiteness of the volume V shifts the critical radius for the 3D nucleation R_{3D} to a larger value R_1 . This radius, R_1 , however, corresponds to unstable configuration: it will either melt or grow further; but in contrast to the infinite liquid where the seed grows infinitely large (upper curve), there is finite radius R_2 at which the configuration is stable. The decrease of the initial oversaturation ΔP_L brings R_1 and R_2 closer together, and at $\Delta P_{L,c} = (8/3)(2\pi\alpha^3\rho_L^2/\beta_L\Delta\rho_{SL}^2V)^{1/4}$ both radii coincide with the critical radius of global instability R_c derived above, $R_{1,\max} = R_{2,\min} = R_c = (\rho_L^2\alpha\beta V/2\pi\Delta\rho_{SL}^2)^{1/4} \approx 0.1$ mm for $V \sim 1$ cm³. The corresponding shift of the equilibrium melting pressure amounts to ≈ 0.5 mbar, and no crystal can be created at lower overpressures. This conclusion agrees very well with all available measurements data of the overpressure needed to nucleate a crystal. For example, in the experiments by Ruutu *et al.* in Helsinki [4,5] the nucleation occurred at overpressures 2.8 ... 3.5 mbar, while in the ³He crystal experiments by Tsepelin *et al.* [6] at 2.5 mbar, and in the experiments in Lancaster at 2 ... 3 mbar [7]; in this work, we have observed 3 ... 15 mbar.

IV. THERMAL EFFECTS AND RELAXATION TIME

Growing or melting of a solid is associated with the release of the latent heat \mathcal{L} on the interface. Due to the finite thermal conductivity κ and due to the Kapitza thermal boundary resistance R_K there will be thermal gradients in the system, $\delta T_\kappa = v\mathcal{L}\rho_S R/\kappa$ and $\delta T_K = v\mathcal{L}\rho_S R_K$, respectively. A typical velocity v of the helium solid-liquid interface in our experiments was about 10^{-5} m/s, and the latent heat at the temperature of the experiment, of 0.15 K, is $\mathcal{L} = 3.6$ mJ/kg [8]. Using $\kappa = 3 \times 10^3$ W/(m² K) [9] we find $\delta T_\kappa \approx 2 \times 10^{-11}$ K, which is negligible. The Kapitza resistance $R_K = 0.78$ J/(m²sK) [10] gives $\delta T_K \approx 10^{-5}$ K. The temperature change δT_K contributes a term $S\delta T_K$ to the variation $\Delta\mu$ of the chemical potential, which should be compared to the term $\delta P/\rho$ due to the variation of the pressure. Thus, we find thermal corrections to the pressure in our experiments $\delta p_T = \rho S\delta T_K \approx 4 \times 10^{-4}$ μ bar, which are negligible.

The relaxation time of the system can be estimated as $\tau = R^2\rho_S/(\alpha k)$ [2], where $k \equiv v/\Delta\mu$ is the growth coefficient of the solid-liquid interface. In the absence of thermal effects the growth coefficient is limited by scattering of phonons on the interface, $k = 33/T^4$ sK⁴/m [11–13], and τ is 0.1 s for a helium sample of radius $R \approx 1$ cm at 0.15 K. We note again that helium at low temperatures presents a unique system where the relaxation is fast enough to carry out measurements on equilibrium crystal shapes [14].

V. EXPERIMENT

We have measured the pressure in superfluid ⁴He in contact with solid helium during slow melting at 0.15 K. The experimental cell is described elsewhere [15]. The capacitive mechanical pressure gauge of Straty-Adams type [16] had a sensitivity of 44 pF/bar allowing measurements with the accuracy of a few microbars in a second using an Andeen-Hagerling 2700A capacitance bridge. Figure 2 shows the pressure during slow melting of the solid sample with the constant extraction rate $\dot{V} = -2.9$ μ mol/s. Zero pressure corresponds to the equilibrium melting pressure P_0 for the flat interface. The solid curve represents $P_L(R) - P_0 = 2\alpha\rho_L/[R(t)\Delta\rho_{SL}]$ assuming spherical shape of the solid with the contact angle of 135° [17–20] at the bottom of the cell and with the initial radius of 3.9 mm. Here we neglect the gravity which has two effects on the measured pressure in the liquid. First, gravity flattens crystals that are larger than the capillary length $\lambda \approx 1$ mm. Second, when the crystal becomes smaller, the hydrostatic pressure difference between the crystal surface and the pressure gauge increases. However, these effects are small: when the crystal is large, $R \gg \lambda$, its height almost does not change with its volume, and the curvature on top is nearly zero; when the crystal is small, $R \lesssim \lambda$, gravity effects are weak compared to the Laplace pressure. Indeed, one can see from Fig. 2 that the simple model of spherical crystal fits our experimental data very well.

The observed system of superfluid-solid helium-4 shows thus negative compressibility: its pressure increases when the number of particles decreases. No spontaneous instability shows up: the pressure increases smoothly without any unusual noise or jumps. In Sec. II we have predicted that the minimum

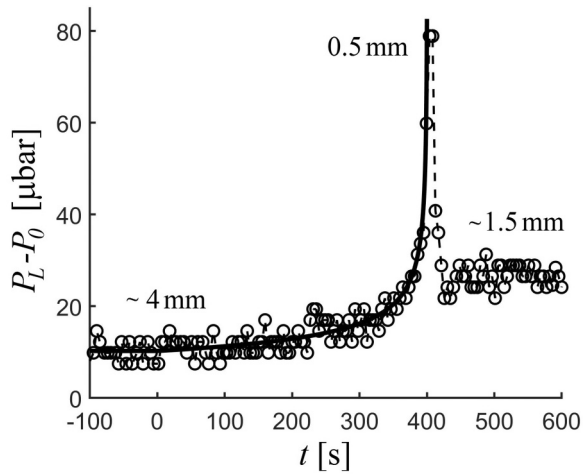


FIG. 2. Pressure in the cell containing superfluid and solid helium-4 during slow unloading with the rate $\dot{V} = -2.9 \mu\text{mol/s}$. The unloading begins at $t = 0$ and stops at $t = 410$ s. During the melting process the size of the crystal decreases which leads to the increase of the melting pressure. The system thus shows negative compressibility. Solid curve represents the theoretical dependence $P_L(R) - P_0 = 2\alpha\rho_L/[R(t)\Delta\rho_{SL}]$, assuming there is no gravity. See text for more details.

stable radius of the solid $R_c = 0.1$ mm and that corresponding shift of the melting pressure $P_L(R_c) - P_0 = 500 \mu\text{bar}$. In the experiment, however, we could not stabilize radii smaller than

0.5 mm with pressure shifts higher than $80 \mu\text{bar}$ because pressure changes became too fast and we had to stop the melting process in order not to lose the crystal. After the helium extraction was stopped, the pressure in the cell dropped because cold helium gas in the filling line warmed, and the crystal grew back slightly due to backflow of helium from the filling line to the cell. After this process the crystal and the pressure in the cell were stable.

VI. CONCLUSIONS

We have shown theoretically that any two-phase system of finite size has negative compressibility, i.e., its internal pressure increases when its volume increases. This effect is inherently originates with the surface tension of the interface between phases, which shifts the co-existence pressure. Moreover, if the size of one of the phases is too small, the phase is unstable and spontaneously melts completely or grows to a large enough size. We have demonstrated the negative compressibility in the system consisting of solid and liquid helium at 0.15 K where the relaxation time is short enough. We have observed the shift of the equilibrium melting pressure up to $80 \mu\text{bar}$ due to the melting of solid.

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