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Time-Varying Reactive Elements for Extreme Accumulation of Electromagnetic Energy

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Accumulation of energy by reactive elements is limited by the amplitude of time-harmonic external sources. In the steady-state regime, all incident power is fully reflected back to the external source, and the stored energy does not increase in time, although the external source continuously supplies energy. Here we show that this limitation can be removed if the reactive element is varying in time, properly modulated by an additional active but lossless device. We show that such time-varying lossless loads of a transmission line or lossless metasurfaces can accumulate electromagnetic energy supplied by a time-harmonic source continuously in time without any theoretical limit. We analytically derive the required time dependence of the load reactance and show that it can, in principle, be realized as a series connection of mixers and filters. Furthermore, we prove that by properly designing time-varying LC circuits, one can arbitrarily engineer the time dependence of the electric current in the circuit fed by a given time-harmonic external source. As an example, we theoretically demonstrate a circuit with a linearly increasing electric current through the inductor. Calculating the energy delivered to the LC circuit and the energy released from it, we show that a modulated LC circuit can accumulate huge energy from both the time-harmonic external source and the modulating device. Finally, we discuss how this stored energy can be released in the form of a time-compressed pulse.

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I. INTRODUCTION

Time-space-modulated structures have recently attracted significant interest especially in realizing magnetless nonreciprocal devices. However, most of the studies have been limited to time-harmonic modulations, extending the classical results on mixers and parametric amplifiers. In this paper we look at alternative possibilities that can open up if we allow *arbitrary* time modulations of structure parameters (circuit reactances, material permittivity, etc.). We expect that this approach may allow a number of important limitations inherent to conventional, harmonically modulated elements to be overcome. As an interesting example for applications we consider capture, accumulation, and storage of electromagnetic energy.

The use of conventional *lossless reactive elements* for energy accumulation is not efficient. From circuit theory, we know that two fundamental reactive elements, the inductance and capacitance, store electromagnetic energy. If a time-varying current source $i(t)$ is connected to an inductor L, the stored magnetic energy is $W_m(t) = Li(t)^2/2$. Replacing the inductor by a capacitor C and using a time-varying voltage source $v(t)$, we can store electric field energy $W_e(t) = Cv(t)^2/2$ [1]. Obviously, the stored energy is limited by the source. To accumulate more energy we need to have sources with a larger output current or voltage. Even more importantly, most practically available sources whose energy we can harvest provide time-periodic (in particular, time-harmonic) output. In the case of time-harmonic sources [$i(t),\ v(t) = A\cos\omega t$, where ω and A represent the angular frequency and the amplitude, respectively], the stored energy fluctuates between zero and $A^2L/2$ (inductor fed by a current source) or $A^2C/2$ (capacitor fed by voltage source). Therefore, the maximum amount of energy we can exploit is at $t = nT$, where $n = 1, 2, 3, \ldots$ (T denotes the period), and it is completely determined by the amplitude of the external source.

The intriguing question we consider here is *if we can exceed this limitation for time-harmonic sources and continuously accumulate the energy supplied by the source in some reactive elements*. The fundamental problem here is to eliminate reflections from a reactive load so that all the incident power will be accumulated in the load and made available at some desired moment in time. If it is possible to control the time dependence of the external source, this problem can be solved by our making the external voltage or current grow exponentially in time [2]. Here we are interested in more-practical scenarios when the external source cannot be controlled (e.g., energy harvesting). Thus, we assume that the external source provides a given time-harmonic output and introduce solutions using

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time-dependent reactive elements $L(t)$ and/or $C(t)$. The discussion here equally applies to circuits, waveguides, or waves incident on lossless boundaries, because in each of these cases the reflection and absorption phenomena can be modeled with an equivalent reactive-load impedance.

To realize the required time modulations of reactance, the load must be active and include some power source. An important question is how much energy does this modulating device supply or receive during the time when the load is accumulating energy from the external time-harmonic source and during the time when this accumulated energy is released. Assuming that there is no dissipation in the load, we show that the energy released can be exactly equal to, greater than, or less than the energy delivered by the time-harmonic source. In the first case, the overall work done by the active time-modulated reactance is zero. When we release more energy than the amount collected, the additional energy is taken from the source inside the load. When we release less energy than the amount collected, the energy stored in the load after the release cycle is greater than the amount initially stored, and it can still be used later. Importantly, we find that in all cases all the energy delivered by the time-harmonic source is captured and stored by the time-modulated load.

Generally speaking, the use of time-varying elements in electronic circuits [3–11] as well as time-varying material properties [12–22] (usually, time-varying permittivity) for engineering wave propagation has been attracting the attention of researchers since the 1960s. These studies mainly focused on achieving nonreciprocity, amplification, and frequency conversion. Here we use time-varying reactive elements for energy accumulation. While in most of the earlier studies, periodic time variations of circuit elements were used, for our goals we need to consider arbitrary time variations of parametric elements.

The paper is organized as follows: In Sec. II, we apply the transmission-line theory and find the required condition to have zero reflection (unlimited accumulation of energy) in a line that is terminated by a single time-modulated reactive load. Subsequently, we elucidate our result by drawing an analogy between our time-modulated load and two different scenarios explained in Secs. II A and II B, respectively. In Sec. III, we go one step further and consider a load that comprises two time-varying reactive elements that are parallel to each other. In contrast to the case studied in Sec. II, we show that it is possible to engineer electric currents flowing through the elements and still obtain zero reflection. We explain how engineering the electric currents affects the amount of energy accumulated by the entire load. Finally, Sec. IV concludes the paper. This paper is our first step to hopefully open a way for further work on systems with arbitrary time modulations of parameters and, in particular, for practical investigations of efficient devices that accumulate energy from time-periodic, low-amplitude sources. We expect that the use of nonharmonic time modulations of system parameters will offer other, more-general means to shape the system response at will.

II. ZERO REFLECTION FROM TIME-MODULATED REACTIVE ELEMENTS

For a linear and time-invariant transmission line fed by a time-harmonic voltage or current source, the energy supplied is fully transferred to the load if the characteristic impedance of the line is equal to the load impedance of that line [23]. Apparently, for the case of a lossless line terminated by a conventional inductance or capacitance, the energy is entirely reflected back as the incident wave arrives at the load, and therefore there is no absorption or accumulation of energy in the load. Mathematically, this property follows from the fact that the characteristic impedance of the line is real, while the load impedance is purely imaginary (reactive).

It is easy to see that by making the load reactance vary in time, we can emulate resistance, although there is no actual power dissipation or generation. The voltage over a capacitor or an inductor and the current flowing through the element are related to each other as

$$\begin{aligned} v_L(t) &= L(t)\frac{di_L(t)}{dt} + \frac{dL(t)}{dt}i_L(t), \\ i_C(t) &= C(t)\frac{dv_C(t)}{dt} + \frac{dC(t)}{dt}v_C(t), \end{aligned} \tag{1}$$

where $v_L(t)$ and $i_L(t)$ denote the instantaneous voltage and current, respectively, in a time-varying inductor and $v_C(t)$ and $i_C(t)$ denote the instantaneous voltage and current, respectively, in a time-varying capacitor. Conventionally, when the element is time independent, the second term in the above equations [$dL(t)/dt$ or $dC(t)/dt$] vanishes, and therefore the voltage and current are proportional by a factor that is purely imaginary in the frequency domain. However, the scenario is completely different when the inductance or capacitance element varies with respect to time. The second term is not zero anymore, and it has the usual form of Ohm's law, where the role of the resistance or conductance is played by the time derivatives of the circuit reactances. Clearly, this virtual resistance or conductance describes virtual absorption of energy, which can actually be accumulated in the reactive element. Next we study how this possibility can be exploited to accept and accumulate incident energy in reactive elements. This study should be done in the time domain. Equation (1) does not allow us to define classical impedance, which is the ratio of the Fourier image of voltage to the Fourier image of electric current. Only when the reactive element is time invariant [$L(t) = L$ or $C(t) = C$] can we define and use the classical impedance $j\omega L$ or $1/j\omega C$. In this study, we cannot use such impedances.

Let us consider a lossless transmission line loaded by a time-varying inductance $L(t)$ and denote the voltage of the wave propagating toward the load as $v^+(t)$. The reflected voltage wave is denoted by $v^-(t)$. The instantaneous voltage over the load and the current flowing through the load are written as

$$
\begin{aligned}
v_L(t) &= v^+(t) + v^-(t), \\
i_L(t) &= \frac{v^+(t) - v^-(t)}{R_0},
\end{aligned}
\tag{2}
$$

where R_0 represents the characteristic impedance of the line. On the other hand, the voltage and the current are related to each other by Eq. (1). Substituting Eq. (1) into Eq. (2), we derive a general formula for the incoming and reflected waves at the load position as

$$
\begin{aligned}
&L(t)\frac{dv^-(t)}{dt} + \left[\frac{dL(t)}{dt} + R_0\right]v^-(t) \\
&\quad = L(t)\frac{dv^+(t)}{dt} + \left[\frac{dL(t)}{dt} - R_0\right]v^+(t).
\end{aligned}
\tag{3}
$$

In this equation, the left-hand side contains the terms measuring the reflected wave, while the right-hand side depends on the incident voltage $v^+(t)$ only. For a given incident voltage $v^+(t)$ and any time-dependent inductance $L(t)$ we can find the reflected wave by solving the above first-order differential equation. Using Eq. (3), we can find ways to manipulate the reflected wave by choosing the proper function for $L(t)$. Here we are interested in accumulating all the incident energy; that is, we are interested in reactive loads that do not reflect.

Assume that the incoming signal is time harmonic, written as $v^+(t) = A\cos\omega t$. In the following, the amplitude and the initial phase of the incident wave are supposed to be unity and zero, respectively, for simplicity of the formulas. Requiring absence of reflection $[v^-(t) = 0]$, we find the corresponding time dependence of $L(t)$ from Eq. (3). It is worth noting that the time-harmonic current in the load is in phase with the incident voltage in the case of zero reflection, as is obvious from Eq. (2). The result reads

$$
L(t) = \frac{R_0}{\omega}\tan\omega t. \tag{4}
$$

Such inductance as a load virtually absorbs all the input energy. Similar considerations can be made for loads in the form of time-varying capacitance or other reactive loads. In the following, we explain the reason why the time-varying inductance described by Eq. (4) gives rise to virtual absorption. To do this, we make an analogy between the inductance and a short-circuited line whose length linearly increases with time. In addition, we also draw another analogy between the inductance and a load consisting of an infinite number of sinusoidal inductances connected in series.

A. Short-circuited line

To understand the physical meaning of the result obtained above, we notice a similarity between this Eq. (4) and the classical formula for the input reactance of a short-circuited transmission line:

$$
Z_{\text{in}} = jR_0\tan kl, \tag{5}
$$

where $k = \omega/c$ (c denotes the phase velocity) is the phase constant and l represents the length of the line. Obviously, the time-varying inductance $L(t)$ given by Eq. (4) is the same as that seen at the input port of a lossless short-circuited transmission line if the length of the transmission line *linearly increases* with constant velocity equal to c, since in this case $Z_{\text{in}} = jR_0\tan kl = j\omega L(t)$ [see Fig. 1(a)]. We see that in this conceptual scenario the reason for having no reflection from a lossless load is that the incident wave never reaches the reflecting termination, since the short is moving away from the input port with the same velocity as the phase front of the incident wave. Thus, varying the load inductance as prescribed by Eq. (4) for enough time, one can accumulate theoretically unlimited field energy in the reactive load. In transmission-line theory, Eq. (5) is derived in the frequency domain as the ratio between the voltage and the electric current at distance l from the short-circuited end. Of course, if the length l is a function of time, as we discussed before, we cannot use the classical impedance. However, this consideration brings a useful analogy for understanding why Eq. (4) results in virtual absorption.

Let us now assume that we would like to release the stored energy. To do that, we would reverse the direction of the velocity of the short (i.e., moving toward the input port) [see Fig. 1(b)]. All the energy stored in the line volume will be compressed in time and released into the feeding line at the moment when the length of the short-circuited line becomes zero. The reactance of the line would correspond to a time-varying inductance given by

$$
L(t) = \frac{R_0}{\omega}\tan(2\omega t_0 - \omega t), \quad t_0 < t < 2t_0, \tag{6}
$$

where t_0 is the moment when the velocity of the short changes direction and, consequently, the short moves backward. On the basis of our analogy between the time-varying inductance and the short-circuited line whose length changes with time, we conclude that the time-modulated inductance is expressed as

$$
\begin{aligned}
L_A(t) &= \frac{R_0}{\omega}\tan\omega t, \quad 0 < t < t_0, \\
L_R(t) &= \frac{R_0}{\omega}\tan(2\omega t_0 - \omega t), \quad t_0 < t < 2t_0,
\end{aligned}
\tag{7}
$$

where $L_A(t)$ and $L_R(t)$ represent the time-modulated inductances in the accumulation and release regimes,

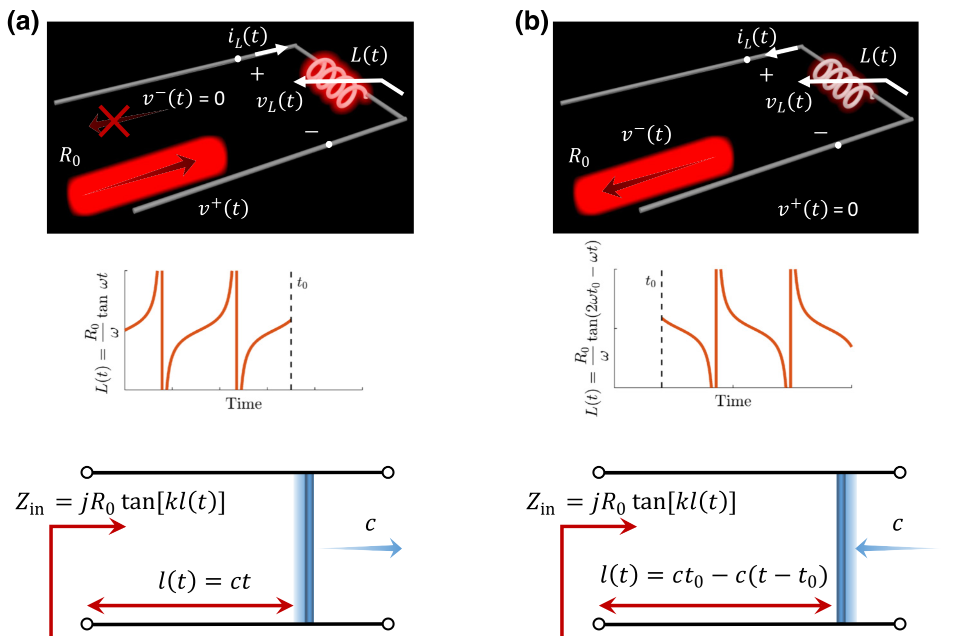

FIG. 1. (a) Transmission line terminated by a time-dependent inductance that absorbs the incoming energy. The white arrow depicted on the time-varying inductance shows that this element is active and contains an internal modulation source that varies the inductance in time. The bottom illustration shows its conceptual realization by a short-circuited line whose length increases with a constant velocity. (b) Energy release by the same line but with different inductance modulation (inverse with respect to moment t_0). This means that in its conceptual realization, the load is moving in the opposite direction.

respectively. It is clear that the function describing $L_R(t)$ is the mirror of the function describing $L_A(t)$ with respect to the moment $t = t_0$.

We simulate the system illustrated at the top of Fig. 1(a) using MathWorks SIMULINK. We assume that our system accumulates the energy until the moment $t_0 = 2.5$ s and subsequently it releases the energy until the moment $t = 2t_0 = 5$ s. The modulation function for the reactive element expressed by Eq. (7) is shown in Fig. 2(a), and the simulated and theoretical results for the reflected wave are shown in Fig. 2(b). The theoretical formula for the reflected wave can be obtained with Eq. (3). The blue line in Fig. 2(b) corresponds to the incident wave [$v^+(t) = 50 \cos \omega t$ V], while the red and yellow lines correspond to the reflected wave. As seen, the reflection is zero until $t_0 = 2.5$ s, meaning that the reactive element stores the electromagnetic energy (virtual absorption). After $t = t_0$, the reflection appears and we are in the release regime. The theoretical and simulated results for the reflected wave are in good agreement.

We next consider the effects of inevitable dissipation losses in the system. Concentrating on the accumulation regime, we add a resistance (R_L) to the load to see the effect of Ohmic losses. This load resistance is connected in series with the time-varying inductance. We simulate again the structure in Fig. 1(a) and observe that if the load resistance is smaller than about 2% of the characteristic impedance of the transmission line (1 Ω for our example of a 50-Ω line), the reflected wave is still near zero and the system works quite well. For resistances larger than 1 Ω ($R_L > 1$ Ω), some reflection appears and the efficiency decreases.

It is possible to eliminate any reflections also for lossy terminations (with arbitrary values of R_L) by a simple modification of the modulation function of the time-varying inductance $L_A(t)$. This way we can compensate the resistive losses completely. Indeed, if we

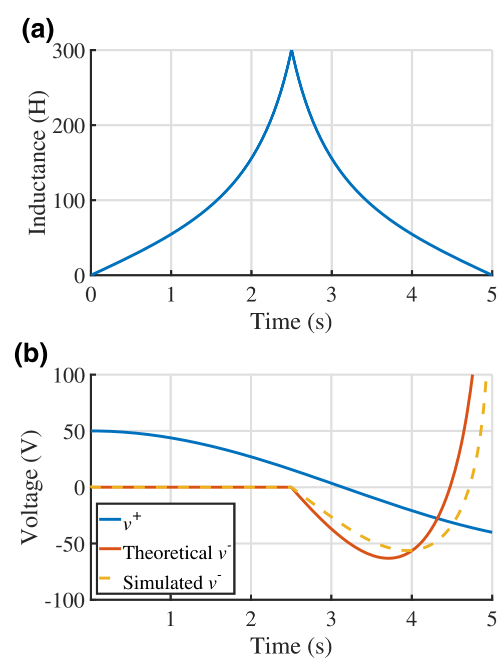

FIG. 2. (a) Time-modulated inductance, as the load of the transmission line, in the accumulation and release regimes. (b) The incident and reflected waves (simulated and theoretical) at the load position. Here $R_0 = 50$ Ω and $v^+(t) = 50 \cos \omega t$ V, where $\omega = 0.5$ rad/s.

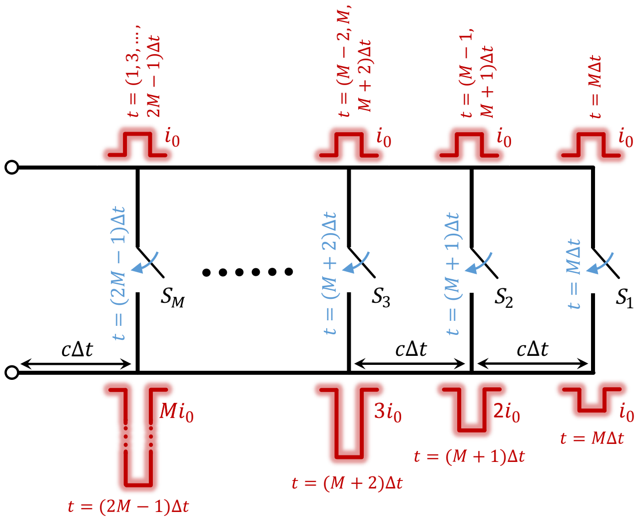

FIG. 3. Space-time-modulated transmission line with M switches. Each switch is closed at specific moments shown in blue. The amplitudes of the signal at different locations of the transmission line at different moments are depicted in red.

rewrite Eq. (3) by assuming that there is also resistance R_L at the termination, we see that the required time-varying inductance to obtain zero reflection becomes

$$L_A(t) = \frac{R_0}{\omega}\left(1 - \frac{R_L}{R_0}\right)\tan\omega t, \quad 0 < t < t_0. \tag{8}$$

According to Eq. 8, if $R_L = 0$, we will achieve the same result as given by Eq. (4) or Eq. (7). If $R_L = R_0$, then the modulated inductance needed to eliminate reflections is zero because in this limiting case the load is already perfectly matched to the transmission line (perfect-absorption condition). Our simulations confirm that the modification of the modulation function given by Eq. (8) indeed results in zero reflection in the accumulation regime for different values of loss resistance R_L.

Similar effects of energy accumulation and release can be achieved with a transmission line periodically loaded with switches that can be switched at appropriate moments, as required by Eq. (5). Figure 3 schematically shows the realization of this concept. We consider an incident signal in the form of periodic pulses of amplitude i_0 and duration Δt. Before moment $t = M\Delta t$, the pulses enter the transmission line and propagate along the line without any reflections because all the switches are open. When the first (leading) pulse approaches switch S_1 at time $t = M\Delta t$, it closes and the pulse is reflected and starts propagating in the opposite direction. Because of the properly adjusted distance between the switches, at moment $t = (M + 1)\Delta t$, the first pulse is summed up (constructive interference) with the second pulse, which has been reflected by switch S_2. The amplitude of the resulting pulse is doubled: $2i_0$. Likewise, at the position of each next switch, the leading-pulse amplitude is increased by i_0, resulting in an output pulse with amplitude Mi_0. In this scenario, the total energy entering the transmission line, proportional to Mi_0^2, is compressed into a single pulse with energy $(Mi_0)^2$. This is not a passive design since, due to the energy conservation, extra work proportional to $(M^2 - M)i_0^2$ is required to close the switches.

B. Load consisting of sinusoidal inductances

Another way to understand why the time-modulated inductance given by Eq. (4) ensures full virtual absorption is to apply Fourier-series analysis. Since the required time-dependent load inductance given by Eq. (4) is a periodic function, we can expand it in a Fourier series. We know that

$$\tan x = 2\sum_{n=1}(-1)^{n-1}\sin 2nx. \tag{9}$$

Using the above equation, we can consider the time-dependent inductance as an infinite collection of harmonically modulated inductances that are connected in series, as illustrated in Fig. 4. Let us assume that the voltage over the whole load in Fig. 4 is $v_L(t) = A\cos\omega t$ and the current flowing through the load is $i_L(t) = A\cos\omega t/R_0$. On the basis of Kirchhoff's voltage law, the voltage $v_L(t) = \sum_{n=1} v_n(t)$, where $v_n(t)$ is the voltage over each time-dependent inductance. Therefore,

$$v_n(t) = L_n(t)\frac{di_L(t)}{dt} + \frac{dL_n(t)}{dt}i_L(t), \tag{10}$$

where $L_n(t) = 2(-1)^{n-1}\frac{R_0}{\omega}\sin 2n\omega t$. Simplifying Eq. (10), we find that

$$\begin{aligned} v_n(t) = A(-1)^{n-1}\{&(2n-1)\cos[(2n-1)\omega t] \\ &+ (2n+1)\cos[(2n+1)\omega t]\}. \end{aligned} \tag{11}$$

Equation (11) shows that the nth time-dependent inductance operates as a mixer in which the input of this mixer is a time-harmonic current signal of frequency ω having amplitude equal to A/R_0 producing as the output two time-harmonic voltage signals of frequency $(2n \pm 1)\omega$ and amplitude $(2n \pm 1)A$. The output signal can be amplified or attenuated depending on the integer number of the inductance element (it is amplified if $2n \pm 1 > 1/R_0$).

By substituting $n = 1, 2, 3, \ldots$ in Eq. (11), we realize that only the first harmonic corresponding to $n = 1$ is not canceled in the series $v_L(t) = \sum_{n=1} v_n(t)$ (which in the usual sense does not converge). The second term of $v_1(t) = A(\cos\omega t + 3\cos 3\omega t)$ cancels out with the first term of $v_2(t) = A(-3\cos 3\omega t - 5\cos 5\omega t)$, and the second

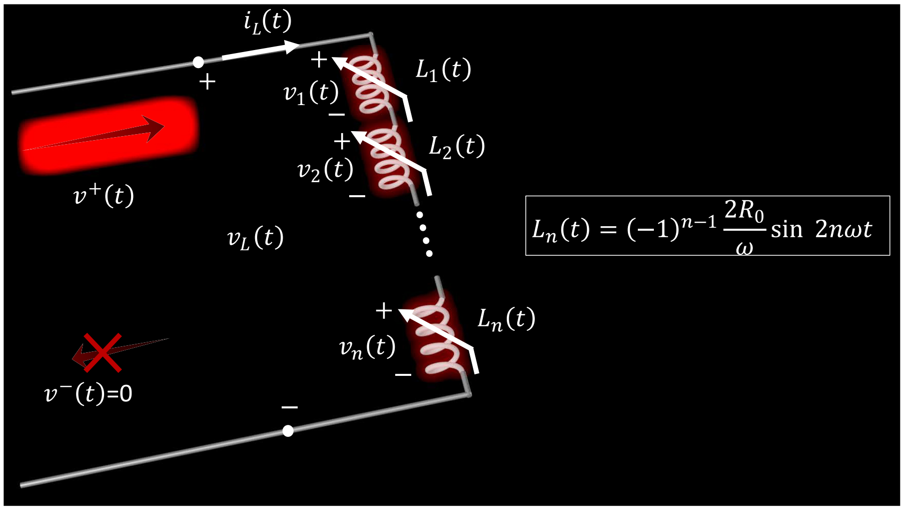

FIG. 4. Transmission line terminated by an infinite number of time-dependent inductances connected in series.

term of $v_2(t)$ is canceled by the first term of $v_3(t)$, and so on. Hence, only the first term $A\cos\omega t$ of $v_1(t)$ survives. Since the amplitude of this term is equal to the amplitude of the total voltage $v_L(t)$, the reflection coefficient is zero. It is worth mentioning that if we have a finite number of time-dependent inductances shown in Fig. 4, we can still emulate full absorption. From the above considerations, we see that only the first harmonic ω and the harmonic $(2n+1)\omega$ remain. The other harmonics automatically vanish. Thus, to emulate full absorption, we need only to remove the $(2n+1)\omega$ harmonic by using a low-pass filter. If we do not filter this harmonic, the reflection is not zero.

III. TIME-DEPENDENT PARALLEL *LC* CIRCUIT

In the previous scenario with one reactive element, the electric current was limited by the characteristic impedance of the line and the amplitude of the incoming wave $[i(t) = A\cos\omega t/R_0]$. The intriguing question is whether *it is possible to realize any (growing) function for the electric current* flowing through the time-varying reactive element. Next we show that it is indeed possible if the time-varying reactive load contains at least two reactive elements, one inductive and one capacitive. Having two connected circuit elements we have an additional degree of freedom to shape the electric current flowing through these components since (assuming a parallel connection) only the sum of the two currents should be equal to $i(t) = A\cos\omega t/R_0$ to ensure zero reflection. In this section we discuss the design of such circuits and investigate the stored energy in the system.

Let us consider a transmission line terminated by a parallel *LC* circuit that is formed by time-dependent components $L(t)$ and $C(t)$. A schematic of the circuit is illustrated in Fig. 5(a). Suppose that the incident voltage wave is $v^+(t) = A\cos\omega t$ and the total electric current is $i(t) = A\cos\omega t/R_0$ (no reflection). Here $i_L(t)$ denotes the current through the inductance and $i_C(t)$ denotes the current through the capacitance. On the basis of Kirchhoff's current law, $i(t) = i_L(t) + i_C(t)$. This condition is fulfilled by

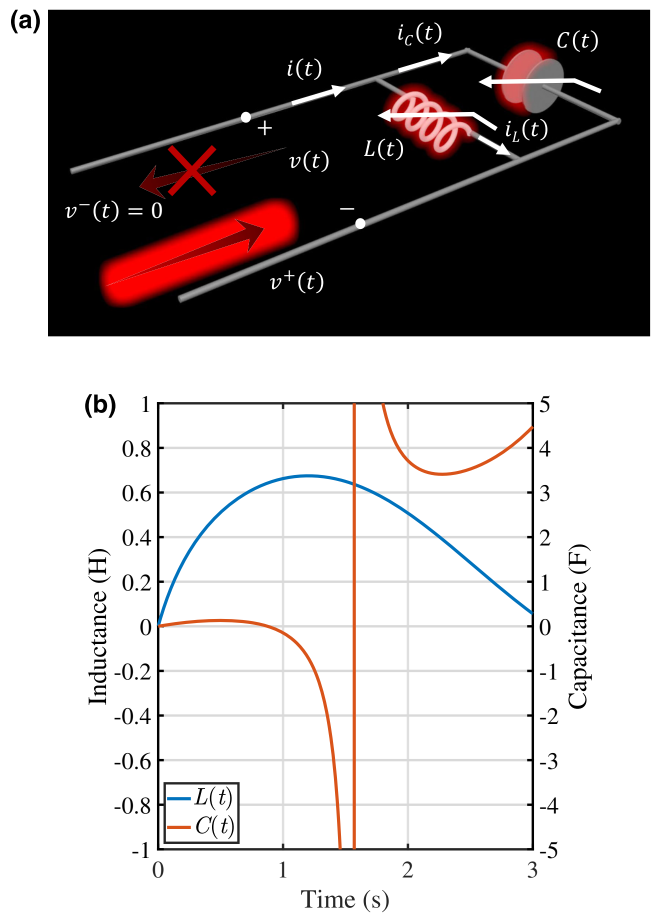

FIG. 5. (a) Transmission line terminated by a parallel $L(t)C(t)$ circuit. (b) Values of $L(t)$ and $C(t)$ required for realization of the zero-reflection regime.

our setting the currents as

$$i_L(t) = \frac{A\cos\omega t}{2R_0} + f(t),$$
$$i_C(t) = \frac{A\cos\omega t}{2R_0} - f(t), \tag{12}$$

where $f(t)$ can be an arbitrary function of time. As an example, we consider $f(t)$ as a linearly growing function $f(t) = I_0 t$ in which $I_0 > 0$ (this is only due to the simplicity of the function, here any differentiable function can be assumed). Applying Kirchhoff's laws and using Eq. (12), we can find the required time dependences of the circuit elements. After some algebraic manipulations, we obtain the following expressions:

$$L(t) = 2R_0\frac{A\sin\omega t}{\omega(A\cos\omega t + 2R_0 I_0 t)},$$
$$C(t) = \frac{\tan\omega t}{2R_0\omega} - \frac{I_0 t^2}{2A\cos\omega t}. \tag{13}$$

As the above equation shows, the capacitance always possess asymptotes due to the tangent function. However, depending on the ratio between $R_0 I_0$ and the angular frequency ω $(R_0 I_0/\omega)$, the inductance can be finite without having an asymptote. For example, Fig. 5(b) shows the functions $L(t)$ and $C(t)$ for $R_0 = 1\ \Omega$, $\omega = 1$ rad/s, $A = 1$ V, and $I_0 = 1$ A/s. At the initial moment, both elements are positive and growing. However, later the inductance decreases and goes to zero, fluctuating around it due to the term $\sin\omega t$ in the numerator.

If the elements are modulated in time as expressed in Eq. (13), no energy is reflected back to the source and all the input energy is continuously accumulated in the LC circuit. However, the reactances exchange energy also with the device that modulates their values in time. Thus, we need to consider the power balance and find how much energy is accumulated in the reactive circuit taking into account also the power exchange with the modulating system. To do this, we assume that the time modulation of the circuit elements stops at a certain time (t_0) and the circuit inductance and capacitance do not depend on time at later times $t > t_0$. This means that at $t > t_0$ there is no power exchange with the system that modulates the reactances. At $t = t_0$ we connect a parallel resistance to the LC circuit as shown in Fig. 6(a) to form a usual RLC circuit with time-invariant elements. We choose the moment t_0 at which the inductance and capacitance are both positive and calculate the energy delivered to the resistor during the relaxation time. This energy is equal to the energy that has been accumulated in the time-modulated circuit during the time $0 < t < t_0$. The rate of releasing the stored energy depends on the value of the resistance. If it is a small resistance, the accumulated energy is consumed in a short time.

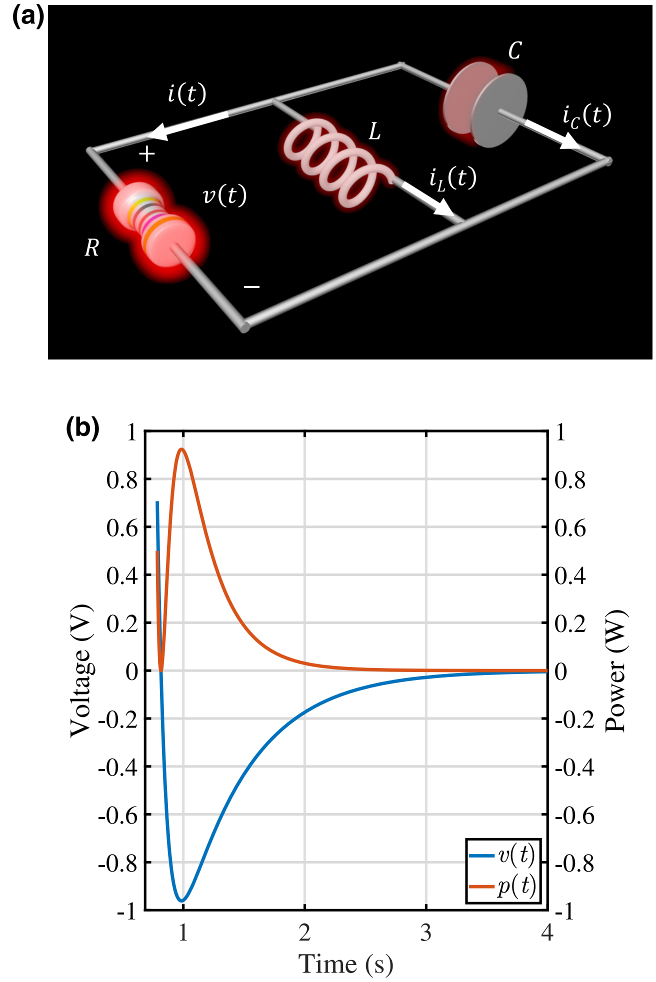

FIG. 6. (a) The corresponding RLC circuit after the stopping of temporal modulation. (b) The instantaneous voltage and the power consumed by the resistance $R = 1\ \Omega$. Here $t_0 = \pi/4\omega$, $\omega = 1$ rad/s, $R_0 = 1\ \Omega$, $A = 1$ V, and $I_0 = 1$ A/s.

Let us choose $t_0 = \pi/4\omega$ as the moment when we stop modulation and energy accumulation. For the RLC circuit in Fig. 6(a), we can write the second-order differential equation in the form

$$LC\frac{d^2 i_L(t)}{dt^2} + \frac{L}{R}\frac{di_L(t)}{dt} + i_L(t) = 0. \tag{14}$$

Regarding the voltage over the elements, we know that $v(t) = L di_L(t)/dt$. Solving the characteristic equation of the RLC circuit, we obtain the electric current $i_L(t)$ as

$$i_L(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}, \tag{15}$$

where

$$S_{1,2} = \frac{-L \pm \sqrt{L^2 - 4R^2 LC}}{2RLC}. \tag{16}$$

In Eq. (15), A_1 and A_2 are unknown coefficients that can be found by our imposing the initial conditions (i.e., the current flowing through the inductance and the voltage over the capacitance should be continuous). In other words,

$$
\begin{aligned}
i_L(t)\Big|_{t=t_0} &= \frac{A\cos\omega t_0}{2R_0} + I_0 t_0 = \alpha, \\
\frac{di_L(t)}{dt}\Big|_{t=t_0} &= \frac{A\cos\omega t_0}{L} = \beta.
\end{aligned} \tag{17}
$$

According to Eqs. (15) and (17), the coefficients A_1 and A_2 can be written as

$$
\begin{aligned}
A_1 &= \frac{\beta-\alpha S_2}{S_1 e^{t_0 S_1} - S_2 e^{t_0 S_1}}, \\
A_2 &= \frac{\beta-\alpha S_1}{S_2 e^{t_0 S_2} - S_1 e^{t_0 S_2}}.
\end{aligned} \tag{18}
$$

Knowing the electric current $i_L(t)$, we can calculate the resistance voltage and finally the instantaneous power as $p(t) = v(t)^2/R$ and the total energy released by integrating the instantaneous power from t_0 to infinity. This energy is the energy that we can extract and use after stopping the modulation. Since at $t_0 = \pi/4\omega$ the values of L and C are such that S_1 and S_2 are real and not equal, the *RLC* circuit is *overdamped*. The time dependence of the instantaneous power is shown in Fig. 6(b). We find that the energy consumed by the resistance W_{ext} is about 0.42 J. Let us compare this value with the energy delivered to the matched *LC* circuit W_{del} from the power source during the accumulation time from $t = 0$ to $t = t_0$, which is approximately 0.64 J. Hence, the time-modulated load not only accumulates all the incident power but also accepts some power from the system that modulates the two reactances.

The above example with $t_0 = \pi/4\omega$ corresponds to a short energy-accumulation time. It is interesting to investigate energy accumulations for longer times. Figure 7 shows that near $t = 20.5$ s, there is an asymptote for the capacitance function and the inductance is positive. Let us stop the reactance modulation at $t_{01} = 20.7$ s, $t_{02} = 21.3$ s, and $t_{03} = 21.9$ s and connect a 0.01-Ω resistance to the time-invariant *LC* circuit at these moments. We choose such a small resistance to release the accumulated energy quickly. On the basis of Eq. (16), because the inductance is very small and the capacitance is very large, we expect that S_1 and S_2 are complex and the conjugate of one another: $S_1 = S_2^*$. In other words, the circuit is *underdamped*. This feature can be seen in Fig. 8, where we show the voltage over the resistance and the instantaneous power for the three different scenarios described above. Calculating the energy released, we find that while the energy delivered does not change much in these three

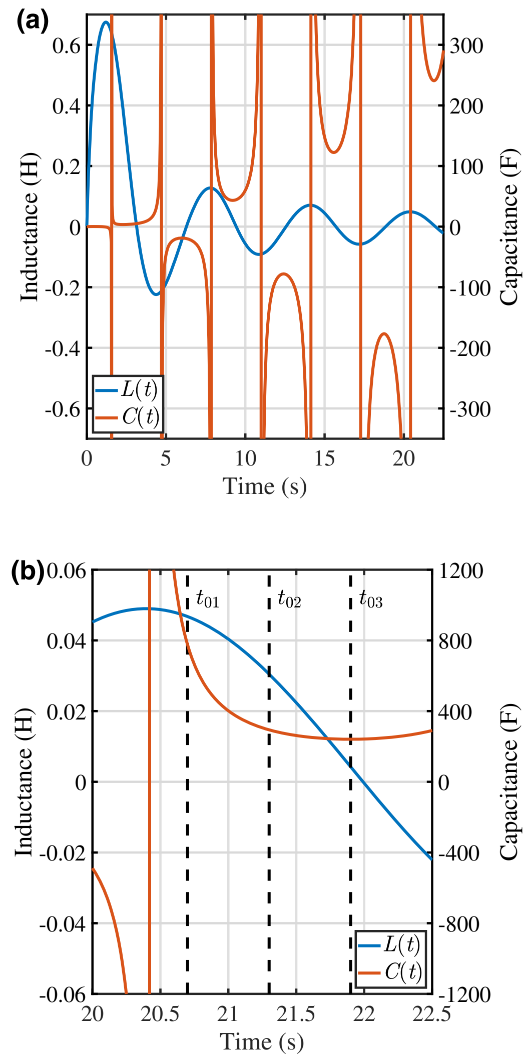

FIG. 7. Time-dependent inductance and capacitance for $R_0 = 1\ \Omega$, $\omega = 1$ rad/s, $A = 1$ V, and $I_0 = 1$ A/s.

cases ($W_{\mathrm{del}} \approx 10.2$, 10.4, and 10.9 J), the energy that is accumulated and then extracted changes dramatically. It is worth noting that the energy extracted can be much greater than the energy delivered, showing that the modulated *LC* circuit accepts and accumulates energy also from the modulation source, in addition to the energy delivered by the source feeding the transmission line. However, stopping modulation and keeping L and C constant in time is only one way of extracting the energy. It is also possible to release the energy without stopping the modulation. We must only change the modulation function. Hence, we can choose a proper modulation function such that all the energy accepted by the *LC* circuit from the modulation source will return to the modulation source. In other words, an equal exchange of energy happens between the *LC* circuit and the modulation source in the accumulation and release regimes. In this scenario, the

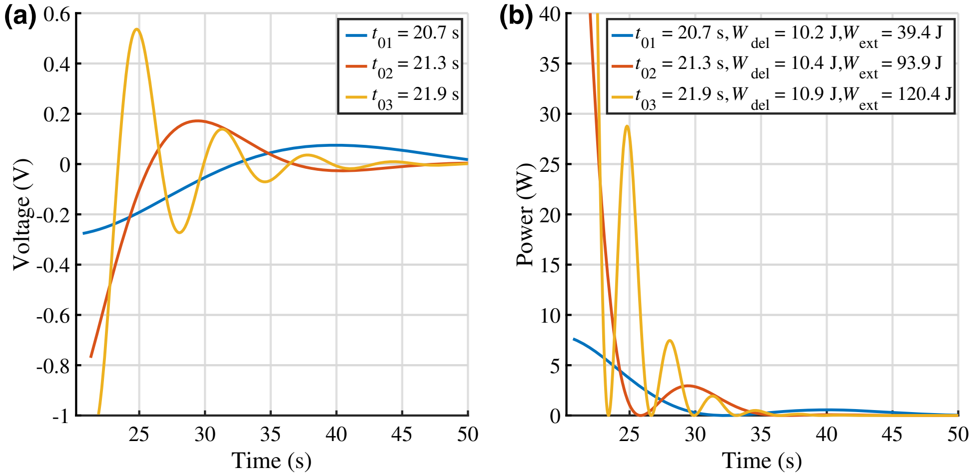

FIG. 8. (a) The resistance voltage and (b) the corresponding instantaneous power consumed by the resistance ($R = 0.01\ \Omega$) after the time modulation is stopped at different moments. Here $R_0 = 1\ \Omega$, $\omega = 1$ rad/s, $A = 1$ V, and $I_0 = 1$ A/s.

energy delivered becomes equal to the energy released (extracted).

IV. CONCLUSIONS

In this paper, we show that properly time-variant reactive elements can continuously accumulate energy from conventional external time-harmonic sources without any reflections of the incident power. We find the required time dependences of reactive elements and discuss possible realizations as time-space-modulated transmission lines and mixer circuits. There is a conceptual analogy of energy-accumulating reactances and short-circuited transmission lines where the short position is moving, which helps to understand the physical mechanism of energy accumulation and release. We show that by properly modulating reactances of two connected elements, one can, in principle, engineer any arbitrary time variations of currents induced by time-harmonic sources. The study of the energy balance reveals that such parametric circuits accept and accumulate power not only from the main power source but also from the pump that modulates the reactances. This is seen from the fact that if we stop energy accumulation at some moment of time and release all the accumulated energy into a resistor, the energy released can be much greater than the energy delivered to the circuit from the primary source. These energy-accumulation properties become possible if the time variations of the reactive elements are not limited to periodic (usually time-harmonic) functions, but other time variations appropriate for desired performance are allowed.

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