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Conjugate-impedance matched metamaterials for super-Planckian radiative heat transfer

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ABSTRACT

A problem of maximization of the radiative heat transfer (at a given wavelength) between a body and its environment is considered theoretically. It is shown that the spectral density of the radiative heat flux is maximized under the formulated conjugate impedance matching condition, in which case the spectral density of radiated power can exceed the black body limit, resulting in a super-Planckian heat exchange at characteristic distances significantly greater than the wavelength. It is demonstrated that the material parameters of the optimal emitters can be deduced from the known material parameters of the environment and represented by closed-form relations, thus, enabling a way for physical realization of such far-field super-Planckian emitters.

Keywords: black body, thermal emission, super-Planckian radiative heat transfer, metamaterials

1. INTRODUCTION

Hot bodies emit electromagnetic radiation known as thermal radiation, the most prominent example of which is the black body radiation. The thermal radiation is responsible for heat exchange between bodies in vacuum, by the process called radiative heat transfer. It is of a great practical importance to be able to control this process. For instance, controlling absorption and emission spectrum enables for efficient outdoor radiative cooling systems which work without using a power source. In thermal photovoltaics, by shaping the thermal radiation spectrum it is theoretically possible to achieve energy conversion efficiencies approaching Carnot's limit.

In recent years, a number of articles devoted to the so-called *super-Planckian radiative heat transfer* has been published. The majority of such works deals with radiation into the electromagnetic *near-field*, when the bodies that exchange radiative heat are separated by distances significantly smaller than the wavelength. Such emission can exceed the black-body limit set by Planck's law, because the oscillators in bodies separated by subwavelength gaps interact through the Coulomb electric field, and such a near field is much stronger than the wave field. However, there have been also publications that claim super-Planckian emission in *the far-field in free space.* These results have raised questions whether such emission is allowed by fundamental laws of physics.[1]

In fact, it is known that the far-field thermal emission from an *optically small* body with characteristic radius $a \ll \lambda$ can be much stronger than from a black body of the same size. For instance, a resonant microwave antenna with the overall size $a \ll \lambda$ loaded with a resistor $R_{\mathrm{load}} = R_{\mathrm{rad}}$, where R_{rad} is the antenna radiation resistance, behaves as a super-Planckian thermal emitter at the resonant frequency of the antenna, i.e., it radiates more spectral power at this frequency than an ideal black body with the radius $r = a$.[2] This is due to the fact that a resonant antenna can have the effective receiving area much greater than the square of the characteristic diameter of the antenna. It is also known that thermal emission from an *optically large* body with the characteristic dimensions $a \gg \lambda$ can be enhanced by placing it in a transparent shell with the radius $r > a$ and with refractive index $n > 1$.[3] In this case the increase in the emitted thermal radiation power is related to the optical magnification effect: The diameter of the emitter as is seen through the transparent shell is larger (about n times larger when $r \gg a$), which results in an increase in emitter's effective area. A shell made of a hyperbolic medium may have even a stronger effect.[4] Nevertheless, the spectral density of thermal radiation at the exterior of the shell is sub-Planckian in these systems.

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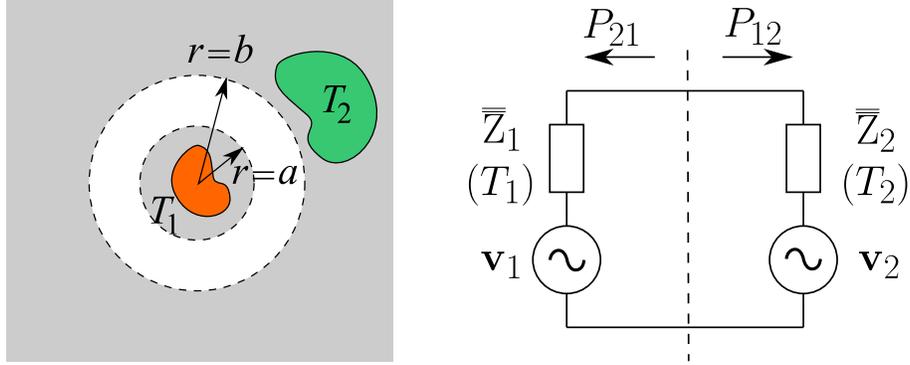
Figure 1. Left: A body fitting the spherical domain $r < a$ exchanging radiative heat with its environment located at $r > b$ (for example, another body and the surrounding space). The body is at the temperature T_1 and the environment is at the temperature T_2. Right: An equivalent circuit representing such heat exchange.

Here, unlike previous works on related subjects, we study a possibility to achieve far-field super-Planckian radiative heat transfer between an optically large body and its environment, by optimizing the emissive properties of the *body itself*. In a nutshell, we question if Kirchhoff-Planck's black body is in fact the best candidate for thermal radiation emitting and receiving *at a given frequency*. The answer to this question is negative.

Unlike in other related works, we do not initially specify the materials of the bodies which exchange heat, instead, we look for hypothetical radiators and sinks with certain electromagnetic properties (which we have to identify) that maximize the transferred spectral power. Only after the maximizing conditions are established, we look for physical realizations. In this way, we arrive at solutions which are conceptually different from typical cases considered in the current literature.

2. RADIATIVE HEAT EXCHANGE BETWEEN A FINITE-SIZE BODY AND ITS ENVIRONMENT

Let us consider a physical body exchanging radiative heat with its remote environment (Fig. 1) at a selected wavelength $\lambda = c/\nu$. The body has the temperature $T = T_1$ and fits a sphere of the radius $r = a$. The environment has the temperature $T = T_2$ and is located in the open domain $r > b$, with $b - a \gg \lambda$. The domain $a < r < b$ is free space. In the following we assume that the intensity of the emitted radiation is low so that the electromagnetic response of all involved media can be considered linear.

The thermal radiation power produced by the body with the temperature T_1 within in a narrow spectral interval $\nu \pm d\nu$ can be expressed as

$$dP(\nu) = \text{Re} \oint_S \mathbf{r}_0 \cdot \overline{[\mathbf{e}_t \times \mathbf{h}_t]} \, dS = \frac{1}{2} \text{Re} \oint_S \mathbf{r}_0 \cdot \overline{[\mathbf{E}_t \times \mathbf{H}_t^{\star}]} \, dS, \tag{1}$$

where $\mathbf{e}_t = \text{Re}\left[\mathbf{E}_t(\mathbf{r}, t)e^{-i2\pi\nu t}\right]$ and $\mathbf{h}_t = \text{Re}\left[\mathbf{H}_t(\mathbf{r}, t)e^{-i2\pi\nu t}\right]$ are the quasi-monochromatic transverse electric and magnetic fields at the selected frequency, $\mathbf{E}_t(\mathbf{r}, t)$ and $\mathbf{H}_t(\mathbf{r}, t)$ are their slowly and randomly varying complex amplitudes, and the integration is taken over the closed spherical surface $r = a$ with the outer unit normal vector $\mathbf{r}_0$. The bar over the vector product stands for statistical averaging and the star symbol * denotes complex conjugate.

In Ref. 5 it is proven that the effect of thermally agitated electromagnetic fluctuations inside a given volume filled with a dissipative magnetodielectric medium with linear response can be equivalently represented with fluctuating electromotive forces concentrated just at the volume boundary. Although Ref. 5 deals with radiated fields expanded over an infinite set of orthogonal plane wave harmonics, the results of Ref. 5 are in fact independent of the expansion basis and can be recast in a compact operator form.

Namely, for any physical body fitting the sphere $r < a$ and satisfying the above conditions, the following operator relation holds for the transverse components of the radiated fields at the surface $r = a$:

$$\mathbf{E}_t = \overline{\overline{Z}}_1 \cdot (\mathbf{r}_0 \times \mathbf{H}_t) + \frac{\mathbf{v}_t}{\sqrt{A_0}}, \tag{2}$$

where $\overline{\overline{Z}}_1$ is the dyadic input impedance operator of the spherical domain $r \leq a$ which encloses the emitting body, and $\mathbf{v_t}/\sqrt{A_0}$ is the fluctuating electromotive force distributed over the boundary of this domain. The source amplitude is defined through its correlation dyadic operator as

$$\overline{\mathbf{v_t v_t^{\star}}} = \frac{4h\nu \, d\nu}{e^{\frac{h\nu}{k_\mathrm{B} T_1}} - 1} \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_1^{\dagger} \right). \tag{3}$$

where h is Planck's constant, k_B is Boltzmann's constant, and the symbol † denotes the conjugate-transpose (Hermitian conjugate) operation. Note that in our case $A_0 = 4\pi a^2$. Eq. (3) follows from the fluctuation-dissipation theorem and is a generalization of Nyquist's formula for noise in electric networks.

On the other hand, the same transverse fields $\mathbf{E_t}$ and $\mathbf{H_t}$ are related through the input impedance operator of the domain $r > a$:

$$\mathbf{E_t} = -\overline{\overline{Z}}_2 \cdot (\mathbf{r_0} \times \mathbf{H_t}). \tag{4}$$

Understanding the transverse electric field $\mathbf{E_t}$ as an analog of voltage in an electric network, and the magnetic field term $-\mathbf{r_0} \times \mathbf{H_t}$ as an analog of current flowing through the impedances $\overline{\overline{Z}}_1$ and $\overline{\overline{Z}}_2$, this situation can be graphically represented with the equivalent circuit shown in Fig. 1 (right), where $\mathbf{v_1} = \mathbf{v_t}$. Note that we do not consider fluctuating sources in the domain $r > a$ when calculating the power emitted from the body with the temperature T_1 (effectively, here we set $T_2 = 0$ and $\mathbf{v_2} = 0$).

Using equations (1)–(4), after some operator algebra presented in Appendix, the spectral density of the emitted power can be expressed as

$$\frac{dP_{12}}{d\nu} = 4\mathrm{Tr}\left[\left(\overline{\overline{Z}}_1^{\dagger} + \overline{\overline{Z}}_2^{\dagger} \right)^{-1} \cdot \overline{\overline{R}}_2 \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_2 \right)^{-1} \overline{\overline{R}}_1 \right] \frac{h\nu}{e^{\frac{h\nu}{k_\mathrm{B} T_1}} - 1}, \tag{5}$$

where $\mathrm{Tr}[\cdot]$ stands for the trace of an operator and $\overline{\overline{R}}_{1,2}$ are Hermitian parts of $\overline{\overline{Z}}_{1,2}$, respectively, as defined in Appendix.

Note that due to complete symmetry of relation (5) with respect to $\overline{\overline{Z}}_1$ and $\overline{\overline{Z}}_2$, the spectral density of power delivered *from* the environment *to* the body at $r < a$ can be expressed as

$$\frac{dP_{21}}{d\nu} = 4\mathrm{Tr}\left[\left(\overline{\overline{Z}}_1^{\dagger} + \overline{\overline{Z}}_2^{\dagger} \right)^{-1} \cdot \overline{\overline{R}}_1 \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_2 \right)^{-1} \overline{\overline{R}}_2 \right] \frac{h\nu}{e^{\frac{h\nu}{k_\mathrm{B} T_2}} - 1}. \tag{6}$$

Hence, when $T_1 = T_2$ the net power flow between the body and the environment vanishes, as mandated by the second law of thermodynamics. Moreover, as is easy to see, the net power flow is always directed from the side with the higher temperature to the side with the lower temperature.

As follows from Eq. (5) the maximum of the spectral density of power emitted by the body at $z < a$ is achieved when $\overline{\overline{Z}}_1 = \overline{\overline{Z}}_2^{\dagger}$, which is essentially the resonant condition for the entire spectrum of spatial harmonics produced by the emitter. We call this condition *conjugate-impedance matching condition* (also note that in the case of reciprocal media $\overline{\overline{Z}}^T = \overline{\overline{Z}}$ and thus $\overline{\overline{Z}}^{\dagger} = \overline{\overline{Z}}^{\star}$, which is the same as the condition of maximum power transfer in reciprocal electric networks). When this condition is fulfilled, the spectral power density satisfies

$$\frac{dP_{12}}{d\nu} = N \times \frac{h\nu}{e^{\frac{h\nu}{k_\mathrm{B} T_1}} - 1}, \tag{7}$$

where N is the rank of $\overline{\overline{R}}_{1,2}$. As, for example, there is infinite number of orthogonal spherical wave harmonics with non-vanishing real part of the wave impedance, there exist emission scenarios in which $N \to \infty$ and thus $dP/d\nu \to \infty$.

Note that in contrast to the conjugate impedance matched body, the Kirchhoff-Planck black body is a simply impedance matched body for which $\overline{\overline{Z}}_1 \approx \overline{\overline{Z}}_2$. In this case, the modes of the radiated field with transverse

wavenumbers not exceeding $k_0 = 2\pi/\lambda$ (the propagating modes) contribute the most to the emission, because for these waves $\overline{\overline{\mathbf{Z}}}_{1,2} \approx \overline{\overline{\mathbf{R}}}_{1,2}$. On the other hand, the modes with transverse wavenumbers higher than k_0 (which are nearly evanescent or dark modes) are characterized with reactive wave impedances and practically do not deliver any power. It can be shown[6] that in this case

$$\mathrm{Tr}\left[\left(\overline{\overline{\mathbf{Z}}}_1^{\dagger} + \overline{\overline{\mathbf{Z}}}_2^{\dagger}\right)^{-1} \cdot \overline{\overline{\mathbf{R}}}_2 \cdot \left(\overline{\overline{\mathbf{Z}}}_1 + \overline{\overline{\mathbf{Z}}}_2\right)^{-1} \cdot \overline{\overline{\mathbf{R}}}_1\right] \approx \frac{\pi a^2}{\lambda^2},\tag{8}$$

and thus, for a black body sphere with the radius $r = a \gg \lambda$ and the surface $S = 4\pi a^2$, we obtain

$$\frac{dP_{12}}{d\nu} = \frac{4\pi a^2}{\lambda^2}\frac{h\nu}{e^{\frac{h\nu}{k_\mathrm{B}T_1}} - 1},\tag{9}$$

which is a known result.

Let us also note that one of the scenarios with the conjugate impedance matched emitter in which $N \to \infty$ is emission into free space, as was first shown in Ref. 6. This fact refutes the widespread belief that Planck's law *itself* limits the spectral density of power emitted by an optically large body in free space. In other words, it is physically allowed to have optically large finite-size emitters which outperform the black body of the same size when emitting into far zone in free space. The effective spectral emissivity of such super-Planckian emitters is greater than unity, moreover, as the above discussion shows, it can be made arbitrary high at any given frequency. Further details regarding such emitters in relation to the fundamental laws of thermodynamics can be found in Ref. 6.

3. REALIZING THE CONJUGATE-IMPEDANCE MATCHING CONDITION

In this section we consider (from a theoretical point of view) whether the conjugate-impedance matching condition is realizable in the case when emitter's environment is not just free space, i.e., when there are other material bodies in the domain $r > b$. As before, the domain $a < r < b$ is free space. In what follows, we consider the case when the electromagnetic properties of the bodies located in the domain $r > b$ are described by reciprocal bi-anisotropic material relations of the form

$$\begin{pmatrix}\mathbf{D} \\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}\overline{\overline{\varepsilon}}(\mathbf{r}) & \overline{\overline{\xi}}(\mathbf{r}) \\ -\overline{\overline{\xi}}^{T}(\mathbf{r}) & \overline{\overline{\mu}}(\mathbf{r})\end{pmatrix} \cdot \begin{pmatrix}\mathbf{E} \\ \mathbf{H}\end{pmatrix}\tag{10}$$

with symmetric permittivity and permeability dyadics: $\overline{\overline{\varepsilon}} = \overline{\overline{\varepsilon}}^{T}$, $\overline{\overline{\mu}} = \overline{\overline{\mu}}^{T}$.

The uniform time-harmonic Maxwell equations for the domain $r > a$ can be written in the so-called six-vector operator notation as

$$\overline{\overline{\partial}} \cdot \mathbf{F} = \omega \overline{\overline{\mathrm{M}}}(\mathbf{r}) \cdot \mathbf{F},\tag{11}$$

where

$$\overline{\overline{\partial}} = \begin{pmatrix}0 & i\nabla \times \overline{\overline{\mathbf{I}}} \\ -i\nabla \times \overline{\overline{\mathbf{I}}} & 0\end{pmatrix}, \quad \overline{\overline{\mathrm{M}}}(\mathbf{r}) = \begin{pmatrix}\overline{\overline{\varepsilon}}(\mathbf{r}) & \overline{\overline{\xi}}(\mathbf{r}) \\ -\overline{\overline{\xi}}^{T}(\mathbf{r}) & \overline{\overline{\mu}}(\mathbf{r})\end{pmatrix}, \quad \text{and} \quad \mathbf{F} = \begin{pmatrix}\mathbf{E} \\ \mathbf{H}\end{pmatrix}.\tag{12}$$

Here, $\overline{\overline{\mathbf{I}}}$ is the dyadic identity operator.

From the uniqueness theorem for Maxwell's equations it follows that Eq. (11) combined with the boundary condition for the transverse magnetic field (we may consider just this case here, without any loss of generality)

$$\begin{pmatrix}0 & \overline{\overline{\mathbf{I}}}_t\end{pmatrix} \cdot \mathbf{F}\big|_{r=a} = \mathbf{H}_t\big|_{r=a}\tag{13}$$

at the surface $r = a$, and the radiation condition

$$\begin{pmatrix}\overline{\overline{\mathbf{I}}}_t & \overline{\overline{\mathbf{Z}}}_\infty \times \mathbf{r}_0 \\ -\overline{\overline{\mathbf{Z}}}_\infty^{-1} \times \mathbf{r}_0 & \overline{\overline{\mathbf{I}}}_t\end{pmatrix} \cdot \mathbf{F}\big|_{r\to\infty} = 0\tag{14}$$

at $r \to \infty$ (where $\bar{\bar{\mathsf{I}}}_{\mathrm{t}} = \bar{\bar{\mathsf{I}}} - \mathbf{r}_0\mathbf{r}_0$, and $\bar{\bar{\mathsf{Z}}}_\infty$ is the wave impedance at $r \to \infty$, for example, $\bar{\bar{\mathsf{Z}}}_\infty = \sqrt{\mu_0/\varepsilon_0}\,\bar{\bar{\mathsf{I}}}_{\mathrm{t}}$ for the case with vacuum at infinity), yields the unique solution for the time-harmonic field $\mathbf{F}(\mathbf{r})$ in the domain $r \geq a$.

In particular, this solution uniquely defines the transverse electric field at the surface $r = a$, which we denote $\mathbf{E}_{\mathrm{t}}\big|_{r=a}$. Considering every possible $\mathbf{H}_{\mathrm{t}}\big|_{r=a}$ in the boundary condition (13) and all corresponding solutions for $\mathbf{E}_{\mathrm{t}}\big|_{r=a}$ we may conclude that Eq. (11), together with the conditions (13) and (14), defines a mapping between the linear vector spaces of the transverse magnetic and electric fields (understood here as functions defined on a surface) at the surface $r = a$. Thus, we may write

$$\mathbf{E}_{\mathrm{t}}\big|_{r=a} = -\bar{\bar{\mathsf{Z}}}_2 \cdot (\mathbf{r}_0 \times \mathbf{H}_{\mathrm{t}})\big|_{r=a}, \tag{15}$$

where $\bar{\bar{\mathsf{Z}}}_2$ is the linear operator of this mapping. The physical meaning of this operator is the input impedance of the domain $r > a$ for the waves incident to it from the domain $r < a$, i.e., it is the same quantity as in Sec. 2. Note that due to the assumed reciprocity of all the media in the domain $r > a$, $\bar{\bar{\mathsf{Z}}}_2 = \bar{\bar{\mathsf{Z}}}_2^T$.

The above result shows that the input impedance operator can be defined for any environment with inhomogeneous constitutive parameters $\bar{\bar{\mathsf{M}}}(\mathbf{r})$. Let us consider one such environment for which the input impedance $\bar{\bar{\mathsf{Z}}}_2$ is known and fixed. In the following, we will look for such a medium (inhomogeneous, in general) filling the domain $r < a$ that maximizes the radiative heat transfer (at a given frequency) from the domain $r < a$ to the domain $r > a$.

As is shown in Sec. 2 the radiative heat flux is maximized when $\bar{\bar{\mathsf{Z}}}_1 = \bar{\bar{\mathsf{Z}}}_2^\dagger$, or in case of reciprocal media, $\bar{\bar{\mathsf{Z}}}_1 = \bar{\bar{\mathsf{Z}}}_2^\star$. In order to find a medium which will realize the latter condition, we apply the coordinate transformation[6] $r \mapsto a^2/r$, which inverts the direction of the radial coordinate ($\mathbf{r}_0 \mapsto -\mathbf{r}_0$), maps the domain $r > a$ into the domain $r < a$, and scales the infinitesimal length elements in the resulting domain proportionally to a^2/r^2:

$$d\mathbf{s} \mapsto \frac{a^2}{r^2} d\mathbf{s}. \tag{16}$$

Thus, the nabla operator under this transformation changes as $\nabla \mapsto (r^2/a^2)\nabla$. Because this transformation inverts the handedness of the coordinate system,[6] all axial vectors such as $\mathbf{H}$, $\mathbf{B}$ and cross-products of polar vectors must flip sign. The same holds for the magnetoelectric coupling dyadic which transforms (at the same physical point in space) as $\bar{\bar{\xi}} \mapsto -\bar{\bar{\xi}}$. Taking into account all mentioned changes, we find that Eq. (11) transforms under the replacement $r \mapsto a^2/r$ as

$$\bar{\bar{\partial}} \cdot \mathbf{F} = \omega \bar{\bar{\mathsf{M}}}' \cdot \mathbf{F}, \tag{17}$$

where (in spherical coordinates)

$$\bar{\bar{\mathsf{M}}}'(r, \theta, \varphi) = \frac{a^2}{r^2} \left[\bar{\bar{\mathsf{M}}}\left(\frac{a^2}{r}, \theta, \varphi\right) \right]^T. \tag{18}$$

Note that the boundary condition (13) stays invariant under the applied transformation if we flip the sign of the given function $\mathbf{H}_{\mathrm{t}}\big|_{r=a}$ at the right-hand side of it in the same way as for the magnetic field component of $\mathbf{F}$ at the left-hand side. Under the same transformation the radiation condition (14) at $r \to \infty$ changes into an absorbing condition at $r \to 0$:

$$\begin{pmatrix} \bar{\bar{\mathsf{I}}}_{\mathrm{t}} & -\bar{\bar{\mathsf{Z}}}_0 \times \mathbf{r}_0 \\ \bar{\bar{\mathsf{Z}}}_0^{-1} \times \mathbf{r}_0 & \bar{\bar{\mathsf{I}}}_{\mathrm{t}} \end{pmatrix} \cdot \mathbf{F}\big|_{r \to 0} = 0. \tag{19}$$

When $\bar{\bar{\mathsf{Z}}}_0 = \bar{\bar{\mathsf{Z}}}_\infty$, the boundary value problem comprising Eq. (17) together with conditions (13) and (19) is equivalent to the problem of finding the input impedance of the domain $r > a$ which we considered earlier, because these problems transform one to another with the chain of substitutions described above. Hence, the input impedance of the domain $r < a$ filled with the medium (18) can be obtained directly from the result (15) by performing the required substitutions. By doing so we find

$$\mathbf{E}_{\mathrm{t}}\big|_{r=a} = \bar{\bar{\mathsf{Z}}}_2 \cdot (\mathbf{r}_0 \times \mathbf{H}_{\mathrm{t}})\big|_{r=a}, \tag{20}$$

that is, the input impedance of the domain $r < a$ filled with the medium (18) coincides with the input impedance of the domain $r > a$ filled with the medium (10).

Thus, we have found with which medium we should fill the domain $r < a$ in order to *impedance-match* it with the given environment in the domain $r > a$. The only remaining task is to understand what we should modify in the chain of substitutions in order to achieve *conjugate impedance* matching. This final task is rather simple. If we complex-conjugate Eq. (17) using that $\overline{\overline{\partial}}^{\star} = -\overline{\overline{\partial}}$ and replacing $\mathbf{F}^{\star} \mapsto \mathbf{F}$ in the result, we obtain the following equation:

$$\overline{\overline{\partial}} \cdot \mathbf{F} = \omega \overline{\overline{\mathrm{M}}}'' \cdot \mathbf{F}, \tag{21}$$

where

$$\overline{\overline{\mathrm{M}}}''(r, \theta, \varphi) = -\frac{a^2}{r^2} \left[\overline{\overline{\mathrm{M}}} \left(\frac{a^2}{r}, \theta, \varphi \right) \right]^{\dagger}. \tag{22}$$

By doing the same in Eq. (19) we obtain

$$\begin{pmatrix} \overline{\overline{\mathrm{I}}}_{\mathrm{t}} & -\overline{\overline{\mathrm{Z}}}_0' \times \mathbf{r}_0 \\ \overline{\overline{\mathrm{Z}}}_0'^{-1} \times \mathbf{r}_0 & \overline{\overline{\mathrm{I}}}_{\mathrm{t}} \end{pmatrix} \cdot \mathbf{F} \big|_{r \to 0} = 0, \tag{23}$$

where $\overline{\overline{\mathrm{Z}}}_0' = \overline{\overline{\mathrm{Z}}}_0^{\star}$. When $\overline{\overline{\mathrm{Z}}}_0' = \overline{\overline{\mathrm{Z}}}_{\infty}^{\star}$, the boundary value problem comprising Eq. (21) together with conditions (13) and (23) is equivalent to the original input impedance problem for the domain $r > a$ with the same chain of substitutions as in the impedance-match case above, but with an additional complex conjugation operation applied to the field vectors. Therefore, when reading the result from Eq. (15), instead of (20) we obtain

$$\mathbf{E}_{\mathrm{t}}^{\star} \big|_{r=a} = \overline{\overline{\mathrm{Z}}}_2 \cdot (\mathbf{r}_0 \times \mathbf{H}_{\mathrm{t}}^{\star}) \big|_{r=a}, \tag{24}$$

or, equivalently,

$$\mathbf{E}_{\mathrm{t}} \big|_{r=a} = \overline{\overline{\mathrm{Z}}}_2^{\star} \cdot (\mathbf{r}_0 \times \mathbf{H}_{\mathrm{t}}) \big|_{r=a}. \tag{25}$$

Thus, the domain $r < a$ filled with medium (22) is conjugate impedance matched with the domain $r > a$ filled with the medium (10). This solves the radiative heat transfer maximization problem.

Note that the medium with parameters (22) is passive whenever the medium (10) is passive. Indeed, the passivity of (10) implies that $i(\overline{\overline{\mathrm{M}}}^{\dagger} - \overline{\overline{\mathrm{M}}})$ is a positive definite matrix: $i(\overline{\overline{\mathrm{M}}}^{\dagger} - \overline{\overline{\mathrm{M}}}) > 0$ at arbitrary $\mathbf{r}$. But this immediately implies that $i(\overline{\overline{\mathrm{M}}}''^{\dagger} - \overline{\overline{\mathrm{M}}}'') > 0$, and therefore (22) is also passive.

However, due to the minus sign in front of the expression (22) the real parts of the material parameters on the main diagonal of the matrix $\overline{\overline{\mathrm{M}}}''$ have the opposite signs as compared to the same of $\overline{\overline{\mathrm{M}}}$, therefore, if the environment is formed by media with positive permittivity and permeability, the optimal emitter must be formed by double-negative media, which is possible only with the use of metamaterials. Various limitations on the performance of such emitters arise from the finiteness of loss and dispersion in metamaterials. These limitations illustrated by a few numerical examples using locally isotropic media are considered in Ref. 6. Examples of using uniaxial metamaterials can be found in Ref. 7.

4. CONCLUSIONS

In this brief article we have presented the theoretical background for the problem of maximization of the radiative heat transfer between a hot body and its environment. Given the known electromagnetic characteristics of the environment — the distribution of complex material parameters such as $\overline{\overline{\varepsilon}}(\mathbf{r})$ and $\overline{\overline{\mu}}(\mathbf{r})$ — we have found a closed form relation for the material parameters of a body which is optimally matched to the environment and achieves maximal spectral density of emitted power at a given wavelength.

The spectral density of emitted power has been expressed in terms of the dyadic input impedance operators of the body and of the environment. These operators relate the transverse components of the electric and magnetic

fields (as functions of the coordinates) at some intermediate boundary separating the domain occupied by the body and its environment. Using an operator-based formalism instead of expanding the fields at this boundary into a series of some orthogonal spatial harmonics (e.g., spherical harmonics) has the obvious benefit: The results obtained with such a formalism are guaranteed to be independent of the expansion basis and are valid for any shape of the body.

Using this approach, we have proven (from the point of view of the fluctuational electrodynamics which deals with bodies kept at thermodynamically equilibrium states and characterized with the infinite internal thermal capacity), that there is no theoretical upper limit on the spectral density of thermal radiation power produced by finite-size bodies, and that it is possible to realize super-Planckian emitters using passive metamaterials. This result refutes the widespread belief that Planck's law itself sets an upper limit for the spectral density of thermal radiation emitted by a macroscopic finite-size object.

Probably the most exciting possible applications of the proposed conjugate-impedance matched superemitters are in creating narrowband super-Planckian thermal emitters for future thermo-photovoltaic systems. Such emitters, if realized, have a potential to revolutionize this technology.

Appendix

Combining equations (2) and (4) we find that

$$\mathbf{E}_{\mathrm{t}} = \overline{\overline{Z}}_2 \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_2 \right)^{-1} \cdot \frac{\mathbf{v}_{\mathrm{t}}}{\sqrt{A_0}}, \tag{26}$$

$$\mathbf{r}_0 \times \mathbf{H}_{\mathrm{t}} = - \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_2 \right)^{-1} \cdot \frac{\mathbf{v}_{\mathrm{t}}}{\sqrt{A_0}}. \tag{27}$$

Therefore, expressing the radial component of the Poynting vector we may write

$$\mathbf{r}_0 \cdot [\mathbf{E}_{\mathrm{t}} \times \mathbf{H}_{\mathrm{t}}^{\star}] = -\mathbf{E}_{\mathrm{t}} \cdot [\mathbf{r}_0 \times \mathbf{H}_{\mathrm{t}}^{\star}] = \mathbf{E}_{\mathrm{t}} \cdot \left(\overline{\overline{Z}}_1^{\star} + \overline{\overline{Z}}_2^{\star} \right)^{-1} \cdot \frac{\mathbf{v}_{\mathrm{t}}^{\star}}{\sqrt{A_0}} = \frac{\mathbf{v}_{\mathrm{t}}^{\star}}{\sqrt{A_0}} \cdot \left(\overline{\overline{Z}}_1^{\dagger} + \overline{\overline{Z}}_2^{\dagger} \right)^{-1} \cdot \mathbf{E}_{\mathrm{t}}, \tag{28}$$

from where, by using Eq. (26) and integrating over the surface $r = a$, we obtain

$$\frac{1}{A_0} \oint_S \mathbf{r}_0 \cdot \overline{[\mathbf{E}_{\mathrm{t}} \times \mathbf{H}_{\mathrm{t}}^{\star}]} \, dS =$$

$$\frac{1}{A_0} \oint_S \overline{\mathbf{v}_{\mathrm{t}}^{\star} \cdot \left(\overline{\overline{Z}}_1^{\dagger} + \overline{\overline{Z}}_2^{\dagger} \right)^{-1} \cdot \overline{\overline{Z}}_2 \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_2 \right)^{-1} \cdot \mathbf{v}_{\mathrm{t}}} \, dS = \mathrm{Tr} \left[\left(\overline{\overline{Z}}_1^{\dagger} + \overline{\overline{Z}}_2^{\dagger} \right)^{-1} \cdot \overline{\overline{Z}}_2 \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_2 \right)^{-1} \cdot \overline{\mathbf{v}_{\mathrm{t}} \mathbf{v}_{\mathrm{t}}^{\star}} \right], \tag{29}$$

where $\mathrm{Tr}[\cdot]$ denotes the trace of an operator. After substituting Eq. (3) into the right hand side of Eq. (29) and considering only the real part of the resulting trace expression as demanded by Eq. (1) we write

$$\mathrm{Re} \left(\mathrm{Tr} \left[\left(\overline{\overline{Z}}_1^{\dagger} + \overline{\overline{Z}}_2^{\dagger} \right)^{-1} \cdot \overline{\overline{Z}}_2 \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_2 \right)^{-1} \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_1^{\dagger} \right) \right] \right) =$$

$$\frac{1}{2} \left(\mathrm{Tr} \left[\left(\overline{\overline{Z}}_1^{\dagger} + \overline{\overline{Z}}_2^{\dagger} \right)^{-1} \cdot \overline{\overline{Z}}_2 \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_2 \right)^{-1} \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_1^{\dagger} \right) \right] + \mathrm{Tr} \left[\left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_1^{\dagger} \right) \cdot \left(\overline{\overline{Z}}_1^{\dagger} + \overline{\overline{Z}}_2^{\dagger} \right)^{-1} \cdot \overline{\overline{Z}}_2^{\dagger} \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_2 \right)^{-1} \right] \right) =$$

$$\frac{1}{2} \mathrm{Tr} \left[\left(\overline{\overline{Z}}_1^{\dagger} + \overline{\overline{Z}}_2^{\dagger} \right)^{-1} \cdot \left(\overline{\overline{Z}}_2 + \overline{\overline{Z}}_2^{\dagger} \right) \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_2 \right)^{-1} \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_1^{\dagger} \right) \right] = 2\mathrm{Tr} \left[\left(\overline{\overline{Z}}_1^{\dagger} + \overline{\overline{Z}}_2^{\dagger} \right)^{-1} \cdot \overline{\overline{R}}_2 \cdot \left(\overline{\overline{Z}}_1 + \overline{\overline{Z}}_2 \right)^{-1} \cdot \overline{\overline{R}}_1 \right], \tag{30}$$

where $\overline{\overline{R}}_{1,2} = \frac{1}{2} \left(\overline{\overline{Z}}_{1,2} + \overline{\overline{Z}}_{1,2}^{\dagger} \right)$. From here we arrive at Eq. (5).

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