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Polarizability and Light Scattering by Subwavelength Graded-Index Plasmonic Spheres

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Abstract—Light scattering by a subwavelength sphere exhibiting radially inhomogeneous permittivity is here presented. The theoretical foundations describing the scattering response are given in a simple manner and the concept of polarizability is generalized incorporating the inhomogeneity effects. Two illustrative examples are briefly discussed, i.e., a power-law and a Drude-like inhomogeneous profile, exposing the physical mechanisms of their scattering response. The presented results can open the path towards the implementation of graded-index subwavelength particles in modern nanoantenna applications.

I. INTRODUCTION

Scattering by a subwavelength homogeneous sphere is a simple, yet fundamental, problem that can be found in the core of many modern power control/harvesting applications [1], especially at the terahertz-optics regime. Its universality emerges from the fact that the scattering response of a small dielectric sphere can be rigorously quantified by a (normalized) polarizability expression \( \alpha = \frac{\varepsilon_1 - \varepsilon_h}{\varepsilon_1 + 2\varepsilon_h} \) [2]. This simple expression reveals a wealth of physical phenomena, such as the position of the localized surface plasmon (LSP) or plasmonic resonances \( (\varepsilon_1 = -2\varepsilon_h) \).

In this work the generalized concept of the polarizability is presented by assuming the case of a sphere with a radially inhomogeneous (graded-index) permittivity profile. This kind of profiles occur either naturally [3] or as a result of sophisticated engineering processes [4]. Previous studies expanded both Mie and electrostatic scattering theory for certain cases of inhomogeneous profiles (see for example [5], [6], [7]). Here we present some simple analytical formulas for several cases of profiles such as the power-law, exponential, and inhomogeneous Drude profile. Special attention to the LSP resonances is given, exposing the underlying scattering mechanisms of the aforementioned spheres.

II. THEORY & RESULTS

Let us assume a sphere (subscript 0 for external and 1 for internal domain) of radius \( r_1 \) (Fig. 2) subject to a uniform electrostatic potential \( \Phi_0(r, \theta) = -E_0 r \), causing a scattering potential of dipolar character, \( \Phi_s(r, \theta) = \frac{P_0}{2 r} \cos \theta \), and an internal potential \( \Phi_1(r) = f(r) \cos \theta \) [8]. Requiring that the electric flux density be divergence-less at the internal region we have, \( \nabla \cdot \mathbf{D}_1 = \nabla \cdot (\varepsilon_r(r)\nabla \Phi_1(r, \theta)) = 0 \), resulting to the following O.D.E

\[
\varepsilon_r(r) f''(r) + \left( \frac{2 + \varepsilon_r'(r)}{r} \right) f'(r) - \frac{2}{r^2} f(r) = 0
\]  
(1)

with \( C = \frac{r_1 A(r_1)}{A(r_1)} \) being the inhomogeneity parameter. Interestingly, this parameter is a function of \( A(r) \) evaluated at the surface of the sphere, \( r = r_1 \), hence an effective permittivity of this inhomogeneous inclusion can be extracted à la Maxwell Garnett [9]. Inspecting Eq. (2) we observe that the inhomogeneity elegantly modifies the polarizability expression, generalizing the plasmonic resonance condition in a simple manner, i.e.,

\[
\varepsilon_r(r_1) = -\frac{2\varepsilon_h}{C}
\]  
(3)
Apparently, the homogeneous case is when \( A(r) = r \) and \( C = 1 \).

### III. Discussion & Conclusions

As an illustrative example we present the results for two different cases: the power-law profile and the inhomogeneous Drude-like profile. Note that the O.D.E. in Eq. (1) obtains a closed-form analytical solution for certain permittivity profiles such as power-law \([7]\), polynomial \([10]\), and exponential profiles. In our analysis the results are validated by implementing an iterative code of a multilayer sphere exhibiting the same profile \([6]\). For the case of the power-law profile, i.e., \( \varepsilon_r(r) = \varepsilon_1 r^n \), the radial function reads

\[
f(r) = A_1 r^{p_1} + B_1 r^{p_2},
\]

with \( p_{1,2} = -\frac{1}{2} \left( n + 1 \mp \sqrt{(n+1)^2 + 8} \right) \), where \( n \) is the power factor \([7]\). The inhomogeneity parameter in this case is \( C = p_1 \). The plot of the polarizability as a function of both permittivity and power factor can be found in Fig. 2 reveals a shift in the polarization enhancement as a function of \( n \).

The second case of inhomogeneous Drude-like profile \([11]\), \( \varepsilon_r(\omega, r) = \varepsilon_\infty - \frac{\omega_p^2}{\omega^2} (1 + br^n) \), where \( \omega_p \) is the plasma frequency, \( n \) is the power factor, and \( b \) is the inhomogeneity factor. For this case function \( A(r) \) can be expressed as a hypergeometric function \(_2F_1(a, \beta; c; z)\) \([12, Ch. 15]\) with the arguments

\[
A(r) = r a_r \_2F_1 \left( \nu_1, \nu_2; 1 + \frac{3}{n}; -\frac{ab}{a - \varepsilon_\infty} r^n \right)
\]

where \( a = \frac{\omega_p^2}{\varepsilon_\infty}, \ a_r = \left( \frac{ab}{n^2(a - \varepsilon_\infty)} \right)^{1/n} r^{2/n}, \) and \( \nu_{1,2} = \frac{1}{2} \left( n + 3 \mp \sqrt{(n+1)^2 + 8} \right) \). The inhomogeneity parameter is a function of \( \omega, \omega_p, n, \) and the inhomogeneity factor \( b \). This compact description enables the study of inhomogeneous structures with dispersive characteristics, such as inhomogeneities in metals or other materials \([13]\) at the optical-IR frequencies.

Summarizing, the expression of Eq. (2) generalizes the concept of a homogeneous polarizability, allowing us to rigorously explore the non-trivial physical mechanisms for a whole new family of graded-index particles. Additionally, this simplified description can be used to reverse-engineer the inhomogeneity coefficient \( C \) fitting the experimental data, suggesting an alternative explanation to the experimentally observed deviations of plasmonic resonances on deeply subwavelength spheres \([14]\).

The introduced description can be also implemented for a wide range of practical cases such as the accurate modeling of inhomogeneous structures (stratified spheres, transformation optics \([13]\), the implementation of temperature gradients (via a varying permittivity profile), the modeling of diffusive effects especially between interfaces or for extremely small particles \([14]\), where interfaces are not hard but rather follow a radially dependent distribution. It is envisioned that the presented study will stimulate novel power control/harvesting ideas for nanophotonic applications, such as the implementation of subwavelength plasmonic particles exhibiting Luneburg, Eaton, or more exotic graded-index profiles.

### References


