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ANTIDERIVATIVE ANTIALIASING, LAGRANGE INTERPOLATION AND SPECTRAL FLATNESS

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ABSTRACT

Aliasing is major problem in any audio signal processing chain involving nonlinearity. The usual approach to antialiasing involves operation at an oversampled rate—usually 4 to 8 times an audio sample rate. Recently, a new approach to antialiasing in the case of memoryless nonlinearities has been proposed, which relies on operations over the antiderivative of the nonlinear function, and which allows for antialiasing at audio or near-audio rates, and without regard to the particular form of the nonlinearity (i.e., polynomial, or hard clipping). Such techniques may be deduced through an application of Lagrange interpolation over unequally-spaced values, and, furthermore, may be constrained to behave as spectrally transparent "throughs" for nonlinearities which reduce to linear at low signal amplitudes. Numerical results are presented.

Index Terms— virtual analog, aliasing, antialiasing, antiderivative antialiasing, nonlinear audio processing

1. INTRODUCTION

Nonlinearities play a central role in virtual analog modeling for audio applications, usually in the context of circuit emulation [1, 2]. An obvious and longstanding problem which emerges, when operating in discrete time, is that of aliasing: under a nonlinear operation, the bandwidth of the signal can be expanded beyond the Nyquist frequency, thus leading to foldover into the baseband. The severity of this aliasing effect depends on the frequency content of the input signal, as well as the form of the nonlinear operation itself, and thus, for audio applications, operation at a standard rate such as 44.1 kHz or 48 kHz can lead to undesirable artefacts. The most basic nonlinear operation on a signal, and the one that will be discussed here is the memoryless nonlinearity.

The brute force and effective remedy is oversampling, usually at four to eight times the audio rate [3, 4, 5]. Specialised techniques, suitable for memoryless nonlinearities of polynomial type [6, 7], and for hard-clipped nonlinearities [8], have been proposed. A recent approach, suitable for memoryless nonlinearities of arbitrary type, is based on a very different idea: that of operation on the antiderivative of the nonlinearity. These methods were first proposed in a signal processing context in [9], and recently extended to higher order in the distinct framework of numerical integration methods in [10], and have been shown to be competitive with oversampling methods, and allow for antialiasing, even for strong nonlinearities such as the hard clipper, at reduced sampling rates (the audio rate, or perhaps twice oversampled rate). Such antiderivative-based antialiasing methods bear some resemblance to methods used in the generation of audio waveforms with reduced aliasing (such as the "integrated wavetable" [11], and "differentiated parabolic waveform" or DPW [12, 13]).

One particular family of such antialiasing methods was proposed in [10]; these methods have the feature that, for nonlinearities which reduce to linear at low amplitudes, a low-pass filtering effect is present, increasing in strength with the order of antidifferentiation, and reducing the utility of such methods. Though one could apply inverse filtering (at the cost of an increased operation count, and boosting of the remaining aliased components), it is natural to look for antiderivative-based antialiasing methods which are, in the low-amplitude limit, free of such a filtering effect. To this end, a distinct family of antialiasing methods is presented here, based on the multiple antidifferentiation framework given in [10], but employing Lagrange interpolation over unequally spaced values, and removing any filtering effect at low signal amplitudes.

Antiderivative antialiasing methods are presented in Section 2. Such methods are reformulated in terms of Lagrange interpolation in Section 4.1. Spectrally flat methods are introduced in Section 4, and derived from an extended Lagrange interpolation formulation. Some concluding remarks and perspectives appear in Section 5.

2. ANTIDERIVATIVE ANTIALIASING

Consider a memoryless nonlinearity defined, in continuous time, as

$$y(t) = f_{(0)}\left(x(t)\right) , \tag{1}$$

where here, $x(t)$ is an input signal dependent on time $t \in \mathbb{R}$, and is assumed bandlimited over a frequency range $[-F_s/2, F_s/2)$, and thus infinitely differentiable, or in $\mathcal{C}^{\infty}$. $y(t)$ is the output signal, which is not in general bandlimited due to the effect of the mapping $f_{(0)}$, assumed here to be nonlinear. Some examples of mappings which occur frequently in virtual analog applications are the hard-clipper, in which case

$$f_{(0)}\left(\alpha\right) = \frac{1}{2}\left(|\alpha+1| - |\alpha-1|\right) , \tag{2}$$

and a soft-clipper

$$f_{(0)}\left(\alpha\right) = \tanh\left(\alpha\right) , \tag{3}$$

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which arises naturally in emulations of analog audio filters such as the Moog VCF [14, 15]. In this paper, the focus is on nonlinear mappings which reduce to linear at low amplitudes:

$$\lim_{\alpha\rightarrow 0} f(\alpha) = \alpha\,. \tag{4}$$

This is true of many nonlinear mappings of interest in audio applications, including (2) and (3), but not all (such as full- of half-wave rectification, for example). Besides this restriction, the nonlinearity may be of a quite general character—in particular, $f_{(0)}(\alpha)$ must be continuous ($\mathcal{C}^0$), but is not necessarily continuously differentiable.

A direct approach to digital emulation is to simply apply the above mapping to a discrete-time sampled and equally-spaced sequence $x^n = x(nT_s)$, with associated sample rate $F_s = 1/T_s$, yielding an output sequence y^n:

$$y^n = f^n_{(0)}\,, \tag{5}$$

where $f^n_{(0)} = f_{(0)}\left(x(nT_s)\right)$. Though implementation is straightforward, aliasing effects are unavoidable, as y^n does not correspond to sampling of a bandlimited function $y(t)$—the standard remedy is operation at an oversampled rate (usually four or eight times above a standard audio rate such as 48 kHz). Here and henceforth in this paper, y^n indicates an approximation to $y(nT_s)$, samples of a $\mathcal{C}^0$ function; in the case above, $y^n = y(nT_s)$, but in general this will not be true.

A recent approach to antialiasing, at audio or near-audio rates, first reported in [9], has relied on the use of antiderivatives of $f_{(0)}$. In [10], the technique is posed in terms of the following reformulation of (1):

$$y = \frac{d^p f_{(p)}}{dx^p}\,, \tag{6}$$

where, for integer $p \geq 1$, $f_{(p)}$ is the pth antiderivative of $f_{(0)}$, to within a polynomial of order $p-1$, assumed henceforth to be 0. In essence, operation over increasingly smooth nonlinearities (i.e., with increasing order p of the antiderivative) leads, in a discrete setting, to increased aliasing suppression. Causal first and second order forms, derived from difference approximations as described in [10], may be written as

$$y^n = \frac{f^n_{(1)} - f^{n-1}_{(1)}}{x^n - x^{n-1}} \tag{7a}$$

$$y^n = \frac{2}{x^n - x^{n-2}}\left(\frac{f^n_{(2)} - f^{n-1}_{(2)}}{x^n - x^{n-1}} - \frac{f^{n-1}_{(2)} - f^{n-2}_{(2)}}{x^{n-1} - x^{n-2}}\right) \tag{7b}$$

where above, $f^n_{(p)} = f_{(p)}\left(x^n\right)$. The first order form above was arrived at through digital filtering considerations in [9], but the second order form does not follow from such analysis. Such approximations introduce bulk delays of $p/2$ samples.

2.1. An Example

The input signals to be used in this short study will be harmonic, and of the form

$$x(t) = \sum_{q=1}^{N} A\sin\left(2\pi q f_0 t\right)\,, \tag{8}$$

for parameters A, f_0 and N. (In this simple example, the amplitude A of each partial is the same.) See Figure 1 for an illustration of the spectral behaviour of the first-order method (7a) under a hard-clip nonlinearity (2), compared to a straightforward implementation (5).

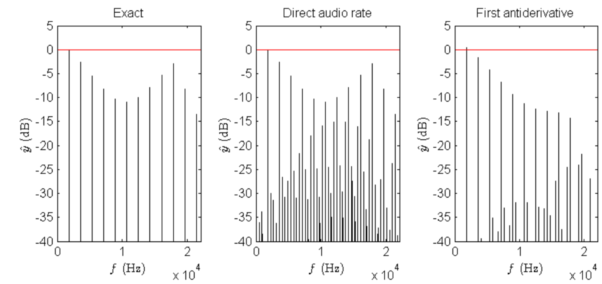

Figure 1: Spectrum $\hat{y}$ of the output y of a hard-clip nonlinearity (2), for an input signal of the form given in (8), with $f_0 = 1783$ Hz, $A = 0.5$ and $N = 10$. Left: exact solution. Middle: solution obtained using (5), at a sample rate of 44.1 kHz. Right: solution obtained using a first order antialiaser as per (7a), also at 44.1 kHz.

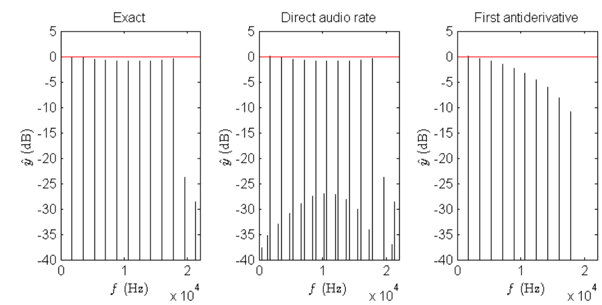

Figure 2: Spectrum $\hat{y}$ of the output y of a soft-clip nonlinearity (3), for an input signal of the form given in (8), with $f_0 = 1783$ Hz, $A = 0.1$ and $N = 10$. Left: exact solution. Middle: solution obtained using (5), at a sample rate of 44.1 kHz. Right: solution obtained using a first order antialiaser as per (7a), also at 44.1 kHz.

Two effects are evident: First, the antiderivative-based method greatly reduces the strength of aliased components with respect to the simple method, particularly in the low to mid frequency range. Second, the antiderivative-based method exhibits a strong degree of suppression of high frequency components in the output signal. This effect persists even in the case of a very low level of nonlinearity—see Figure 2, illustrating a similar effect under the soft-clip nonlinearity (3), and at a much lower input signal amplitude.

3. LAGRANGE INTERPOLATION

Another approach to the discrete-time approximation of (6) is through interpolation of the continuous function $f_{(p)}\left(\alpha\right)$ at unequally spaced locations $\alpha = x^n, \ldots, x^{n-p}$. To this end, define a causal discrete time approximation to (6) as

$$y^n = \sum_{k=0}^{p} a^n_k f^{n-k}_{(p)}\,, \tag{9}$$

where the coefficients a^n_k are to be determined, and will necessarily be dependent on the input signal values $x^n, \ldots, x^{n-p}$. Given that $f_{(0)}$ is in $\mathcal{C}^0$, then $f_{(p)}$ is in $\mathcal{C}^p$, and a Taylor expansion about an arbitrary expansion point x^* to $p+1$ terms yields

$$f_{(p)}^{n-k} \approxeq \sum_{q=0}^{p} \frac{(\Delta_k^n)^q}{q!} f_{(p-q)}^{n} \qquad \Delta_k^n = x^{n-k} - x^* . \tag{10}$$

The $p+1$ constraints on a_k^n such that (9) approximates (1) may then be formulated as the linear system of equations

$$\begin{bmatrix} 1 & \dots & 1 \\ \Delta_0^n & \dots & \Delta_p^n \\ \vdots & \vdots & \vdots \\ (\Delta_0^n)^p & \dots & (\Delta_p^n)^p \end{bmatrix} \begin{bmatrix} a_0^n \\ a_1^n \\ \vdots \\ a_p^n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ p! \end{bmatrix} . \tag{11}$$

(Here and henceforth, the superscript p in $\left(\Delta_p^n\right)^p$ indicates the pth power of Δ_p^n, and should not be confused with the superscript n indicating a time index.) This Vandermonde system corresponds to Lagrange interpolation of the unequally spaced values $x^n, \dots, x^{n-p}$ in order to yield an approximation to (1) using values of the pth antiderivative $f_{(p)}(x^n), \dots, f_{(p)}(x^{n-p})$. Assuming that the $p+1$ values $x^n, \dots, x^{n-p}$ are distinct, it has the solution

$$a_k^n = \frac{p!}{\prod_{l=0, l\neq k}^{p} \left(\Delta_l^n - \Delta_k^n\right)} . \tag{12}$$

Notice that the solution is independent of the expansion point x^*, and thus, without loss of generality, we will set $x^* = 0$, and thus $\Delta_l^n = x^{n-l}$. The case for which the values $x^n, \dots, x^{n-p}$ are not distinct has been discussed in [9, 10], leading to special rules which must be employed, and will not be considered henceforth here.

This new family of antialiasing methods produces the first and second order forms given in (7); it is important to point out, however, that the family of approximants given above is distinct from those given in [10] for $p > 2$. The antialiasing properties of these methods, for $p = 1$ and $p = 2$ have been illustrated in [10], and will be compared against newer methods in the following sections.

3.1. Linear Regime: Spectral Shaping

It is useful to examine the behaviour of the approximations (9), using the coefficients a_k^n given in (12), under low amplitude or linear conditions. Under such conditions, $f_{(0)}(\alpha) = \alpha$, and $f_{(p)}(\alpha) = \alpha^{p+1}/(p+1)!$. Under such conditions, (9) reduces to the linear filtering operation

$$y^n = \frac{1}{(p+1)} \sum_{k=0}^{p} x^{n-k} , \tag{13}$$

with transfer function

$$H\left(z^{-1}\right) = \frac{1}{(p+1)} \sum_{k=0}^{p} z^{-k} , \tag{14}$$

where $z = e^{j\omega T_s}$, for angular frequencies $\omega \in [-\pi F_s, \pi F_s)$. The transfer functions of such $p+1$ averaging operations are illustrated in Figure 3. Such a filtering effect limits the use of such methods in the audio range, particularly given that there is a filter zero at $F_s/(p+1)$, in Hz; note also that it increases with increasing order p of the antidifferentiation. It is thus desirable to seek antialiasing methods which do not produce such a filtering effect, and which behave as simple "throughs" when the input signal is of low amplitude.

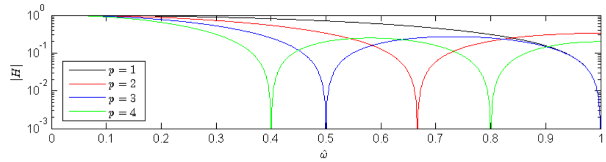

Figure 3: Filter response magnitudes $|H|$, from (14), for different orders p, as indicated, as a function of normalised frequency $\hat{\omega} = \omega/\pi F_s$.

4. SPECTRALLY FLAT METHODS

We present here some novel techniques for ensuring spectrally flat output at low signal amplitudes. One obvious approach to obtaining a spectrally flat method, worth mentioning briefly, for a nonlinearity $f_{(0)}(\alpha)$ which reduces to $f_{(0)}(\alpha) = \alpha$ in the limit of small signal values α, is to rewrite the antiderivative form (6) as

$$y = x + \frac{d^p \tilde{f}_{(p)}}{dx^p} , \tag{15}$$

where $\tilde{f}_{(p)}$ is the pth antiderivative of $\tilde{f}_{(0)} = f_{(0)} - x$. A spectrally flat method, corresponding to a delay of $d \geq 0$ samples at low signal amplitudes then follows immediately as

$$y^n = x^{n-d} + \sum_{k=0}^{p} a_k^n \tilde{f}_{(p)}^{n-k} , \tag{16}$$

where the coefficients a_k^n are as defined in (12). This approach will henceforth be referred to as "simple."

4.1. Extended Lagrange Interpolation

Clearly, for a given order p, a $p+1$ point approximation is fully determined by the interpolation constraints given in (11). By increasing the number of values over which the approximation is calculated, it is possible to arrive at antialiasing methods for which there is no filtering effect in the linear regime—without extracting the linear term in $f_{(0)}$, as above. To this end, consider an approximation over $p+2$ points, as

$$y^n = \sum_{k=0}^{p+1} a_k^n f_{(p)}^{n-k} . \tag{17}$$

Using Taylor expansions about an arbitrary point gives the $p+1$ constraints

$$\sum_{k=0}^{p+1} a_k^n \left(x^{n-k}\right)^q = \begin{cases} 0, & 0 \leq q \leq p-1 \\ p!, & q = p \end{cases} \tag{18}$$

As a final constraint, suppose that, under low amplitude conditions, we constrain constant a_k^n such that (17) evaluates to

$$y^n = \bar{x} , \tag{19}$$

for some value $\bar{x}$. This leads, using $f_{(p)}(\alpha) = \alpha^{p+1}/(p+1)!$ to the further constraint

$$\sum_{k=0}^{p+1} a_k^n \left(x^{n-k}\right)^{p+1} = (p+1)!\bar{x} . \tag{20}$$

The complete set of constraints (18) and (20) may then be written as the augmented Vandermonde system

$$\begin{bmatrix} 1 & \dots & 1 \\ x^n & \dots & x^{n-p-1} \\ \vdots & \vdots & \vdots \\ (x^n)^p & \dots & \left(x^{n-p-1}\right)^p \\ (x^n)^{p+1} & \dots & \left(x^{n-p-1}\right)^{p+1} \end{bmatrix} \begin{bmatrix} a_0^n \\ a_1^n \\ \vdots \\ a_p^n \\ a_{p+1}^n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ p! \\ (p+1)!\bar{x} \end{bmatrix} \tag{21}$$

which has solution

$$a_k^n = \frac{p! \sum_{l=0, l\neq k}^{p+1} \left(\bar{x} - x^{n-l}\right)}{\prod_{l=0, l\neq k}^{p+1} \left(x^{n-l} - x^{n-k}\right)} \quad k = 0, \dots, p+1 \tag{22}$$

If $\bar{x} = x^{n-d}$, for integer $d \geq 0$, then (19) reduces to a pure delay of d samples, and the method is thus spectrally transparent or flat under low amplitude conditions.

4.2. Examples

As a first example, return to the case of the hard clip nonlinearity under a harmonic excitation signal, as described in Figure 1. See Figure 4. The Lagrange-interpolated spectrally flat first order approximation described in the previous section clearly is a better match to the general spectral contour of the exact solution—but with increased aliasing present with respect to the simple (non spectrally-flat) first order antialiaser. The simple approach to spectrally-flat antialiasing, as given in (16), while exhibiting the same antialiasing behaviour as the non spectrally-flat first order antialiaser, shows large deviations in the amplitudes of the primary signal components. Thus spectral flatness in the linear regime does not correspond, in a simple way, to relative flatness of the response once the signal has undergone a nonlinear operation. In this case, the Lagrange-interpolated form outperforms the simple method.

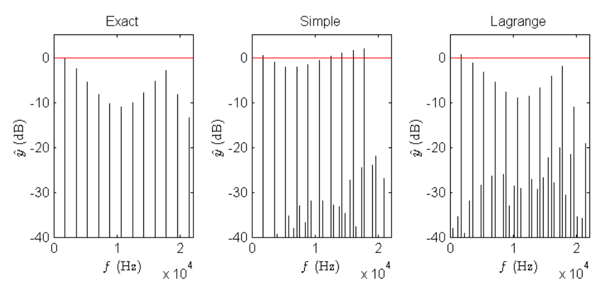

Figure 4: Output as per the input/nonlinearity pair as described in the caption to Figure 1. Left: exact solution. Middle: solution obtained using a simple spectrally-flat first-order antialiaser, as per (16). Right: solution obtained using a spectrally-flat first-order Lagrange interpolated antialiaser, as per (22). In both cases, the number of samples of delay is $d = 1$.

A further complication is that, for the Lagrange-interpolated spectrally flat methods as outlined in Section 4.1, the aliasing characteristics are not independent of the number d of samples of pure delay that the schemes reduce to under linear conditions. See Figure 5, illustrating aliasing characteristics under different choices of d—a roughly 5dB difference in the strength of aliased components may be observed in this case, with better behaviour when $d = 1$—the case for which the range of values employed by the method (x^n, x^{n-1} and x^{n-2}) is centered about the delayed sample x^{n-1}.

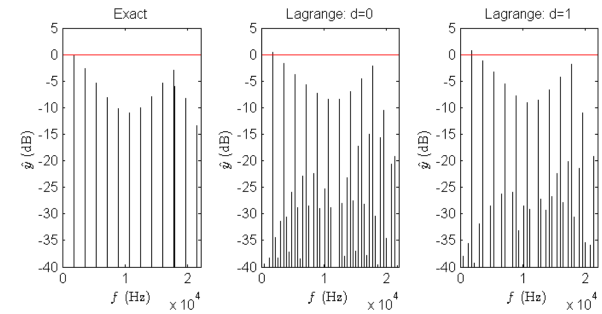

Figure 5: Output as per the input/nonlinearity pair as described in the caption to Figure 1. Left: exact solution. Solution obtained using a simple spectrally-flat first-order antialiaser, as per (16), with pure delay $d = 0$ samples (middle) and $d = 1$ sample (right) under linear conditions.

5. CONCLUDING REMARKS

Antiderivative-based antialiasing methods have been reformulated here as an interpolation problem, allowing an approach to at least understanding some of the deficiencies of such methods, and, in particular, the spectral shaping effect inherent in such methods, at least in the manner in which they have previously been posed. Along with such understanding comes design flexibility with regard to removing such effects at least in the linear regime. What is clear, however, is that such behaviour in the linear regime, while related to the general spectral contour in the nonlinear regime, does not yield a complete picture of the rather complex spectral effects of using such antiderivative-based antialiasers. In particular, phase effects come into play, as seen in the choice of the number of samples of delay in the spectrally flat methods, which has a non-trivial effect on the amount of aliasing suppression. The situation is more complex and less conclusive in the case of higher order antialiasers, not illustrated here. Furthermore, only harmonic tones have been examined here—a case more general than that of isolated sine tones, as examined in previous work [10], but not approaching the complexity of realistic audio input signals.

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