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LOWER SEMICONTINUOUS OBSTACLES FOR THE POROUS MEDIUM EQUATION

RIIKKA KORTE, PEKKA LEHTELÄ, AND STEFAN STURM

Abstract. We deal with the obstacle problem for the porous medium equation in the slow diffusion regime $m > 1$. Our main interest is to treat fairly irregular obstacles assuming only boundedness and lower semicontinuity. In particular, the considered obstacles are not regular enough to work with the classical notion of variational solutions, and a different approach is needed. We prove the existence of a solution in the sense of the minimal supersolution lying above the obstacle. As a consequence, we can show that non-negative weak supersolutions to the porous medium equation can be approximated by a sequence of supersolutions which are bounded away from zero.

1. Introduction

In this paper, we study the obstacle problem for the porous medium equation

$$\partial_t u - \Delta u^m = 0 \quad \text{in } \Omega_T, \tag{1.1}$$

where $\Omega_T = \Omega \times (0, T)$ denotes the space-time cylinder of height $T > 0$ over a bounded open set $\Omega \subset \mathbb{R}^n$. We concentrate on the degenerate regime $m > 1$, also known as the slow diffusion case. Our main goal here is to prove the existence of a solution to the obstacle problem for fairly irregular obstacles which are only bounded and lower semicontinuous. As some general references to the theory of the porous medium equation, we mention [7, 18, 19].

We define the solution to the obstacle problem as the least supersolution above the given obstacle, see Definition 2.5 for the details. The supersolutions we have in mind are defined in terms of a parabolic comparison principle, analogously to supercaloric functions in classical potential theory. For that reason, we refer to them as m-supercaloric functions.

Obstacle problems particularly provide a method of approximation in the following sense. Since (super)solutions are lower semicontinuous by [2, Thm. 1.1], there exists a nondecreasing approximating sequence of smooth functions which can be used as obstacles to get an approximating sequence of supersolutions.

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Generally, in the case of the porous medium equation, the set where the solution vanishes often causes technical difficulties. Therefore, one would like to have a tool of approximation by supersolutions which are bounded away from zero. This issue has already been recognized by DiBenedetto, Gianazza, and Vespri in their discussion of the proof of [8, Thm. V.17.1]. We address this question constructing such an approximating sequence by using perturbed supersolutions as obstacles.

Previously, the obstacle problem for the porous medium equation has usually been studied from the point of view of variational inequalities, see [1, 4, 5, 6]. The existence of such variational solutions was established in [5, Thm. 2.7]. However, this approach requires sufficiently regular obstacles. In particular, the existence of the time derivative of the obstacle is needed to guarantee the existence of variational solutions. Thus, for instance a supersolution is not an admissible obstacle in this context.

A different approach to obstacle problems based on minimal supersolutions above the obstacle has been employed in the case of evolutionary p-Laplace type equations for instance in [13]. In [16, Cor. 3.16], it was established that the notion of variational solution coincides with the least supersolution as long as the involved obstacle possesses adequate regularity. By contrast, the connection between those two concepts is not well understood for the porous medium equation. By [3, Thm. 4.12], we know that the least supersolution is a variational solution, provided that the obstacle is regular enough, whereas the other implication remains an open question.

To the authors' knowledge, the existence question in the sense of least supersolutions has not been addressed in the literature yet. As our main result, Theorem 4.1, we prove the existence of a solution to the obstacle problem for bounded lower semicontinuous obstacles by approximation with continuous obstacles. For the latter, we construct a solution to the obstacle problem by iteratively building a sequence of functions, analogously to the Schwarz alternating method (see [12, 15], for instance). However, following the approach in [13], we need to slightly modify the technique to ensure that the limiting function stays above the obstacle. Note that the function we construct is often called "Balayage" in potential theory.

Finally, as a consequence of our main theorem, we will show in Theorem 5.1 that a bounded m-supercaloric function can be approximated by a nonincreasing sequence of m-supercaloric functions which are bounded away from zero. Our result can be applied to improve earlier Harnack type estimates which were often established under additional assumptions like $u > 0$ or $u \geq \varepsilon > 0$. An approximation tool that allows to bypass such assumptions was previously suggested in [8], however a rigorous proof was missing up to now.

2. Preliminaries

Before we list the definitions and basic tools used in this paper, we clarify the notation. By $U_{t_1,t_2} \Subset \Omega_T$, we describe the fact that the space-time cylinder

$U_{t_1,t_2} = U \times (t_1, t_2)$ is compactly contained in Ω_T, i.e. $\overline{U_{t_1,t_2}} \subset \Omega_T$. The parabolic boundary of U_{t_1,t_2} is

$$\partial_p U_{t_1,t_2} = (\partial U \times (t_1, t_2)) \cup (\overline{U} \times \{t_1\}),$$

and we will write $|U_{t_1,t_2}|$ for the $(n+1)-$dimensional Lebesgue measure of such a set. On the contrary, space-time boxes

$$\prod_{i=1}^{n} (a_i, b_i) \times (t_1, t_2)$$

will usually be called Q. Here, $\prod_{i=1}^{n}(a_i, b_i)$ indicates the set $(a_1, b_1) \times ... \times (a_n, b_n)$. Moreover, we will refer to the superlevel set

$$\{(x,t) \in \Omega_T \colon u(x,t) > \psi(x,t)\}$$

where the function u lies above the obstacle ψ by $\{u > \psi\}$. Finally, for a function u, we will use the abbreviation $u_+ = \max\{u, 0\}$.

Next, we define our notion of weak (super)solutions.

Definition 2.1. A non-negative function $u \colon \Omega_T \to [0, \infty]$ is a *weak supersolution to the porous medium equation* (1.1) if $u^m \in L^2_{\mathrm{loc}}(0, T; H^1_{\mathrm{loc}}(\Omega))$ and u satisfies

$$\iint_{\Omega_T} \Big(- u\partial_t\varphi + \nabla u^m \cdot \nabla\varphi \Big)\, dx\, dt \geq 0 \tag{2.1}$$

for any test function $\varphi \in C_0^\infty(\Omega_T)$ with $\varphi \geq 0$. Similarly, u is a *weak subsolution* if the above inequality holds reversed. Moreover, u is a *weak solution* if it is a weak sub- and supersolution.

As a tool, we need the boundary value problem

$$\begin{cases} \partial_t u - \Delta u^m = 0 & \text{in } \Omega_T, \\ u(\cdot, 0) = g(\cdot, 0) & \text{in } \Omega, \\ u^m - g^m \in L^2(0, T; H_0^1(\Omega)) \end{cases} \tag{2.2}$$

with boundary and initial values determined by a sufficiently regular function g. We continue by giving the rigorous definition of weak solutions to the boundary value problem.

Definition 2.2. A non-negative function $u \in C^0([0,T]; L^{m+1}(\Omega))$ is a *weak solution to the boundary value problem* (2.2) with sufficiently regular boundary and initial values g if $u^m - g^m \in L^2(0, T; H_0^1(\Omega))$ and u satisfies

$$\iint_{\Omega_T} \Big(- u\partial_t\varphi + \nabla u^m \cdot \nabla\varphi \Big)\, dx\, dt = \int_\Omega g(x,0)\varphi(x,0)\, dx$$

for all smooth test functions φ having compact support with respect to space and vanishing at $t = T$.

The next defintion concerns the notion of m-supercaloric functions.

Definition 2.3. A function $u \colon \Omega_T \to [0, \infty]$ is *m-supercaloric* if

(1) u is lower semicontinuous;

(2) u is finite in a dense subset of Ω_T;
(3) u satisfies the following comparison principle in every interior cylinder $U_{t_1,t_2} \Subset \Omega_T$: If $w \in C^0(\overline{U_{t_1,t_2}})$ is a weak solution to the porous medium equation (1.1) in U_{t_1,t_2} and $u \geq w$ on $\partial_p U_{t_1,t_2}$, then $u \geq w$ in U_{t_1,t_2}.

Remark 2.4. The weak (super)solutions according to Definition 2.1 are m-supercaloric. Indeed, the lower semicontinuity follows from [2, Thm. 1.1], and the comparison principle is due to [18, Thm. 6.5]. Moreover, in the locally bounded case, m-supercaloric functions are weak supersolutions (see [9, Thm. 1.3]).

Finally, we state the precise definition of a solution to the obstacle problem.

Definition 2.5. Let $\psi\colon \Omega_T \to [0,\infty)$ be a bounded and lower semicontinuous obstacle. A function $u\colon \Omega_T \to [0,\infty]$ is a *solution to the obstacle problem for the porous medium equation* with obstacle ψ if

(1) $u \geq \psi$ in Ω_T;
(2) u is an m-supercaloric function in Ω_T;
(3) u is the smallest m-supercaloric function in Ω_T which lies above ψ, i. e. if v is an m-supercaloric function in Ω_T with $v \geq \psi$ in Ω_T, then $v \geq u$ in Ω_T.

Remark 2.6. We mention the following properties of solutions to the obstacle problem, which can be deduced from the previous definition.

(i) Note that solutions to the obstacle problem in the above sense are unique by their definition.
(ii) We will prove in Lemma 2.10 that solutions u to the obstacle problem are weak solutions to the porous medium equation in the set $\{u > \psi\}$.
(iii) Due to the boundedness assumption on the obstacle function ψ, we also know that solutions u to the obstacle problem in the sense of the previous definition are locally bounded by $\sup_{\Omega_T} |\psi|$ (see [3, Thm. 3.1] and Lemma 2.10). Therefore, since locally bounded m-supercaloric functions are weak supersolutions by [9, Thm. 1.3], we conclude that solutions to the obstacle problem are in fact weak supersolutions to the porous medium equation in Ω_T, too.

To overcome the fact that constants cannot be added to solutions to the porous medium equation, we need the following result, which can be retrieved from [11, Lemma 3.2].

Lemma 2.7. *Let $g \in C^0(\overline{\Omega_T})$ be such that $g^m \in L^2(0,T;H^1(\Omega))$ and $0 \leq g \leq M$. Suppose that $\varepsilon > 0$ and let u and u_ε be the solutions to the boundary value problem (2.2) with boundary and initial values given by g and $g_\varepsilon = (g^m + \varepsilon^m)^{1/m}$, respectively. Then, we have*

$$\iint_{\Omega_T} (u_\varepsilon - u)(u_\varepsilon^m - u^m)\,dx\,dt \leq \varepsilon^m |\Omega_T|(M+1) + \varepsilon |\Omega_T|(M+1)^m. \tag{2.3}$$

For the interested reader, we mention that the proof utilizes an Oleĭnik type test function

$$\varphi(x,t) = \begin{cases} \int_t^T (u_\varepsilon^m - u^m - \varepsilon^m)\, ds & \text{for } 0 < t < T, \\ 0 & \text{for } t \geq T. \end{cases}$$

We refer to [11] for the actual proof. Observe that the argument does not work for general porous medium type equations with coefficients.

Remark 2.8. The previous lemma implies that, for u_ε and u as above, there exists a (nonrelabeled) subsequence of u_ε such that $u_\varepsilon \to u$ a. e. in Ω_T. This can be deduced by the elementary inequality $|u_\varepsilon - u|^{m+1} \leq (u_\varepsilon - u)(u_\varepsilon^m - u^m)$, which can be found in [5, Cor. 3.11].

Next, we consider the Poisson modification $u_P\colon \Omega_T \to [0,\infty]$ of a locally bounded m-supercaloric function u. It is defined with respect to some (sufficiently regular) space-time cylinder $U_{t_1,t_2} \Subset \Omega_T$ as

$$u_P = \begin{cases} u & \text{in } \Omega_T \setminus \overline{U_{t_1,t_2}}, \\ v & \text{in } \overline{U_{t_1,t_2}}, \end{cases} \tag{2.4}$$

where v is the continuous weak solution in U_{t_1,t_2} with boundary values given by u on $\partial_p U_{t_1,t_2}$. The existence of such a weak solution v will be justified in the following lemma.

Lemma 2.9. *Let u be a locally bounded m-supercaloric function in Ω_T. The Poisson modification u_P exists and it is an m-supercaloric function.*

Proof. First, we show the existence of the Poisson modification by constructing the function v appearing in (2.4). As u is lower semicontinuous, there exists a nondecreasing sequence of smooth functions $\eta_k \geq 0$ converging to u pointwise in Ω_T. Let v_k be the continuous weak solution to the boundary value problem (2.2) in U_{t_1,t_2} with boundary and initial values given by η_k on $\partial_p U_{t_1,t_2}$. By the comparison principle, we obtain $v_k \leq v_{k+1} \leq u$ in U_{t_1,t_2} for every $k \in \mathbb{N}$. Thus, due to the local boundedness of u and the Harnack type convergence theorem [9, Lemma 3.4], $v = \lim v_k$ is a continuous weak solution in U_{t_1,t_2}.

It remains to prove that u_P is m-supercaloric. From the above construction of v, it follows that u_P satisfies the finiteness condition (2) in Definition 2.3. Next, we will argue that u_P is lower semicontinuous. Obviously, it suffices to consider the points $(x_0,t_0) \in \partial U_{t_1,t_2}$. For those, we calculate

$$\liminf_{\substack{(x,t)\to(x_0,t_0) \\ (x,t)\in U_{t_1,t_2}}} u_P(x,t) = \liminf_{\substack{(x,t)\to(x_0,t_0) \\ (x,t)\in U_{t_1,t_2}}} v(x,t) = v(x_0,t_0) = u_P(x_0,t_0).$$

Hence, we can conclude the lower semicontinuity of u_P. Finally, we will show that u_P satisfies the comparison principle. For that purpose, we let $V_{\tau_1,\tau_2} \Subset \Omega_T$ and consider a weak solution $w \in C^0(\overline{V_{\tau_1,\tau_2}})$. Suppose that

$$w \leq u_P \quad \text{on } \partial_p V_{\tau_1,\tau_2}.$$

By construction, we know that $u_P \le u$, which implies

$$w \le u \quad \text{on } \partial_p V_{\tau_1,\tau_2}.$$

Therefore, by the comparison principle for u, we find that $w \le u$ in V_{τ_1,τ_2}. This takes care of the points $(x,t) \in V_{\tau_1,\tau_2} \setminus U_{t_1,t_2}$, and it remains to consider the set $V_{\tau_1,\tau_2} \cap U_{t_1,t_2}$. The previous consideration particularly shows that

$$w \le u = v \quad \text{on } \partial_p U_{t_1,t_2} \cap V_{\tau_1,\tau_2}.$$

On the other hand, we have

$$w \le u_P = v \quad \text{on } \partial_p V_{\tau_1,\tau_2} \cap U_{t_1,t_2}$$

by assumption. Thus, there holds

$$w \le v \quad \text{on } \partial_p (U_{t_1,t_2} \cap V_{\tau_1,\tau_2}),$$

and the comparison principle implies

$$w \le v \quad \text{in } U_{t_1,t_2} \cap V_{\tau_1,\tau_2}.$$

Together, we can conclude $w \le u_P$ in V_{τ_1,τ_2}. □

To complete this section, we establish the following property of solutions to the obstacle problem.

Lemma 2.10. *Let $\psi\colon \Omega_T \to [0,\infty)$ be a bounded and lower semicontinuous obstacle. Suppose that u is a solution to the obstacle problem in Ω_T. Then, u is a weak solution to the porous medium equation in the set $\{u > \psi\}$.*

Proof. Let $(x_0,t_0) \in \{u > \psi\}$. Since the set $\{u > \psi\}$ is open by the lower semicontinuity of u, we can find a number $\lambda > 0$ and a neighbourhood $U_{t_1,t_2} \Subset \{u > \psi\}$ of (x_0,t_0) such that

$$u > \lambda > \psi \quad \text{in } \overline{U_{t_1,t_2}}.$$

Recall that u is locally bounded by Remark 2.6. Therefore, as the set U can be chosen arbitrarily smooth, we may consider the Poisson modification u_P of u with respect to U_{t_1,t_2}, defined as in (2.4). Since $u_P = u$ on $\partial_p U_{t_1,t_2}$, the comparison principle implies

$$u \ge u_P > \lambda > \psi \quad \text{in } U_{t_1,t_2}.$$

By Lemma 2.9, the function u_P is m-supercaloric, hence, we know $u = u_P$ in U_{t_1,t_2} from property (3) in Definition 2.5. Therefore, u is a weak solution to the porous medium equation in U_{t_1,t_2}. Finally, since being a solution is a local property, we conclude that u is a weak solution in $\{u > \psi\}$. □

3. Existence of solutions for continuous obstacles

In this section, we describe how to construct a solution to the obstacle problem with a bounded continuous obstacle. Therefore, we assume throughout this section that $\psi \in C^0(\Omega_T) \cap L^\infty(\Omega_T)$.

Construction 3.1. Our aim is to construct the unique solution to the obstacle problem as the limit of a sequence $(f_j)_{j\in\mathbb{N}_0}$ using a modified Schwarz alternating method. The functions f_j are obtained recursively by solving boundary value problems on a dense countable collection $\mathcal{F} = \{Q_j \subset \Omega_T : j \in \mathbb{N}_0\}$ of space-time boxes Q_j ending at $t = T$. For instance, one could consider the collection

$$\mathcal{F} = \left\{ \prod_{i=1}^{n} (a_i, b_i) \times (t, T) \subset \Omega_T : a_i < b_i,\ a_i, b_i, t \in \mathbb{Q} \right\}.$$

Next, we describe the construction in detail. We choose $f_0 = \psi$ in Ω_T, and, for any $j \in \mathbb{N}_0$, we define f_{j+1} as

$$f_{j+1} = \begin{cases} \max\{g_j, f_j\} & \text{in } Q_j, \\ f_j & \text{in } \Omega_T \setminus Q_j, \end{cases}$$

where g_j is the continuous weak solution to the boundary value problem (2.2) in Q_j with boundary and initial values given by f_j. Roughly, the idea there is to redefine f_j as g_j in the box Q_j like in the Schwarz alternating method. However, in order to guarantee that the functions f_j stay above the obstacle, we modify the approach by taking the maximum. Finally, we denote the pointwise limit by

$$u(x,t) = \lim_{j\to\infty} f_j(x,t). \tag{3.1}$$

The limit exists and is well-defined for any $(x,t) \in \Omega_T$, which will be shown in Lemma 3.3 and Lemma 3.5.

Next, we collect the basic properties of the previous construction in the following two lemmata.

Lemma 3.2. *Let $\psi \in C^0(\Omega_T) \cap L^\infty(\Omega_T)$. Then, the following statements for the generating sequence $(f_j)_{j\in\mathbb{N}_0}$ from Construction 3.1 hold:*

(i) *The sequence $(f_j)_{j\in\mathbb{N}_0}$ is nondecreasing. More precisely, we have*

$$f_{j+1} \geq f_j \geq \psi \quad \text{in } \Omega_T$$

for any $j \in \mathbb{N}_0$.

(ii) *The sequence $(f_j)_{j\in\mathbb{N}_0}$ is uniformly bounded. More precisely, we have*

$$|f_j| \leq \sup_{\Omega_T} |\psi| \quad \text{in } \Omega_T$$

for any $j \in \mathbb{N}_0$.

(iii) *The function f_j is continuous for any $j \in \mathbb{N}_0$.*

(iv) *The function f_j is a weak subsolution to the porous medium equation in the set $\{f_j > \psi\}$ for any $j \in \mathbb{N}$.*

Proof. By definition, we have $f_0 = \psi$ and

$$f_{j+1} = \max\{g_j, f_j\} \geq f_j$$

for any $j \in \mathbb{N}_0$. This proves assertion (i). Next, we will show (ii). Obviously, there holds

$$|f_0| \leq \sup_{\Omega_T} |\psi| \quad \text{in } \Omega_T.$$

Suppose now that the claim is true for some $j \in \mathbb{N}_0$. Since the functions f_{j+1} are constructed inductively by solving boundary value problems with boundary and initial values given by f_j, the comparison principle gives (ii). Since the obstacle ψ and the weak solutions g_j used in the construction are continuous, also the functions f_j are continuous as a maximum of continuous functions such that (iii) is proved. Finally, (iv) follows since in the set $\{f_j > \psi\}$, the function f_j is obtained as a maximum of a finite number of subsolutions g_i, $i \in \{0, ..., j-1\}$, and therefore it is a subsolution. □

Lemma 3.3. *Take Construction 3.1 with an obstacle function $\psi \in C^0(\Omega_T) \cap L^\infty(\Omega_T)$. Then, the following statements for the limit function u hold:*

(i) *The limit u exists and satisfies $u \geq \psi$ in Ω_T.*
(ii) *If v is an m-supercaloric function in Ω_T with $v \geq \psi$ in Ω_T, then $v \geq u$ in Ω_T.*
(iii) *The function u is lower semicontinuous and the set $\{u > \psi\}$ is open.*

Proof. Concerning (i), we observe that the limit u exists at every point $(x,t) \in \Omega_T$ since the sequence $(f_j)_{j\in\mathbb{N}_0}$ is nondecreasing and uniformly bounded by Lemma 3.2. The assertion (ii) follows once we have shown that $v \geq f_j$ in Ω_T for any $j \in \mathbb{N}_0$. For $j = 0$, there is nothing to prove since $f_0 = \psi$ in Ω_T. Assume now that there exists some $j \in \mathbb{N}_0$ such that $v \geq f_j$ holds. Then, by construction and the comparison principle, we obtain $v \geq g_j$ in Q_j, where Q_j denotes the space-time box used in the construction. Therefore, we conclude that $v \geq f_{j+1}$ in Ω_T, and induction proves that $v \geq u$ in Ω_T. Moreover, as a limit of a nondecreasing sequence of continuous functions, u is lower semicontinuous. This directly implies the openness of the set $\{u > \psi\}$. □

We proceed by gathering further properties of the limit function u. The next result contains a comparison principle.

Lemma 3.4. *The limit u from (3.1) satisfies the comparison principle in all space-time boxes $Q \subset \Omega_T$, i. e. if $w \in C^0(\overline{Q})$ is a weak solution to the porous medium equation in Q satisfying $w \leq u$ on $\partial_p Q$, then we have $w \leq u$ in Q.*

Proof. We fix a space-time box

$$Q = \prod_{i=1}^{n} (a_i, b_i) \times (t_1, t_2).$$

Let $w \in C^0(\overline{Q})$ be a weak solution in Q with $w \leq u$ on $\partial_p Q$. Further, let $\varepsilon > 0$ and define the sets

$$E_j = \overline{Q} \cap \{f_j > w - \varepsilon\}.$$

We observe that E_j is open with respect to the relative topology by the continuity of f_j and w. Moreover, we have $\partial_p Q \subset \bigcup_j E_j$ since $u \geq w$ on $\partial_p Q$. Therefore, we can choose $j_0 \in \mathbb{N}$ such that

$$f_{j_0} > w - \varepsilon \quad \text{on } \partial_p Q.$$

As the set E_{j_0} is a neighbourhood of $\partial_p Q$ in the relative topology, there exists a number $j_1 \geq j_0$ such that $Q_{j_1} \in \mathcal{F}$ satisfies

$$\partial_p Q_{j_1} \cap \{t < t_2\} \subset E_{j_0} \quad \text{and} \quad Q \setminus E_{j_0} \subset Q_{j_1}.$$

The definition of E_{j_0} and the monotonicity of the sequence f_j guarantee that

$$w < f_{j_0} + \varepsilon \leq f_{j_1} + \varepsilon \quad \text{on } \partial_p Q_{j_1} \cap \{t < t_2\}. \tag{3.2}$$

In order to prove that

$$w \leq f_{j_1+1} \quad \text{in } Q_{j_1} \cap \{t < t_2\},$$

we will show that $w \leq g_{j_1}$. Since g_{j_1} is a weak solution in Q_{j_1} with boundary and initial values given by f_{j_1}, the inequality (3.2) gives us

$$w \leq g_{j_1} + \varepsilon \quad \text{on } \partial_p Q_{j_1} \cap \{t < t_2\}.$$

Let $g_{j_1,\varepsilon}$ be a weak solution in $Q_{j_1} \cap \{t < t_2\}$ with boundary and initial values $g_{j_1} + \varepsilon$. The comparison principle implies

$$w \leq g_{j_1,\varepsilon} \quad \text{in } Q_{j_1} \cap \{t < t_2\}.$$

Therefore, we know

$$(w - g_{j_1})_+ (w^m - g_{j_1}^m)_+ \leq (g_{j_1,\varepsilon} - g_{j_1})(g_{j_1,\varepsilon}^m - g_{j_1}^m).$$

By Lemma 2.7, we conclude

$$\begin{aligned} 0 &\leq \iint_{Q_{j_1} \cap \{t<t_2\}} (w - g_{j_1})_+ (w^m - g_{j_1}^m)_+ \, dx \, dt \\ &\leq \iint_{Q_{j_1} \cap \{t<t_2\}} (g_{j_1,\varepsilon} - g_{j_1})(g_{j_1,\varepsilon}^m - g_{j_1}^m) \, dx \, dt \\ &\leq C(\varepsilon) \end{aligned}$$

with a constant $C(\varepsilon)$ as in (2.3). Letting $\varepsilon \to 0$, we have proven that

$$w \leq g_{j_1} \quad \text{in } Q_{j_1} \cap \{t < t_2\}.$$

Thus, it follows

$$w \leq g_{j_1} \leq f_{j_1+1}$$

and, by construction, also $w \leq u$. □

The comparison principle from Lemma 3.4 allows us to prove that the limit function u is independent of the choice of the collection $\mathcal{F}$ used in Construction 3.1. Thus, u is well-defined.

Lemma 3.5. *Let $\psi \in C^0(\Omega_T) \cap L^\infty(\Omega_T)$. Then, the limit u is unique. In particular, it does not depend on the choice of space-time boxes used in Construction 3.1.*

Proof. Assume that $u^{(1)}$ and $u^{(2)}$ are two limits of the construction with generating functions $(f_j^{(i)})_{j\in\mathbb{N}_0}$ and $(g_j^{(i)})_{j\in\mathbb{N}_0}$ for $i\in\{1,2\}$. Moreover, we denote the corresponding collections of space-time boxes by $\{Q_j^{(i)}\colon j\in\mathbb{N}_0\}$. Obviously, we have

$$u^{(1)}\geq f_0^{(1)}=\psi=f_0^{(2)}.$$

Now, suppose that

$$u^{(1)}\geq f_j^{(2)}\quad\text{in } \Omega_T$$

for some $j\in\mathbb{N}_0$. By construction, the function $g_j^{(2)}$ solves the boundary value problem (2.2) in $Q_j^{(2)}$ with boundary and initial values $f_j^{(2)}$ on $\partial_p Q_j^{(2)}$. Since

$$u^{(1)}\geq f_j^{(2)}=g_j^{(2)}\quad\text{on } \partial_p Q_j^{(2)},$$

the comparison principle from Lemma 3.4 gives us

$$u^{(1)}\geq g_j^{(2)}\quad\text{in } Q_j^{(2)}.$$

Therefore, we have proven that

$$u^{(1)}\geq \max\{g_j^{(2)},f_j^{(2)}\}=f_{j+1}^{(2)}\quad\text{in } Q_j^{(2)},$$

and induction yields $u^{(1)}\geq u^{(2)}$ in Ω_T. Eventually, interchanging the roles of $u^{(1)}$ and $u^{(2)}$, the claim follows. □

We conclude this section by establishing a comparison principle which displays that, if two obstacles $\psi^{(1)}$ and $\psi^{(2)}$ satisfy $\psi^{(1)}\leq\psi^{(2)}$, then, the associated limits are ordered in the same way.

Lemma 3.6. *Suppose that $\psi^{(1)},\psi^{(2)}\in C^0(\Omega_T)\cap L^\infty(\Omega_T)$ are obstacles. If $\psi^{(1)}\leq\psi^{(2)}$ in Ω_T, then the corresponding limits $u^{(1)}$ and $u^{(2)}$ of Construction 3.1 satisfy $u^{(1)}\leq u^{(2)}$ in Ω_T.*

Proof. By Lemma 3.5, the limits $u^{(1)}$ and $u^{(2)}$ do not depend on the choice of the collection $\mathcal{F}$. Thus, we may use the same family $\{Q_j\colon j\in\mathbb{N}_0\}$ of space-time boxes in the constructions of $u^{(1)}$ and $u^{(2)}$. Suppose that $u^{(i)}$ is generated by functions $f_j^{(i)}$ and $g_j^{(i)}$, where $i\in\{1,2\}$. By assumption, we have

$$f_0^{(1)}=\psi_1\leq\psi_2=f_0^{(2)}\quad\text{in } \Omega_T.$$

Therefore, we may proceed by induction to show that the claim holds. Assume that $f_j^{(1)}\leq f_j^{(2)}$ in Ω_T for some $j\in\mathbb{N}_0$. In particular, this implies

$$f_j^{(1)}\leq f_j^{(2)}\quad\text{on } \partial_p Q_j.$$

Hence, by the comparison principle, we know that $g_j^{(1)}\leq g_j^{(2)}$ in Q_j, which in turn shows that $f_{j+1}^{(1)}\leq f_{j+1}^{(2)}$ in Ω_T. By induction, we conclude $u^{(1)}\leq u^{(2)}$ in Ω_T. □

4. Existence of solutions for lower semicontinuous obstacles

In this section, we prove our main result, Theorem 4.1. We remark that, unlike in Section 3, we do not assume continuity for the obstacle ψ anymore.

Theorem 4.1. *Let ψ be a non-negative, bounded, and lower semicontinuous obstacle function in Ω_T. Then, there exists a unique solution u to the obstacle problem in the sense of Definition 2.5.*

Proof. Let ψ be as in the statement of the theorem. There exists a nondecreasing sequence $(\psi_k)_{k\in\mathbb{N}}$ of continuous functions such that

$$\psi_k \to \psi \quad \text{a. e. in } \Omega_T.$$

Our aim is to construct a sequence $(u_k)_{k\in\mathbb{N}}$ of solutions to the obstacle problems with obstacles ψ_k by using the results of Section 3. We will show that the functions u_k converge to a solution u to the obstacle problem with obstacle ψ. In order to argue that the limit function is indeed a solution to the obstacle problem, we have to verify the properties for u which are listed in Definition 2.5.

First, Lemma 3.3 (i) yields $u_k \geq \psi_k$ in Ω_T for any $k \in \mathbb{N}$, and this property persists in the limit. Therefore, we have ensured (1) in Definition 2.5. Next, suppose that v is an m-supercaloric function satisfying

$$v \geq \psi \geq \psi_k \quad \text{in } \Omega_T.$$

Then, by Lemma 3.3 (ii), we know that $u_k \leq v$ in Ω_T. Letting $k \to \infty$ shows that $u \leq v$ in Ω_T, i. e. (3) of Definition 2.5.

It remains to verify (2). For that purpose, we first observe that the local boundedness of u_k follows from Lemma 3.2 (ii). Therefore, Lemma 3.3 (iii) and Lemma 3.4 show that u_k is m-supercaloric for each $k \in \mathbb{N}$. We point out that by [15, Lemma 4.5] (see also [12, Lemma 4.1]), the comparison principle for general subcylinders follows from the one proved for space-time boxes in Lemma 3.4. By the comparison principle from Lemma 3.6, the sequence $(u_k)_{k\in\mathbb{N}}$ is nondecreasing, and therefore, we may apply the Harnack type convergence theorem [10, Prop. 6.8] to conclude that u_k converges to an m-supercaloric function u. Note that the finiteness condition (2) in Definition 2.3 holds due to the boundedness of ψ. This proves that u is a solution to the obstacle problem in the sense of Definition 2.5. □

5. Applications

In Theorem 5.1, we will show that Theorem 4.1 gives us a method of approximating bounded m-supercaloric functions by a nonincreasing sequence of m-supercaloric functions which are bounded away from zero. In a sense, this complements the work of DiBenedetto et al. in [8], where the existence of such a sequence was assumed. Moreover, we discuss the consequences of this result.

Theorem 5.1. *Let $u \geq 0$ be a bounded m-supercaloric function in Ω_T. Then, there exists a sequence of uniformly bounded m-supercaloric functions $\{u_\varepsilon\}_\varepsilon$ satisfying $u_\varepsilon \geq \varepsilon$ in Ω_T for any $\varepsilon > 0$ such that, in the limit $\varepsilon \to 0$, we have*

$$\begin{cases} u_\varepsilon \to u \quad \text{pointwise in } \Omega_T, \\ u_\varepsilon^q \to u^q \quad \text{in } L^p(\Omega_T) \quad \text{for } q > 0,\ p \geq 1, \\ u_\varepsilon^q(\cdot, t_0) \to u^q(\cdot, t_0) \quad \text{in } L^p(\Omega) \quad \text{for } t_0 \in (0,T),\ q > 0,\ p \geq 1, \\ \nabla u_\varepsilon^m \rightharpoonup \nabla u^m \quad \text{weakly in } L^2_{\mathrm{loc}}(\Omega_T, \mathbb{R}^n), \\ \nabla u_\varepsilon^m \to \nabla u^m \quad \text{a. e. in } \Omega_T. \end{cases}$$

Proof. Without loss of generality, we may assume $\varepsilon \in (0,1)$. For such a fixed number ε, we define

$$\psi_\varepsilon = (u^m + \varepsilon^m)^{1/m}.$$

Due to the fact that u is a bounded, lower semicontinuous function in Ω_T, we know that ψ_ε has those properties, too. Therefore, by Theorem 4.1, there exists a solution u_ε to the obstacle problem with obstacle ψ_ε. Since the sequence ψ_ε is bounded and nonincreasing, also u_ε is bounded and nonincreasing. Thus, we conclude the pointwise convergence $u_\varepsilon \to u$ as $\varepsilon \to 0$. Using the dominated convergence theorem, this implies the convergence of u_ε^q in $L^p(\Omega_T)$. The same argument works on time-slices. Now, being a bounded m-supercaloric function, u_ε is also a weak supersolution (see Remark 2.4), and therefore, the Caccioppoli inequality [9, Lemma 2.15] shows that ∇u_ε^m is locally bounded in $L^2(\Omega_T)$ with an estimate independent of ε. Hence, there exists a (nonrelabeled) subsequence of ∇u_ε^m which converges weakly in $L^2_{\mathrm{loc}}(\Omega_T)$. By the arguments of [17, Section 3.6], the above mentioned convergences imply the pointwise convergence of the gradients ∇u_ε^m a. e. in Ω_T. We remark that weak solutions to doubly nonlinear equations were treated there so that the porous medium equation is covered as a special case. Moreover, the same reasoning applies also for weak supersolutions. □

This result has several useful consequences. For instance, it gives us a method to improve reverse Hölder inequalities and weak Harnack estimates (see [14, Thm. 1.1] and also [8, Thm. 17.1]). In such estimates, assumptions of the form $u > 0$ or $u \geq \varepsilon > 0$ are often imposed. Usually, they are of merely technical nature as they arise from the application of a Moser type iteration method, where Caccioppoli estimates with possibly negative powers are utilized. Nevertheless, since constants cannot be added to (super)solutions to the porous medium equation, bypassing these assumptions is more involved in comparison to p-Laplace type equations. In [8], an approximation by supersolutions which are bounded away from zero is suggested to overcome this issue. However, to the authors' knowledge, no method for such an approximation has been presented until now.

By applying Theorem 5.1, we can show that the weak Harnack estimate [14, Thm. 1.1] holds for non-negative supersolutions to the porous medium equation. More precisely, an approximation by supersolutions $u_\varepsilon \geq \varepsilon > 0$

allows us to discard the assumption $u > 0$. As a consequence, the estimate takes the following form.

Corollary 5.2. *Let u be a non-negative weak supersolution to (1.1) in*

$$\Omega_T \supset B(x_0, 8\rho) \times (0, T).$$

Then, there exist constants C_1, $C_2 > 0$ depending on m and n such that, for almost every $t_0 \in (0,T)$, the following inequality holds

$$\fint_{B(x_0,\rho)} u(x,t_0)\,dx \leq \left(\frac{C_1\rho^2}{T-t_0}\right)^{1/(m-1)} + C_2 \operatorname*{ess\,inf}_V u,$$

where

$$V = B(x_0, 4\rho) \times (t_0 + \tau/2,\, t_0 + \tau) \quad \textit{and}$$

$$\tau = \min\left\{T - t_0,\, C_1\rho^2 \left(\fint_{B(x_0,\rho)} u(x,t_0)\,dx\right)^{-(m-1)}\right\}.$$

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