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THE POSET OF PROPER DIVISIBILITY*

DAVIDE BOLOGNINI†, ANTONIO MACCHIA†, EMANUELE VENTURA‡, AND VOLKMAR WELKER†

Abstract. We study the partially ordered set $P(a_1,\dots,a_n)$ of all multidegrees $(b_1,\dots,b_n)$ of monomials $x_1^{b_1}\cdots x_n^{b_n}$, which properly divide $x_1^{a_1}\cdots x_n^{a_n}$. We prove that the order complex $\Delta(P(a_1,\dots,a_n))$ of $P(a_1,\dots a_n)$ is (nonpure) shellable by showing that the order dual of $P(a_1,\dots,a_n)$ is CL-shellable. Along the way, we exhibit the poset $P(4,4)$ as a new example of a poset with CL-shellable order dual that is not CL-shellable itself. For $n=2$, we provide the rank of all homology groups of the order complex $\Delta(P(a_1,a_2))$. Furthermore, we give a succinct formula for the Euler characteristic of $\Delta(P(a_1,a_2))$.

Key words. proper division, posets, CL-shellability, simplicial homology, Euler characteristic

AMS subject classifications. 06A07, 06A11, 05E45

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1. Introduction.

In this paper, we study proper divisibility of monomials in the polynomial ring in n variables $x_1,\dots,x_n$. Since any monomial $x_1^{a_1}\cdots x_n^{a_n}$ is determined by its exponent vector $(a_1,\dots,a_n)\in\mathbb{N}^n$, we phrase all concepts in terms of exponent vectors.

For every $(a_1,\dots,a_n),(b_1,\dots,b_n)\in\mathbb{N}^n$, we say that $(a_1,\dots,a_n)$ *properly divides* $(b_1,\dots,b_n)$ if for every $1\le i\le n$, either $a_i=b_i=0$ or $a_i<b_i$. Proper divisibility of monomials appears naturally in the context of the Buchberger algorithm from Gröbner basis theory, and it plays an important role in the combinatorics of free resolutions of monomial ideals, as shown by Miller and Sturmfels [4, 5] (see also [6]).

Here we consider proper divisibility as an order relation, setting $(a_1,\dots,a_n)\le(b_1,\dots,b_n)$ if and only if either $(a_1,\dots,a_n)=(b_1,\dots,b_n)$ or $(a_1,\dots,a_n)$ properly divides $(b_1,\dots,b_n)$. For an arbitrary $(a_1,\dots,a_n)\in\mathbb{N}^n$, we set

$$P(a_1,\dots,a_n)=\{(b_1,\dots,b_n)\in\mathbb{N}^n:(b_1,\dots,b_n)\le(a_1,\dots,a_n)\}$$

and consider $P(a_1,\dots,a_n)$ as a partially ordered set, poset for short, ordered by proper divisibility. With this order, $P(a_1,\dots,a_n)$ has a unique minimal element $\widehat{0}=(0,\dots,0)$ and a unique maximal element $\widehat{1}=(a_1,\dots,a_n)$. The poset of all divisors of $(a_1,\dots,a_n)$ with respect to the usual divisibility relation is well understood, and it is a direct product of n chains of length $a_1,\dots,a_n$, respectively. Our approach to study $P(a_1,\dots,a_n)$ is topological. We associate to $P(a_1,\dots,a_n)$ its order complex $\Delta(P(a_1,\dots,a_n))$, which is the simplicial complex consisting of all chains in $P(a_1,\dots,a_n)\setminus\{\widehat{0},\widehat{1}\}$. Through the order complex and its geometric realization, we can study a poset in topological terms, and, in particular, we can talk about homotopy

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†Fachbereich Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Strasse 6, 35032 Marburg, Germany (davide.bolognini@yahoo.it, macchia.antonello@gmail.com, welker@mathematik.uni-marburg.de).
‡Department of Mathematics and Systems Analysis, Aalto University, Helsinki, Finland (emanuele.ventura@aalto.fi).

equivalence and homology of posets. Note that, in general, not all maximal chains in $P(a_1, \dots, a_n)$ have equal length, and hence its order complex is not pure. A central tool for describing the topology of posets is (nonpure) shellability (see [2, 3]). In our first result, we prove shellability for $\Delta(P(a_1, \dots, a_n))$.

THEOREM 1.1. *For all* $(a_1, \dots, a_n) \in \mathbb{N}^n$*, the order complex* $\Delta(P(a_1, \dots, a_n))$ *is shellable. In particular,* $\Delta(P(a_1, \dots, a_n))$ *is homotopy equivalent to a wedge of spheres, and its reduced simplicial homology groups* $\widetilde{H}_\bullet(\Delta(P(a_1, \dots, a_n)); \mathbb{Z})$ *are torsion-free.*

Indeed, we prove a slightly stronger statement. In Theorem 2.2, we show that the poset $P(a_1, \dots, a_n)^*$, which is the order dual of $P(a_1, \dots, a_n)$, admits a recursive atom ordering. By [2, Thm. 5.11], this is equivalent to CL-shellability, which in turn, by [3, Thm. 11.6] and [2, Thm. 5.8], implies vertex-decomposability, and hence shellability, of the order complex $\Delta(P(a_1, \dots, a_n)^*)$. Then Theorem 1.1 follows by the isomorphism $\Delta(P(a_1, \dots, a_n)^*) \cong \Delta(P(a_1, \dots, a_n))$.

CL-shellability and recursive atom orderings are concepts defined for nonpure posets in [2, 3]. The posets $P(a_1, \dots, a_n)$ provide a good source of counterexamples in this context.

PROPOSITION 1.2. *The poset* $P(4,4)$ *is not* CL*-shellable, but its dual* $P(4,4)^*$ *is. In particular,* $\Delta(P(4,4))$ *is shellable, but* $P(4,4)$ *is not* CL*-shellable.*

In [11], Walker provides an example of a pure poset whose order complex is shellable but which is not CL-shellable. Our example, $P(4,4)$, is not pure but is smaller both in the number of elements and in dimension than the example from [11]. The question of whether there is a non-CL-shellable poset whose dual is CL-shellable was posed as an open question by Wachs in [10, p. 71]. Already in 2008, it was answered by Schweig [8], who provided a counterexample of the same dimension as ours and of almost equal size but that is even pure. Note that the example by Schweig also provides a poset that is shellable but not CL-shellable.

Using the CL-shelling from Theorem 2.2, one can in principle read off the homotopy type and the homology groups. Since the process is technically involved, we present a pleasing solution for the case $n = 2$ only. To simplify the notation, in this case we set $a = a_1$ and $b = a_2$, and, without loss of generality, we may assume $a \le b$.

THEOREM 1.3. *Let* $2 \le a \le b$*. Then* $\widetilde{H}_i(\Delta(P(a,b)); \mathbb{Z}) = 0$ *for* $i > a-2$ *and*

(1)
$$\operatorname{rank} \widetilde{H}_i(\Delta(P(a,b)); \mathbb{Z}) = 2 \sum_{t=0}^{i} \binom{a-3-i}{t-1} \left[\binom{i}{t} \binom{b-2-i}{i-t} + \binom{i}{t-1} \binom{b-3-i}{i-t} \right],$$

for $0 \le i \le a-2$*, where we set* $\binom{-1}{-1} = 1$*.*

A remarkable property of these posets is the following persistence theorem, which is a phenomenon rarely observed in posets defined *naturally* on combinatorial objects.

PROPOSITION 1.4. *For every* a, b*, with* $2 \le a \le b$*, there exists an integer* $t_{(a,b)} \ge 0$ *such that*

$$H_i(\Delta(P(a,b)); \mathbb{Z}) \neq 0 \quad \text{if and only if } 0 \le i \le t_{(a,b)},$$

where H_i *denotes the ith (nonreduced) simplicial homology group.*

Moreover, in Corollary 3.7, we show that the only poset $P(a,b)$, with $2 \le a \le b$, whose order complex is contractible, and indeed collapsible, is $P(3,3)$.

One of the most important numerical invariants of a poset P with unique minimal element $\widehat{0}$ and unique maximal element $\widehat{1}$ is its Möbius number $\mu(P)$ [7]. It is well known that $\mu(P)$ is the alternating sum of the ranks of the homology groups of the order complex $\Delta(P)$. In particular, it equals the reduced Euler characteristic $\widetilde{\chi}(\Delta(P))$ of $\Delta(P)$. Hence, as a consequence of Theorem 1.3, we can derive the following formula for the reduced Euler characteristic of $\Delta(P(a,b))$.

THEOREM 1.5. *For every $2 \leq a \leq b$, the reduced Euler characteristic of $\Delta(P(a,b))$ is*

$$\widetilde{\chi}(\Delta(P(a,b))) = (-1)^a \cdot 2 \sum_{h=0}^{\lfloor \frac{a}{2} \rfloor - 1} (-1)^h \binom{a-2}{h} \binom{b-a}{a-2-2h}. \tag{2}$$

The structure of the paper is as follows. In section 2, we recall the relation between recursive atom ordering and CL-shellability and prove Theorem 1.1 and Proposition 1.2. The rest of the paper deals with the case $n = 2$, i.e., the posets $P(a,b)$. In section 3, we study the homology of $\Delta(P(a,b))$ by characterizing and counting the falling chains of $P(a,b)^*$. A first qualitative result concerns the vanishing of the homology; see Propositions 3.4 and 3.6. Our proofs of Theorem 1.3 and Proposition 1.4 depend heavily on the labeling induced by the recursive atom ordering from the proof of Theorem 2.2. Moreover, in Corollary 3.7, we show that the only poset $P(a,b)$, with $2 \leq a \leq b$, whose order complex is contractible, and indeed collapsible, is $P(3,3)$. In section 4, we prove Theorem 1.5 using generating function techniques. Note that the formula from Theorem 1.5 is much simpler than the alternating sum of the rank of the homologies given in Theorem 1.3. From this, in Corollary 4.1, we deduce that $\widetilde{\chi}(\Delta(P(a,b))) = 0$ if $a = b$ and a is odd. The paper closes with the definition of generalized proper divisibility posets and open questions.

2. CL-Shellability.

In this section, we prove that the order complex of the poset $P(a_1,\ldots,a_n)$ is vertex decomposable, hence shellable, by showing that the dual poset $P(a_1,\ldots,a_n)^*$ is CL-shellable. Indeed, we will show that $P(a_1,\ldots,a_n)^*$ admits a recursive atom ordering which, by [2, Thm. 5.11], is equivalent to show that the poset is CL-shellable.

Before defining the recursive atom ordering, we need to introduce some more poset terminology. Let P be a poset with order relation $\leq$. We say that $p \in P$ *covers* $q \in P$ and use the notation $q \to p$ if $q < p$ and there is no $q' \in P$ with $q < q' < p$. The *atoms* of a poset P with unique minimal element $\widehat{0}$ are the elements of P that cover $\widehat{0}$. For $q \leq p$ in P, we define the *interval* $[q,p] := \{q' \in P \ : \ q \leq q' \leq p\}$, which is a poset with the induced order, unique minimal element q and unique maximal element p. Finally, the *length* of a chain in P is the number of its elements minus one, and the *length* of P, denoted $\ell(P)$, is the maximal length of its chains.

Some immediate properties of $P(a_1,\ldots,a_n)$ are the following:

- If $a_1,\ldots,a_n \geq 1$, then $P(a_1,\ldots,a_n)$ has $a_1 \cdots a_n + 1$ elements since the elements of $P(a_1,\ldots,a_n)$, except the top element, are exactly the elements of the classical divisibility poset with top element $(a_1 - 1,\ldots,a_n - 1)$ but with a different partial order.
- $\ell(P(a_1,\ldots,a_n)) = \max_{1 \leq i \leq n}\{a_i\}$. In fact, assume that $\max_{1 \leq i \leq n}\{a_i\} = a_n$. Then the chain

$$(0,\ldots,0,0) \to (0,\ldots,0,1) \to (0,\ldots,0,2) \to \cdots \\ \to (0,\ldots,0,a_n - 1) \to (a_1,\ldots,a_{n-1},a_n)$$

  has length a_n. There are no longer chains since every covering relation is of the form $(c_1,\ldots,c_n) \to (d_1,\ldots,d_n)$, where for every $1 \leq i \leq n$, either $c_i = d_i = 0$ or $c_i < d_i$ and $c_j = d_j - 1$ for some $1 \leq j \leq n$.

Definition 2.1. *Let P be a poset with unique minimal element $\widehat{0}$ and unique maximal element $\widehat{1}$. The poset P admits a recursive atom ordering if $\ell(P) \le 1$ or $\ell(P) > 1$, and there is a linear ordering $\preceq$ of the atoms of P satisfying the following two conditions:*

(i) *for all atoms p of P, the interval $[p, \widehat{1}]$ admits a recursive atom ordering in which the atoms of $[p, \widehat{1}]$ that belong to $[q, \widehat{1}]$, for some $q \prec p$, come first;*

(ii) *for all atoms $p \prec p'$ and elements $p, p' < q$ of P, there exist an atom $p'' \prec p'$ of P and an atom q' of $[p', \widehat{1}]$ such that $p'' < q' \le q$.*

Theorem 1.1 is a consequence of the following key result.

Theorem 2.2. *For every $(a_1, \ldots, a_n) \in \mathbb{N}^n$, the poset $P(a_1, \ldots, a_n)^*$ admits a recursive atom ordering and hence is* CL*-shellable.*

Proof. We denote by $\le_*$ the order on $P(a_1, \ldots, a_n)^*$, which is the dual of the order on $P(a_1, \ldots, a_n)$.

We proceed by induction on n and $\ell(P(a_1, \ldots, a_n))$. If $n = 1$, the poset is a single chain, which is easily checked to admit a recursive atom ordering. Assume $n > 1$. Suppose that $a_i \le 1$ for some i. Then $P(a_1, \ldots, a_n)^*$ can be identified with $P(a_1, \ldots, \widehat{a_i}, \ldots, a_n)^*$, in which we remove the ith component from all elements. In fact, all the elements of $P(a_1, \ldots, a_n)^*$, except possibly the bottom element, have the ith coordinate equal to zero. Thus, by induction on n, we know that $P(a_1, \ldots, a_n)^*$ admits a recursive atom ordering.

Hence, we may assume that $a_h \ge 2$ for all h.

We will frequently use the following fact.

($\star$) For every $(b_1, \ldots, b_n) \in P(a_1, \ldots, a_n)^* \setminus \{(0, \ldots, 0)\}$ and any atom $(c_1, \ldots, c_n)$ of the interval $[(b_1, \ldots, b_n), (0, \ldots, 0)]$, there is an index $1 \le j \le n$ such that $c_j = b_j - 1$.

Now, for every element $(b_1, \ldots, b_n)$ of $P(a_1, \ldots, a_n)^*$, we order the atoms of $[(b_1, \ldots, b_n), (0, \ldots, 0)]$ by the dual $\preceq$ of the lexicographic order: for any two atoms $(c_1, \ldots, c_n)$ and $(d_1, \ldots, d_n)$, we set $(c_1, \ldots, c_n) \preceq (d_1, \ldots, d_n)$ if and only if either $c_1 > d_1$ or there exists $1 \le i \le n$ such that $c_h = d_h$ for every $1 \le h \le i$ and $c_{i+1} > d_{i+1}$. In particular, the least atom of the interval $[(b_1, \ldots, b_n), (0, \ldots, 0)]$ is $(\bar{b}_1, \ldots, \bar{b}_n)$, where $\bar{b}_h = b_h - 1$ if $b_h \ne 0$ and $\bar{b}_h = 0$ if $b_h = 0$.

For every $(b_1, \ldots, b_n) \in P(a_1, \ldots, a_n)^* \setminus \{(a_1, \ldots, a_n)\}$, the interval $[(b_1, \ldots, b_n), (0, \ldots, 0)]$ is easily identified with the poset $P(b_1, \ldots, b_n)^*$, and hence, by induction on the length, we may assume that the dual lexicographic order is a recursive atom ordering for all intervals $[(b_1, \ldots, b_n), (0, \ldots, 0)]$ in $P(a_1, \ldots, a_n)^*$, with $(b_1, \ldots, b_n) \ne (a_1, \ldots, a_n)$. Thus, it suffices to verify conditions (i) and (ii) from Definition 2.1 for the ordering of the atoms of $P(a_1, \ldots, a_n)^*$ only.

(i) Let p_1 be the least atom of $P(a_1, \ldots, a_n)^*$. Then $p_1 = (a_1 - 1, \ldots, a_n - 1)$. Let $(b_1, \ldots, b_n) \ne p_1$ be another atom of $P(a_1, \ldots, a_n)^*$. Notice that every atom $(c_1, \ldots, c_n)$ of the interval $[(b_1, \ldots, b_n), (0, \ldots, 0)]$ satisfies $c_h < b_h \le a_h - 1$ if $b_h \ne 0$, and $c_h = b_h = 0 \le a_h - 1$ if $b_h = 0$. Hence, $(c_1, \ldots, c_n)$ also belongs to the interval $[p_1, (0, \ldots, 0)]$. Thus, $p_1 < (c_1, \ldots, c_n)$, and hence condition (i) is fulfilled.

(ii) Let $p \prec p'$ be atoms of $P(a_1, \ldots, a_n)^*$. Then $p' = (a_1 - k_1, \ldots, a_n - k_n)$, with $k_h \ge 1$ for every $h = 1, \ldots, n$. Let q be another element of $P(a_1, \ldots, a_n)^*$ such that $p, p' <_* q$. Then

$$q = (b_1, \ldots, b_n) = (a_1 - k_1 - k'_1, \ldots, a_n - k_n - k'_n),$$

where $k'_h = 0$ if and only if $a_h - k_h = 0$ and $k'_h \geq 1$ otherwise. Since $a_h \geq 2$ for every h, there exists s such that $a_s - k_s > 0$, and hence $k'_s \geq 1$. We distinguish between two cases.

If $k'_t = 1$ for some t, then we set $q' = q$. Clearly, by $(\star)$, q is an atom of $[p', (0, \dots, 0)]$ and $p_1 = (a_1 - 1, \dots, a_n - 1) <_* q' \leq_* q$.

Otherwise, if for every h, either $k'_h = 0$ or $k'_h > 1$, then we set $q' = (c_1, \dots, c_n)$, where for every $h = 1, \dots, n$,

$$c_h = \begin{cases} 0 & \text{if } k'_h = 0 \\ a_h - k_h - k'_h + \min_{1 \leq r \leq n}\{k'_r : k'_r \neq 0\} - 1 & \text{if } k'_h > 1. \end{cases}$$

Notice that $k'_s \neq 0$. Again q' is an atom of $[p', (0, \dots, 0)]$ by $(\star)$ since $c_h = 0$ when $a_h - k_h = 0$, $c_h \leq a_h - k_h - 1$ when $k'_h > 1$, and $c_t = a_t - k_t - 1$, where t is such that $k'_t = \min_{1 \leq r \leq n}\{k'_r : k'_r \neq 0\}$.

On the other hand, $q' \leq_* q$ since $c_h = b_h = 0$ if $b_h = 0$, and $\min_{1 \leq r \leq n}\{k'_r : k'_r \neq 0\} - 1 > 0$; hence, $b_h < c_h$ if $b_h > 0$. Furthermore, $p'' := p_1 \prec p'$ because $c_h < a_h - 1$ for every h (since either $c_h = 0$, if $k'_h = 0$, or $-k_h - k'_h + \min_{1 \leq r \leq n}\{k'_r : k'_r \neq 0\} \leq -1$, if $k'_h > 1$) and $p'' <_* q'$. This concludes the proof. ∎

Now we can prove Proposition 1.2 showing that, indeed, Theorem 2.2 does not hold for $P(a_1, \dots, a_n)$.

Proof of Proposition 1.2. We prove that the poset $P(4, 4)$ in Figure 1 does not admit any recursive atom ordering by showing that no ordering on the atoms of $P(4, 4)$ does fulfill condition (ii) of Definition 2.1. Let $\preceq$ be a linear order on the atoms $(1, 0), (1, 1), (0, 1)$ of $P(4, 4)$. Since $P(4, 4)$ is invariant under switching coordinates, we may assume that $p = (1, 0) \preceq p' = (0, 1)$. We consider $q = (2, 3)$. Clearly, $p, p' < q$. For every $p'' \prec p'$ and for every atom q' of $[p', \widehat{1}]$, either $p'' \not< q'$ or $q' \not\leq q$.

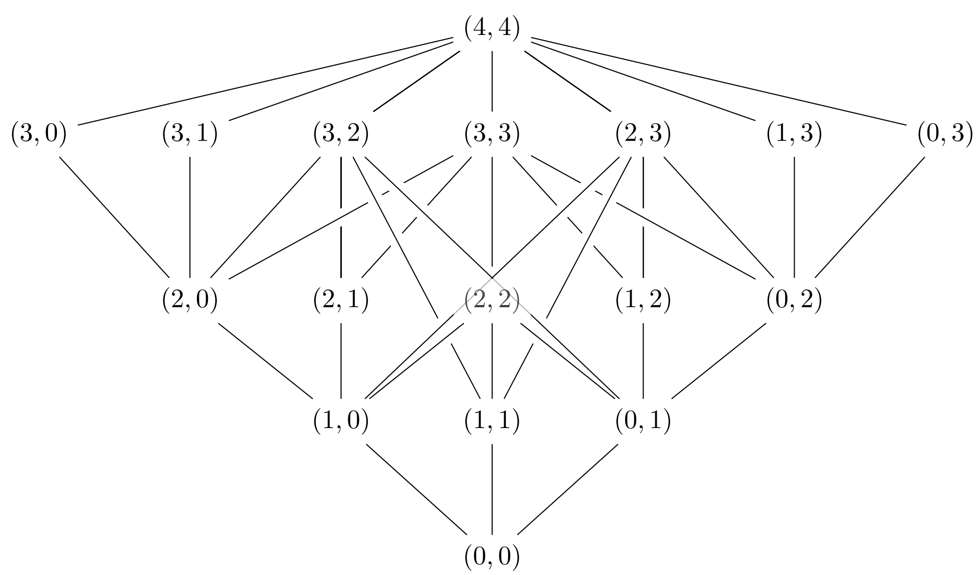

FIG. 1. *The poset* $P(4, 4)$.

Thus, $P(4,4)$ does not admit any recursive atom ordering and hence is not CL-shellable. Nevertheless, by Theorem 2.2, we know that $P(4,4)^*$ is CL-shellable, and hence its order complex $\Delta(P(4,4)^*) \cong \Delta(P(4,4))$ is shellable. ∎

3. Homology of the order complex of $\boldsymbol{P(a,b)}$. In this section, we study the simplicial homology groups of $\Delta(P(a,b)) \cong \Delta(P(a,b)^*)$ with coefficients in $\mathbb{Z}$. Without loss of generality, we may assume $a \leq b$.

For this we construct, from a recursive atom ordering of a poset P with unique minimal element $\widehat{0}$ and unique maximal element $\widehat{1}$, a labeling $\lambda(q \to p)$ of the cover relations of P by integers. This construction is contained in the proof of [1, Thm. 3.2], showing that a poset with recursive atom ordering is CL-shellable. Note that [1] deals with pure posets only, but, as noted in [2], the same construction goes through in the nonpure case. For our purposes, we only need a part of this construction.

CONSTRUCTION 3.1. *Let P be a poset with unique minimal element $\widehat{0}$ and unique maximal element $\widehat{1}$. Assume that P admits a recursive atom ordering and $\preceq$ is the linear ordering of the atoms of P. First, we choose a labeling λ of the cover relations $\widehat{0} \to p$ by integers such that $\lambda(\widehat{0} \to p) < \lambda(\widehat{0} \to p')$ if $p \prec p'$. If $\ell(P) > 1$, let $F(p')$ be the set of all atoms of $[p', \widehat{1}]$ that cover some atom $p \prec p'$ of P. Then we choose the labeling as follows:*

$$\text{if } q \in F(p'), \text{ then } \lambda(p' \to q) < \lambda(\widehat{0} \to p'),$$
$$\text{if } q \notin F(p'), \text{ then } \lambda(p' \to q) > \lambda(\widehat{0} \to p').$$

By [2, Thm. 5.9], given a labeling λ on P, from Construction 3.1 it follows that (FCH), the rank of the ith reduced homology group of $\Delta(P)$ with integer coefficients equals the number of chains $\widehat{0} = p_0 \to p_1 \to \cdots \to p_{i+1} \to p_{i+2} = \widehat{1}$ of length $i+2$ in P for which $\lambda(p_0 \to p_1) > \cdots > \lambda(p_{i+1} \to p_{i+2})$.

The latter chains are called *falling*. We use this principle for determining the reduced homology groups of a poset with recursive atom ordering, and we refer to it as the *Falling-Chain-Homology principle*, or (FCH) for short.

In particular, rank $H_0(\Delta(P);\mathbb{Z})$ is one more than the number of falling chains of length 2.

From now on, we consider $P(a,b)^*$ equipped with the recursive atom ordering from Theorem 2.2. Let λ be the labeling of the edges of $P(a,b)^*$ induced through Construction 3.1 by this ordering.

We call $(c,d) \in \mathbb{N}^2$ a *border element* of $P(a,b)^*$ if it has one of the forms $(1,k)$, $(k,1)$, $(0,k)$, or $(k,0)$ for some $k \geq 2$.

LEMMA 3.2. *Let $m : \widehat{0} = (a,b) = p_0 \to p_1 \to \cdots \to p_k = (0,0) = \widehat{1}$ be a chain of length $k \geq 2$ in $P(a,b)^*$. Then m is falling with respect to λ if and only if it satisfies the following conditions:*

(i) *p_{i+1} is not the lexicographically least atom of $[p_i,(0,0)]$ for $0 \leq i \leq k-2$;*

(ii) *p_i is not a border element for $1 \leq i \leq k-2$.*

In particular, by condition (i)*, if m is falling, then $p_i = (c_i,d_i)$ is not a border element for $i \leq k-2$, and p_{i+1} cannot be the element $(c_i - 1, d_i - 1)$.*

Proof. Let m be a falling chain of length k in $P(a,b)^*$. We have to verify (i) and (ii).

(i) Suppose that, for some $0 \leq i \leq k-2$, p_{i+1} is the least atom of $[p_i,(0,0)]$. Then $\lambda(p_{i+1} \to p_{i+2}) > \lambda(p_i \to p_{i+1})$ by Construction 3.1. Then m is not a falling chain.

(ii) Suppose that p_i is a border element for some $1 \leq i \leq k-2$. Then p_{i+1} is the only atom of $[p_i, (0,0)]$. In particular, it is the least atom, which contradicts (i).

Conversely, suppose that m satisfies conditions (i) and (ii). Then it is immediate from Construction 3.1 that $\lambda(p_{i+1} \to p_{i+2}) < \lambda(p_i \to p_{i+1})$ for every $0 \leq i \leq k-2$. Hence, m is a falling chain. ∎

We now describe the homology of $\Delta(P(a,b))$. Let

$$t_{(a,b)} = \max\left\{i : H_i(\Delta(P(a,b)); \mathbb{Z}) \neq 0\right\}.$$

By convention, we set $t_{(a,b)} = -1$ if $H_i(\Delta(P(a,b)); \mathbb{Z}) = 0$ for every $i \geq 0$; i.e., $\Delta(P(a,b))$ is empty.

Example 3.3. First we describe the order complex $\Delta(P(a,b))$ in some simple cases, when $0 \leq a \leq 3$.

1. Let $a = 0$ or $a = 1$.
   If $b \leq 1$, then $\Delta(P(a,b)) = \emptyset$ and $t_{(a,b)} = -1$. Hence, assume $b \geq 2$. Then $P(a,b)$ is a single chain of length b
   $$(0,0) \to (0,1) \to (0,2) \to \cdots \to (0,b-1) \to (a,b),$$
   and its order complex is a simplex of dimension $b-2$. Thus, $t_{(a,b)} = 0$ if $b \geq 2$.
2. Let $a = 2$.
   In Figure 2, we draw the poset $P(2,b)^*$. For $b = 2$, the poset $P(2,2)^*$ has only three maximal chains: $(2,2) \to (1,1) \to (0,0)$, $(2,2) \to (1,0) \to (0,0)$, and $(2,2) \to (0,1) \to (0,0)$. By Lemma 3.2, the first chain is not falling, and the other two are falling. Moreover, the order complex $\Delta(P(2,2)^*)$ consists of three isolated points.
   Let $b > 2$. There are exactly two maximal chains of length 2 in $P(2,b)^*$, $m : (2,b) \to (1,0) \to (0,0)$ and $m' : (2,b) \to (1,1) \to (0,0)$. Note that the

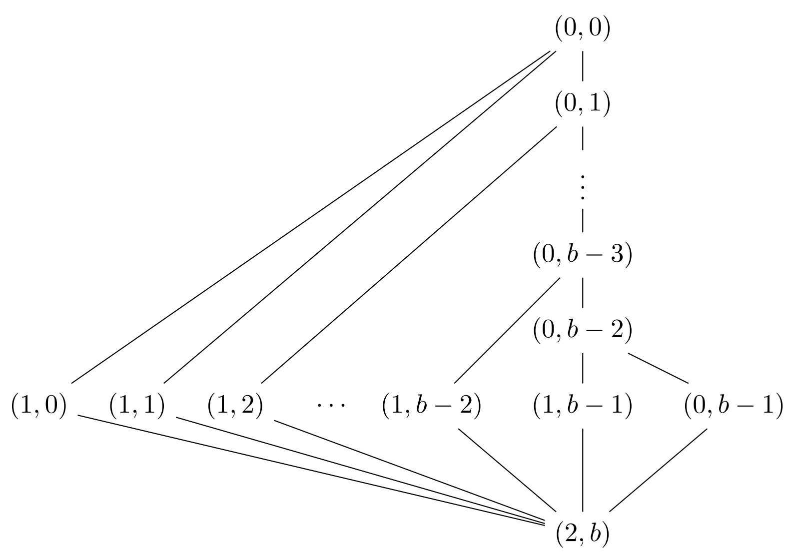

FIG. 2. *The poset* $P(2,b)^*$.

least atom of $[(2,b),(0,0)]$ is $(1,b-1)$. Since $b-1>1$, neither of $(1,0)$ and $(1,1)$ is the least atom of $[(2,b),(0,0)]$. Hence, both m and m' are falling chains.

Now we show that no other maximal chain in $P(2,b)^*$ is falling. Consider the maximal chain $m_t : (2,b) \to (1,t) \to (0,t-1) \to \cdots \to (0,1) \to (0,0)$, with $2 \le t \le b-1$. Since $(1,t)$ is a border element, the chain m_t is not falling by Lemma 3.2. Again, by Lemma 3.2, the remaining chain $(2,b) \to (0,b-1) \to (0,b-2) \to \cdots \to (0,1) \to (0,0)$ is not falling since $(0,b-1)$ is a border element. Thus, by (FCH), for every $b \ge 2$,

$$\operatorname{rank} H_0(\Delta(P(2,b)^*);\mathbb{Z}) = 3 \text{ and } \operatorname{rank} H_i(\Delta(P(2,b)^*);\mathbb{Z}) = 0, \text{ for } i \ge 1.$$

Hence, $t_{(2,b)} = 0$, for every $b \ge 2$.

3. Let $a = 3$.
If $b = 3$, there are no falling chains by Lemma 3.2. Hence, by (FCH),

$$\operatorname{rank} H_i(\Delta(P(3,3)^*);\mathbb{Z}) = 0 \quad \text{if and only if } i \ne 0.$$

Moreover, $\operatorname{rank} H_0(\Delta(P(3,3)^*);\mathbb{Z}) = 1$. Thus, $t_{(3,3)} = 0$.

We may assume $b > 3$. Notice that there are no maximal chains of length 2. Therefore, (FCH) implies $\operatorname{rank} H_0(\Delta(P(3,b)^*);\mathbb{Z}) = 1$. In fact, a chain of the form $(3,b) \to (c,d) \to (0,0)$ is not maximal for any $0 \le c \le 2$ with $d = b-1$ and for any $0 \le d < b-1$ with $c = 2$.

We consider the maximal chains of length ≥ 3. By Lemma 3.2, if a chain $m : p_0 = (3,b) \to p_1 \to \cdots \to p_k = (0,0)$ is falling, then p_1 is not one of the elements $(2,b-1)$ (least atom of $[(3,b),(0,0)]$) and $(1,b-1),(0,b-1),(2,1),(2,0)$ (border elements). In other words, if m is falling, then it is of the form $m : p_0 = (3,b) \to p_1 = (2,t) \to \cdots \to p_k = (0,0)$, with $2 \le t \le b-2$. If $t = 2$, there are three maximal chains containing $(2,2)$: $m_1 : (3,b) \to (2,2) \to (1,1) \to (0,0)$, $m_2 : (3,b) \to (2,2) \to (1,0) \to (0,0)$, and $m_3 : (3,b) \to (2,2) \to (0,1) \to (0,0)$. By Lemma 3.2, m_1 is not falling since $(1,1)$ is the least atom of $[(2,2),(0,0)]$, while m_2 and m_3 are falling.

Let $t \ge 3$. If the third element p_2 in the chain m is of the form $(0,t-1)$, it is a border element, and m is not falling. By Lemma 3.2, we can exclude $p_2 = (1,t-1)$ since it is the least atom of $[(2,t),(0,0)]$. Assume p_2 is of the form $(1,h)$, with $0 \le h \le t-2$. Thus, $h = 0$ or $h = 1$ since for $h \ge 2$ the element $(1,h)$ is a border element. The element p_3 of m is now forced to be $(0,0)$. By Lemma 3.2, all maximal chains of the form $(3,b) \to (2,t) \to (1,h) \to (0,0)$, with $3 \le t \le b-2$ and $h = 0,1$, are falling. Then we have $2+2(b-2-3+1) = 2(b-3)$ falling chains of length 3. Moreover, there are no falling chains of length ≥ 4. Thus, (FCH) implies $\operatorname{rank} H_1(\Delta(P(3,b)^*);\mathbb{Z}) = 2(b-3)$ and $\operatorname{rank} H_i(\Delta(P(3,b)^*);\mathbb{Z}) = 0$, for $i \ge 2$. Hence, $t_{(3,b)} = 1$ for every $b \ge 4$.

PROPOSITION 3.4. *For every $2 \le a \le b$, the reduced homology $\widetilde{H}_i(\Delta(P(a,b));\mathbb{Z})$ is zero whenever $i > a-2$.*

Proof. If $a = 2,3$, the claim follows from Example 3.3. Hence, we may assume $a \ge 4$. By (FCH), it suffices to show that there are no falling chains of length $\ell \ge a+1$. If $b = a$, then there are no chains of length $\ell \ge a+1$ since $\ell(P(a,b)^*) = \ell(P(a,b)) = a$ (see discussion before Definition 2.1).

Let $b \ge a+1$ and $m : p_0 = (a,b) \to p_1 \to \cdots \to p_{\ell-1} \to p_\ell = (0,0)$ be a maximal chain of length $\ell \ge a+1$. We denote the ith element of m by $p_i = (c_i,d_i)$. Notice that for every i, if $c_i \neq 0$, then $c_{i+1} < c_i$; otherwise, if $c_i = 0$, then $c_{i+1} = 0$. Moreover, $0 \leq c_{i+1} \leq c_i \leq \max\{0, a-i\}$, for every i. In particular, $c_{\ell-1} \leq 0$; hence, $c_{\ell-1} = 0$. Thus, $d_{\ell-1} = 1$; otherwise, m is not maximal. On the other hand, $0 \leq c_{\ell-2} \leq 1$; hence, $d_{\ell-2} \geq 2$; otherwise, $d_{\ell-2} \nleq d_{\ell-1}$. Therefore, $p_{\ell-2}$ is a border element, and m is not a falling chain. ∎

LEMMA 3.5. *Let* $m : \widehat{0} = (a,b) = p_0 \to \cdots \to p_i = (c_i, d_i) \to \cdots \to p_\ell = (0,0) = \widehat{1}$ *be a falling chain in* $P(a,b)^*$. *Then* $p_{\ell-1}$ *is one of the elements* $(1,0), (1,1)$ *and* $(0,1)$. *Moreover, if, for every* $i = 1, \ldots, \ell$, *we set* $u_i = c_{i-1} - c_i$ *and* $v_i = d_{i-1} - d_i$, *then*

(i) *if* $p_{\ell-1} = (1,0)$, *then* $u_{\ell-1} = 1$ *and* $v_{\ell-1} \geq 2$;

(ii) *if* $p_{\ell-1} = (0,1)$, *then* $u_{\ell-1} \geq 2$ *and* $v_{\ell-1} = 1$.

Proof. Clearly, any falling chain contains exactly one of the elements $(1,1), (1,0)$ and $(0,1)$; otherwise, it is not maximal in $P(a,b)^*$.

(i) Let $p_{\ell-1} = (1,0)$. If $u_{\ell-1} \geq 2$, then $v_{\ell-1} = 1$; otherwise, the chain is not maximal. This implies that $p_{\ell-2} = (c_{\ell-2}, 1)$, with $c_{\ell-2} \geq 3$; hence, the chain is not maximal because $(c_{\ell-2}, 1) < (2,0) < (1,0)$. Thus, $u_{\ell-1} = 1$ and $v_{\ell-1} \geq 2$.

Similarly one shows (ii). ∎

In particular, notice that in both cases of Lemma 3.5 there is no further restriction on the other increments u_i and v_i, with $i \leq \ell - 2$.

The following result, together with Corollary 3.9, shows that the highest degree in which the homology of $\Delta(P(a,b))$ is nonzero depends on the value of b relative to a.

PROPOSITION 3.6. *Let* $4 \leq a \leq b$. *If* $2a - 3k - 2 \leq b \leq 2a - 3k$ *for some* $k \geq 1$, *then*

$$t_{(a,b)} = a - 2 - k.$$

Proof. Let $2a - 3k - 2 \leq b \leq 2a - 3k$ for some $k \geq 1$. Then the chain

$$\begin{aligned} m : \widehat{0} = (a,b) &\to (a-1, b-2) \to (a-2, b-4) \to \cdots \to (2k+2, b-2a+4k+4) \to \\ &\to (2k, b-2a+4k+3) \to (2k-2, b-2a+4k+2) \to \cdots \\ &\to (2, b-2a+3k+4) \to (1,0) \to (0,0) = \widehat{1} \end{aligned}$$

is maximal since $2k \geq 2$ and $2k+2 \leq a$. To show the second inequality, it is enough to notice that, since $b \leq 2a - 3k$ and $a \leq b$, we have $3k \leq 2a - b \leq a$; hence, $k \leq \lfloor \frac{a}{3} \rfloor$. Moreover, m is falling by Lemma 3.2 and has length $a-k$. Note that $b-2a+3k+4 \geq 2$ since $b \geq 2a - 3k - 2$ by assumption. Thus, by (FCH), $t_{(a,b)} \geq a - 2 - k$.

Conversely, we show that $t_{(a,b)} \leq a-2-k$. Again by (FCH), it is enough to show that no maximal chain of length $\ell \geq a - k + 1$ in $P(a,b)^*$ is falling.

Let $m : (a,b) = p_0 \to \cdots \to p_\ell = (0,0)$ be a falling chain in $P(a,b)^*$ of length $\ell \geq a - k + 1$. We denote the ith element of m by $p_i = (c_i, d_i)$. Furthermore, for $i = 1, \ldots, \ell$, let us denote by $u_i = c_{i-1} - c_i$ and $v_i = d_{i-1} - d_i$ the increment on the first and on the second component, respectively. Since m is a maximal chain, $p_{\ell-1}$ is one of the elements $(1,1), (1,0)$, and $(0,1)$. Hence, we have one of the following three cases:

(i) $u_\ell = 1$ and $v_\ell = 1$;

(ii) $u_\ell = 0$ and $v_\ell = 1$;

(iii) $u_\ell = 1$ and $v_\ell = 0$.

Note that the cases (ii) and (iii) are symmetric; hence, it is enough to discuss (i) and (ii).

Claim: For every $i = 1, \dots, \ell - 1$, $u_i > 0$ and $v_i > 0$.

$\triangleleft$ In fact, if $u_i = 0$ for some $1 \le i \le \ell - 1$, then $v_i \ge 1$ and $c_{i-1} = c_i$. This only happens if $c_i = 0$; hence, $p_{i-1} = (0, d_{i-1})$ is a border element, and m is not a falling chain by Lemma 3.2. By symmetry, also $v_i > 0$ for $i = 1, \dots, \ell - 1$. $\triangleright$

First assume that (ii) holds; thus, $u_\ell = 0$ and $v_\ell = 1$. By Lemma 3.5 (ii), it follows that $u_{\ell-1} \ge 2$ and $v_{\ell-1} = 1$. Hence,

$$\sum_{i=1}^{\ell-2} u_i \le a - 2 \text{ and } \sum_{i=1}^{\ell-2} v_i = b - 2.$$

Summing the two expressions, we have

$$\sum_{i=1}^{\ell-2} (u_i + v_i) \le a + b - 4 \le 3a - 3k - 4 < 3(a - k - 1) \le 3(\ell - 2).$$

If $u_i + v_i \ge 3$ for every $i = 1, \dots, \ell - 2$, then we have

$$3(\ell - 2) \le \sum_{i=1}^{\ell-2} (u_i + v_i) < 3(\ell - 2),$$

which is a contradiction. Thus, since both u_i and v_i are positive, there exists $1 \le j \le \ell - 2$ such that $u_j + v_j = 2$ and then $u_j = v_j = 1$. This implies that the element p_j in m is the least atom of $[p_{j-1}, (0, 0)]$, and thus m is not a falling chain by Lemma 3.2.

For the case (i), using a similar argument, one shows that there exists $1 \le j \le \ell - 2$ such that $u_j + v_j = 2$ and then $u_j = v_j = 1$. This concludes the proof. $\square$

Corollary 3.7. *Let $2 \le a \le b$. Then $\Delta(P(a, b))$ is contractible if and only if $a = b = 3$. Moreover, $\Delta(P(3, 3))$ is collapsible.*

Proof. The assertion follows from Example 3.3, Proposition 3.4, and Proposition 3.6. Indeed, the order complex $\Delta(P(3, 3))$ is a connected acyclic graph, as shown in Figure 3. Hence, it is collapsible.

Now we are in position to prove Theorem 1.3.

Proof of Theorem 1.3. Let $2 \le a \le b$ and $0 \le i \le a - 2$. By Lemma 3.5, any falling chain contains exactly one of the elements $(1, 1), (1, 0)$ and $(0, 1)$. Hence, by (FCH), rank $\widetilde{H}_i(\Delta(P(a, b)); \mathbb{Z}) = F_{(1,1)} + F_{(1,0)} + F_{(0,1)}$, where $F_{(c,d)}$ denotes the number of falling chains in $P(a, b)$ of length $i + 2$ containing the element (c, d). Before computing the three contributions, we need one more general fact.

Let $m : p_0 = (a, b) \to p_1 \to \cdots \to p_{i+1} \to p_{i+2} = (0, 0)$ be a falling chain of length $i + 2$. We set $p_\ell = (c_\ell, d_\ell)$ for $0 \le \ell \le i + 2$ and $(u_\ell, v_\ell) = p_{\ell-1} - p_\ell$ for $1 \le \ell \le i + 2$.

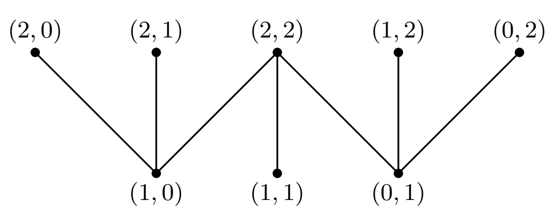

Fig. 3. *The order complex of $P(3, 3)$.*

Clearly,

$$\sum_{\ell=1}^{i+2} u_\ell = a \text{ and } \sum_{\ell=1}^{i+2} v_\ell = b. \tag{3}$$

We recall that, for every $1 \leq \ell \leq i+1$, $u_\ell > 0$ and $v_\ell > 0$ (as in the proof of Proposition 3.6).

We set

$$S = \{\ell : u_\ell = 1\},\ T = \{\ell : u_\ell \geq 2\} \text{ and } U = \{\ell : v_\ell = 1\},\ V = \{\ell : v_\ell \geq 2\}$$

and $s = |S|$, $t = |T|$. Notice that $s + 2t \leq a$. By Lemma 3.2, we have

$$S \cap U \subseteq \{i+2\}. \tag{4}$$

Since, for $1 \leq \ell \leq i+1$, $p_{\ell-1} < p_\ell$ is a cover relation, it follows that for all $1 \leq \ell \leq i+1$, either $u_\ell = 1$ or $v_\ell = 1$ (see also ($\star$)). Thus,

$$\begin{aligned} S \cap \{1, \ldots, i+1\} &= V \\ U \cap \{1, \ldots, i+1\} &= T. \end{aligned} \tag{5}$$

Now we compute the numbers $F_{(1,1)}, F_{(1,0)}$, and $F_{(0,1)}$.

$\rightarrow$ $F_{(1,1)}$

For contributions to $F_{(1,1)}$, we have $u_{i+2} = 1$ and $v_{i+2} = 1$. Then by definition of S, T, U, V,

$$s + t = i + 2 = |U| + |V|.$$

By (4), it follows that $S \cap U = \{i+2\}$. By (5), we have $S = V \cup \{i+2\}$ and $T \cup \{i+2\} = U$. Thus, fixing S fixes the other sets.

For S, we have $\binom{i+1}{t}$ choices. Once S is fixed, we have $\binom{a-i-3}{t-1}$ choices for the u_ℓ, with $\ell \in T$. Now we are left with $\binom{b-i-3}{i-t}$ choices for the v_ℓ, with $\ell \in V$. By Lemma 3.2, each choice corresponds to a falling chain.

This sums up to

$$F_{(1,1)} = \sum_{t=0}^{a-i-2} \binom{b-i-3}{i-t}\binom{a-i-3}{t-1}\binom{i+1}{t}.$$

$\rightarrow$ $F_{(1,0)}$

For contributions to $F_{(1,0)}$, we have $u_{i+2} = 1$ and $v_{i+2} = 0$. Thus,

$$s + t = i + 2 = |U| + |V| + 1.$$

By (4), we have $S \cap U = \emptyset$, and by (5), we have $S \setminus \{i+2\} = V$ and $T = U$. Again, fixing S fixes the other sets. But there is one additional constraint. By Lemma 3.5 (i), we have $u_{i+1} = 1$ and $v_{i+1} \geq 2$. Thus, $i + 1 \in S \cap V$.

For S, we have $\binom{i}{t}$ choices. Once we have S fixed, we have $\binom{a-i-3}{t-1}$ choices for the u_ℓ, with $\ell \in T$. Now we are left with $\binom{b-i-2}{i-t}$ choices for the v_ℓ, with $\ell \in V$.

This sums up to

$$F_{(1,0)} = \sum_{t=0}^{a-i-2} \binom{b-i-2}{i-t}\binom{a-i-3}{t-1}\binom{i}{t}.$$

$\to$ $F_{(0,1)}$

For contributions to $F_{(0,1)}$, we have $u_{i+2} = 0$ and $v_{i+2} = 1$. Thus,

$$s + t + 1 = i + 2 = |U| + |V|.$$

By (4), we have $S \cap U = \emptyset$, and by (5), we have $S = V$ and $T \cup \{i+2\} = U$. Again, fixing S fixes the other sets. But also here, there is one additional constraint. By Lemma 3.5 (ii), we have $u_{i+1} \geq 2$ and $v_{i+1} \geq 1$. Thus, $i+1 \in T \cap U$.

For S, we have $\binom{i}{t-1}$ choices. Once we have S fixed, we have $\binom{a-i-2}{t-1}$ choices for the u_ℓ, with $\ell \in T$. Now we are left with $\binom{b-i-3}{i-t}$ choices for the v_ℓ, with $\ell \in V$.

Since $t \geq 1$, this sums up to

$$F_{(0,1)} = \sum_{t=1}^{a-i-1} \binom{b-i-3}{i-t}\binom{a-i-2}{t-1}\binom{i}{t-1} = \sum_{t=0}^{a-i-2} \binom{b-i-3}{i-t-1}\binom{a-i-2}{t}\binom{i}{t}.$$

Then, summing these three contributions, we obtain

$$\begin{aligned} \operatorname{rank} \widetilde{H}_i(\Delta(P(a,b));\mathbb{Z}) &= F_{(1,1)}+F_{(1,0)}+F_{(0,1)} \\ &= \sum_{t=0}^{a-i-2} \left[\binom{b-i-3}{i-t}\binom{a-i-3}{t-1}\binom{i+1}{t}\right. \\ &\quad \left.+\binom{b-i-2}{i-t}\binom{a-i-3}{t-1}\binom{i}{t}+\binom{b-i-3}{i-t-1}\binom{a-i-2}{t}\binom{i}{t}\right]. \end{aligned} \tag{6}$$

We observe that in (6), we can replace the upper summation index $a-i-2$ by i. In fact, if $i > a-i-2$, then for every $t \geq a-i-1$ the tth summand is zero since $\binom{a-i-3}{t-1}=\binom{a-i-2}{t}=0$. If $i < a-i-2$, we show that the tth summand is zero for every $t \geq i+1$. In fact, $\binom{i}{t}=0$ and, if $t > i+1$, also $\binom{i+1}{t}=0$. We only need to show that, if $t = i+1$, the first summand in (6) is zero. Notice that $\binom{b-i-3}{-1}\neq 0$ if and only if $b-i-3=-1$. This means that $i = b-2$. On the other hand, $2i < a-2$; hence, $2b-4 < a-2$, that is, $2b-2 < a \leq b$. Thus, $b < 2$, in contradiction with $2 \leq a < b$. Therefore, $b-i-3 \neq -1$ and $\binom{b-i-3}{-1}=0$. Hence,

$$\begin{aligned} &\operatorname{rank} \widetilde{H}_i(\Delta(P(a,b));\mathbb{Z}) \\ &= F_{(1,1)}+F_{(1,0)}+F_{(0,1)} = \sum_{t=0}^{i} \left[\binom{b-i-3}{i-t}\binom{a-i-3}{t-1}\binom{i+1}{t}\right. \\ &\quad \left.+\binom{b-i-2}{i-t}\binom{a-i-3}{t-1}\binom{i}{t}+\binom{b-i-3}{i-t-1}\binom{a-i-2}{t}\binom{i}{t}\right] \\ &= \sum_{t=0}^{i} \left[\binom{a-i-3}{t-1}\left[\binom{b-i-3}{i-t}\binom{i+1}{t}+\binom{b-i-2}{i-t}\binom{i}{t}\right]+\binom{b-i-3}{i-t-1}\binom{a-i-2}{t}\binom{i}{t}\right] \\ &= \sum_{t=0}^{i} \left[\binom{a-i-3}{t-1}\left[\binom{b-i-3}{i-t}\binom{i}{t}+\binom{b-i-3}{i-t}\binom{i}{t-1}+\binom{b-i-3}{i-t}\binom{i}{t}\right.\right. \\ &\quad \left.\left.+\binom{b-i-3}{i-t-1}\binom{i}{t}\right]+\binom{a-i-3}{t}\binom{b-i-3}{i-t-1}\binom{i}{t}+\binom{a-i-3}{t-1}\binom{b-i-3}{i-t-1}\binom{i}{t}\right] \\ &= \sum_{t=0}^{i} \left[2\binom{a-i-3}{t-1}\left[\binom{b-i-3}{i-t}\binom{i}{t}+\binom{b-i-3}{i-t-1}\binom{i}{t}\right]+\left[\binom{a-i-3}{t-1}\binom{b-i-3}{i-t}\binom{i}{t-1}\right.\right. \end{aligned}$$

$$+\binom{a-i-3}{t}\binom{b-i-3}{i-t-1}\binom{i}{t}\Bigg]\Bigg] = \sum_{t=0}^{i}\Bigg[2\binom{a-i-3}{t-1}\binom{b-i-2}{i-t}\binom{i}{t}$$
$$+\binom{a-i-3}{t-1}\binom{b-i-3}{i-t}\binom{i}{t-1}+\binom{a-i-3}{t}\binom{b-i-3}{i-t-1}\binom{i}{t}\Bigg].$$

Consider the three summands in the brackets above. The second summand is zero if $t = 0$. The third one is zero if $t = i$. This follows since, by convention, $\binom{b-i-3}{-1}=1$ if and only if $b - i - 3 = -1$. In this case, $b - 2 = i$ and, on the other hand, $i \leq a - 2$; hence, $a = b$, $t = i = a - 2$ and $\binom{a-a+2-3}{a-2}=0$ for $a \geq 2$. Thus, summing over the second and third summand yields

$$\sum_{t=1}^{i}\binom{a-i-3}{t-1}\binom{b-i-3}{i-t}\binom{i}{t-1}+\sum_{t=0}^{i-1}\binom{a-i-3}{t}\binom{b-i-3}{i-t-1}\binom{i}{t}$$
$$= 2\sum_{t=1}^{i}\binom{a-i-3}{t-1}\binom{b-i-3}{i-t}\binom{i}{t-1} = 2\sum_{t=0}^{i}\binom{a-i-3}{t-1}\binom{b-i-3}{i-t}\binom{i}{t-1}.$$

Substituting in the above expression, we get the desired formula. ∎

Remark 3.8. Notice that, for $4 \leq a \leq b$, by Theorem 1.3 it follows that rank $\widetilde{H}_1(\Delta(P(a, b)); \mathbb{Z}) = 4$.

COROLLARY 3.9. *For every* $2 \leq a \leq b$, rank $\widetilde{H}_{a-2}(\Delta(P(a, b)); \mathbb{Z}) = 2\binom{b-a}{a-2}$. *In particular, if* $b \geq 2a - 2$, *then* $t_{(a,b)} = a - 2$.

Proof. If $i = a - 2$, the expression (1) has only the summand for $t = 0$; hence, rank $\widetilde{H}_{a-2}(\Delta(P(a, b)); \mathbb{Z}) = 2\binom{b-a}{a-2}$. In particular, if $2 \leq a \leq b < 2a - 2$, then $\widetilde{H}_{a-2}$ $(\Delta(P(a, b)); \mathbb{Z}) = 0$. The second part of the claim follows from Proposition 3.4. ∎

Another noteworthy property of the posets of proper divisibility is that the nonreduced simplicial homology is nonzero for *every* degree between 0 and $t_{(a,b)}$.

Proof of Proposition 1.4. Let $2 \leq a \leq b$. First of all, notice that rank H_0 $(\Delta(P(a, b)); \mathbb{Z}) \neq 0$ since $\Delta(P(a, b)) \neq \emptyset$. Moreover, the claim is true for $a = 2, 3$ by Example 3.3.

Hence, by (FCH), it suffices to show that there exists a falling chain of length $i + 2$ in $P(a, b)^*$, for every $1 \leq i \leq t_{(a,b)}$. First assume $i = 1$. By Lemma 3.2, the chain $\widehat{0} = (a, b) \to (a-1, 2) \to (1, 0) \to (0, 0) = \widehat{1}$ has length 3 in $P(a, b)^*$ and is falling. Now let $i \geq 2$.

If $b \geq 2a - 2$, then $t_{(a,b)} = a - 2$ by Corollary 3.9. The chain

$$m : \widehat{0} = (a, b) \to (a-1, b-2) \to (a-2, b-4) \to \cdots \to (a+1-i, b-2i+2)$$
$$\to (a-i, 2) \to (1, 0) \to (0, 0) = \widehat{1}$$

has length $i + 2$. Since $i \geq 2$, we obtain $a + 1 - i < a$ and $b - 2i + 2 < b$. Notice that $a - i \geq 2$ by assumption; then $b - 2i + 2 \geq 2a - 2 - 2i + 2 \geq 4$. By Lemma 3.2, m is falling.

Let $4 \leq a \leq b < 2a-2$. We consider the cases $a = 4$ and $a = 5$ separately. If $a = 4$, then $b \in \{4, 5\}$; if $a = 5$, then $b \in \{5, 6, 7\}$. By Proposition 3.6, $t_{(4,4)} = t_{(4,5)} = 1$ and $t_{(5,5)} = t_{(5,6)} = t_{(5,7)} = 2$. For $i = 1$, a falling chain of length 3 is given above, and for $i = t_{(a,b)}$, a falling chain of length $t_{(a,b)}$ is provided in the proof of Proposition 3.6.

Finally, we assume $a \geq 6$. Notice that $a-2 < b-1$. It suffices to show that there exists a falling chain

$$c : (a-2, b-1) \to p_1 \to \cdots \to p_{i+1} = (0,0)$$

of length $i+1$ in $P(a-2, b-1)^*$, for every $2 \leq i \leq t_{(a,b)}$. In fact, by Lemma 3.2, it follows that the chain

$$m : \widehat{0} = (a,b) \to (a-2, b-1) \to p_1 \to \cdots \to p_{i+1} = (0,0) = \widehat{1}$$

obtained by adding the element (a,b) to the chain c is a falling chain of length $i+2$ in $P(a,b)^*$.

Since $b \leq 2a-3$, there exists $k \geq 1$ such that $2a-3k-2 \leq b \leq 2a-3k$; then $t_{(a,b)} = a-2-k$ by Proposition 3.6. Hence, $2(a-2)-3(k-1)-2 \leq b-1 \leq 2(a-2)-3(k-1)$. If $k = 1$, then $b-1 \geq 2(a-2)-2$; thus, $t_{(a-2,b-1)} = a-4$ by Corollary 3.9. On the other hand, if $k > 1$, then $t_{(a-2,b-1)} = a-2-2-(k-1) = a-3-k$ by Proposition 3.6. Hence, $t_{(a-2,b-1)} = a-3-k$, for every $k \geq 1$. By induction on $a \geq 4$, for every $1 \leq h \leq a-3-k$, there exists a falling chain of length $h+2$ in $P(a-2, b-1)^*$. Since $1 \leq i-1 \leq a-3-k$ by assumption, we conclude the proof. ∎

Remark 3.10. Even though Theorem 1.3 provides an explicit formula for the rank of all homology groups, we do not see how to get Proposition 1.4, Proposition 3.4, and Proposition 3.6 as direct consequences of Theorem 1.3.

4. Euler characteristic of the order complex of $P(a,b)$.

In this section, we give the proof of Theorem 1.5. We will mainly use generating functions techniques.

Proof of Theorem 1.5. First we set $f(u,v)$ as the generating series

$$f(u,v) = \sum_{a=2}^{\infty} \sum_{b=a}^{\infty} \widetilde{\chi}(\Delta(P(a,b)))u^a v^b$$

of the reduced Euler-characteristic of $\Delta(P(a,b))$

$$\widetilde{\chi}(\Delta(P(a,b))) = \sum_{i \geq 0} (-1)^i \operatorname{rank} \widetilde{H}_i(\Delta(P(a,b)); \mathbb{Z}).$$

By Theorem 1.3, we have that, for $2 \leq a \leq b$,

$$\widetilde{\chi}(\Delta(P(a,b))) = \sum_{i=0}^{\infty} (-1)^i 2 \sum_{t=0}^{i} \binom{a-3-i}{t-1} \left[\binom{i}{t} \binom{b-2-i}{i-t} + \binom{i}{t-1} \binom{b-3-i}{i-t} \right].$$

The expression on the right-hand side makes sense for all $a, b \geq 0$; therefore, we can formally write the series

$$\bar{f}(u,v) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i 2 \sum_{t=0}^{i} \binom{a-3-i}{t-1} \left[\binom{i}{t} \binom{b-2-i}{i-t} + \binom{i}{t-1} \binom{b-3-i}{i-t} \right] u^a v^b. \tag{7}$$

Using elementary generating function identities (see, e.g., identity **d.** in [9, p. 209]), we obtain

$$\sum_{a=0}^{\infty} \binom{a-3-i}{t-1} u^a = \frac{u^{i+t+2}}{(1-u)^t}, \sum_{b=0}^{\infty} \binom{b-2-i}{i-t} v^b = \frac{v^{2i-t+2}}{(1-v)^{i-t+1}}, \sum_{b=0}^{\infty} \binom{b-3-i}{i-t} v^b = \frac{v^{2i-t+3}}{(1-v)^{i-t+1}}.$$

Interchanging the summation in (7) and using the above identities, we get

$$\begin{aligned}\bar{f}(u,v)&=2\sum_{i=0}^{\infty}(-1)^i\left[\frac{u^{i+2}v^{2i+2}}{(1-v)^{i+1}}\sum_{t=0}^{i}\binom{i}{t}\left(\frac{u(1-v)}{(1-u)v}\right)^t+\frac{u^{i+2}v^{2i+3}}{(1-v)^{i+1}}\sum_{t=0}^{i}\binom{i}{t-1}\left(\frac{u(1-v)}{(1-u)v}\right)^t\right]\\
&=2\sum_{i=0}^{\infty}(-1)^i\left[\frac{u^{i+2}v^{2i+2}}{(1-v)^{i+1}}\left(1+\frac{u(1-v)}{(1-u)v}\right)^i+\frac{u^{i+2}v^{2i+3}}{(1-v)^{i+1}}\frac{u(1-v)}{(1-u)v}\left(\left(1+\frac{u(1-v)}{(1-u)v}\right)^i\right.\right.\\
&\left.\left.-\left(\frac{u(1-v)}{(1-u)v}\right)^i\right)\right]=2\sum_{i=0}^{\infty}(-1)^i\left[\frac{u^{i+2}v^{i+2}(u+v-2uv)^i}{(1-u)^i(1-v)^{i+1}}+\frac{u^{i+3}v^{i+2}(u+v-2uv)^i}{(1-u)^{i+1}(1-v)^{i+1}}\right.\\
&\left.-\frac{u^{2i+3}v^{i+2}}{(1-u)^{i+1}}\right]=2\sum_{i=0}^{\infty}(-1)^i\left[\frac{u^{i+2}v^{i+2}(u+v-2uv)^i(1-uv)}{(1-u)^{i+1}(1-v)^{i+1}}-\frac{u^{2i+3}v^{i+2}}{(1-u)^{i+1}}\right]\\
&=2\left(\frac{u^2v^2}{2uv-u-v+1}-\frac{u^3v^2}{u^2v-u+1}\right).\end{aligned}$$

Notice that, since

$$\frac{1}{u^2v-u+1}=\frac{1}{1-u(1-uv)}=\sum_{n=0}^{\infty}(u^n(1-uv)^n),$$

in the Taylor expansion of $\frac{u^3v^2}{u^2v-u+1}$ at $(0,0)$ only monomials u^av^b, where $a>b$, appear with nonzero coefficient. Hence, for every $2\le a\le b$, the coefficient of u^av^b in the Taylor expansion of $\frac{2u^2v^2}{2uv-u-v+1}$ at $(0,0)$ is $\widetilde{\chi}(\Delta(P(a,b)))$, which in turn is also the coefficient of u^av^b in $f(u,v)$.

On the other hand, let us compute the generating function of the right-hand side of (2). Using a similar approach as above, we consider the series

$$\bar{g}(u,v)=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}(-1)^a\cdot 2\left[\sum_{h=0}^{\lfloor\frac{a}{2}\rfloor-1}(-1)^h\binom{a-2}{h}\binom{b-a}{a-2-2h}\right]u^av^b.$$

Notice that, if $h\ge\lfloor\frac{a}{2}\rfloor$, then $\binom{b-a}{a-2-2h}=0$. Hence, we may extend the sum on h up to infinity. Interchanging the order of summation and using the identity **d.** in [9, p. 209], we have

$$\begin{aligned}\bar{g}(u,v)&=2\sum_{a=0}^{\infty}(-1)^a\sum_{h=0}^{\infty}(-1)^h\binom{a-2}{h}\left[\sum_{b=0}^{\infty}\binom{b-a}{a-2-2h}v^b\right]u^a\\
&=2\sum_{a=0}^{\infty}(-1)^a\sum_{h=0}^{\infty}(-1)^h\binom{a-2}{h}\frac{v^{2a-2-2h}u^a}{(1-v)^{a-1-2h}}\\
&=2\sum_{a=0}^{\infty}(-1)^a\frac{v^{2a-2}u^a}{(1-v)^{a-1}}\sum_{h=0}^{\infty}(-1)^h\binom{a-2}{h}\left(\frac{1-v}{v}\right)^{2h}\\
&=2\sum_{a=0}^{\infty}(-1)^a\frac{v^{2a-2}u^a}{(1-v)^{a-1}}\left(1-\frac{(1-v)^2}{v^2}\right)^{a-2}=\frac{2(1-v)v^2}{(2v-1)^2}\sum_{a=0}^{\infty}(-1)^a\left(\frac{u(2v-1)}{(1-v)}\right)^a\\
&=\frac{2(1-v)v^2}{(2v-1)^2}\frac{u^2(2v-1)^2}{(1-v)(2uv-u-v+1)}=\frac{2u^2v^2}{2uv-u-v+1}.\end{aligned}$$

Since we already know that, for $2\le a\le b$, the coefficient of u^av^b in the Taylor expansion of $\frac{2u^2v^2}{2uv-u-v+1}$ is $\widetilde{\chi}(\Delta(P(a,b)))$, the assertion follows. ∎

COROLLARY 4.1. *For every* $2 \leq a \leq b$, $\widetilde{\chi}(\Delta(P(a,b))) = 0$ *if* $a = b$ *is odd. Moreover, if* $a = b$ *is even,*

$$\widetilde{\chi}(\Delta(P(a,b))) = (-1)^{\frac{a-2}{2}} \cdot 2\binom{a-2}{\frac{a-2}{2}}. \tag{8}$$

Proof. Notice that, using the formula; (2), when $a = b$, the second binomial coefficient is $\binom{0}{a-2-2h}$. This is zero if a is odd; hence, $\widetilde{\chi}(\Delta(P(a,a))) = 0$, and it is 1 when a is even and $h = \frac{a-2}{2}$. In the last case, we get the formula (8). ◻

5. Open problems. The posets $P(a_1, \ldots, a_n)$ can be seen as examples of the following general construction. Let $P_1, \ldots, P_n$ be posets. Assume that, for $1 \leq i \leq n$, the poset P_i has unique minimal element $\widehat{0}_i$ and unique maximal element $\widehat{1}_i$. For every two elements $(a_1, \ldots, a_n), (b_1, \ldots, b_n)$ in the Cartesian product $P_1 \times \cdots \times P_n$, we set $(a_1, \ldots, a_n) \leq_p (b_1, \ldots, b_n)$ if $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ or, for every i, either $a_i = b_i = \widehat{0}_i$ or $a_i < b_i$ in P_i. We write $P_1 \times_p \cdots \times_p P_n$ for the set of all $(a_1, \ldots, a_n) \in P_1 \times \cdots \times P_n$ with $(a_1, \ldots, a_n) \leq_p (\widehat{1}_1, \ldots, \widehat{1}_n)$. If we denote by C_{k+1} a chain of length k, then it is easily seen that $P(a_1, \ldots, a_n) \cong C_{a_1+1} \times_p \cdots \times_p C_{a_n+1}$. For this reason, we call $\times_p$ the *proper division product*. Note that $(P_1 \times_p P_2) \times_p P_3 = P_1 \times_p P_2 \times_p P_3$. A natural question is whether Theorem 1.1 and Proposition 1.4 can be extended to this setting.

Let P and Q be two (pure) shellable posets with unique maximal and unique minimal element.

QUESTION 5.1. *Is* $\Delta(P \times_p Q)$ *shellable?*

QUESTION 5.2. *Assume* $\Delta(P \times_p Q)$ *is nonempty. Is there an integer* $t_{P;Q} \geq 0$ *such that* $H_i(\Delta(P \times_p Q); \mathbb{Z}) \neq 0$ *if and only if* $0 \leq i \leq t_{P;Q}$*?*

We have tested both questions when P and Q are Boolean lattices on a reasonably sized set of examples. For all those examples, the answer to both questions is affirmative. If we denote by B_i the Boolean lattice on i elements, then we have the following:

$P \times_p Q$	(rank $H_i(\Delta(P \times_p Q)) : i \geq 0$)
$B_2 \times_p B_6$	(15,30,40,30,13,0,...)
$B_2 \times_p B_7$	(17,42,70,70,42,15,0,...)
$B_3 \times_p B_6$	(1,1461,1275,705,172,0,...)
$B_3 \times_p B_7$	(1,3381,3822,2940,1218,232,0,...)

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