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MIXED METHODS FOR ELASTODYNAMICS WITH WEAK SYMMETRY*

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Abstract. We analyze the application to elastodynamic problems of mixed finite element methods for elasticity with weakly imposed symmetry of stress. Our approach leads to a semidiscrete method which consists of a system of ordinary differential equations without algebraic constraints. Our error analysis, which is based on a new elliptic projection operator, applies to several mixed finite element spaces developed for elastostatics. The error estimates we obtain are robust for nearly incompressible materials.

Key words. mixed finite element, elastodynamics, weak symmetry

AMS subject classifications. 65N30, 74H15, 74S05

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1. Introduction.

The linear elastodynamics equation describes wave propagation in an elastic medium. It has the form

$$\rho \ddot{u} - \operatorname{div} C\epsilon(u) = f \quad \text{in } \Omega, \tag{1.1}$$

where $u : \Omega \to \mathbb{R}^n$ is the unknown displacement vector field, $\epsilon(u)$ the corresponding linearized strain tensor, C the stiffness tensor of the elastic medium, ρ the mass density, and f an external body force. In (1.1) we have suppressed the dependence on the independent variables for simplicity, but all the quantities appearing in the equation may depend on $x \in \Omega$, and $t \in [0, T_0]$ (for some positive T_0), and the equation is supposed to hold for all such t. Combining (1.1) with initial conditions $u = u_0$, $\dot{u} = v_0$ at time $t = 0$ and with appropriate boundary conditions, we obtain a well-posed problem (see, e.g., [15, Theorem 4.1], or section 3).

Mixed finite element methods, in which the stress $\sigma = C\epsilon(u)$ and displacement u are approximated independently, are popular for the numerical approximation of elastostatic problems. The application of mixed methods to elastodynamic problems has also been studied by various researchers, and in the previous research there are two basic approaches for introducing time-dependence. The *displacement–stress formulation* retains the same variables as for elastostatics, namely the displacement and the stress, introducing a momentum term into the equilibrium equation. Since the constitutive equation, which does not involve time differentiation, is unchanged, this leads to a system of differential–algebraic equations in time. The second approach, the *velocity–stress formulation*, uses the velocity and the stress as the fundamental variables and this leads to a standard hyperbolic system, a system of differential equations in time after semidiscretization in space. The displacement is not a primary unknown, but is recoverable as the time-integral of the velocity.

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In [14], Douglas and Gupta studied plane linear elastodynamics using the displacement–stress formulation and the mixed finite elements developed in [4]. In [22], Makridakis analyzed both the displacement–stress and velocity–stress formulations of the problem including higher order time discretization with the elements in [4, 19, 23, 25]. In [6], Bécache, Joly, and Tsogka developed a new family of rectangular mixed finite elements for elastostatics and applied it to the problem using the velocity–stress formulation. See Table 1 for a summary.

Table 1

Comparison with previous results, showing for each the number of space dimensions treated, whether the displacement–stress (DS) or velocity–stress (VS) formulation is used, whether symmetry is imposed strongly or weakly, and the mixed finite elements and temporal discretization scheme considered (IN = implicit Newmark, CN = Crank–Nicolson).

	Douglas, Gupta [14]	Makridakis [22]	Bécache, Joly, Tsogka [6]	Boulaajine, Farhloul, Paquet [7]	This paper
Dimension	2D	2D, 3D	2D, 3D	2D	2D, 3D
Formulation	DS	DS/VS	VS	DS	VS
Element	[4]	[4, 19, 23, 25]	[6]	[16]	[5, 10, 17, 26]
Symmetry	strong	strong	strong	weak	weak
Time scheme	–	Padé	–	IN	CN

In the standard mixed method approach to linear elasticity, the mixed finite element spaces incorporate the symmetry of the stress tensor into the finite element space, and as a result are rather complicated. This has led to a great deal of interest in mixed finite elements for elasticity in which the symmetry of the stress is imposed only weakly. This idea was first suggested in [13] and elements based upon it were first developed in [1] and [3]. Recently, a great deal of progress has been made in stable mixed finite elements for elasticity with weak symmetry. However, the application of such methods to elastodynamics has not yet been well studied. An exception is the paper [7] which uses a displacement–stress formulation and the dual hybrid method, introduced in [16] to analyze the two-dimensional problem for a homogeneous isotropic elastic media. In contrast, we use the velocity–stress formulation, which we believe leads to a more standard discrete problem as well as a more natural error analysis. Moreover, our analysis covers general linear elastic media in two and three dimensions and a larger class of finite elements. In particular, we treat in a unified fashion the elements of Arnold, Falk, and Winther [5] and the variant introduced by Cockburn, Gopalakrishnan, and Guzmán [10], as well as another method of Gopalakrishnan and Guzmán [17] and a related older method of Stenberg [26]. In fact, the same analysis applies to other elements satisfying the conditions (A0), (A1), and (A2) stated in the next section. Finally, although we only consider the case of elastodynamics, we point out that one advantage of mixed finite elements is that they can be easily extended to materials with more complex constitutive equations, such as viscoelasticity [20, 24], and likely also to plasticity and poroelasticity.

Since symmetry of the stress tensor is an algebraic condition, the most obvious formulation of elastodynamics with weak symmetry leads, after spatial discretization, to a system of differential–algebraic equations. Indeed, that is the approach taken in [24] for quasi-static viscoelasticity. However, in this paper we propose a different mixed variational formulation for elastodynamics with weak symmetry (see (3.9)–(3.11)). Our approach leads simply to a system of ordinary differential equations in time after spatial discretization. Therefore, standard time stepping methods can be applied, and the analysis of the temporal discretization is relatively standard. For that reason we mainly focus on the analysis of semidiscrete solutions and discuss temporal discretization only briefly and without seeking full generality.

The remainder of this paper is organized as follows. In section 2, we set out notation and describe the features of mixed finite elements for elasticity with weak symmetry of stress which we will need for analysis of elastodynamic problems. In section 3, we prove well-posedness of linear elastodynamics using the Hille–Yosida theorem and derive the weak formulation of it which we will use for discretization. In section 4, we analyze the semidiscretization, and obtain a priori error estimates for the elements of [5] and [10]. In this context, we also prove that numerical solution is free from locking in the nearly incompressible regime, i.e., the constants in the error bounds do not grow unboundedly as the Lamé coefficient λ tends to infinity. In section 5, we give an improved error analysis for the elements of [17] and [26]. Temporal discretization and its analysis is discussed in section 6, and finally numerical results supporting the analysis are presented in section 7.

2. Notation and preliminaries.

2.1. Notation. Let Ω be a bounded smooth domain in $\mathbb{R}^n$ with $n = 2$ or 3. We use $\mathbb{V}$ to denote the space $\mathbb{R}^n$ of n-vectors and $\mathbb{M}$, $\mathbb{S}$, and $\mathbb{K}$ to denote the space of all, symmetric, and skew-symmetric $n \times n$ matrices, respectively. The components of a vector field $u : \Omega \to \mathbb{V}$ and a matrix field $\sigma : \Omega \to \mathbb{M}$ are denoted by u_i and σ_{ij}, respectively. The L^2 inner products on vector and matrix fields are given by

$$(v, w) = \int_\Omega v \cdot w \, dx = \int_\Omega \sum_{i=1}^n v_i w_i \, dx, \quad v, w : \Omega \to \mathbb{V},$$
$$(\sigma, \tau) = \int_\Omega \sigma : \tau \, dx = \int_\Omega \sum_{1 \le i,j \le n} \sigma_{ij} \tau_{ij} \, dx, \quad \sigma, \tau : \Omega \to \mathbb{M}.$$

We denote the corresponding norms by $\|\sigma\|$, $\|u\|$ and the corresponding Hilbert spaces by $L^2(\Omega; \mathbb{M})$, $L^2(\Omega; \mathbb{V})$. For $\sigma : \Omega \to \mathbb{M}$ and $u : \Omega \to \mathbb{V}$, $\operatorname{div} \sigma$ and $\operatorname{grad} u$ are defined as the rowwise divergence and the rowwise gradient

$$(\operatorname{div} \sigma)_i = \sum_j \partial_j \sigma_{ij}, \qquad (\operatorname{grad} u)_{ij} = \partial_j u_i,$$

respectively, where ∂_j denotes the jth partial derivative, applied in the sense of distributions. For $\sigma : \Omega \to \mathbb{M}$, the skew-symmetric part of σ is $\operatorname{skw} \sigma = (\sigma - \sigma^T)/2$.

We use standard notation for the Sobolev space $H^m(\Omega)$, $m \ge 0$, with norm $\| \cdot \|_m$, and the space $\mathring{H}^1(\Omega)$ of $H^1(\Omega)$ functions with vanishing trace. For $\mathbb{X} = \mathbb{V}, \mathbb{M}, \mathbb{K}$, or $\mathbb{S}$, we write $H^m(\Omega; \mathbb{X})$ for the space of $\mathbb{X}$-valued fields such that each component belongs to $H^m(\Omega)$. If $\mathbb{X}$ is clear in context, we may write $H^m(\Omega)$ instead of $H^m(\Omega; \mathbb{X})$. For $\mathbb{X}$ being a subspace of $\mathbb{M}$, let

$$H(\operatorname{div}, \Omega; \mathbb{X}) = \{ \sigma \in L^2(\Omega; \mathbb{X}) \mid \operatorname{div} \sigma \in L^2(\Omega; \mathbb{V}) \},$$

which is a Hilbert space with the norm $\|\sigma\|_{\operatorname{div}}^2 = \|\sigma\|^2 + \|\operatorname{div} \sigma\|^2$. We abbreviate

$$M = H(\operatorname{div}, \Omega; \mathbb{M}), \quad S = H(\operatorname{div}, \Omega; \mathbb{S}), \quad V = L^2(\Omega; \mathbb{V}), \quad K = L^2(\Omega; \mathbb{K}). \tag{2.1}$$

Let $\mathcal{X}$ be a Banach space (e.g., one of the Hilbert spaces defined above), T_0 a positive real number, m a nonnegative integer, and $1 \le p \le \infty$. We denote by

$L^p([0,T_0];\mathcal{X})$ or $L^p\mathcal{X}$ the space of functions $f:[0,T_0]\to\mathcal{X}$ for which

$$\|f\|_{L^p\mathcal{X}}^p := \int_0^{T_0} \|f\|_{\mathcal{X}}^p\,dt < \infty$$

(with the usual modification for $p=\infty$), and by $W^{m,p}([0,T_0];\mathcal{X})$ or $W^{m,p}\mathcal{X}$ the space for which

$$\|u\|_{W^{m,p}\mathcal{X}}^p := \sum_{l=0}^{m} \|\partial^l u/\partial t^l\|_{L^p\mathcal{X}}^p < \infty.$$

We shall also use the space $C^m([0,T_0];\mathcal{X})$ of m-times continuously differentiable functions.

For brevity of notation, we write $\|f,g\|_{\mathcal{X}}$ to denote $\|f\|_{\mathcal{X}}+\|g\|_{\mathcal{X}}$ when f and g both belong to some Banach space $\mathcal{X}$, and, as we have seen, we use $\dot f$, $\ddot f$, ..., to denote $\partial f/\partial t$, $\partial^2 f/\partial t^2$, etc.

2.2. Mixed formulations of linear elastostatics. In this section we review the discretization of *stationary* linear elasticity using mixed finite elements with weak symmetry. For details, see [5]. The constitutive equation of linear elasticity is $\sigma = C\epsilon(u)$, where, for a given displacement vector field u, the linearized strain tensor $\epsilon(u)$ is given by $\epsilon(u) = (\operatorname{grad} u + (\operatorname{grad} u)^T)/2$, and at each point x the elasticity tensor $C(x)$ is a symmetric positive definite linear operator from $\mathbb{S}$ to $\mathbb{S}$, bounded above and below. The same then holds for the compliance tensor $A := C^{-1}$. For a homogeneous isotropic elastic material

$$C\tau = 2\mu\tau + \lambda\operatorname{tr}(\tau)I, \quad A\tau = \frac{1}{2\mu}\left(\tau - \frac{\lambda}{2\mu+n\lambda}\operatorname{tr}(\tau)I\right), \tag{2.2}$$

where μ, λ are positive scalars called the Lamé coefficients, and $\operatorname{tr}(\tau)$ is the trace of τ.

The boundary value problem of linear elastostatics consists of the constitutive equation, the equilibrium equation $-\operatorname{div}\sigma = f$, where f is a given body force density, and boundary conditions. For simplicity, we only consider problems with the homogeneous displacement boundary conditions, although it is not difficult to extend our approach to more general boundary conditions. Thus the elastostatic problem is

$$A\sigma = \epsilon(u), \quad -\operatorname{div}\sigma = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Integrating by parts, we obtain a weak formulation of linear elasticity with strongly imposed symmetry. It seeks (σ,u) in $S\times V$ so that

$$(A\sigma,\tau) + (\operatorname{div}\tau, u) = 0, \qquad \tau \in S, \tag{2.3}$$

$$-(\operatorname{div}\sigma, w) = (f,w), \qquad w \in V. \tag{2.4}$$

For any $f\in L^2(\Omega;\mathbb{V})$ this system admits a unique solution.

We now modify this formulation to impose the symmetry of stress weakly. For this, we extend the operator A, originally defined only on symmetric tensors, to map $\mathbb{M}\to\mathbb{M}$, by setting it equal to the identity map on skew-symmetric tensors (or a positive multiple of the identity map). Next, we introduce the rotation field, $r := \operatorname{skw}\operatorname{grad} u$. Then the triple (σ,u,r) in $M\times V\times K$ satisfies

$$(A\sigma,\tau) = (\operatorname{grad} u - r, \tau) = -(u, \operatorname{div}\tau) - (r,\tau), \quad \tau\in M.$$

Now we seek (σ, u, r) in $M \times V \times K$ satisfying

$$(A\sigma, \tau) + (\operatorname{div} \tau, u) + (r, \tau) = 0, \qquad \tau \in M, \tag{2.5}$$

$$-(\operatorname{div} \sigma, w) = (f, w), \qquad w \in V, \tag{2.6}$$

$$(\sigma, q) = 0, \qquad q \in K, \tag{2.7}$$

where the third equation expresses the symmetry of the stress. The formulation (2.5)–(2.7) admits a unique solution, for which (σ, u) coincides with the solution of (2.3–2.4) and $r = \operatorname{skw} \operatorname{grad} u$. The triple (σ, u, r) may be characterized variationally as the unique critical point of the functional

$$(\tau, w, q) \mapsto \frac{1}{2}(A\tau, \tau) + (\operatorname{div} \tau, w) + (\tau, q) + (f, w)$$

over $M \times V \times K$.

2.3. Mixed finite elements for elastostatics with weak symmetry. A mixed method for elastostatics with weak symmetry is a Galerkin method based on this weak formulation. Thus we make a choice of finite element subspaces $M_h \subset M$, $V_h \subset V$, $K_h \subset K$, and seek (σ_h, u_h, r_h) in $M_h \times V_h \times K_h$ so that

$$(A\sigma_h, \tau) + (\operatorname{div} \tau, u_h) + (r_h, \tau) = 0, \qquad \tau \in M_h, \tag{2.8}$$

$$-(\operatorname{div} \sigma_h, w) = (f, w), \qquad w \in V_h, \tag{2.9}$$

$$(\sigma_h, q) = 0, \qquad q \in K_h. \tag{2.10}$$

Of course the spaces must be suitably chosen to ensure that this finite dimensional problem is nonsingular, and to obtain error estimates. In this paper, we shall consider four families of such spaces, each based on a simplicial triangulation of Ω into elements, and a choice of polynomial degree $k > 0$. The simplest choice is the Arnold–Falk–Winther (AFW) family defined in [5]. For any $k \geq 1$, the spaces V_h and K_h are simply defined as the fields in V and K which are piecewise polynomial of degree at most $k-1$ on each element. That is, the shape function space for each component is $\mathcal{P}_{k-1}$, with no interelement continuity imposed. The stress space M_h consists of all matrix fields in M which belong piecewise to $\mathcal{P}_k$. For these elements, all three variables, σ, u, and r, are approximated with an error $O(h^k)$ in L^2. This is clearly the best permitted by the subspaces for u and r, but the inclusion of $\mathcal{P}_k$ in the shape functions for σ suggests the possibility of $O(h^{k+1})$ for σ. This, however, does not hold. This observation motivated the Cockburn–Gopalakrishnan–Guzmán (CGG) elements of [10], which take the same spaces for V_h and K_h, but replace the shape function space for M_h with a space which is strictly smaller than $\mathcal{P}_k$ but which still contains $\mathcal{P}_{k-1}$. These elements were shown to satisfy the same error estimates as the AFW elements.

The elements in [17] go in the other direction, increasing the AFW spaces to obtain a higher rate of convergence. The displacement space remains piecewise $\mathcal{P}_{k-1}$, but the rotation space is increased to piecewise $\mathcal{P}_k$, and the stress space on each element consists of $\mathcal{P}_k$ plus a number of higher degree bubble functions. The same approach was taken by Stenberg [26], although a larger number of bubble functions were used and he required $k \geq 2$. The four methods are summarized in Table 2.

These methods share a number of common features which will allow us to analyze them in a unified fashion. Each satisfies the following stability conditions:

(A0) $\operatorname{div} M_h = V_h$.

TABLE 2
Complete polynomial degree included in the shape function spaces for various mixed methods.

Elements	Approximability σ	u	r	Order
AFW [5]	k	$k-1$	$k-1$	$k \geq 1$
CGG [10]	$k-1$	$k-1$	$k-1$	$k \geq 2$
Stenberg [26]	k	$k-1$	k	$k \geq 2$
GG [17]	k	$k-1$	k	$k \geq 1$

(A1) There exists $c > 0$ so that for any $(u, r) \in V_h \times K_h$, there is $\tau \in M_h$ with

$$\operatorname{div} \tau = u, \quad (\tau, q) = (r, q) \quad \forall q \in K_h, \qquad \|\tau\|_{\mathrm{div}} \leq c(\|u\| + \|r\|).$$

These conditions imply that the mixed method is stable in the sense of Brezzi, and so admits a unique solution. In order to get the best estimates, however, more structure is used. For each of the methods there is a natural interpolation operator $\Pi_h : H^1(\Omega; \mathbb{M}) \to M_h$ which satisfies the commutativity condition

(A2) $\operatorname{div} \Pi_h \sigma = P_h \operatorname{div} \sigma, \quad \sigma \in H^1(\Omega; \mathbb{M}),$

where P_h is the L^2 projection onto V_h. We also denote by P_h' the L^2 projection onto K_h. The projection operator Π_h is defined element by element and preserves the finite element space, and so satisfies the error estimates

$$\|\sigma - \Pi_h \sigma\| \leq ch^m \|\sigma\|_m, \quad 1 \leq m \leq \begin{cases} k, & \text{CGG}, \\ k+1, & \text{AFW, Stenberg, GG}. \end{cases} \tag{2.11}$$

The conditions (A0), (A1), and (A2) imply the following error estimates, which improve on the basic stability estimates:

$$\|\sigma - \sigma_h\| + \|P_h u - u_h\| + \|r - r_h\| \leq c(\|\sigma - \Pi_h \sigma\| + \|r - P_h' r\|). \tag{2.12}$$

These improved estimates appeared in [5, 17, 10], and an equivalent result for $\|\sigma - \sigma_h\|$ and $\|r - r_h\|$ is obtained in [26]. We refer the reader to [18] for a unified analysis. Combining this estimate with the approximation rates implied by Table 2, we obtain error bounds

$$\|\sigma - \sigma_h\| + \|P_h u - u_h\| + \|r - r_h\| \leq ch^m (\|\sigma\|_m + \|r\|_m), \qquad 1 \leq m \leq \bar{k},$$

where $\bar{k} = k$ for the AFW and CGG elements, and $\bar{k} = k + 1$ for the Stenberg and Gopalakrishnan–Guzmán (GG) elements.

2.4. The weakly symmetric elliptic projection operator. Our error analysis for linear elastodynamics will depend on a bounded projection $\tilde{\Pi}_h : M \to M_h$ which we now define. Let M_h, V_h, K_h be one of the choices of finite element spaces discussed in the previous section, and $\Pi_h : H^1(\Omega; \mathbb{M}) \to M_h$ the corresponding interpolant. Given $\sigma \in M$, there exists a unique triple $(\sigma_h, u_h, r_h) \in M_h \times V_h \times K_h$ such that

$$(\sigma_h, \tau) + (\operatorname{div} \tau, u_h) + (\tau, r_h) = (\sigma, \tau), \qquad \tau \in M_h, \tag{2.13}$$

$$(\operatorname{div} \sigma_h, w) = (\operatorname{div} \sigma, w), \qquad w \in V_h, \tag{2.14}$$

$$(\sigma_h, q) = (\sigma, q), \qquad q \in K_h. \tag{2.15}$$

In other words, (σ_h, u_h, r_h) is the mixed method approximation of $(\sigma, 0, 0)$. We define $\tilde{\Pi}_h \sigma = \sigma_h$. If $\sigma \in M_h$, we clearly have $\sigma_h = \sigma$, $u_h = 0$, $r_h = 0$, so $\tilde{\Pi}_h$ is a projection. We now establish some additional properties.

LEMMA 2.1. *For one of the mixed methods given in Table 2, let M_h be the stress space, $\Pi_h : H^1(\Omega; \mathbb{M}) \to M_h$ the corresponding projection satisfying* (A2), *and $\tilde{\Pi}_h : M \to M_h$ the elliptic projection just defined. Then*

$$\operatorname{div} \tilde{\Pi}_h \sigma = P_h \operatorname{div} \sigma, \quad (\tilde{\Pi}_h \sigma, q) = (\sigma, q), \quad \sigma \in M,\ q \in K_h. \tag{2.16}$$

Moreover, there exists a constant c such that

$$\|\tilde{\Pi}_h \sigma\|_{\text{div}} \le c \|\sigma\|_{\text{div}}, \quad \|\sigma - \tilde{\Pi}_h \sigma\| \le c \|\sigma - \Pi_h \sigma\|, \quad \sigma \in H^1(\Omega; \mathbb{M}). \tag{2.17}$$

Proof. The properties in (2.16) are immediate from (2.14) and (2.15) in the definition of the elliptic projection, and the fact (A0) that $\operatorname{div} M_h = V_h$. The first estimate in (2.17) is a consequence of the Brezzi stability, and the second estimate is just the error estimate (2.12) in the case $u = 0$ and $r = 0$. ∎

3. Weak formulation of elastodynamics with weak symmetry. In this section we derive a velocity–stress formulation of linear elastodynamics with weakly imposed symmetry of stress and show that it is well-posed. For simplicity, we only consider homogeneous displacement boundary conditions.

In order to have a mixed form with velocity and stress, we set $v = \dot{u}$, $\sigma = C\epsilon(u)$ in (1.1), and get a system of equations

$$\rho \dot{v} - \operatorname{div} \sigma = f, \qquad A\dot{\sigma} = \epsilon(v), \tag{3.1}$$

where $A = C^{-1}$. For boundary conditions we take $v = 0$, implied by the vanishing of u on $\partial\Omega$, and, for initial conditions, $\sigma(0) = \sigma_0 := C\epsilon(u_0)$, $v(0) = v_0$. We assume that the mass density ρ satisfies $0 < \rho_0 \le \rho \le \rho_1 < \infty$ for constants ρ_0, ρ_1.

To establish well-posedness of this system, we recall the Hille–Yosida theorem. For a Hilbert space $\mathcal{X}$ and a closed, densely defined operator $\mathcal{L}$ on $\mathcal{X}$ with domain $D(\mathcal{L})$, we consider an evolution equation $\dot{U} = \mathcal{L}U + F$ with initial condition $U(0) = U_0$. The operator $\mathcal{L}$ is dissipative if $(\mathcal{L}u, u)_{\mathcal{X}} \le 0$ for $u \in D(\mathcal{L})$, and it is m-dissipative if, further, $I - \mathcal{L} : D(\mathcal{L}) \to \mathcal{X}$ is surjective (see [9, Definition 2.2.2, Proposition 2.2.6, and Proposition 2.4.2]). The Hille–Yosida theorem states that, if $\mathcal{L}$ is an m-dissipative operator, $F \in W^{1,1}([0, T_0]; \mathcal{X})$, and $U_0 \in D(\mathcal{L})$, then the initial value problem has a unique solution $U \in C^0([0, T_0]; D(\mathcal{L})) \cap C^1([0, T_0]; \mathcal{X})$ (see [9, Proposition 4.1.6]). We now apply this to (3.1), which we rewrite as

$$\begin{pmatrix} \dot{\sigma} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & C\epsilon \\ \rho^{-1} \operatorname{div} & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ \rho^{-1} f \end{pmatrix}.$$

Let $\mathcal{X} = L^2(\Omega; \mathbb{S}) \times V$ be the Hilbert space with the inner product

$$((\sigma, v), (\tau, w))_{\mathcal{X}} := (\sigma, \tau)_A + (v, w)_\rho = (A\sigma, \tau) + (\rho v, w).$$

We define the linear operator $\mathcal{L}$ as $\mathcal{L}(\sigma, v) = (C\epsilon(v), \rho^{-1} \operatorname{div} \sigma)$. Note that $\mathcal{L}$ is an unbounded operator on $\mathcal{X}$ and its domain $D(\mathcal{L}) = S \times \mathring{H}^1(\Omega; \mathbb{V})$ is dense in $\mathcal{X}$.

To apply the Hille–Yosida theorem, we verify that $\mathcal{L}$ is m-dissipative. Let $(\sigma, v) \in D(\mathcal{L})$. Then

$$\begin{aligned} (\mathcal{L}(\sigma, v), (\sigma, v))_{\mathcal{X}} &= ((C\epsilon(v), \rho^{-1} \operatorname{div} \sigma), (\sigma, v))_{\mathcal{X}} \\ &= (\epsilon(v), \sigma) + (\operatorname{div} \sigma, v) = 0, \end{aligned}$$

where the last equality comes from the integration by parts. Thus $\mathcal{L}$ is dissipative. To show that it is m-dissipative, it remains to prove that $I-\mathcal{L}$ is surjective. We shall show that, for any given $(\eta, p) \in \mathcal{X}$, the weakly formulated problem

$$((I-\mathcal{L})(\sigma, v), (\tau, w))_{\mathcal{X}} = ((\eta, p), (\tau, w))_{\mathcal{X}}, \qquad (\tau, w) \in D(\mathcal{L}),$$

has a solution $(\sigma, v) \in D(\mathcal{L})$. If (σ, v) satisfies this weak formulation, then $(I-\mathcal{L})(\sigma, v) = (\eta, p)$, since $D(\mathcal{L})$ is dense in $\mathcal{X}$. The weak problem may be restated as

$$(\sigma - C\epsilon(v), \tau)_A = (\eta, \tau)_A, \qquad \tau \in S, \tag{3.2}$$

$$(v - \rho^{-1} \operatorname{div} \sigma, w)_\rho = (p, w)_\rho, \qquad w \in \mathring{H}^1(\Omega; \mathbb{V}). \tag{3.3}$$

Rewriting (3.3) using integration by parts and the symmetry of σ, we get

$$(\rho v, w) + (\sigma, \epsilon(w)) = (\rho p, w), \quad w \in \mathring{H}^1(\Omega; \mathbb{V}). \tag{3.4}$$

Equation (3.2) gives a constraint $\sigma - C\epsilon(v) = \eta$, and substituting σ in (3.4) by $C\epsilon(v) + \eta$, we obtain

$$(\rho v, w) + (C\epsilon(v), \epsilon(w)) = (\rho p, w) - (\eta, \epsilon(w)), \quad w \in \mathring{H}^1(\Omega; \mathbb{V}).$$

By Korn's inequality and the Lax–Milgram lemma, this equation has a unique solution $v \in \mathring{H}^1(\Omega; \mathbb{V})$. One can easily see that $\sigma = C\epsilon(v) + \eta$ is in $L^2(\Omega; \mathbb{S})$, and also in $M \cap L^2(\Omega; \mathbb{S}) = S$ because (3.4) implies that $\operatorname{div} \sigma$ is well-defined in the sense of distributions. This completes the verification that $\mathcal{L}$ is m-dissipative.

We may therefore apply the Hille–Yosida theorem, and obtain the following result. Given $\sigma_0 \in S$, $v_0 \in \mathring{H}^1(\Omega; \mathbb{V})$, and $f \in W^{1,1}([0, T_0]; \mathbb{V})$, there exist

$$\sigma \in C^0([0, T_0]; S) \cap C^1([0, T_0]; L^2(\Omega; \mathbb{S})),$$
$$v \in C^0([0, T_0]; \mathring{H}^1(\Omega; \mathbb{V})) \cap C^1([0, T_0]; V),$$

satisfying the evolution equations (3.1) and assuming the given initial data.

Now we describe a weak formulation of (3.1) with weak symmetry of stress. We assume that $\sigma_0 = C\epsilon(u_0)$ for some $u_0 \in \mathring{H}^1(\Omega; \mathbb{V})$. If we define

$$u(t) = u_0 + \int_0^t v(s)\, ds, \tag{3.5}$$

then, using $A\dot{\sigma} = \epsilon(v)$ and the fundamental theorem of calculus, we get $A\sigma = \epsilon(u)$. If we set

$$r = \operatorname{skw} \operatorname{grad} u, \tag{3.6}$$

then $\dot{r} = \operatorname{skw} \operatorname{grad} v$. Integrating the second equation of (3.1) by parts with the boundary conditions $v \equiv 0$ on $\partial\Omega$, we get $(A\dot{\sigma}, \tau) = (\epsilon(v), \tau) = (\operatorname{grad} v - \dot{r}, \tau) = -(v, \operatorname{div} \tau) - (\dot{r}, \tau)$ for all $\tau \in M$, i.e.,

$$(A\dot{\sigma}, \tau) + (\operatorname{div} \tau, v) + (\dot{r}, \tau) = 0, \qquad \tau \in M. \tag{3.7}$$

From the first equation of (3.1), we get $(\rho \dot{v}, w) - (\operatorname{div} \sigma, w) = (f, w)$ for $w \in V$. Finally, the symmetry of σ gives $(\dot{\sigma}, q) = 0$ for $q \in K$. The equations together constitute our weak formulation with weak symmetry of stress. We seek

$$\begin{gathered} \sigma \in C^0([0, T_0]; M) \cap C^1([0, T_0]; L^2(\Omega; \mathbb{M})), \\ v \in C^1([0, T_0]; V), \quad r \in C^1([0, T_0]; K), \end{gathered} \tag{3.8}$$

such that

$$(A\dot\sigma, \tau) + (\operatorname{div}\tau, v) + (\dot r, \tau) = 0, \qquad \tau \in M, \tag{3.9}$$

$$(\rho \dot v, w) - (\operatorname{div}\sigma, w) = (f, w), \qquad w \in V, \tag{3.10}$$

$$(\dot\sigma, q) = 0, \qquad q \in K, \tag{3.11}$$

with given initial data $(\sigma_0, v_0, r_0) = (C\epsilon(u_0), v_0, \operatorname{skw}\operatorname{grad} u_0)$. We now show that this problem is well-posed.

THEOREM 3.1. *Let* $f \in W^{1,1}([0,T_0];\mathbb{V})$ *and* $u_0, v_0 \in \mathring{H}^1(\Omega;\mathbb{V})$. *Set* $\sigma_0 = C\epsilon(u_0)$, $r_0 = \operatorname{skw}\operatorname{grad} u_0$. *Then the system* (3.8)–(3.11) *has a unique solution assuming the given initial data.*

Proof. By the Hille–Yosida theorem, (3.1) has a solution (σ, v) with the initial data (σ_0, v_0). We define u by (3.5) and r by (3.6). The resulting triple (σ, v, r) then satisfies (3.9)–(3.11) and takes on the desired initial values. We have thus proved existence of a solution.

For uniqueness, suppose that there are two solutions of (3.9)–(3.11) with the same initial data, and denote their difference by (σ^d, v^d, r^d). Then this triple satisfies

$$\begin{aligned} (A\dot\sigma^d, \tau) + (\operatorname{div}\tau, v^d) + (\dot r^d, \tau) &= 0, && \tau \in M,\\ (\rho \dot v^d, w) - (\operatorname{div}\sigma^d, w) &= 0, && w \in V,\\ (\dot\sigma^d, q) &= 0, && q \in K, \end{aligned} \tag{3.12}$$

with $\sigma^d(0)$, $v^d(0)$, and $r^d(0)$ all zero. Now we set $\tau = \sigma^d$, $w = v^d$ in the first two equations and add them. Since $\sigma^d \perp K$ and $\dot r \in K$, we have $(\dot r, \sigma^d) = 0$, so the sum of two equations gives

$$\frac{1}{2}\frac{d}{dt}\|\sigma^d\|_A^2 + \frac{1}{2}\frac{d}{dt}\|v^d\|_\rho^2 = 0.$$

Therefore, $\|\sigma^d(t)\|_A^2 + \|v^d(t)\|_\rho^2 = \|\sigma^d(0)\|_A^2 + \|v^d(0)\|_\rho^2 = 0$, so $\sigma^d \equiv 0 \equiv v^d$. From (3.12), one then sees that $\dot r^d \equiv 0$ as well. Since $r^d(0) = 0$, we have $r^d \equiv 0$, so uniqueness is proved. ∎

We close this section by pointing out the straightforward changes needed to handle mixed displacement–traction boundary conditions in our velocity–stress formulation. Suppose Γ_D and Γ_N are two disjoint open subsets of $\partial\Omega$ with $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$ and Γ_D nonempty. We consider the boundary conditions $v = g$ on Γ_D, $\sigma\nu = \kappa$ on Γ_N, where

$$g : [0, T_0] \times \Gamma_D \to \mathbb{R}^n, \qquad \kappa : [0, T_0] \times \Gamma_N \to \mathbb{R}^n$$

are given. We define $M_{\Gamma_N} = \{\tau \in M \mid \tau\nu = 0 \text{ on } \Gamma_N\}$. Then a velocity–stress formulation with weak symmetry seeks (σ, v, r) satisfying (3.8) with $\sigma\nu = \kappa$ on Γ_N and

$$\begin{aligned} (A\dot\sigma, \tau) + (\operatorname{div}\tau, v) + (\dot r, \tau) &= \int_{\Gamma_D} g \cdot \tau\nu \, ds, && \tau \in M_{\Gamma_N},\\ (\rho \dot v, w) - (\operatorname{div}\sigma, w) &= (f, w), && w \in V,\\ (\dot\sigma, q) &= 0, && q \in K. \end{aligned} \tag{3.13}$$

The initial data must satisfy the compatibility conditions $\sigma_0\nu = \kappa(0)$ on Γ_N and $v_0 = g(0)$ on Γ_D.

4. Semidiscrete error analysis for the AFW and CGG elements. In this section we consider spatial discretization of problem (3.9)–(3.11) with given initial data. We show existence and uniqueness of semidiscrete solutions and discuss the semidiscrete error analysis. Although the main result of this section is stated for the AFW and CGG elements, the results in this section are valid for all elements in Table 2. We will discuss improved results for the Stenberg and GG elements in section 5.

4.1. The semidiscrete problem. Let $M_h \times V_h \times K_h$ be one of the elements in Table 2. Given initial data $(\sigma_{h0}, v_{h0}, r_{h0}) \in M_h \times V_h \times K_h$, the semidiscretization of (3.9)–(3.11) seeks

$$\sigma_h \in C^1([0,T_0]; M_h), \quad v_h \in C^1([0,T_0]; V_h), \quad r_h \in C^1([0,T_0]; K_h), \tag{4.1}$$

satisfying the equations

$$(A\dot{\sigma}_h, \tau) + (\operatorname{div} \tau, v_h) + (\dot{r}_h, \tau) = 0, \qquad \tau \in M_h, \tag{4.2}$$

$$(\rho \dot{v}_h, w) - (\operatorname{div} \sigma_h, w) = (f, w), \qquad w \in V_h, \tag{4.3}$$

$$(\dot{\sigma}_h, q) = 0, \qquad q \in K_h, \tag{4.4}$$

for all time $t \in [0, T_0]$, and assuming the given initial data.

THEOREM 4.1. *The semidiscrete system has a unique solution.*

Proof. Let $\{\phi_i\}$, $\{\psi_i\}$, $\{\chi_i\}$ be bases of M_h, V_h, and K_h, respectively. We use $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{M}$ to denote the matrices whose (i,j)-entries are

$$(A\phi_j, \phi_i), \quad (\operatorname{div} \phi_j, \psi_i), \quad (\phi_j, \chi_i), \quad (\rho\psi_j, \psi_i),$$

respectively. We write $\sigma_h = \sum_i \alpha_i \phi_i$, $v_h = \sum_i \beta_i \psi_i$, $r_h = \sum_i \gamma_i \chi_i$, and set $\zeta_i = (f, \psi_i)$, and use α, β, γ, ζ to denote the corresponding vectors. Then we may rewrite (4.2)–(4.4) in a matrix equation form

$$\begin{pmatrix} \mathscr{A} & 0 & \mathscr{C}^T \\ 0 & \mathscr{M} & 0 \\ \mathscr{C} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} = \begin{pmatrix} 0 & -\mathscr{B}^T & 0 \\ \mathscr{B} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} + \begin{pmatrix} 0 \\ \zeta \\ 0 \end{pmatrix}.$$

The above matrix equation is a linear system of ordinary differential equations. Note that the coefficient matrix on the left-hand side is invertible because $\mathscr{A}$ and $\mathscr{M}$ are positive definite and $\mathscr{C}^T$ is injective from the inf-sup condition (A1). By standard ODE theory (see [11, p. 75]), the matrix equation is well-posed as an initial value problem, so the existence and uniqueness of solutions of (4.2)–(4.4) follow. ∎

Next, we discuss the construction of initial data for the semidiscretization, starting from the initial data $u_0, v_0 \in \mathring{H}^1(\Omega; \mathbb{V})$ for the continuous problem. As initial data for the velocity we simply take

$$v_{h0} = P_h v_0. \tag{4.5}$$

Recall that we obtained initial data for σ and r as $\sigma_0 = C\epsilon(u_0)$ and $r_0 = \operatorname{skw} \operatorname{grad} u_0$. Consequently, $(A\sigma_0, \tau) + (\operatorname{div} \tau, u_0) + (r_0, \tau) = 0$ for $\tau \in M$, and $\sigma_0 \perp q$ for $q \in K$. We compute the initial data for σ_h, u_h, and r_h, from a mixed elliptic problem: $(\sigma_{h0}, u_{h0}, r_{h0}) \in M_h \times V_h \times K_h$ of the system

$$(A\sigma_{h0}, \tau) + (\operatorname{div} \tau, u_{h0}) + (r_{h0}, \tau) = 0, \qquad \tau \in M_h, \tag{4.6}$$

$$(\operatorname{div} \sigma_{h0}, w) = (\operatorname{div} \sigma_0, w), \qquad w \in V_h, \tag{4.7}$$

$$(\sigma_{h0}, q) = 0, \qquad q \in K_h, \tag{4.8}$$

for which we know, by section 2.3, that there exists a unique solution and we have the error estimate

$$\|\sigma_0 - \sigma_{h0}, r_0 - r_{h0}\| \le ch^m \|\sigma_0, r_0\|_m, \qquad 1 \le m \le \begin{cases} k & \text{AFW, CGG,} \\ k+1 & \text{Stenberg, GG.} \end{cases} \tag{4.9}$$

4.2. Decomposition of semidiscrete errors. For the error analysis, we follow a standard approach: representatives of (σ, v, r) are used to split the semidiscrete error into the projection error and the approximation error, and bounds are obtained by a priori error analysis.

We now state the main convergence result for the AFW and CGG elements.

THEOREM 4.2. *Let (M_h, V_h, K_h) be the AFW or CGG elements in Table* 2 *of order $k \ge 1$, and let m be a real number such that $1 \le m \le k$. Suppose that $\sigma, v, r \in W^{1,1}([0, T_0]; H^m)$, and let (σ_h, v_h, r_h) be the solution of* (4.1)–(4.4) *with initial data $(\sigma_{h0}, v_{h0}, r_{h0})$ defined as in* (4.5)–(4.8). *Then we have*

$$\|\sigma - \sigma_h, v - v_h, r - r_h\|_{L^\infty L^2} \le ch^m \|\sigma, v, r\|_{W^{1,1} H^m},$$

where c depends on the compliance tensor A and the lower and upper bounds of the mass density ρ_0, ρ_1.

For our error analysis, we denote the semidiscrete errors, i.e., the difference of the exact solution (σ, v, r) and the semidiscrete solution (σ_h, v_h, r_h), by

$$e_\sigma = \sigma - \sigma_h, \quad e_v = v - v_h, \quad e_r = r - r_h.$$

Then, by taking the differences of (3.9)–(3.11) and (4.2)–(4.4), we get

$$(A\dot{e}_\sigma, \tau) + (\operatorname{div} \tau, e_v) + (\dot{e}_r, \tau) = 0, \qquad \tau \in M_h, \tag{4.10}$$

$$(\rho \dot{e}_v, w) - (\operatorname{div} e_\sigma, w) = 0, \qquad w \in V_h, \tag{4.11}$$

$$(\dot{e}_\sigma, q) = 0, \qquad q \in K_h. \tag{4.12}$$

Recall that $\tilde{\Pi}_h$ is the weakly symmetric elliptic projection in Lemma 2.1 and P_h, P_h' are the orthogonal L^2 projections onto V_h and K_h, respectively. We decompose the semidiscrete errors (e_σ, e_v, e_r) into

$$e_\sigma = e_\sigma^P + e_\sigma^h := (\sigma - \tilde{\Pi}_h \sigma) + (\tilde{\Pi}_h \sigma - \sigma_h), \tag{4.13}$$

$$e_v = e_v^P + e_v^h := (v - P_h v) + (P_h v - v_h), \tag{4.14}$$

$$e_r = e_r^P + e_r^h := (r - P_h' r) + (P_h' r - r_h). \tag{4.15}$$

We call the e^P terms the projection errors and the e^h terms the approximation errors, respectively. We shall prove Theorem 4.2 by bounding the projection errors in section 4.3 and the approximation errors in section 4.4. First, we remark that

$$\begin{aligned} (\operatorname{div} \tau, e_v^P) &= 0, \qquad \tau \in M_h, \\ (\operatorname{div} e_\sigma^P, w) &= 0, \qquad w \in V_h, \end{aligned} \tag{4.16}$$

as follows from (A0) in section 2.3 and (2.16).

4.3. Projection error estimates for the AFW and CGG elements. A priori estimates of the $L^\infty L^2$ norms of the projection errors follow from the approximability of $M_h \times V_h \times K_h$.

THEOREM 4.3. *There exists a constant $c > 0$ such that*

$$\|e_\sigma^P\| \le ch^m \|\sigma\|_{H^m}, \quad 1 \le m \le k, \tag{4.17}$$

$$\|e_v^P\| \le ch^m \|v\|_{H^m}, \quad 0 \le m \le k, \tag{4.18}$$

$$\|e_r^P\| \le ch^m \|r\|_{H^m}, \quad 0 \le m \le k, \tag{4.19}$$

at each time $t \in [0, T_0]$. Furthermore, similar inequalities hold with σ, v, and r, replaced by their time derivatives.

Proof. For any $t \in [0, T_0]$ and $1 \le m \le k$, by (2.17) and (2.11), we have

$$\|e_\sigma^P(t)\| = \|\sigma(t) - \tilde{\Pi}_h \sigma(t)\| \le c\|\sigma(t) - \Pi_h \sigma(t)\| \le ch^m \|\sigma(t)\|_m,$$

and (4.17) is proved. Similarly, from definitions of e_v^P and e_r^P, we have

$$\|e_v^P(t)\| \le ch^m \|v(t)\|_m, \qquad \|e_r^P(t)\| \le ch^m \|r(t)\|_m$$

for any $t \in [0, T_0]$, $0 \le m \le k$. The same argument applies to time derivatives of the projection errors because the projections $\tilde{\Pi}_h$, P_h, P_h' commute with time differentiation. ∎

4.4. Approximation error estimates for the AFW and CGG elements. Now we estimate the $L^\infty L^2$ norms of the approximation errors.

THEOREM 4.4. *For $1 \le m \le k$,*

$$\|e_\sigma^h, e_v^h, e_r^h\|_{L^\infty L^2} \le ch^m \|\sigma, v, r\|_{W^{1,1} H^m}, \tag{4.20}$$

where c depends on ρ_0, ρ_1, and A.

Proof. The proof is based on two estimates:

$$\|e_\sigma^h, e_v^h\|_{L^\infty L^2} \le ch^m (\|\sigma_0, r_0\|_m + \|\dot\sigma, \dot v, \dot r\|_{L^1 H^m}), \tag{4.21}$$

$$\|e_r^h\|_{L^\infty L^2} \le c\|e_\sigma^h, e_\sigma^P, e_r^P\|_{L^\infty L^2} \tag{4.22}$$

for $1 \le m \le k$. Theorem 4.4 follows from these estimates and Theorem 4.3, since $\|\sigma, r\|_{L^\infty H^m} \le c\|\sigma, r\|_{W^{1,1} H^m}$ by Sobolev embedding.

To prove (4.21)–(4.22), we first remark that $\sigma_{h0} \perp K_h$ from (4.8) and $\tilde{\Pi}_h \sigma_0 \perp K_h$ from the definition of $\tilde{\Pi}_h$, and so $e_\sigma^h(0) \perp K_h$. Similarly, $\dot\sigma_h \perp K_h$ from (4.4), and $\tilde{\Pi}_h \dot\sigma \perp K_h$ from the definition of $\tilde{\Pi}_h$ and its commutativity with time differentiation, so $\dot e_\sigma^h \perp K_h$. Combining these facts, we deduce that $e_\sigma^h \perp K_h$, as well. To show (4.21), we rewrite (4.10)–(4.11), using the notation in (4.13)–(4.15) and the reductions in (4.16), as

$$(A\dot e_\sigma^h, \tau) + (\operatorname{div} \tau, e_v^h) + (\dot e_r^h, \tau) = -(A\dot e_\sigma^P, \tau) - (\dot e_r^P, \tau), \qquad \tau \in M_h, \tag{4.23}$$

$$(\rho \dot e_v^h, w) - (\operatorname{div} e_\sigma^h, w) = -(\rho \dot e_v^P, w), \qquad w \in V_h. \tag{4.24}$$

We take $\tau = e_\sigma^h$, $w = e_v^h$ in the above two equations, add them, and use the fact that $e_\sigma^h \perp \dot e_r^h$ from $\dot e_r^h \in K_h$, obtaining

$$\frac{1}{2}\frac{d}{dt}\|e_\sigma^h\|_A^2 + \frac{1}{2}\frac{d}{dt}\|e_v^h\|_\rho^2 = -(A\dot e_\sigma^P, e_\sigma^h) - (\dot e_r^P, e_\sigma^h) - (\rho \dot e_v^P, e_v^h). \tag{4.25}$$

Bounding the right-hand side of this inequality using the Cauchy–Schwarz inequality and the bounds on A and ρ, we get

$$\frac{1}{2}\frac{d}{dt}(\|e_\sigma^h\|_A^2 + \|e_v^h\|_\rho^2) \le c\|\dot e_\sigma^P, \dot e_r^P, \dot e_v^P\|\,(\|e_\sigma^h\|_A^2 + \|e_v^h\|_\rho^2)^{\frac{1}{2}}. \tag{4.26}$$

Dividing both sides by $(\|e_\sigma^h\|_A^2 + \|e_v^h\|_\rho^2)^{1/2}$ and integrating in time on $[0,t]$, we have

$$\left(\|e_\sigma^h(t)\|_A^2 + \|e_v^h(t)\|_\rho^2\right)^{\frac{1}{2}} \le \left(\|e_\sigma^h(0)\|_A^2 + \|e_v^h(0)\|_\rho^2\right)^{\frac{1}{2}} + c\int_0^t \|\dot e_\sigma^P, \dot e_r^P, \dot e_v^P\|\,ds. \tag{4.27}$$

Since A is coercive and ρ has a positive lower bound, in order to establish (4.21), it suffices to show that the right-hand side of (4.27) is bounded by $ch^m(\|\sigma_0, r_0\|_m + \|\dot\sigma, \dot v, \dot r\|_{L^1H^m})$. For the integral term, this follows directly from Theorem 4.3. We also have that $e_v^h(0) = 0$, from the choice of v_{h0}. Finally, we use the boundedness of A and ρ, the triangle inequality, (4.9), (2.17), (2.11), to get

$$\|e_\sigma^h(0)\|_A \le c(\|\sigma_{h0} - \sigma_0\| + \|\sigma_0 - \tilde\Pi_h\sigma_0\|) \le ch^m\|\sigma_0, r_0\|_m \tag{4.28}$$

for $1 \le m \le k$. This completes the proof of (4.21).

To complete the proof of the theorem, we now verify (4.22). Since $\sigma = C\epsilon(u)$ and $r = \operatorname{skw}\operatorname{grad} u$, we have

$$(A\sigma, \tau) + (u, \operatorname{div}\tau) + (r, \tau) = 0, \qquad \tau \in M,$$

and therefore $(A\sigma, \tau) + (r, \tau) = 0$ for $\tau \in M$ divergence-free. Similarly, from (4.6) we have $(A\sigma_{h0}, \tau) + (r_{h0}, \tau) = 0$ for $\tau \in M_h$ divergence-free. Subtracting, we see that $(Ae_\sigma(0), \tau) + (e_r(0), \tau) = 0$ for such τ. Next, we may take a divergence-free τ in (4.10) for $\tau \in M_h$ to find that $(A\dot e_\sigma, \tau) + (\dot e_r, \tau)$ vanishes as well. Combining, we conclude that

$$(Ae_\sigma, \tau) + (e_r, \tau) = 0, \quad \tau \in M_h, \quad \operatorname{div}\tau = 0,$$

or, equivalently,

$$(e_r^h, \tau) = -(A(e_\sigma^h + e_\sigma^P), \tau) + (e_r^P, \tau), \quad \tau \in M_h, \quad \operatorname{div}\tau = 0$$

for all $t \in [0, T_0]$. Now fix t, and choose $\tau \in M_h$ such that $\operatorname{div}\tau = 0$, $(\tau, e_r^h(t)) = \|e_r^h(t)\|^2$, and $\|\tau\| \le c\|e_r^h(t)\|$, which is possible by the stability condition (A1) in section 2.2. It follows that $\|e_r^h(t)\| \le c\|e_\sigma^h(t), e_\sigma^P(t), e_r^P(t)\|$, from which (4.22) follows. ◻

Combining Theorems 4.3 and 4.4, we complete the proof of Theorem 4.2, the error estimates for the semidiscrete solutions.

4.5. Robustness for nearly incompressible materials. Throughout this section, we assume that the elastic medium is homogeneous and isotropic, i.e., the compliance tensor A has the form (2.2) with Lamé coefficients μ and λ which are constant. We continue to consider homogeneous displacement boundary conditions. In nearly incompressible elastic materials, λ is very large, and, in the incompressible limit, $\lambda = +\infty$. Many standard discretizations of elasticity suffer from locking, which means that the errors, while they decay with the mesh size, grow as λ increases. A robust or locking-free method is one in which the error estimates hold uniformly as $\lambda \to +\infty$. In contrast to many displacement methods, mixed methods for stationary elasticity problems are typically locking-free (see [4, 8]). In this section, we show that our mixed method for linear elastodynamics is likewise free of locking. Again, we focus on semidiscretization in space, which is the essential aspect. For an analysis taking into account temporal discretization, we refer the reader to [20].

We require the following lemmas, proved in [4]. Let $\tau^D := \tau - (1/n)\operatorname{tr}(\tau)I$ denote the deviatoric part of τ in $L^2(\Omega;\mathbb{M})$.

LEMMA 4.5. *Let $\tau \in M$ satisfy $\int_\Omega \operatorname{tr}(\tau)\,dx = 0$. Then the estimate*

$$\|\tau\| \le c(\|\tau^D\| + \|\operatorname{div}\tau\|_{-1}) \tag{4.29}$$

holds with $c > 0$ independent of τ.

LEMMA 4.6. *For $\tau \in L^2(\Omega;\mathbb{M})$ and A of the form in (2.2), the inequality*

$$\|\tau^D\|^2 \le c\|\tau\|_A^2 \tag{4.30}$$

holds with c depending only on μ and n.

THEOREM 4.7. *Let $M_h \times V_h \times K_h$ be one of the elements in Table 2 of order $k \ge 1$ and assume that A has the form of (2.2) with μ and λ constant. We assume that the exact solution σ, v, and r belong to $W^{2,1}H^k$. Then there exists a constant $c > 0$ independent of λ such that*

$$\|v - v_h\|_{L^\infty L^2} \le ch^k\|\sigma, v, r\|_{W^{1,1}H^k}, \tag{4.31}$$

$$\|\sigma - \sigma_h\|_{L^\infty L^2} \le ch^k\|\sigma, v, r\|_{W^{2,1}H^k}. \tag{4.32}$$

Proof. The projection error estimates in Theorem 4.3 certainly hold with a constant c independent of λ, because $\tilde{\Pi}_h$, P_h, P_h' do not depend on λ. Furthermore, the inequality $\|e_\sigma^P\|_A \le c\|e_\sigma^P\|$ holds uniformly in λ, since A remains uniformly bounded as $\lambda \to +\infty$.

The proof is based on the following estimates, in which the constant c does not depend on λ:

$$(\|e_\sigma^h(t)\|_A^2 + \|e_v^h(t)\|_\rho^2)^{\frac{1}{2}} \le ch^k\|\sigma, v, r\|_{W^{1,1}H^k}, \tag{4.33}$$

$$\|e_\sigma(t)\| \le c(\|e_\sigma(t)\|_A + \|\operatorname{div} e_\sigma(t)\|_{-1}), \tag{4.34}$$

$$\|\operatorname{div} e_\sigma(t)\|_{-1} \le c(\|\dot{e}_v(t)\| + h^k\|\sigma(t)\|_k), \tag{4.35}$$

$$\|\dot{e}_v^h(t)\| \le ch^k\|\sigma, v, r\|_{W^{2,1}H^k}. \tag{4.36}$$

We first show that (4.31) and (4.32) follow from these estimates. The estimate (4.31) is a consequence of (4.33), the estimate on $\|e_v^P\|$ in Theorem 4.3, and the triangle inequality. To show (4.32), observe that (4.34), (4.35), and the triangle inequality give

$$\|e_\sigma(t)\| \le c(\|e_\sigma^h(t)\|_A + \|e_\sigma^P(t)\|_A + \|\dot{e}_v^h(t)\| + \|\dot{e}_v^P(t)\| + h^k\|\sigma(t)\|_k).$$

Then (4.32) is obtained by (4.33), (4.36), and Theorem 4.3.

To prove (4.33), observe that $\operatorname{tr}(\dot{e}_r^P) = 0$ because $\dot{e}_r^P$ is skew-symmetric, so $\dot{e}_r^P = A(2\mu\dot{e}_r^P)$ holds for A of the form (2.2). We may therefore rewrite (4.25) as

$$\frac{1}{2}\frac{d}{dt}(\|e_\sigma^h\|_A^2 + \|e_v^h\|_\rho^2) = -(A(\dot{e}_\sigma^P + 2\mu\dot{e}_r^P), e_\sigma^h) - (\rho\dot{e}_v^P, e_v^h),$$

and repeating the argument in (4.25)–(4.27), we have

$$(\|e_\sigma^h(t)\|_A^2 + \|e_v^h(t)\|_\rho^2)^{\frac{1}{2}} \le (\|e_\sigma^h(0)\|_A^2 + \|e_v^h(0)\|_\rho^2)^{\frac{1}{2}} + \int_0^t (\|\dot{e}_\sigma^P + 2\mu\dot{e}_r^P\|_A^2 + \|\dot{e}_v^P\|_\rho^2)^{\frac{1}{2}}\, ds.$$

Since A is uniformly bounded in λ, (4.33) follows from (4.28), $e_v^h(0) = 0$, Theorem 4.3, and Sobolev embedding $\|\sigma, r\|_{L^\infty H^k} \le c\|\sigma, r\|_{W^{1,1}H^k}$.

To show (4.34), it is enough to show that $\int_\Omega \operatorname{tr}(e_\sigma(t))\, dx = 0$ by Lemmas 4.5 and 4.6. For $\tau = I$ in (4.10), satisfying $\operatorname{div}\tau = 0$ and $(\dot{e}_r, \tau) = 0$ due to the skew-symmetry of $\dot{e}_r$, we have $(A\dot{e}_\sigma(t), I) = 0$ for $t \in [0, T_0]$. From (4.6) we see that $(Ae_\sigma(0), I) = 0$, whence $(Ae_\sigma(t), I) = 0$ for all $t \in [0, T_0]$. By the form of A in (2.2),

$$\int_\Omega \operatorname{tr}(e_\sigma(t))\, dx = (e_\sigma(t), I) = (2\mu + n\lambda)(Ae_\sigma(t), I) = 0.$$

For (4.35), by the triangle inequality,

$$\|\operatorname{div} e_\sigma(t)\|_{-1} \le \|\operatorname{div} e_\sigma^h(t)\|_{-1} + \|\operatorname{div} e_\sigma^P(t)\|_{-1} \le \|\operatorname{div} e_\sigma^h(t)\| + \|\operatorname{div} e_\sigma^P(t)\|_{-1},$$

so we only estimate $\|\operatorname{div} e_\sigma^h(t)\|$ and $\|\operatorname{div} e_\sigma^P(t)\|_{-1}$, separately. In (4.24), $\operatorname{div} e_\sigma^h(t) = P_h(\rho\dot{e}_v(t))$, so $\|\operatorname{div} e_\sigma^h(t)\| \le c\|\dot{e}_v(t)\|$. For the estimate of $\|\operatorname{div} e_\sigma^P(t)\|_{-1}$ it is enough to show $\|\operatorname{div} e_\sigma^P(t)\|_{-1} \le ch\|\operatorname{div} e_\sigma^P(t)\|$ because

$$\|\operatorname{div} e_\sigma^P(t)\| = \|\operatorname{div}\sigma(t) - P_h \operatorname{div}\sigma(t)\| \le ch^{k-1}\|\sigma(t)\|_k, \qquad k \ge 1.$$

For $w \in \mathring{H}^1(\Omega; \mathbb{V})$ let $\bar{w}$ denote the L^2-orthogonal projection of w into the space of $\mathbb{V}$-valued piecewise constant functions associated to the triangulation $\mathcal{T}_h$. By the definition of the $\|\cdot\|_{-1}$ norm and the orthogonality $\operatorname{div} e_\sigma^P \perp V_h$,

$$\|\operatorname{div} e_\sigma^P(t)\|_{-1} = \sup_{w\in\mathring{H}^1(\Omega;\mathbb{V})} \frac{(\operatorname{div} e_\sigma^P(t), w)}{\|w\|_1} = \sup_{w\in\mathring{H}^1(\Omega;\mathbb{V})} \frac{(\operatorname{div} e_\sigma^P(t), w - \bar{w})}{\|w\|_1}.$$

By the Cauchy–Schwarz and Poincaré inequalities,

$$|(\operatorname{div} e_\sigma^P(t), w - \bar{w})| \le ch\|\operatorname{div} e_\sigma^P(t)\|\|w\|_1$$

holds, and it gives $\|\operatorname{div} e_\sigma^P(t)\|_{-1} \le ch\|\operatorname{div} e_\sigma^P(t)\|$ with the previous identity.

For (4.36) we will show a stronger result which is similar to (4.33) for $\dot{e}_\sigma^h$ and $\dot{e}_v^h$. If we use the energy estimate argument, presented in (4.23)–(4.27), for time derivatives of (4.23) and (4.24) with $\tau = \dot{e}_\sigma^h$ and $w = \dot{e}_v^h$, then

$$(\|\dot{e}_\sigma^h(t)\|_A^2 + \|\dot{e}_v^h(t)\|_\rho^2)^{\frac{1}{2}} \le (\|\dot{e}_\sigma^h(0)\|_A^2 + \|\dot{e}_v^h(0)\|_\rho^2)^{\frac{1}{2}} + c\int_0^t (\|\ddot{e}_\sigma^P, \ddot{e}_r^P\|_A^2 + \|\ddot{e}_v^P\|_\rho^2)^{\frac{1}{2}}\, ds.$$

The integral term is handled by Theorem 4.3 with $ch^k\|\sigma, v, r\|_{W^{2,1}H^k}$. To estimate $(\|\dot{e}_\sigma^h(0)\|_A^2 + \|\dot{e}_v^h(0)\|_\rho^2)^{1/2}$, take $t = 0$ in (4.23), (4.24), and use $A(2\mu\dot{e}_r^P(0)) = \dot{e}_r^P(0)$ to have

$$\begin{aligned}(A\dot{e}_\sigma^h(0), \tau) + (\operatorname{div}\tau, e_v^h(0)) + (\dot{e}_r^h(0), \tau) &= -(A(\dot{e}_\sigma^P(0) + 2\mu\dot{e}_r^P(0)), \tau), \qquad &\tau \in M_h,\\ (\rho\dot{e}_v^h(0), w) - (\operatorname{div} e_\sigma^h(0), w) &= -(\rho\dot{e}_v^P(0), w), &w \in V_h.\end{aligned}$$

Recall that $e_v^h(0) = \operatorname{div} e_\sigma^h(0) = 0$ from the choice of v_{h0} in (4.5) and the property of σ_{h0} in (4.7). Furthermore, $(\dot e_r^h(0), \dot e_\sigma^h(0)) = 0$ because $\dot e_\sigma^h(0) \perp K_h$. Thus, taking $\tau = \dot e_\sigma^h(0)$, $w = \dot e_v^h(0)$, and adding the above equations, we have

$$\|\dot e_\sigma^h(0)\|_A^2 + \|\dot e_v^h(0)\|_\rho^2 = -(A(\dot e_\sigma^P(0) + 2\mu \dot e_r^P(0)), \dot e_\sigma^h(0)) - (\rho \dot e_v^P(0), \dot e_v^h(0)).$$

By the Cauchy–Schwarz inequality and Theorem 4.3,

$$(\|\dot e_\sigma^h(0)\|_A^2 + \|\dot e_v^h(0)\|_\rho^2)^{\frac{1}{2}} \le ch^k \|\dot\sigma(0), \dot r(0), \dot v(0)\|_k,$$

and (4.36) follows from $\|\sigma, v, r\|_{W^{1,\infty}H^k} \le c\|\sigma, v, r\|_{W^{2,1}H^k}$. ∎

5. Improved error analysis for the Stenberg and GG elements. The AFW elements have the simplest shape functions of those shown in Table 2, in that they use the space $\mathcal{P}_k$ for stress shape functions, without any additional functions, and for the displacement and rotation shape functions they use $\mathcal{P}_{k-1}$. The Stenberg and GG elements maintain the space $\mathcal{P}_{k-1}$ for the displacement, but use $\mathcal{P}_k$ for the rotation r, and a space somewhat larger than $\mathcal{P}_k$ for the stress. For these elements we can prove one higher order of convergence for σ and r than is obtained by the AFW and CGG elements with the same displacement space. Moreover, a better numerical solution of u can be obtained for these elements via a local postprocessing.

5.1. Improved a priori error estimates. Since the error analysis for the GG and Stenberg elements parallels that for the AFW and CGG elements, we avoid repetition and focus only on the steps that require modification. While the convergence theory for the AFW and CGG elements required only that the density ρ be bounded above and below, in order to obtain the improved estimates for the Stenberg and GG elements, we require that the density have bounded derivatives, at least on each element separately (it may jump across element boundaries). More precisely, letting grad_h denote the piecewise gradient operator adapted to the triangulation $\mathcal{T}_h$, we require that

$$\|\rho\|_{W_h^{1,\infty}} := \|\rho\|_{L^\infty} + \|\operatorname{grad}_h \rho\|_{L^\infty} < \infty. \tag{5.1}$$

Theorem 5.1 gives the main result for the Stenberg and GG elements from Table 2.

THEOREM 5.1. *Let (M_h, V_h, K_h) be the Stenberg or GG elements of order $k \ge 1$. Suppose that*

$$\sigma, r \in W^{1,1}([0, T_0]; H^m), \qquad v \in W^{1,1}([0, T_0]; H^{m-1}) \tag{5.2}$$

for some integer m with $1 \le m \le k+1$, that (5.1) *holds, and that the initial data is chosen by* (4.5)–(4.8). *Then the semidiscrete solution (σ_h, v_h, r_h) in* (4.2)–(4.4) *satisfies*

$$\|\sigma - \sigma_h, P_h v - v_h, r - r_h\|_{L^\infty L^2} \le ch^m(\|\sigma, r\|_{W^{1,1}H^m} + \|\rho\|_{W_h^{1,\infty}} \|v\|_{W^{1,1}H^{m-1}}), \tag{5.3}$$

where c depends on A and ρ_0.

Note that, in this theorem, m may be as large as $k+1$, while in Theorem 4.2, $m \le k$. When $m = k+1$, the estimate (5.3) shows that v_h is *superclose* to $P_h v$; that is, they are nearer to each other than either is to v. As we show in the next section, this can be exploited to define a higher order approximation to u via a local postprocess.

To prove the theorem, we decompose the errors into $(e^P_\sigma, e^P_v, e^P_r)$ and $(e^h_\sigma, e^h_v, e^h_r)$, as in (4.13)–(4.15), and estimate the two contributions separately.

THEOREM 5.2. *Under the hypotheses of Theorem* 5.1 *the following estimates hold:*

$$\|e^P_\sigma\|_{L^\infty L^2} \le ch^m \|\sigma\|_{L^\infty H^m}, \qquad 1 \le m \le k+1, \tag{5.4}$$

$$\|e^P_v\|_{L^\infty L^2} \le ch^m \|v\|_{L^\infty H^m}, \qquad 0 \le m \le k, \tag{5.5}$$

$$\|e^P_r\|_{L^\infty L^2} \le ch^m \|r\|_{L^\infty H^m}, \qquad 0 \le m \le k+1. \tag{5.6}$$

Furthermore, similar inequalities hold with σ, v, r replaced by their time derivatives.

The proof is similar to that of Theorem 4.3 and so will be omitted. Note that a better approximation (5.6) in K_h is obtained because the shape functions of K_h for the Stenberg and GG elements of order k are one degree higher than the ones for the AFW and CGG elements of order k.

Now we prove a priori estimates of the approximation errors.

THEOREM 5.3. *Under the hypotheses of Theorem* 5.1,

$$\|e^h_\sigma, e^h_v, e^h_r\|_{L^\infty L^2} \le ch^m (\|\sigma, r\|_{W^{1,1}H^m} + \|\rho\|_{W^{1,\infty}_h} \|v\|_{W^{1,1}H^{m-1}}) \tag{5.7}$$

holds for $1 \le m \le k+1$ with c depending on A and ρ_0.

Proof. Arguing as in the proof of Theorem 4.4, we obtain (4.25). Let $\bar\rho$ be the L^2-orthogonal projection of ρ into the space of piecewise constant functions associated to the triangulation $\mathcal{T}_h$. Then $\bar\rho \dot e^P_v$ is L^2-orthogonal to V_h, and therefore $(\rho \dot e^P_v, e^h_v) = ((\rho - \bar\rho)\dot e^P_v, e^h_v)$, so we may obtain from (4.25) that

$$\frac{1}{2}\frac{d}{dt}(\|e^h_\sigma\|^2_A + \|e^h_v\|^2_\rho) \le c\|\dot e^P_\sigma, \dot e^P_r, (\rho-\bar\rho)\dot e^P_v\| \, (\|e^h_\sigma\|^2_A + \|e^h_v\|^2_\rho)^{\frac{1}{2}},$$

which leads to

$$(\|e^h_\sigma(t)\|^2_A + \|e^h_v(t)\|^2_\rho)^{\frac{1}{2}} \le (\|e^h_\sigma(0)\|^2_A + \|e^h_v(0)\|^2_\rho)^{\frac{1}{2}} + c\int_0^t \|\dot e^P_\sigma, \dot e^P_r, (\rho-\bar\rho)\dot e^P_v\| ds.$$

The integral $\int_0^T \|\dot e^P_\sigma, \dot e^P_r\| ds$ and the term $\|e^h_\sigma(0)\|_A$ may be bounded as before, and again, $e^h_v(0) = 0$ because of our choice of initial data. By the Hölder inequality, we have

$$\|(\rho - \bar\rho)\dot e^P_v\| \le \|\rho - \bar\rho\|_{L^\infty} \|\dot e^P_v\| \le ch\|\rho\|_{W^{1,\infty}_h} \|\dot e^P_v\|.$$

Combining these estimates, we obtain the bound on e^h_σ and e^h_v in (5.7). The bound on e^h_r then follows just as in Theorem 4.4. ∎

Theorem 5.1 follows from Theorems 5.7 and 5.2. Note that the bound on $P_h v - v_h = e^h_v$ comes directly from (5.7).

5.2. Postprocessing. Let V_h^* be the space of (possibly discontinuous) piecewise polynomials associated to $\mathcal{T}_h$ of degree k (one degree higher than for V_h), and denote by $\tilde V_h$ the orthogonal complement of V_h in V_h^*. Denote by P_h^* and $\tilde P_h$ the L^2-orthogonal projections onto V_h^* and $\tilde V_h$, respectively. With (σ_h, v_h, r_h) the semidiscrete solution and u_h defined by

$$u_h(t) = u_{h0} + \int_0^t v_h(s)\, ds, \tag{5.8}$$

we define $u_h^* \in V_h^*$ at each time $t \in [0, T_0]$ by

$$(5.9) \qquad (\operatorname{grad}_h u_h^*, \operatorname{grad}_h w) = (A\sigma_h + r_h, \operatorname{grad}_h w), \qquad w \in \tilde{V}_h,$$

$$(5.10) \qquad (u_h^*, w) = (u_h, w), \qquad w \in V_h.$$

Note that V_h^* is a discontinuous piecewise polynomial space, so u_h^* can be computed elementwise at relatively little computational cost.

THEOREM 5.4. *Let* (σ_h, v_h, r_h) *be the semidiscrete solution for the Stenberg or GG method of order* $k \geq 1$, *and let* u_h *be defined by* (5.8) *with* u_{h0} *chosen so that* $\|P_h u_0 - u_{h0}\| \leq ch^{k+1}$ (*e.g.,* $u_{h0} = P_h u_0$). *Let* u_h^* *be defined by* (5.9)–(5.10). *Then*

$$(5.11) \qquad \|u - u_h^*\|_{L^\infty L^2} \leq ch^{k+1} \|\sigma, v, r\|_{W^{1,1} H^{k+1}}$$

holds with c *depending on* A, ρ_0, $\|\rho\|_{W_h^{1,\infty}}$.

The proof of this theorem is similar to the postprocessing of stationary elasticity problem in [17]. A detailed proof can be found in [20].

6. Time discretization.

We have thus far only considered semidiscretization in space, where the main novelty of this paper lies. In this section, we carry out the a priori error analysis of a fully discrete solution of (3.9)–(3.11) and discuss its implementation with the hybridization technique in [13, 2]. For brevity, we restrict ourselves to the case of the AFW elements and the Crank–Nicolson time discretization. However, the same approach can be adapted to the other elements in Table 2 (with modifications as for the semidiscrete error analysis in section 4) and other stable time discretization schemes for linear hyperbolic systems such as the RadauIIA scheme which will be illustrated in Example 7.3.

6.1. Analysis of fully discrete solutions.

Suppose that $T_0 = N\Delta t$ for $\Delta t > 0$ and a positive integer N, and set $t_j = j\Delta t$ for $j = 0, 1, \ldots, N$. For a continuous function g defined on $[0, T_0]$, we define $g^j = g(t_j)$ and $g^{j+1/2} = g(t_j + \Delta t/2)$. For example, σ^j and $e_\sigma^{P,j}$ denote $\sigma(t_j)$ and $e_\sigma^P(t_j)$, respectively. For a sequence g^j we define

$$(6.1) \qquad \partial_t g^{j+\frac{1}{2}} = \frac{g^{j+1} - g^j}{\Delta t}, \qquad \hat{g}^{j+\frac{1}{2}} = \frac{g^{j+1} + g^j}{2}.$$

Note that $\hat{g}^{j+1/2} \neq g^{j+1/2}$ in general.

Let $(\Sigma^0, V^0, R^0) \in M_h \times V_h \times K_h$ be the initial data and denote by (Σ^j, V^j, R^j) the fully discrete approximation of (σ, v, r) at time t_j. In the Crank–Nicolson scheme, $(\Sigma^{j+1}, V^{j+1}, R^{j+1})$ is defined inductively for $0 \leq j \leq N-1$ by

$$(6.2) \qquad \left(A\partial_t \Sigma^{j+\frac{1}{2}}, \tau\right) + (\operatorname{div} \tau, \hat{V}^{j+\frac{1}{2}}) + (\partial_t R^{j+\frac{1}{2}}, \tau) = 0, \qquad \tau \in M_h,$$

$$(6.3) \qquad \left(\rho \partial_t V^{j+\frac{1}{2}}, w\right) - (\operatorname{div} \hat{\Sigma}^{j+\frac{1}{2}}, w) = (\hat{f}^{j+\frac{1}{2}}, w), \qquad w \in V_h,$$

$$(6.4) \qquad (\partial_t \Sigma^{j+\frac{1}{2}}, q) = 0, \qquad q \in K_h.$$

This system can be rewritten as

$$\begin{aligned}\big(A\Sigma^{j+1},\tau\big)+\frac{\Delta t}{2}(\operatorname{div}\tau,V^{j+1})+(R^{j+1},\tau)&\\ =(A\Sigma^{j},\tau)-\frac{\Delta t}{2}(\operatorname{div}\tau,V^{j})+(R^{j},\tau),&\\ (\rho V^{j+1},w)-\frac{\Delta t}{2}(\operatorname{div}\Sigma^{j+1},w)=(\rho V^{j},w)+\frac{\Delta t}{2}(\operatorname{div}\Sigma^{j},w)+\Delta t(\hat f^{j+\frac12},w),&\\ (\Sigma^{j+1},q)=(\Sigma^{j},q)&\end{aligned}$$

for $(\tau,w,q)\in M_h\times V_h\times K_h$. This may be viewed as a square system of linear equations for the unknowns $(\Sigma^{j+1},V^{j+1},R^{j+1})$ in terms of Σ^j, V^j, R^j, f^j, and f^{j+1}. To see that it is nonsingular, we assume that all of the latter vanish and show that then Σ^{j+1}, V^{j+1}, and R^{j+1} must vanish. If we take $\tau=\Sigma^{j+1}$, $w=V^{j+1}$, $q=-R^{j+1}$ in the above equations and add them together, then we have $(A\Sigma^{j+1},\Sigma^{j+1})+(\rho V^{j+1},V^{j+1})=0$, so $\Sigma^{j+1}=0$, $V^{j+1}=0$. This implies that $(R^{j+1},\tau)=0$ for all $\tau\in M_h$, and so, by (A1), $R^{j+1}=0$.

We now state the main result of this section.

THEOREM 6.1. *Let $M_h\times V_h\times K_h$ be the AFW elements of degree $k\ge 1$, and let (Σ^j,V^j,R^j) be defined by the Crank–Nicolson scheme* (6.2)–(6.4). *Assuming that*

$$\sigma,v,r\in W^{1,1}([0,T_0];H^m)\cap W^{3,1}([0,T_0];L^2),\tag{6.5}$$

and that the initial data (Σ^0,V^0,R^0) satisfies the conditions (4.5) *and* (4.9), *we have*

$$\|\sigma^j-\Sigma^j,v^j-V^j,r^j-R^j\|\le c(\Delta t^2+h^m)\|\sigma,v,r\|_{W^{1,1}H^m\cap W^{3,1}L^2},\tag{6.6}$$

with $c>0$ which depends on A, ρ_0, and ρ_1 but is independent of h and Δt.

As in the semidiscrete error analysis, we based the proof on a decomposition of errors

$$E^j_\sigma:=\sigma^j-\Sigma^j=(\sigma^j-\sigma_h^{P,j})+(\sigma_h^{P,j}-\Sigma^j)=:e_\sigma^{P,j}+\theta^j_\sigma,\tag{6.7}$$

$$E^j_v:=v^j-V^j=(v^j-v_h^{P,j})+(v_h^{P,j}-V^j)=:e_v^{P,j}+\theta^j_v,\tag{6.8}$$

$$E^j_r:=r^j-R^j=(r^j-r_h^{P,j})+(r_h^{P,j}-R^j)=:e_r^{P,j}+\theta^j_r.\tag{6.9}$$

We already proved error bounds for the projection errors (e^P_σ,e^P_v,e^P_r) in the semidiscrete error analysis, so Theorem 6.1 follows from the a priori estimates of $(\theta^j_\sigma,\theta^j_v,\theta^j_r)$ given in the following theorem.

THEOREM 6.2. *Suppose the assumptions of Theorem* 6.1 *hold and that θ^i_σ, θ^i_v, θ^i_r are defined as in* (6.7)–(6.9). *Then there exists a constant $c>0$ such that*

$$\|\theta^i_\sigma,\theta^i_v,\theta^i_r\|\le c(\Delta t^2+h^m)\|\sigma,v,r\|_{W^{1,1}H^m\cap W^{3,1}L^2}\tag{6.10}$$

for $1\le i\le N$, $1\le m\le k$, where the constant c depends on A, ρ_0, ρ_1 but is independent of h and Δt.

Proof. Setting $t=t_j$ and $t=t_{j+1}$ in (3.9)–(3.10) and taking the arithmetic mean we obtain

$$\begin{aligned}(A\hat\sigma^{j+\frac12},\tau)+(\operatorname{div}\tau,\hat v^{j+\frac12})+(\hat r^{j+\frac12},\tau)&=0, && \tau\in M_h,\\ (\rho\hat v^{j+\frac12},w)-(\operatorname{div}\hat\sigma^{j+\frac12},w)&=(\hat f^{j+\frac12},w), && w\in V_h.\end{aligned}$$

Subtracting (6.2)–(6.3) from the above two equations, the difference of equations can be written, using (E_σ, E_v, E_r) in (6.7)–(6.9), as

$$(A(\hat{\sigma}^{j+\frac{1}{2}} - \partial_t \Sigma^{j+\frac{1}{2}}), \tau) + (\operatorname{div} \tau, \hat{E}_v^{j+\frac{1}{2}}) + (\hat{r}^{j+\frac{1}{2}} - \partial_t R^{j+\frac{1}{2}}, \tau) = 0, \qquad \tau \in M_h,$$
$$(\rho \hat{v}^{j+\frac{1}{2}} - \rho \partial_t V^{j+\frac{1}{2}}, w) - (\operatorname{div} \hat{E}_\sigma^{j+\frac{1}{2}}, w) = 0, \qquad w \in V_h.$$

Algebraic manipulations of these with equalities

$$\hat{\sigma}^{j+\frac{1}{2}} - \partial_t \Sigma^{j+\frac{1}{2}} = \hat{\sigma}^{j+\frac{1}{2}} - \partial_t \sigma^{j+\frac{1}{2}} + \partial_t E_\sigma^{j+\frac{1}{2}},$$
$$\hat{r}^{j+\frac{1}{2}} - \partial_t R^{j+\frac{1}{2}} = \hat{r}^{j+\frac{1}{2}} - \partial_t r^{j+\frac{1}{2}} + \partial_t E_r^{j+\frac{1}{2}},$$
$$\hat{v}^{j+\frac{1}{2}} - \partial_t V^{j+\frac{1}{2}} = \hat{v}^{j+\frac{1}{2}} - \partial_t v^{j+\frac{1}{2}} + \partial_t E_v^{j+\frac{1}{2}}$$

yield

$$\begin{aligned}(A\partial_t E_\sigma^{j+\frac{1}{2}}, \tau) + (\operatorname{div} \tau, \hat{E}_v^{j+\frac{1}{2}}) + (\partial_t E_r^{j+\frac{1}{2}}, \tau) \\ = (A(\partial_t \sigma^{j+\frac{1}{2}} - \hat{\sigma}^{j+\frac{1}{2}}), \tau) + (\partial_t r^{j+\frac{1}{2}} - \hat{r}^{j+\frac{1}{2}}, \tau),\end{aligned} \tag{6.11}$$

$$(\rho \partial_t E_v^{j+\frac{1}{2}}, w) - (\operatorname{div} \hat{E}_\sigma^{j+\frac{1}{2}}, w) = (\rho \partial_t v^{j+\frac{1}{2}} - \rho \hat{v}^{j+\frac{1}{2}}, w). \tag{6.12}$$

Considering the decomposition of errors in (6.7)–(6.9) and the reductions due to (4.16), we have

$$\begin{aligned}(A\partial_t \theta_\sigma^{j+\frac{1}{2}}, \tau) + (\operatorname{div} \tau, \hat{\theta}_v^{j+\frac{1}{2}}) + (\partial_t \theta_r^{j+\frac{1}{2}}, \tau) \\ = (A(\omega_1^{j+\frac{1}{2}} + \omega_2^{j+\frac{1}{2}}) + \omega_3^{j+\frac{1}{2}} + \omega_4^{j+\frac{1}{2}}, \tau),\end{aligned} \tag{6.13}$$

$$(\rho \partial_t \theta_v^{j+\frac{1}{2}}, w) - (\operatorname{div} \hat{\theta}_\sigma^{j+\frac{1}{2}}, w) = (\omega_5^{j+\frac{1}{2}} + \omega_6^{j+\frac{1}{2}}, w), \tag{6.14}$$

where

$$\begin{aligned}&\omega_1^{j+\frac{1}{2}} = \partial_t \sigma^{j+\frac{1}{2}} - \hat{\sigma}^{j+\frac{1}{2}}, \quad \omega_2^{j+\frac{1}{2}} = -\partial_t e_\sigma^{P,j+\frac{1}{2}}, \quad \omega_3^{j+\frac{1}{2}} = \partial_t r^{j+\frac{1}{2}} - \hat{r}^{j+\frac{1}{2}}, \\ &\omega_4^{j+\frac{1}{2}} = -\partial_t e_r^{P,j+\frac{1}{2}}, \quad \omega_5^{j+\frac{1}{2}} = \rho(\partial_t v^{j+\frac{1}{2}} - \hat{v}^{j+\frac{1}{2}}), \quad \omega_6^{j+\frac{1}{2}} = -\rho \partial_t e_v^{P,j+\frac{1}{2}}.\end{aligned} \tag{6.15}$$

Letting $\tau = \hat{\theta}_\sigma^{j+1/2}$, $w = \hat{\theta}_v^{j+1/2}$ in (6.13) and (6.14) and adding gives

$$\begin{aligned}&(\|\theta_\sigma^{j+1}\|_A^2 + \|\theta_v^{j+1}\|_\rho^2) - (\|\theta_\sigma^{j}\|_A^2 + \|\theta_v^{j}\|_\rho^2) \\ &= 2\Delta t(A(\omega_1^{j+\frac{1}{2}} + \omega_2^{j+\frac{1}{2}}) + \omega_3^{j+\frac{1}{2}} + \omega_4^{j+\frac{1}{2}}, \hat{\theta}_\sigma^{j+\frac{1}{2}}) + 2\Delta t(\omega_5^{j+\frac{1}{2}} + \omega_6^{j+\frac{1}{2}}, \hat{\theta}_v^{j+\frac{1}{2}}).\end{aligned}$$

If we divide this by $(\|\theta_\sigma^{j+1}\|_A^2 + \|\theta_v^{j+1}\|_\rho^2)^{1/2} + (\|\theta_\sigma^{j}\|_A^2 + \|\theta_v^{j}\|_\rho^2)^{1/2}$ and apply a weighted Cauchy–Schwarz inequality, then we get

$$(\|\theta_\sigma^{j+1}\|_A^2 + \|\theta_v^{j+1}\|_\rho^2)^{\frac{1}{2}} \le (\|\theta_\sigma^{j}\|_A^2 + \|\theta_v^{j}\|_\rho^2)^{\frac{1}{2}} + c\Delta t \sum_{l=1}^{6} \|\omega_l^{j+\frac{1}{2}}\| \tag{6.16}$$

for $0 \le j \le N-1$, with c independent of h, Δt, and j. By induction, (6.16) yields

$$(\|\theta_\sigma^{i}\|_A^2 + \|\theta_v^{i}\|_\rho^2)^{\frac{1}{2}} \le (\|\theta_\sigma^{0}\|_A^2 + \|\theta_v^{0}\|_\rho^2)^{\frac{1}{2}} + c\Delta t \sum_{j=0}^{i-1} \sum_{l=1}^{6} \|\omega_l^{j+\frac{1}{2}}\|, \quad 1 \le i \le N. \tag{6.17}$$

Since A is coercive and $0 < \rho_0 \le \rho$ for a constant ρ_0, there exists $c > 0$ depending only on A and ρ_0 such that

$$\|\theta_\sigma^i\| + \|\theta_v^i\| \le c(\|\theta_\sigma^i\|_A^2 + \|\theta_v^i\|_\rho^2)^{\frac{1}{2}}. \tag{6.18}$$

Note that $(\|\theta_\sigma^0\|_A^2 + \|\theta_v^0\|_\rho^2)^{\frac{1}{2}} = \|\theta_\sigma^0\|_A \le ch^m\|\sigma(0), r(0)\|_m$ holds by an argument similar to that of the semidiscrete error analysis with (4.5) and (4.9) of (Σ^0, V^0, R^0). Hence, if we show

$$c\Delta t \sum_{j=0}^{i-1}\sum_{l=1}^{6} \|\omega_l^{j+\frac{1}{2}}\| \le c(\Delta t^2 + h^m), \quad 1 \le m \le k, \tag{6.19}$$

then the estimate (6.10) for $\|\theta_\sigma^i\| + \|\theta_v^i\|$ is proved by (6.17) and (6.18).

To show (6.19), we first recall Taylor expansions for $g \in C^3([-a, a])$,

$$|g(a) - g(-a) - 2ag'(0)| \le ca^2\|g'''\|_{L^1(-a,a)}, \tag{6.20}$$

$$|g(a) + g(-a) - 2g(0)| \le ca\|g''\|_{L^1(-a,a)}, \tag{6.21}$$

$$|g(a) - g(-a) - a(g'(a) + g'(-a))| \le ca^2\|g'''\|_{L^1(-a,a)}. \tag{6.22}$$

From the definitions of $\omega_l^{j+1/2}$, $l = 1, 3, 5$ in (6.15), we can use (6.22) by substituting a by $\Delta t/2$, 0 by t_j, and g by σ, v, r. Then we have

$$\Delta t\|\omega_1^{j+\frac{1}{2}}\| = \frac{1}{2}\|2\sigma^{j+1} - 2\sigma^j - \Delta t\dot\sigma^{j+1} - \Delta t\dot\sigma^j\| \le c\Delta t^2 \int_{t_j}^{t_{j+1}} \|\dddot\sigma\|\, ds, \tag{6.23}$$

$$\Delta t\|\omega_3^{j+\frac{1}{2}}\| = \frac{1}{2}\|2r^{j+1} - 2r^j - \Delta t\dot r^{j+1} - \Delta t\dot r^j\| \le c\Delta t^2 \int_{t_j}^{t_{j+1}} \|\dddot r\|\, ds, \tag{6.24}$$

$$\Delta t\|\omega_5^{j+\frac{1}{2}}\| = \frac{1}{2}\|\rho(2v^{j+1} - 2v^j - \Delta t\dot v^{j+1} - \Delta t\dot v^j)\| \le c\Delta t^2 \int_{t_j}^{t_{j+1}} \|\dddot v\|\, ds. \tag{6.25}$$

By the definitions of $\omega_l^{j+1/2}$, $l = 2, 4, 6$, in (6.15), and Theorem 4.3, we obtain

$$\Delta t\|\omega_2^{j+\frac{1}{2}}\| = \Delta t\|\partial_t e_\sigma^{P,j+\frac{1}{2}}\| = \left\|\int_{t_j}^{t_{j+1}} \dot e_\sigma^P\, ds\right\| \le ch^m \int_{t_j}^{t_{j+1}} \|\dot\sigma\|_m\, ds, \tag{6.26}$$

$$\Delta t\|\omega_4^{j+\frac{1}{2}}\| = \Delta t\|\partial_t e_r^{P,j+\frac{1}{2}}\| = \left\|\int_{t_j}^{t_{j+1}} \dot e_r^P\, ds\right\| \le ch^m \int_{t_j}^{t_{j+1}} \|\dot r\|_m\, ds, \tag{6.27}$$

$$\Delta t\|\omega_6^{j+\frac{1}{2}}\| = \Delta t\|\rho\partial_t e_v^{P,j+\frac{1}{2}}\| = \left\|\int_{t_j}^{t_{j+1}} \rho\dot e_v^P\, ds\right\| \le ch^m \int_{t_j}^{t_{j+1}} \|\dot v\|_m\, ds \tag{6.28}$$

for $1 \le m \le k$. The estimate (6.19) is obtained by combining (6.23)–(6.28), and as we remarked before, the estimate (6.10) is proved for $\|\theta_\sigma^i, \theta_v^i\|$.

To complete the proof of (6.10), we show an estimate of $\|\theta_r^i\|$. We first note that $(A\sigma^j, \tau) + (r^j, \tau) = 0$ for $0 \le j \le N$ for all $\tau \in M_h$ such that $\operatorname{div}\tau = 0$. In (4.6) for the condition of initial data, we have $(A\Sigma^0, \tau) + (R^0, \tau) = 0$ for $\tau \in M_h$ satisfying $\operatorname{div}\tau = 0$. Combining it with (6.2), we have $(A\Sigma^j, \tau) + (R^j, \tau) = 0$, $0 \le j \le N$, for any $\tau \in M_h$ satisfying $\operatorname{div}\tau = 0$ by induction. Therefore, we have an error equation $(AE_\sigma^j, \tau) + (E_r^j, \tau) = 0$ for $\tau \in M_h$ such that $\operatorname{div}\tau = 0$, which is equivalent to

$$(\theta_r^j, \tau) = -(A(\theta_\sigma^j + e_\sigma^{P,j}), \tau) - (e_r^{P,j}, \tau), \quad \tau \in M_h, \quad \operatorname{div}\tau = 0, \quad 0 \le j \le N.$$

If we take $\tau \in M_h$ such that $\operatorname{div}\tau = 0$, $(\tau, q) = \|\theta_r^j\|^2$, and $\|\tau\| \le c\|\theta_r^j\|$ by (A2), then we obtain

$$\|\theta_r^j\| \le c\|\theta_\sigma^j, e_\sigma^{P,j}, e_r^{P,j}\| \le c(\|\theta_\sigma^j\| + h^m\|\sigma, r\|_{L^\infty H^m}).$$

Combining this with (6.10) for $\|\theta_\sigma^j\|$, the estimate (6.10) for $\|\theta_r^j\|$ is proved. ∎

To obtain an approximation of the displacement from the fully discrete solution, we integrate the velocity approximation using the trapezoidal rule. Let $U^0 \in V_h$ be an approximation of initial displacement $u(0)$. Then we define the approximate displacement at time $t = t_i$ by

$$U^i = U^0 + \Delta t \sum_{j=1}^{i} \frac{V^j + V^{j-1}}{2} = U^0 + \Delta t \sum_{j=1}^{i} \hat{V}^{j+\frac{1}{2}}. \tag{6.29}$$

The next result is an error bound for this approximation.

THEOREM 6.3 (numerical solutions of displacement). *Let $U^0 \in V_h$ be an approximation of initial displacement $u(0)$ with $\|u(0) - U^0\| \le ch^m\|u(0)\|_m$, $1 \le m \le k$. Suppose that U^i is defined by (6.29). Then, for $0 \le i \le N$, $1 \le m \le k$,*

$$\|u^i - U^i\| \le c(h^m + \Delta t^2)(\|u(0)\|_m + \|\sigma, v, r\|_{W^{1,1}H^m \cap W^{3,1}L^2}).$$

Proof. Noting that $u^i = u^0 + \int_0^{t_i} v\, ds$ and using the triangle inequality, we have

$$\|u^i - U^i\| \le \|u(0) - U^0\| + \sum_{j=1}^{i} \left\| \int_{t_{j-1}}^{t_j} v\, ds - \Delta t \hat{V}^{j+\frac{1}{2}} \right\|. \tag{6.30}$$

We recall an error estimate for the trapezoidal rule:

$$\left| \int_{-a}^{a} g(s)\, ds - a(g(a) + g(-a)) \right| \le ca^3\|g''\|_{L^\infty(-a,a)}.$$

Combining the triangle inequality, the above estimate, and $\|v^j - V^j\| \le c(h^m + \Delta t^2)$ in Theorem 6.1 yields

$$\begin{aligned} \left\| \int_{t_{j-1}}^{t_j} v\, ds - \Delta t \hat{V}^{j+\frac{1}{2}} \right\| &\le \left\| \int_{t_{j-1}}^{t_j} v\, ds - \Delta t \hat{v}^{j+\frac{1}{2}} \right\| + \Delta t \|\hat{v}^{j+\frac{1}{2}} - \hat{V}^{j+\frac{1}{2}}\| \\ &\le c\Delta t^3 \|\ddot{v}\|_{L^\infty L^2} + c(h^m + \Delta t^2)\Delta t. \end{aligned} \tag{6.31}$$

The conclusion follows from the assumption of U^0, (6.30), and (6.31). ∎

Remark 6.4. An analogue of Theorem 5.4 holds when the GG or Stenberg elements are used and $\|P_h u(0) - U^0\| \le ch^{k+1}\|u(0)\|_{k+1}$ holds.

6.2. Implementation with the hybridization. Finally, we discuss implementation of the system (6.2)–(6.4) via hybridization as proposed in [12, 2]. The main advantage of this is transforming the saddle point problem to a linear system with a symmetric positive definite matrix such that efficient iterative solvers are available. The hybridization technique was already discussed in [10, 17] for elastostatics and in [7] for elastodynamics, so we only provide a brief overview of its application in our context.

We only consider a model problem. At each time step, the fully discrete scheme requires finding $(\sigma_h, v_h, r_h) \in M_h \times V_h \times K_h$ such that

$$(A\sigma_h, \tau) + (\operatorname{div} \tau, v_h) + (r_h, \tau) = (g, \tau), \qquad \tau \in M_h, \tag{6.32}$$

$$(v_h, w) - (\operatorname{div} \sigma_h, w) = (f, w), \qquad w \in V_h, \tag{6.33}$$

$$(\sigma_h, q) = 0, \qquad q \in K_h. \tag{6.34}$$

Let $\mathcal{E}_h$ denote the codimension 1 skeleton of the triangulation $\mathcal{T}_h$ and set $\Xi = L^2(\mathcal{E}_h; \mathbb{R}^n)$. For the elements in Table 2 of order k, we define

$$\Xi_h = \begin{cases} \{\xi \in \Xi \mid \xi|_e \in \mathcal{P}_k(e; \mathbb{R}^n) \quad \text{for } e \in \mathcal{E}_h\} & \text{for AFW, GG, Stenberg,} \\ \{\xi \in \Xi \mid \xi|_e \in \mathcal{P}_{k-1}(e; \mathbb{R}^n) \quad \text{for } e \in \mathcal{E}_h\} & \text{for CGG.} \end{cases}$$

We also define $M_h' \subset L^2(\Omega; \mathbb{M})$ as the space of piecewise polynomials obtained by breaking the $H(\operatorname{div})$ continuity of M_h. In other words, the restriction of M_h' on each element $T \in \mathcal{T}_h$ coincides with the space of local shape functions of M_h on the same T. Then it is not difficult to see that (6.32)–(6.34) is equivalent to seeking $(\sigma_h', v_h, r_h, \xi_h) \in M_h' \times V_h \times K_h \times \Xi_h$ such that

$$(A\sigma_h', \tau') + (\operatorname{div} \tau, v_h) + (r_h, \tau) + \langle \xi_h, \tau' \rangle = (g, \tau'), \qquad \tau' \in M_h', \tag{6.35}$$

$$(v_h, w) - (\operatorname{div} \sigma_h', w) = (f, w), \qquad w \in V_h, \tag{6.36}$$

$$(\sigma_h', q) = 0, \qquad q \in K_h, \tag{6.37}$$

$$\langle \sigma_h', \eta \rangle = 0, \qquad \eta \in \Xi_h, \tag{6.38}$$

where

$$\langle \xi_h, \tau' \rangle = \sum_{T \in \mathcal{T}_h} \int_{\partial T} \xi_h \cdot \tau' \nu \, ds, \qquad \xi_h \in \Xi_h, \tau' \in M_h',$$

with ν the unit outward normal vector field on ∂T for each $T \in \mathcal{T}_h$. Let $\phi_h = (r_h, \xi_h)$ and consider the matrix equation form of the above system. For simplicity we use σ_h', v_h, r_h, and ξ_h to denote the vectors corresponding to the unknowns. Then the matrix equation is

$$\mathbb{A}\sigma_h' + \mathbb{B}^T v_h + \mathbb{C}^T \phi_h = \mathbb{E} g_h, \tag{6.39}$$

$$\mathbb{D} v_h - \mathbb{B}\sigma_h' = \mathbb{D} f_h, \tag{6.40}$$

$$\mathbb{C}\sigma_h' = 0, \tag{6.41}$$

in which the matrices $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$, $\mathbb{E}$ are defined by the bilinear forms

$$(A\sigma_h', \tau'), \qquad (\operatorname{div} \sigma_h', w), \qquad (\sigma_h', q) + \langle \sigma_h', \eta \rangle, \qquad (v_h, w), \qquad (\sigma_h', \tau')$$

for $\sigma_h', \tau' \in M_h'$, $v_h, w \in V_h$, $q \in K_h$, and $\eta \in \Xi_h$. Since M_h' and V_h do not have any interelement continuity, the matrices $\mathbb{A}$ and $\mathbb{D}$ are block diagonal with symmetric positive definite block matrices, so their inverses can be obtained at cheap computational cost.

Note that $\sigma_h' = -\mathbb{A}^{-1}\mathbb{B}^T v_h - \mathbb{A}^{-1}\mathbb{C}^T \phi_h + \mathbb{A}^{-1}\mathbb{E} g_h$ holds by (6.39). Substituting σ_h' in (6.40), (6.41) with this equality, we have

$$(\mathbb{D} + \mathbb{B}\mathbb{A}^{-1}\mathbb{B}^T) v_h + \mathbb{B}\mathbb{A}^{-1}\mathbb{C}^T \phi_h = \mathbb{B}\mathbb{A}^{-1}\mathbb{E} g_h + \mathbb{D} f_h,$$
$$\mathbb{C}\mathbb{A}^{-1}\mathbb{B}^T v_h + \mathbb{C}\mathbb{A}^{-1}\mathbb{C}^T \phi_h = \mathbb{C}\mathbb{A}^{-1}\mathbb{E} g_h.$$

Using the Schur complement, we have

$$(\mathbb{C}\mathbb{A}^{-1}\mathbb{C}^T - \mathbb{C}\mathbb{A}^{-1}\mathbb{B}^T(\mathbb{D} + \mathbb{B}\mathbb{A}^{-1}\mathbb{B}^T)^{-1}\mathbb{B}\mathbb{A}^{-1}\mathbb{C}^T)\phi_h \\ = \mathbb{C}\mathbb{A}^{-1}\mathbb{E}g_h - \mathbb{C}\mathbb{A}^{-1}\mathbb{B}^T(\mathbb{D} + \mathbb{B}\mathbb{A}^{-1}\mathbb{B}^T)^{-1}(\mathbb{B}\mathbb{A}^{-1}\mathbb{E}g_h + \mathbb{D}f_h),$$

and the coefficient matrix of this system is symmetric positive definite. Since K_h does not have interelement continuity one may reduce this system further to a system which has only ξ_h as its unknown, by eliminating r_h with the Schur complement.

7. Numerical results. In this section, we present some numerical results supporting the analysis above. As the domain we take the unit square $(0,1)\times(0,1)$, and as finite elements we use the AFW element with $k = 2$ in the first two examples, and with $k = 3$ in the third. In all three examples, we take the material to be homogeneous and isotropic with constant density and, for simplicity, we simply set $\mu = \lambda = \rho = 1$. In each example, we use a temporal discretization method with the same order as the spatial discretization and with $\Delta t = h$. All results were implemented using the FEniCS project software [21].

Example 7.1. In the first example, we take a smooth displacement field which satisfies the homogeneous displacement boundary conditions,

$$u(t,x,y) = \begin{pmatrix} \sin(\pi x)\sin(\pi y)\sin t \\ x(1-x)y(1-y)\sin t \end{pmatrix}, \tag{7.1}$$

and define f accordingly. Table 3 displays the error at time $t = 1$, for a sequence of meshes, and the observed rates of convergence. For the numerical method we take the AFW elements with $k = 2$ for spatial discretization, and the Crank–Nicolson scheme with $\Delta t = h$ for time discretization which is also second order. As predicted by Theorem 4.2, the L^2 errors for all variables converge to zero with second order.

Table 3
Errors and observed convergence rates for the test problem with exact solution given in (7.1).

$\frac{1}{h}$	$\Vert\sigma-\sigma_h\Vert$ Error	Order	$\Vert v-v_h\Vert$ Error	Order	$\Vert u-u_h\Vert$ Error	Order	$\Vert r-r_h\Vert$ Error	Order
4	5.73e-02	–	1.03e-02	–	1.61e-02	–	2.42e-02	–
8	1.19e-02	1.99	2.62e-03	1.98	4.06e-03	1.99	6.09e-03	1.99
16	2.78e-03	2.00	6.57e-04	2.00	1.02e-03	2.00	1.52e-03	2.00
32	6.77e-04	2.00	1.64e-04	2.00	2.54e-04	2.00	3.80e-04	2.00
64	1.67e-04	2.00	4.10e-05	2.00	6.35e-05	2.00	9.51e-05	2.00

Example 7.2. In this example, the displacement boundary conditions are inhomogeneous, and so we use the formulation (3.13). We take an exact solution with limited regularity,

$$u(t,x,y) = \begin{pmatrix} (1+t^2)x^\alpha y^2 \\ (1+\cos t)x^2 y^\alpha \end{pmatrix}, \tag{7.2}$$

and again define the load accordingly. The fields v and σ then belong to $H^{\alpha+1/2-\delta}$ and $H^{\alpha-1/2-\delta}$, respectively, for arbitrary $\delta > 0$. Numerical results for several different values of α are shown in Table 4. We see that the convergence rates are somewhat decreased due to the decreased regularity of the solution (but perhaps not as much as might be expected).

Example 7.3. In the final example, we consider a third order method. For spatial discretization we use the AFW method with $k = 3$, and for time discretization we use

Table 4
Order of convergence for the exact solution with displacement as in (7.2) ($\lambda = 1$, $\mu = 1$, $h = \Delta t$, and $T_0 = 1$).

α	$\frac{1}{h}$	$\|\sigma - \sigma_h\|$ Error	Order	$\|v - v_h\|$ Error	Order	$\|u - u_h\|$ Error	Order	$\|r - r_h\|$ Error	Order
2.2	4	4.92e-02	–	4.04e-02	–	4.22e-02	–	1.28e-02	–
	8	2.23e-02	1.14	1.07e-02	1.91	1.04e-02	2.02	4.28e-03	1.58
	16	7.37e-03	1.60	3.67e-03	1.55	2.52e-03	2.05	1.45e-03	1.56
	32	2.37e-03	1.63	1.26e-03	1.54	6.31e-04	2.00	4.18e-04	1.79
	64	7.60e-04	1.64	4.13e-04	1.61	1.58e-04	2.00	1.22e-04	1.78
2.7	4	6.92e-02	–	6.72e-02	–	4.92e-02	–	1.61e-02	–
	8	2.85e-02	1.28	9.20e-03	2.87	1.19e-02	2.05	4.63e-03	1.80
	16	7.50e-03	1.92	1.76e-03	2.39	2.94e-03	2.01	1.21e-03	1.93
	32	1.90e-03	1.98	4.12e-04	2.09	7.34e-04	2.00	3.02e-04	2.01
	64	4.82e-04	1.98	1.08e-04	1.93	1.83e-04	2.00	8.03e-05	1.91
3.2	4	1.14e-01	–	1.05e-01	–	5.70e-02	–	2.55e-02	–
	8	4.41e-02	1.36	1.49e-02	2.81	1.37e-02	2.05	7.65e-03	1.73
	16	1.17e-02	1.91	2.68e-03	2.47	3.42e-03	2.01	2.09e-03	1.87
	32	2.96e-03	1.99	6.07e-04	2.15	8.51e-04	2.00	5.21e-04	2.01
	64	7.39e-04	2.00	1.57e-04	1.95	2.13e-04	2.00	1.33e-04	1.97

Table 5
The Butcher tableau for the 2-stage RadauIIA Runge–Kutta scheme.

1/3	5/12	−1/12
1	3/4	1/4
	3/4	1/4

Table 6
Order of convergence for the exact solution with displacement in (7.1) ($\lambda = 1$, $\mu = 1$, $h = \Delta t$, and $T_0 = 1$). The AFW elements with $k = 3$ and the 2-stage RadauIIA time discretization are used.

$\frac{1}{h}$	$\|\sigma - \sigma_h\|$ Error	Order	$\|v - v_h\|$ Error	Order	$\|u - u_h\|$ Error	Order	$\|r - r_h\|$ Error	Order
4	1.31e-02	–	1.38e-03	–	2.10e-03	–	3.77e-03	–
8	1.02e-03	3.68	1.78e-04	2.96	2.58e-04	3.02	4.18e-04	3.17
16	9.88e-05	3.37	2.23e-05	3.00	3.25e-05	2.99	5.05e-05	3.05
32	1.14e-05	3.12	2.78e-06	3.00	4.09e-06	2.99	6.28e-06	3.01
64	1.40e-06	3.03	3.46e-07	3.00	5.13e-07	2.99	7.85e-07	3.00

the 2-stage RadauIIA method which is a third order implicit Runge–Kutta method with the Butcher tableau shown in Table 5.

In the previous examples, u_h is obtained by a simple numerical time integration of v_h based on the trapezoidal rule. However, the trapezoidal rule gives only second order convergence in Δt, which is lower than the convergence rates of other unknowns. To achieve third order convergence of $\|u - u_h\|$, a numerical integration of v_h, exploiting additional numerical data generated by the RadauIIA method, is needed. In Table 5, the RadauIIA method at the ith time step ($t = i\Delta t$) generates an auxiliary numerical data approximating $\dot{v}((i + 1/3)\Delta t)$, which will be denoted by $V_t^{i+1/3}$. Let V^i be the ith numerical velocity obtained by the RadauIIA method and U^0 be the numerical initial displacement such that $\|u(0) - U^0\| \le ch^3$. Then, regarding Taylor expansion

$$g(a) = g(0) + ag'(0) + \frac{a^2}{2} g''\Big(\frac{a}{3}\Big) + o(a^3),$$

the numerical integration for reconstruction of u_h is inductively defined by

$$U^{i+1} = U^i + \Delta t V^i + \frac{\Delta t^2}{2} V_t^{i+\frac{1}{3}}, \qquad i \geq 0.$$

The numerical results in Table 6 show that the expected third order convergence rates are obtained for all errors.

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