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SHEAR-CORRECTED REISSNER-MINDLIN PLATE MODEL

J. Freund*

Aalto University, School of Engineering, Department of Mechanical Engineering

Puumiehenkuja 5 A, 00076 Aalto, Finland*Tel: +358 50 430 0665

Email address: jouni.freund@aalto.fi (J. Freund)

Abstract

A new shear-corrected Reissner-Mindlin model is presented. The method reduces the modeling error of the classical model without affecting the form of the classical plate equations. Therefore, implementation on an existing software for the Reissner-Mindlin plate model is simple. The principle of virtual work and an explicit set of kinematic and kinetic assumptions is used in derivation. There, displacement assumption of the classical model is enhanced by a warping part which is eliminated to end up with equations for the classical part only. The equations differ from the classical ones in shear correction factors, modification in the source term, and stress expressions.

Keywords: Reissner-Mindlin, refined plate model, shear correction factor, layered plate

1 Introduction

The classical Kirchhoff and Reissner-Mindlin plate models assume linearity of displacement for the in-plane component, the through-the-thickness one being constant. The rather severe assumption is the key to a simple and practical model, but it is also the source for the modeling error compared to the full 3D elasticity model. In particularly, prediction of the transverse stress components is poor. With plates composed of orthotropic layers, classical models tend also to overestimate the stiffness and thereby underestimate the transverse displacement. The obvious way to reduce the modeling error of the classical models is to modify the assumptions of classical models into a less restrictive direction.

The number of different refinements of the classical plate models in literature is impressive [1], [2], [3], [4], [5], and [6]. The equivalent single-layer theories in [7], [8], and [9] use a cubic or a higher order polynomial displacement in the thin direction without introducing additional unknowns over the classical model. Although modification of a kinematic assumption into a less restrictive direction reduces the modeling error, lack of regularity of the exact solution, e.g., due to discontinuities in material properties, may become a restrictive factor with this refinement type as finding a good continuous polynomial representation of a discontinuous function is difficult. Examples are the various layered sandwich and plywood plates. Layer-wise theories in [10], [2], [3], and [11] overcome this by the use of piecewise continuous displacements adapted to the plate structure. The choice is very good for structures of a few layers but becomes impractical when the number of layers is large. The layer-wise theory in [12] combines physical plies into numerical layers to reduce the computational cost. In literature, less attention is paid to the kinetic assumption of the classical models. For example, refinements in [8] and [11] assume that the transverse normal stress vanishes whereas the assumption is rejected in [13] and [14].

The shear-corrected Reissner-Mindlin plate model of this work aims to reduce the modeling error of the classical plate model without adding to the complexity of plate equations. Following the convention of the engineering model, the principle of virtual work and an explicit set of assumptions are used in derivation. Displacement assumption consists of the classical and warping parts both considered as unknowns of a plate problem. Although plate equations for the two parts are connected through the stress expression, elimination of the warping part from equations of the classical part turns out to be possible. The outcome differs from the classical plate equations in shear correction factors, modification in the source term, and stress expressions. The integral expression for the two-by-two shear correction matrix implied by the model is new. Reduction in the modeling error is indicated by comparing the exact model, classical model, and shear-corrected model solutions to isotropic, sandwich, and plywood plates.

2 Plate models

In what follows, the domain occupied by plate is denoted by $\Omega \times Z \subset \mathbb{R}^3$ in which $\Omega \subset \mathbb{R}^2$ is the reference plane and $Z = (z_-, z_+)$ the domain for the thin directions to be called also the transverse domain. Top and bottom surfaces and the boundary surface of the plate are denoted by $\Omega \times \partial Z \subset \mathbb{R}^3$ and $\partial\Omega \times Z \subset \mathbb{R}^3$, respectively. The external loading of the plate in the transverse direction consists of the volume force g, the surface load q_+ acting on the top surface, and surface load q_- acting on the bottom surface. The interpolant q of the surface loads, which satisfies $q = q_+$ at z_+ and $q = q_-$ at z_-, is linear in the transverse coordinate. To shorten the expressions, the boundary conditions on the boundary surface are assumed to be homogeneous.

According to the principle of virtual work, the in-plane displacement components u_x, u_y and the transverse displacement u_z of a linearly elastic body satisfy

$$\delta W = -\int_{\Omega \times Z} \left[\{\delta\varepsilon\}^{\mathrm{T}} \{\sigma\} + \{\delta\gamma\}^{\mathrm{T}} \{\tau\} + \delta\varepsilon_z \sigma_z \right] dV + \int_{\Omega \times Z} \delta u_z g dV + \int_{\Omega \times \partial Z} \delta u_z q dA = 0 \tag{1}$$

for all variations δu_x, δu_y, and δu_z. In the virtual work expression, the in-plane stress and in-plane strain are defined by

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix}, \ \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_x / \partial y + \partial u_y / \partial x \end{Bmatrix}, \tag{2}$$

and the transverse shear stress and the transverse shear strain by

$$\{\tau\} = \begin{Bmatrix} \tau_x \\ \tau_y \end{Bmatrix}, \ \text{and} \ \{\gamma\} = \begin{Bmatrix} \gamma_{zx} \\ \gamma_{zy} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial z + \partial u_z / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \end{Bmatrix}. \tag{3}$$

The model for an orthotropic linearly elastic material is given by (Appendix A)

$$\{\sigma\} = [E]\{\varepsilon\} + \{C\}\,\varepsilon_z, \tag{4}$$

$$\{\tau\} = [G]\{\gamma\}, \tag{5}$$

$$\sigma_z = \{C\}^{\mathrm{T}}\{\varepsilon\} + E_z \varepsilon_z \tag{6}$$

in which the elasticity matrices

$$[E] = \begin{bmatrix} E_1 & E_4 & E_5 \\ E_4 & E_2 & E_6 \\ E_5 & E_6 & E_3 \end{bmatrix}, [G] = \begin{bmatrix} G_1 & G_3 \\ G_3 & G_2 \end{bmatrix}, \text{and} \ \{C\} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \end{Bmatrix} \tag{7}$$

are assumed to depend on the transverse coordinate only. The principle of virtual work and the expressions in Eqs. (1) to (7) comprise the exact plate model whose solution will be called as the exact solution to the plate problem.

2.1 Equivalent single-layer theory

In equivalent single-layer theories, one tacitly assumes that displacement is smooth so a polynomial series representation, like a truncated Taylor series, with respect to the transverse coordinate suffices. The kinematic assumption is the key to the simple Kirchhoff and Reissner-Mindlin models and the refined third order model by [7] and [8]. In terms of the in-plane translation components, transverse translation component, and the rotation components

$$\{u\}_0 = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix},\ w(x,y),\text{ and } \{\omega\}_0 = \begin{Bmatrix} \theta(x,y) \\ -\phi(x,y) \end{Bmatrix}, \tag{8}$$

the in-plane displacement assumptions of these models can be written as

$$\{u\} = \{u\}_0 - \{\nabla w\}_0 z, \tag{9}$$

$$\{u\} = \{u\}_0 + \{\omega\}_0 z, \tag{10}$$

$$\{u\} = \{u\}_0 + \{\omega\}_0 z - \frac{4z^3}{3t^2}(\{\omega\}_0 + \{\nabla w\}_0) \tag{11}$$

in which $\{u\} = \{u_x\ u_y\}^{\mathrm{T}}$. Transverse displacement $u_z = w$ of the models coincides. Expression of the Reissner-Mindlin model in Eq. (10) follows from the assumption that the normal line segments to the reference plane $z = 0$ move as rigid bodies in deformation. Expression of the Kirchhoff model in Eq. (9) assumes additionally that the line segments remain normal to the reference plane in deformation, which implies the Kirchhoff constraint $\{\omega\}_0 = -\{\nabla w\}_0$ on Eq. (10). Expression of Reddy's third order model [6] in Eq. (11) contains an additional warping term, which is chosen so that the transverse shear stress vanishes at the top and bottom surfaces $z = \pm t/2$.

All these models assume additionally that $\sigma_z = 0$ which is taken into account by eliminating the transverse normal strain ε_z from constitutive Eq. (4) with Eq. (6). This gives the modified form of Eq. (4)

$$\{\sigma\} = ([E] - \frac{1}{E_z}\{C\}\{C\}^{\mathrm{T}})\{\varepsilon\} = [E]_p\{\varepsilon\} \tag{12}$$

in which $[E]_p$ is the well-known elasticity matrix for plane stress.

2.2 Shear-corrected plate model

The shear-corrected plate model suggested here is motivated by the displacement assumption in Eq. (11). However, the model aims to reduce the modeling error further by considering the warping term as an unknown to be determined as part of a plate problem. To keep the mathematical form of the classical Reissner-Mindlin equations, the derivatives of the warping term with respect to the in-plane coordinates are assumed to be negligible in the displacement gradient expression. The warping part is also restricted by orthogonality to the classical part which means that the integral of their product vanishes when calculated over the thickness. The condition ensures uniqueness of the warping part.

In more precise forms, the kinematic assumptions of the shear-corrected model are

$$\{u\}=\{u\}_0+\{\omega\}_0 z+\{\Delta u\} \text{ and } u_z=w+\Delta u_z, \tag{13}$$

$$\{\varepsilon\}=\{\varepsilon\}_0+z\{\kappa\}_0,\ \{\gamma\}=\{\gamma\}_0+\frac{\partial}{\partial z}\{\Delta u\}, \text{ and } \varepsilon_z=\frac{\partial}{\partial z}\Delta u_z, \tag{14}$$

$$\int_Z \{\Delta u\}\,dz=0,\ \int_Z z\{\Delta u\}\,dz=0, \text{ and } \int_Z \Delta u_z dz=0 \text{ on } \Omega \tag{15}$$

in which $\{\varepsilon\}_0$, $\{\kappa\}_0$ and $\{\gamma\}_0$ are the well-known strain measures of the classical Reissner-Mindlin model. The strain expressions in Eq. (14) follow the from displacement assumption in Eq. (13) and definitions in Eq. (2) when derivatives of the warping part, except those with respect to the transverse coordinate, are omitted. It is important that the assumption does not restrict the warping displacement to being a function of the transverse coordinate only.

In contrast to the classical models, the shear-corrected model does not use any kinetic assumptions and the constitutive equations

$$\{\sigma\}=[E](\{\varepsilon\}_0+z\{\kappa\}_0)+\{C\}\frac{\partial}{\partial z}\Delta u_z, \tag{16}$$

$$\{\tau\}=[G](\{\gamma\}_0+\frac{\partial}{\partial z}\{\Delta u\}), \tag{17}$$

$$\sigma_z = \{C\}^{\mathrm{T}}(\{\varepsilon\}_0 + z\{\kappa\}_0) + E_z \frac{\partial}{\partial z}\Delta u_z \tag{18}$$

follow directly from Eqs. (4), (5), (6) and strain expression in Eq. (14). It is noteworthy that the first term of Eq. (16) is not the expression for the classical model, as the elasticity matrix is not modified to take into account the kinetic assumption of the theory. Elimination of the transverse warping displacement from Eqs. (16) and (18) gives the form

$$\{\sigma\} = [E]_p(\{\varepsilon\}_0 + z\{\kappa\}_0) + \frac{1}{E_z}\{C\}\sigma_z \tag{19}$$

which indicates that the transverse normal stress affects the in-plane components of stress, whereas the effect is not present in the classical model.

3 Plate equations

Derivation of the plate equations for the shear-corrected model follows the usual lines of dimension reduction. After substituting the displacement and strain expressions in Eqs. (13) and (14) into the virtual work expression in Eq. (1), the principle of virtual work, integration by parts, and the fundamental lemma of variation calculus imply the equilibrium equations and the pairs of work-conjugate boundary conditions.

With the Lagrange multiplier method for the orthogonality conditions in Eq. (15), the virtual work expression takes the form $\delta W = \delta W^{\mathrm{int}} + \delta W^{\mathrm{ext}} + \delta W^{\mathrm{lag}}$ in which

$$\delta W^{\mathrm{int}} = -\int_{\Omega}\int_{Z}\left[\{\delta\varepsilon\}_0^{\mathrm{T}}\{\sigma\} + \{\delta\kappa\}_0^{\mathrm{T}} z\{\sigma\} + (\{\delta\gamma\}_0 + \frac{\partial}{\partial z}\{\delta\Delta u\})^{\mathrm{T}}\{\tau\} + \frac{\partial}{\partial z}\delta\Delta u_z\sigma_z\right]dzdA, \tag{20}$$

$$\delta W^{\mathrm{ext}} = \int_{\Omega}\delta w(\int_{Z} gdz + \sum\nolimits_{\partial Z} q)dA + \int_{\Omega}(\int_{Z}\delta\Delta u_z gdz)dA + \int_{\Omega}(\sum\nolimits_{\partial Z}\delta\Delta u_z q)dA, \tag{21}$$

$$\delta W^{\mathrm{lag}} = \delta\int_{\Omega}\left[\{\alpha\}^{\mathrm{T}}\int_{Z}\{\Delta u\}dz + \{\beta\}^{\mathrm{T}}\int_{Z} z\{\Delta u\}dz + \eta\int_{Z}\Delta u_z dz\right]dA. \tag{22}$$

The Lagrange multipliers $\{\alpha\}$, $\{\beta\}$, and η, which correspond to $\{u\}_0$, $\{\omega\}_0$, and w of the Reissner-Mindlin model, depend on the planar coordinates but not on the transverse coordinate. The consequences of the principle of virtual work can be deduced in two steps.

3.1 Classical equations

First, by selecting $\{\delta\Delta u\}$, $\delta\Delta u_z$, $\{\delta\alpha\}$, $\{\delta\beta\}$, and $\delta\eta$ to vanish, the principle of virtual work implies the condition

$$-\int_\Omega \int_Z \left[\{\delta\varepsilon\}_0^{\mathrm{T}}\{\sigma\}+\{\delta\kappa\}_0^{\mathrm{T}} z\{\sigma\}+\{\delta\gamma\}_0^{\mathrm{T}}\{\tau\}\right]dzdA+\int_\Omega \delta w\left(\int_Z gdz+\sum\nolimits_{\partial Z} q\right)dA=0 \tag{23}$$

for all $\{\delta u\}_0$, $\{\delta\omega\}_0$, and δw. With the usual stress resultant definitions of the internal forces

$$\{N\}=\int_Z \{\sigma\}dz\,,\ \{M\}=\int_Z z\{\sigma\}dz\,,\text{ and }\{Q\}=\int_Z \{\tau\}dz\,, \tag{24}$$

and contribution coming from the external volume and surface forces

$$f=\int_Z gdz+\sum\nolimits_{\partial Z} q\,, \tag{25}$$

Eq. (23) takes the form

$$-\int_\Omega \left[\{\delta\varepsilon\}_0^{\mathrm{T}}\{N\}+\{\delta\kappa\}_0^{\mathrm{T}}\{M\}+\{\delta\gamma\}_0^{\mathrm{T}}\{Q\}\right]dA+\int_\Omega \delta wfdA=0\,. \tag{26}$$

The variational equation indicates that the shear-corrected model preserves the form of the classical plate equations. The steps for finding the equilibrium equations in terms of the stress resultants from Eq. (26) are well-known and they are not repeated here.

3.2 Warping equations

Second, by choosing $\{\delta u\}_0$, $\{\delta\omega\}_0$, and δw to vanish, the outcome is the condition

$$\int_{\Omega}\int_{Z}\left[-\frac{\partial}{\partial z}\{\delta\Delta u\}^{\mathrm{T}}\{\tau\}-\frac{\partial}{\partial z}\delta\Delta u_z\sigma_z+\delta\Delta u_z g\right]dzdA+\int_{\Omega}\left[\sum_{\partial Z}\delta\Delta u_z q\right]dA+$$

$$\int_{\Omega}\delta\left[\{\alpha\}^{\mathrm{T}}\int_{Z}\{\Delta u\}dz+\{\beta\}^{\mathrm{T}}\int_{Z}z\{\Delta u\}dz+\eta\int_{Z}\Delta u_z dz\right]dA=0 \tag{27}$$

for all $\{\delta\Delta u\}$, $\delta\Delta u_z$, $\{\delta\alpha\}$, $\{\delta\beta\}$, and $\delta\eta$. This condition is clearly satisfied if

$$\int_{Z}\left[-\frac{\partial}{\partial z}\{\delta\Delta u\}^{\mathrm{T}}\{\tau\}-\frac{\partial}{\partial z}\delta\Delta u_z\sigma_z+\delta\Delta u_z g\right]dz+\sum_{\partial Z}\delta\Delta u_z q+$$

$$\delta\left[\{\alpha\}^{\mathrm{T}}\int_{Z}\{\Delta u\}dz+\{\beta\}^{\mathrm{T}}\int_{Z}z\{\Delta u\}dz+\eta\int_{Z}\Delta u_z dz\right]=0 \tag{28}$$

for all $\{\delta\Delta u\}$, $\delta\Delta u_z$, $\{\delta\alpha\}$, $\{\delta\beta\}$, and $\delta\eta$ at all points of Ω. Condition in Eq. (28), which is stronger than that in Eq. (27), is the key to a practical implementation.

Assuming that the warping displacement components are continuous in the transverse coordinate and have continuous derivatives up to and including second order except on I (material layer interfaces, for example), integration by parts in Eq. (28) gives an equivalent form

$$\int_{Z\setminus I}\{\delta\Delta u\}^{\mathrm{T}}\left[\frac{\partial}{\partial z}\{\tau\}+\{\alpha\}+z\{\beta\}\right]dz+\sum_{I}\{\delta\Delta u\}^{\mathrm{T}}[\![\{\tau\}]\!]-\sum_{\partial Z}n_z\{\delta\Delta u\}^{\mathrm{T}}\{\tau\}+$$

$$\int_{Z}\delta\Delta u_z\left[\frac{\partial}{\partial z}\sigma_z+g+\eta\right]dz+\sum_{I}\delta\Delta u_z[\![\sigma_z]\!]-\sum_{\partial Z}\delta\Delta u_z(n_z\sigma_z-q)+$$

$$\{\delta\alpha\}^{\mathrm{T}}\int_{Z}\{\Delta u\}dz+\{\delta\beta\}^{\mathrm{T}}\int_{Z}z\{\Delta u\}dz+\delta\eta\int_{Z}\Delta u_z dz=0\,. \tag{29}$$

The unit outward normal $n_z=1$ at the top surface, $n_z=-1$ at the bottom surface, and the jump bracket $[\![a(z)]\!]=\lim_{\varepsilon\to 0}[a(z+\varepsilon)-a(z-\varepsilon)]$. Finally, assuming a traction condition on the top and bottom surfaces, the fundamental lemma of variation calculus implies the boundary value problem

$$\frac{\partial}{\partial z}\{\tau\}+\{\alpha\}+\{\beta\}z=0 \text{ in } Z\setminus I, \tag{30}$$

$$[\![\{\tau\}]\!]=0 \text{ on } I \text{ and } \{\tau\}=0 \text{ on } \partial Z, \tag{31}$$

$$\int_Z \{\Delta u\}\, dz = 0 \text{ and } \int_Z z\{\Delta u\}\, dz = 0 \tag{32}$$

for the in-plane warping displacement components. The boundary value problem for the transverse displacement is given by

$$\frac{\partial}{\partial z}\sigma_z + g + \eta = 0 \text{ in } Z \setminus I, \tag{33}$$

$$\sigma_z = 0 \text{ on } I \text{ and } n_z\sigma_z - q = 0 \text{ on } \partial Z, \tag{34}$$

$$\int_Z \Delta u_z dz = 0. \tag{35}$$

From the physical viewpoint, the Lagrange multipliers represent distributed constraint volume forces needed to enforce the orthogonality constraints. The jump conditions for the transverse stress components indicate that stress needs to be continuous even when material properties change abruptly at the layer interfaces.

4 Warping solution

Equilibrium Eqs. (30) to (35) and constitutive Eqs. (16) to (18) are simple enough to allow an analytical integral solution. In the first step, equilibrium equations are used to find the transverse stress components. Thereafter, integration of the constitutive equations gives the warping displacement. In what follows, the notation is simplified by being explicit only with the dependency on the transverse coordinate, although the Lagrange multipliers and the warping displacement components depend on the in-plane coordinates too.

Integration of the equilibrium equations gives the transverse stress expressions

$$\{\tau\} = -\frac{1}{2}(z-z_-)(z-z_+)\{\beta\} = -\frac{6}{t^3}(z-z_-)(z-z_+)\{Q\}, \tag{36}$$

$$\sigma_z = \frac{z-z_-}{z_+-z_-}q_+ + \frac{z-z_+}{z_+-z_-}q_- \equiv \pi q. \tag{37}$$

The second form of Eq. (36) follows after integration of the first form over the thickness, using the definition of the transverse shear stress resultant in Eq. (24), and elimination of the Lagrange multiplier $\{\beta\}$ by using the integrated form. Therefore, irrespective of the material properties, the transverse shear stress is quadratic in the transverse coordinate and vanishes at the bottom and top surfaces. The solution to the transverse normal stress is linear interpolation to q_+ and $-q_-$ acting on the top and bottom surfaces with the shorthand notation πq (notice the sign change in the bottom surface traction vector and the difference between q and πq, the former being the linear interpolant to q_+ and q_-). To clarify the difference: if the surface forces are due to external pressure so that $q_+ = -p$ and $q_- = p$, $\pi q = -p$ is constant, whereas q depends on the transverse coordinate.

The transverse stress expressions in Eqs. (36) and (37) and the transverse stress constitutive equations in Eqs. (17) and (18) imply the equations

$$\frac{\partial}{\partial z}\{\Delta u\} = -\frac{6}{t^3}(z-z_-)(z-z_+)\left[G\right]^{-1}\{Q\} - \{\gamma\}_0, \tag{38}$$

$$\frac{\partial}{\partial z}\Delta u_z = \frac{1}{E_z}\left[\pi q - \{C\}^{\mathrm{T}}\{\varepsilon\}_0 - z\{C\}^{\mathrm{T}}\{\kappa\}_0\right] \tag{39}$$

for the warping displacement components. The way to find the integral solutions

$$\{\Delta u(z)\} = -\int_{z_-}^{z_+} H(\xi,z)\left[\frac{6}{t^3}(\xi-z_-)(\xi-z_+)\left[G(\xi)\right]^{-1}\{Q\} + \{\gamma\}_0\right]d\xi, \tag{40}$$

$$\Delta u_z(z) = \int_{z_-}^{z_+} H(\xi,z)\frac{1}{E_z(\xi)}\left[\pi q(\xi) - \{C(\xi)\}^{\mathrm{T}}\{\varepsilon\}_0 - \xi\{C(\xi)\}^{\mathrm{T}}\{\kappa\}_0\right]d\xi \tag{41}$$

and the definition of kernel $H(\xi,z)$ therein are given in Appendix B. Finally, the second orthogonality condition in Eq. (32) and solution in Eq. (40) imply the constitutive equation

$$\{Q\} = \frac{t^6}{36}\left[\int_{z_-}^{z_+} (z-z_-)^2(z-z_+)^2\left[G(z)\right]^{-1}dz\right]^{-1}\{\gamma\}_0 \tag{42}$$

for the resultant transverse shear stress (Appendix B). The constitutive equation of the shear-corrected model given by Eq. (42) differs substantially from that of the classical model

$$\{Q\} = \int_{z_-}^{z_+} [G(z)] dz \{\gamma\}_0 = (z_+ - z_-)[\overline{G}]\{\gamma\}_0 \tag{43}$$

in which the overbar denotes the mean value over the thickness of the plate.

5 Implementation

Implementation of the shear-corrected model assumes a numerical or an analytical method for the classical plate problem in its variational form: Find $\{u\}_0$, $\{\omega\}_0$, and w such that

$$\delta W = \int_\Omega \left[-\{\delta\varepsilon\}_0^{\mathrm{T}} \{N\} \right] dA + \int_\Omega \left[-\{\delta\kappa\}_0^{\mathrm{T}} \{M\} - \{\delta\gamma\}_0^{\mathrm{T}} \{Q\} + \delta w f \right] dA = 0 \tag{44}$$

for all $\{\delta u\}_0$, $\{\delta\omega\}_0$, and δw. The definitions of the stress resultants in Eq. (24), the stress expressions in Eqs. (19) and (37) give the constitutive equations

$$\{N\} = [A]\{\varepsilon\}_0 + [C]\{\kappa\}_0 + \{D\}, \tag{45}$$

$$\{M\} = [C]\{\varepsilon\}_0 + [B]\{\kappa\}_0 + \{E\}, \tag{46}$$

in which the 3-by-3 matrices

$$[A] = \int_{z_-}^{z_+} [E]_p dz, \; [C] = \int_{z_-}^{z_+} z[E]_p dz, \; [B] = \int_{z_-}^{z_+} z^2 [E]_p dz, \tag{47}$$

and the 3-by-1 matrices

$$\{D\} = \int_{z_-}^{z_+} \frac{1}{E_z}\{C\} \pi q dz, \; \{E\} = \int_{z_-}^{z_+} z \frac{1}{E_z}\{C\} \pi q dz \tag{48}$$

depend on the plate properties. The definitions for the classical and shear-corrected models differ in the last terms of Eqs. (45) and (46) and in the constitutive equation for the resultant shear stress implied by Eqs. (42) and (43)

$$\{Q\} = [K](z_+ - z_-)\overline{[G]}\{\gamma\}_0 \tag{49}$$

in which the shear correction factor (2-by-2 matrix here)

$$[K] = \frac{(z_+ - z_-)^5}{36}\left[\overline{[G]}\int_{z_-}^{z_+} (z - z_-)^2 (z - z_+)^2 [G]^{-1} dz\right]^{-1} \tag{50}$$

depends on the matrix of the transverse shear module.

5.1 Stress expression

The solution to the plate problem gives directly the translation and rotation components $\{u\}_0$, w, and $\{\omega\}_0$ thereby the strain measures $\{\varepsilon\}_0$, $\{\kappa\}_0$, and $\{\gamma\}_0$ of the classical model. The stress resultants follow from the constitutive Eqs. (45) and (46). Expression

$$\{\sigma\} = [E]_p(\{\varepsilon\}_0 + z\{\kappa\}_0) + \frac{1}{E_z}\{C\}\pi q \tag{51}$$

for the in-plane stress components follow from Eqs. (19) and (37) and expressions

$$\{\tau\} = -\frac{6}{(z_+ - z_-)^2}(z - z_-)(z - z_+)[K]\overline{[G]}\{\gamma\}_0, \tag{52}$$

$$\sigma_z = \pi q\,, \tag{53}$$

for the transverse stress components from Eqs. (19), (36), and (49). It is noteworthy that the transverse stress components are always continuous whereas the in-plane components may be discontinuous.

5.2 Warping displacement expressions

The warping displacement solution in terms of the strain measures of the classical model follow from Eqs. (40) and (41)

$$\{\Delta u(z)\} = -\int_{z_-}^{z_+} H(\xi,z)\left[\frac{6}{t^2}(\xi - z_-)(\xi - z_+)\left[G(\xi)\right]^{-1}\left[K\right]\left[\overline{G}\right] + \left[I\right]\right] d\xi \{\gamma\}_0 , \tag{54}$$

$$\Delta u_z(z) = \int_{z_-}^{z_+} H(\xi,z)\frac{1}{E_z(\xi)}\left[\pi q(\xi) - \{C(\xi)\}^{\mathrm{T}}\{\varepsilon\}_0 - \xi\{C(\xi)\}^{\mathrm{T}}\{\kappa\}_0\right] d\xi . \tag{55}$$

However, these expressions are seldom needed in practice due to the orthogonality conditions in Eq. (15), which imply that, e.g., the mean value of the displacement is given by the classical part. In practical applications, this gives a precise enough picture about displacement of a plate.

6 Application examples

A rectangular simply supported plate of side length L and thickness t is used as an application example. The origin of the transverse coordinate is placed at the mid-plane so $z_+ = t/2$ and $z_- = -t/2$. In addition, the elasticity matrices in Eq. (7) are restricted by conditions $E_5 = E_6 = C_3 = G_3 = 0$ to allow a simple solution to the exact model [15]. Symmetry of the material properties with respect to the mid-plane is assumed to disconnect the in-plane and bending deformation modes of plate.

With the surface load

$$q_+ = q_0 \sin(\frac{\pi x}{L})\sin(\frac{\pi y}{L}) \tag{56}$$

acting on the top surface, the displacement solutions to the exact, classical, and shear-corrected models are all

$$u_x = L\frac{q_0}{E}(\frac{L}{t})^2 \tilde{u}_x(z)\cos(\frac{\pi x}{L})\sin(\frac{\pi y}{L}) , \tag{57}$$

$$u_y = L\frac{q_0}{E}(\frac{L}{t})^2 \tilde{u}_y(z)\sin(\frac{\pi x}{L})\cos(\frac{\pi y}{L}) , \tag{58}$$

$$u_z = L\frac{q_0}{E}(\frac{L}{t})^3\tilde{u}_z(z)\sin(\frac{\pi x}{L})\sin(\frac{\pi y}{L}). \tag{59}$$

The transverse stress solutions are

$$\sigma_{zx} = q_0(\frac{L}{t})\tilde{\sigma}_{zx}(z)\cos(\frac{\pi x}{L})\sin(\frac{\pi y}{L}), \tag{60}$$

$$\sigma_{zy} = q_0(\frac{L}{t})\tilde{\sigma}_{zy}(z)\sin(\frac{\pi x}{L})\cos(\frac{\pi y}{L}), \tag{61}$$

$$\sigma_{zz} = q_0\tilde{\sigma}_{zz}(z)\sin(\frac{\pi x}{L})\sin(\frac{\pi y}{L}) \tag{62}$$

and the in-plane stress solutions

$$\sigma_{xx} = q_0(\frac{L}{t})^2\tilde{\sigma}_{xx}(z)\sin(\frac{\pi x}{L})\sin(\frac{\pi y}{L}), \tag{63}$$

$$\sigma_{yy} = q_0(\frac{L}{t})^2\tilde{\sigma}_{yy}(z)\sin(\frac{\pi x}{L})\sin(\frac{\pi y}{L}), \tag{64}$$

$$\sigma_{xy} = q_0(\frac{L}{t})^2\tilde{\sigma}_{xy}(z)\cos(\frac{\pi x}{L})\cos(\frac{\pi y}{L}). \tag{65}$$

Solutions to the models differ only in the 9 dimensionless functions $\tilde{u}_x(z)$, $\tilde{u}_y(z)$, $\tilde{u}_z(z)$, $\tilde{\sigma}_{zx}(z)$, $\tilde{\sigma}_{zy}(z)$, $\tilde{\sigma}_{zz}(z)$, $\tilde{\sigma}_{xx}(z)$, $\tilde{\sigma}_{yy}(z)$, and $\tilde{\sigma}_{xy}(z)$. The combinations of the geometrical parameters t and L, the reference Young's modulus E, and loading q_0 have been chosen to allow comparison in a dimensionless form with a convenient scaling. All solutions in the comparison are exact to the models.

6.1 Isotropic material

The first example assumes an isotropic material, so elasticity matrices take the forms

$$[E]_p = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}, \ [G] = G\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } \{C\} = \frac{\nu E}{(1+\nu)(1-2\nu)}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}. \tag{66}$$

The diagonal elements $5/6$ of the shear correction matrix given by Eq. (50) coincide with the classical values in literature. The stress expressions

$$\{\sigma\} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}(\{\varepsilon\}_0 + z\{\kappa\}_0) + \frac{\nu}{1-\nu}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}\pi q, \tag{67}$$

$$\{\tau\} = -\frac{5}{t^2}(z^2 - \frac{1}{4}t^2)G\{\gamma\}_0, \tag{68}$$

$$\sigma_z = \pi q, \tag{69}$$

follow from Eqs. (51) to (53) by substituting the elasticity matrices there. The solution to the warping displacement

$$\{\Delta u\} = \frac{1}{12}z(3 - 20\frac{z^2}{t^2})\{\gamma\}_0, \tag{70}$$

$$\Delta u_z = -\frac{\nu}{1-\nu}z(\{1\}^{\mathrm{T}}\{\varepsilon\}_0) + \frac{\nu}{1-\nu}\frac{t^2-12z^2}{24}(\{1\}^{\mathrm{T}}\{\kappa\}_0) - \frac{(1+\nu)(1-2\nu)}{1-\nu}\frac{t^2-12tz-12z^2}{24t}\frac{q_0}{E}, \tag{71}$$

where $\{1\}$ is a 2 by 1 matrix of ones, follows from Eqs. (54) and (55). It is noteworthy that the warping displacement expression in Eq. (70) differs from the a priori expression in Eq. (11) [7], [8]. Also, the shear-corrected model gives a warping displacement in the transverse direction, whereas theory in [7] and [8] does not. There, the two first terms are due to the Poisson effect and the last term represents shrinkage or elongation of the line segments caused by the external surface load.

Fig. 1 compares the exact model, shear-corrected model, and classical model solutions to displacement and stress when $L = 5t$, $E = 10q_0$, and $\nu = 1/3$. For a homogeneous simply supported plate, displacements by the classical and shear-corrected models almost coincide with the exact solution even when the plate is thick. Solutions to the transverse stress components by the classical model are poor but the shear-corrected model predicts all the components well.

Fig.1

6.2 Symmetric sandwich

The second example is a symmetric three-layer sandwich plate composed of isotropic skin (*s*), layers of thickness t_s, and core (*c*) of thickness t_c. The elasticity matrices are the same as in the previous example, the only difference being the different values of the Young's module and Poisson's ratio for the skin and core materials. The shear correction matrix in Eq. (50) simplifies to

$$[K] = k\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{72}$$

in which the shear correction factor is given by

$$k = \frac{5}{6}\frac{G_c G_s (t_c + 2t_s)^6}{2G_c t_s^3 (10t_c^2 + 25t_c t_s + 16t_s^2) + G_s t_c (t_c^4 + 10t_c^3 t_s + 40t_c^2 t_s^2 + 60t_c t_s^3 + 30t_s^4)}\frac{1}{G_c t_c + 2G_s t_s}. \tag{73}$$

If the shear module coincides or thickness of the core or skin vanishes, the shear correction factor takes the classical value $5/6$. For the other typical values $0 < t_s / t < 1$ and $0 < G_c / G_s < 1$, the value is smaller than $5/6$ and can become very small for soft cores.

Tab. 1 and Tab. 2 show the effect of the relative skin layer thickness t_s / t and module ratio G_c / G_s to the transverse displacement modeling error at $z = 0$. Modeling errors of the classical and shear-corrected models increase in the module ratio and decrease in the thickness ratio. Predictions of both models are poor for a sandwich plate composed of a very soft core and rigid skin layer of about the same thickness. In all cases, the shear-corrected model overestimates and the classical model underestimates the transverse displacement. Overall, the modification in the shear-corrected model reduces the modeling error of the classical plate model substantially.

Tab.1

Tab.2

Fig. 2 compares solutions to displacement and stress when $L = 10t$, $t_c = 2t_s$, $E_s = 10q_0$, $E_s = 100E_c$, and $\nu_s = \nu_c = 1/3$. Due to the rather soft core, the transverse displacement, as predicted by the classical model, is way too small. The prediction by the shear-corrected model is more accurate and on the safe side concerning the design of plate structures. Again, solutions to the transverse stress components by the classical model are poor. The shear-corrected model gives a closer fit to the exact solution although the exact transverse stress is not close to quadratic in the transverse coordinate.

Fig.2

6.3 Cross-ply laminate

A symmetric cross-ply structure of orthotropic layers is a good model for plywood and also a popular test problem for refined plate models. In the example, the stiff direction of $n+1$ layers is aligned with the $x-$axis whereas the remaining n layers are aligned with the $y-$axis. Thickness of the layers is constant $\Delta t = t/(2n+1)$. In the orthotropy $\xi\eta\zeta-$system of the layers, in which $\xi-$

axis is aligned with the stiff direction, Young's module, Poisson's ratios, and shear module are taken to be $E_\xi = E$, $E_\eta = E_\zeta = \alpha E$, $\nu_{\zeta\eta} = \nu_{\eta\zeta} = \nu_{\xi\eta} = \nu_{\xi\zeta} = \nu$, $\nu_{\eta\xi} = \nu_{\zeta\xi} = \alpha\nu$, $G_{\zeta\xi} = G_{\xi\eta} = G$, and $G_{\eta\zeta} = \alpha G$ (see Appendix A.). With notation

$$E^* = \frac{E}{(1+\nu)(1-\nu-2\alpha\nu^2)} \ , \tag{74}$$

elasticity matrices of the layers in the $x-$ and $y-$directions are

$$[E]_p = \begin{bmatrix} E^*(1-\nu^2) & E^*\alpha\nu(1+\nu) & 0 \\ E^*\alpha\nu(1+\nu) & E^*\alpha(1-\alpha\nu^2) & 0 \\ 0 & 0 & G \end{bmatrix}, [G] = G\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}, \{C\} = E^*\alpha\nu\begin{Bmatrix} 1+\nu \\ 1+\alpha\nu \\ 0 \end{Bmatrix}, \tag{75}$$

$$[E]_p = \begin{bmatrix} E^*\alpha(1-\alpha\nu^2) & E^*\alpha\nu(1+\nu) & 0 \\ E^*\alpha\nu(1+\nu) & E^*(1-\nu^2) & 0 \\ 0 & 0 & G \end{bmatrix}, [G] = G\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}, \{C\} = E^*\alpha\nu\begin{Bmatrix} 1+\alpha\nu \\ 1+\nu \\ 0 \end{Bmatrix}, \tag{76}$$

respectively. The diagonal elements of the shear correction matrix according to Eq. (50)

$$[K] = \begin{bmatrix} k_{xx} & 0 \\ 0 & k_{yy} \end{bmatrix} \tag{77}$$

are given by

$$k_{xx} = \frac{10}{6}\frac{\alpha(1+2n)}{(1+\alpha-\alpha a_n + a_n)(1+n+\alpha n)}, \quad k_{yy} = \frac{10}{6}\frac{\alpha(1+2n)}{(1+\alpha-a_n+\alpha a_n)(\alpha+n+\alpha n)}, \tag{78}$$

in which

$$a_n = \frac{20(1+n)n-1}{(1+2n)^5}. \tag{79}$$

The value of a_n reduces fast in the number of layers, so one may simplify the setting by using $a_n \approx 0$. If $\alpha = 1$ or $n = 0$, the classical values $k_{xx} = k_{yy} = 5/6$ are obtained.

Fig. 3, Fig. 4, and Fig. 5 compare solutions to displacement and stress when $L = 10t$, $E = 10q_0$, $\alpha = 1/100$, and $\nu = 1/3$. Results for the single-layer structure in Fig. 3 are qualitatively much the same as for the isotropic material. For the five-layer structure in Fig. 4 and the 21 layer structure in Fig. 5 transverse displacement solution by the classical model indicates a way too stiff behaviour. In both cases, the transverse stress solutions by the classical model are poor.

Fig.3

Fig.4

Fig.5

7 Conclusions

The shear-corrected plate model combines the benefits of the equivalent single-layer and layer-wise models in a new manner in which the aim is to reduce the modeling error of the classical plate model without compromising its simplicity. Although the solution to the warping displacement part is layer-wise, plate equations of the shear-corrected model correspond to the single-layer theory. Therefore, the computational complexity of the method is independent in the number of layers whereas the computational complexity of a layer-wise theory is proportional to the number of layers. This gives huge savings, e.g., in plywood applications. For sandwich structures of a thick, soft core and thin, stiff skin layers, a layer-wise theory is still a very good choice.

The additional condition of uniform warping allows a simple integral solution to the warping displacement part, but it excludes the effect of warping constraints like clamped edges. Rejecting the assumption is possible but only with the prize of nonstandard plate equations of increased order. The kinetic assumption of the vanishing transverse normal stress is not used. In the classical model, the assumption counteracts the rather severe kinematic assumption, but the effect is likely to be opposite with a less restrictive kinematics. The integral solution to the warping displacement is exact to the plate model used. There, loading of the plate is in the transverse direction, orthotropy coordinate systems and the plate coordinate system differ in rotation along the transverse direction, material properties were assumed to depend on only the transverse coordinate, etc. These assumptions simplify the derivation and the final outcome but do not restrict the method itself.

Implementation of the method in terms of shear correction factors and a modified source term uses equations of the classical mathematical form so application of the well-established numerical methods and software designed for the Reissner-Mindlin model is possible. The generic integral expression for the shear correction factor differs substantially from the ones in literature [16], [17], and it is given by the model itself. The stress expressions of the shear-corrected model are of the same type as those of the classical model. Therefore, e.g., integration of the full elasticity equations for an improved transverse stress prediction is not needed [18]. This is important in practice as the calculation requires derivatives of the in-plane stress components whose evaluation out of a numerical solution to the plate equations is tricky.

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Appendix A. Orthotropic material model

The material model for an orthotropic layer of plate, which is adapted from the classical representation in [19],

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \tau_{zx} \\ \tau_{zy} \\ \sigma_z \end{Bmatrix} = [T]^{\mathrm{T}} \begin{bmatrix} 1/E_\xi & -\nu_{\eta\xi}/E_\eta & 0 & 0 & 0 & -\nu_{\zeta\xi}/E_\zeta \\ -\nu_{\xi\eta}/E_\xi & 1/E_\eta & 0 & 0 & 0 & -\nu_{\zeta\eta}/E_\zeta \\ 0 & 0 & 1/G_{\xi\eta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{\zeta\xi} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{\zeta\eta} & 0 \\ -\nu_{\xi\zeta}/E_\xi & -\nu_{\eta\zeta}/E_\eta & 0 & 0 & 0 & 1/E_\zeta \end{bmatrix}^{-1} [T] \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{zx} \\ \gamma_{zy} \\ \varepsilon_{zz} \end{Bmatrix} \tag{A.1}$$

depends on Young's module E_ξ, E_η, E_ζ, Poisson's ratio $\nu_{\zeta\eta}$, $\nu_{\eta\zeta}$, $\nu_{\xi\eta}$, $\nu_{\eta\xi}$, $\nu_{\xi\zeta}, \nu_{\zeta\xi}$, shear module $G_{\xi\eta}, G_{\eta\zeta}$, $G_{\zeta\xi}$, and orientation angle α of plate $xyz-$coordinate system relative to the orthotropy $\xi\eta\zeta-$coordinate system. Due to the symmetry, the number of independent elasticity parameters is 9. The transformation matrix

$$[T] = \begin{bmatrix} \cos^2(\alpha) & \sin^2(\alpha) & \frac{1}{2}\sin(2\alpha) & 0 & 0 & 0 \\ \sin^2(\alpha) & \cos^2(\alpha) & -\frac{1}{2}\sin(2\alpha) & 0 & 0 & 0 \\ -\sin(2\alpha) & \sin(2\alpha) & \cos(2\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\alpha) & \sin(\alpha) & 0 \\ 0 & 0 & 0 & -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A.2}$$

assumes that the $\zeta-$axis and $z-$axis are aligned. The more concise representation of the material model in Eq. (A.1)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \tau_{zx} \\ \tau_{zy} \\ \sigma_z \end{Bmatrix} = \begin{bmatrix} E_1 & E_4 & E_5 & 0 & 0 & C_1 \\ E_4 & E_2 & E_6 & 0 & 0 & C_2 \\ E_5 & E_6 & E_3 & 0 & 0 & C_3 \\ 0 & 0 & 0 & G_1 & G_3 & 0 \\ 0 & 0 & 0 & G_3 & G_2 & 0 \\ C_1 & C_2 & C_3 & 0 & 0 & E_z \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{zx} \\ \gamma_{zy} \\ \varepsilon_{zz} \end{Bmatrix} \tag{A.3}$$

contains 13 expressions of Young's module, shear module, Poisson's ratio and the orientation angle of a layer.

Appendix B. Integral solution

Consider the solution $a(x, y, z)$ to equations

$$\frac{\partial a(x, y, z)}{\partial z} = \mu(x, y, z) \quad z_- < z < z_+ , \tag{B.1}$$

$$\int_{z_-}^{z_+} a(x, y, z)dz = 0 \text{ and } \int_{z_-}^{z_+} za(x, y, z)dz = 0 \tag{B.2}$$

in which $\mu(x, y, z)$ is given. The equation set overdetermines $a(x, y, z)$ and, therefore, existence of the solution is possible only under a constraint on data $\mu(x, y, z)$. Integral identity

$$\int_{z_-}^{z_+} H(\xi, z)\frac{\partial a(x, y, \xi)}{\partial \xi} d\xi = a(x, y, z) - \frac{1}{z_+ - z_-}\int_{z_-}^{z_+} a(x, y, z)dz , \tag{B.3}$$

where the kernel

$$H(\xi, z) = \frac{1}{z_+ - z_-}\begin{cases} \xi - z_-, & \xi \le z \\ \xi - z_+, & \xi > z \end{cases} \tag{B.4}$$

implies the solution

$$a(x, y, z) = \int_{z_-}^{z_+} H(\xi, z)\mu(x, y, \xi)d\xi \tag{B.5}$$

to the differential equation and the first of the integral constraints. The second integral constraint, the differential equation, and the integral identity imply that

$$\int_{z_-}^{z_+} za(x,y,z)dz = -\int_{z_-}^{z_+} \frac{1}{2}(z-z_-)(z-z_+)\mu(x,y,z)dz = 0\,. \tag{B.6}$$

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Fig. 1. Displacement and stress by the exact (broken), shear-corrected (dotted), and classical (solid) models as the functions of $\tilde{z} = z/t$. Isotropic plate with $L = 5t$, $E = 10q_0$, and $\nu = 1/3$.

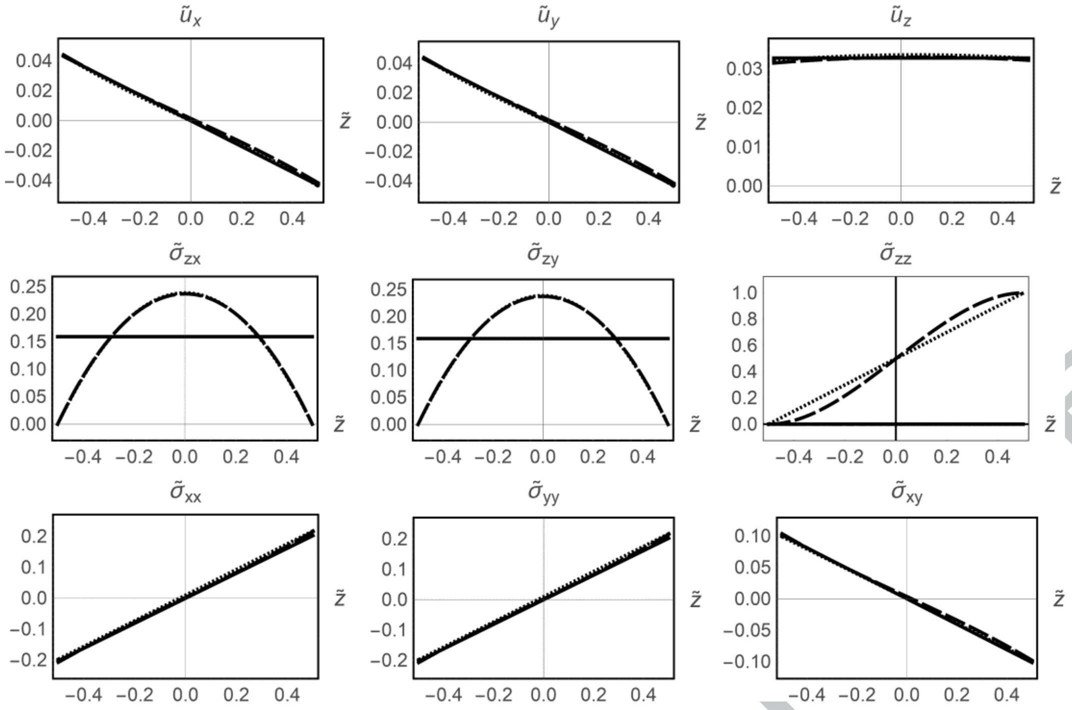

Fig. 2. Displacement and stress by the exact (broken), shear-corrected (dotted), and classical (solid) models as the functions of $\tilde{z} = z/t$. Sandwich plate problem with $L = 10t$, $t_c = 2t_s$, $E_s = 10q_0$, $E_s = 100E_c$, and $\nu_s = \nu_c = 1/3$.

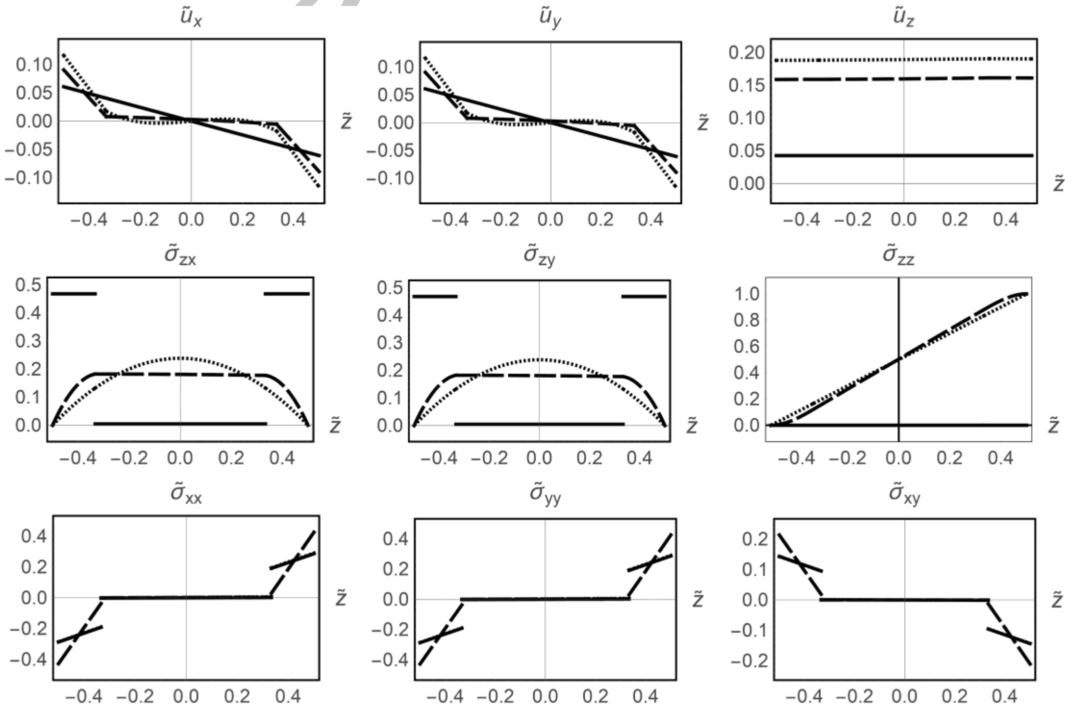

Fig. 3. Displacement and stress by the exact (broken), shear-corrected (dotted), and classical (solid) models as the functions of $\tilde{z} = z/t$. Single layer plywood plate with $L = 10t$, $E = 10q_0$, $\alpha = 1/100$, and $\nu = 1/3$.

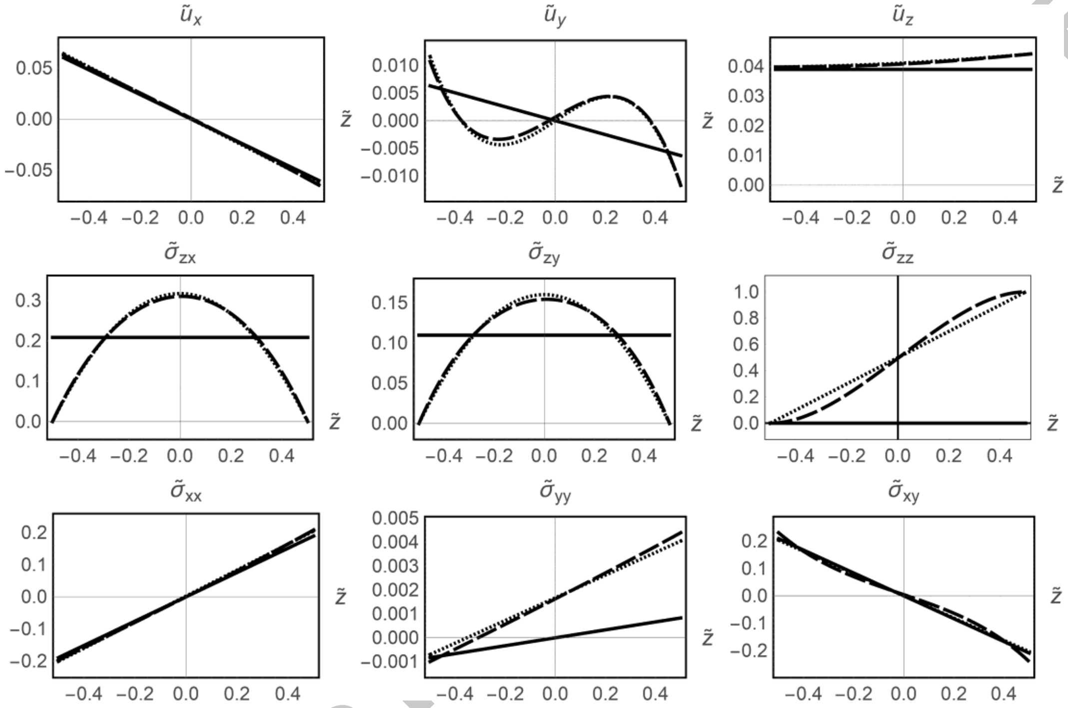

Fig. 4. Displacement and stress by the exact (broken), shear-corrected (dotted), and classical (solid) models as the functions of $\tilde{z} = z/t$. Five layer plywood plate with $L = 10t$, $E = 10q_0$, $\alpha = 1/100$, and $\nu = 1/3$.

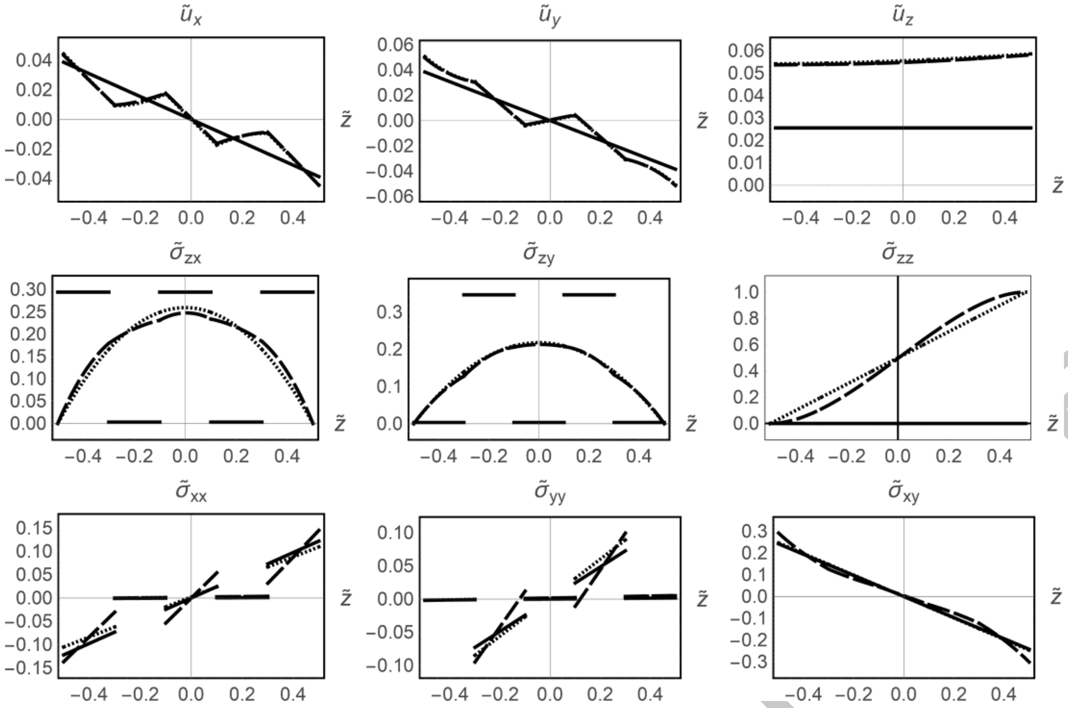

Fig. 5. Displacement and stress by the exact (broken), shear-corrected (dotted), and classical (solid) models as the functions of $\tilde{z} = z/t$. Plywood plate of 21 layers with $L = 10t$, $E = 10q_0$, $\alpha = 1/100$, and $\nu = 1/3$.

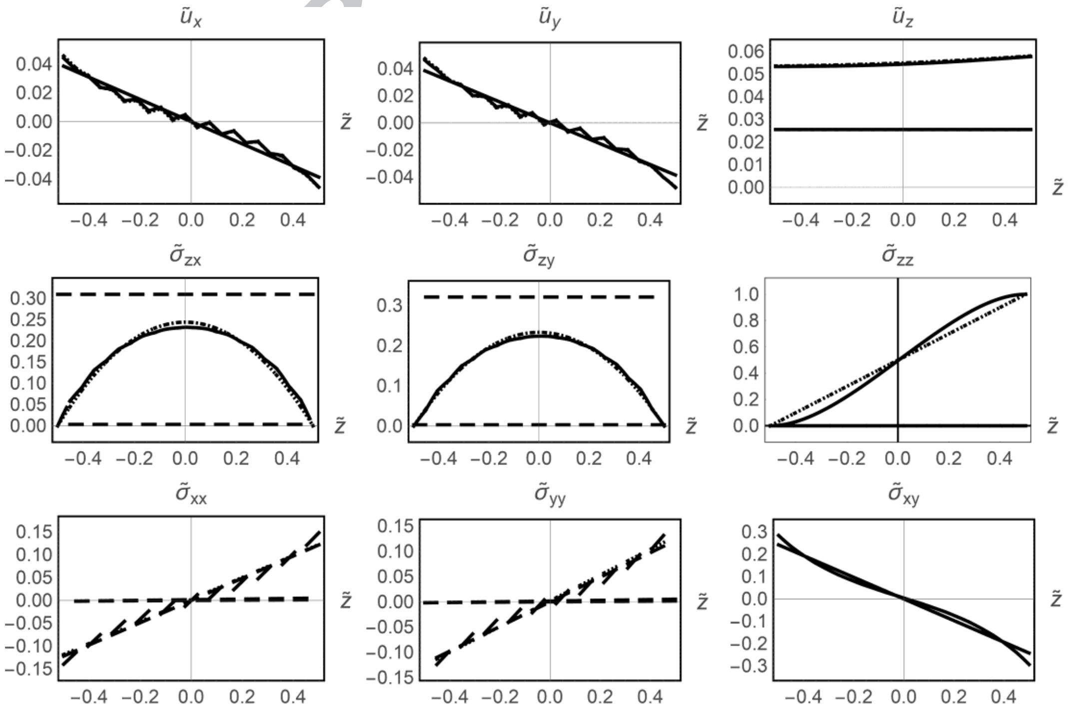

Table 1. Effects of the relative skin layer thickness t_s / t and rigidity ratio E_s / E_c on the transverse displacement modeling error at the mid-surface $z = 0$ ($\tilde{u}_z^c$ ~ classical model, $\tilde{u}_z^s$ ~shear-corrected model, $\tilde{u}_z^e$ ~ exact). Thick plate with $t / L = 1/10$.

	$t_s / t = 1/8$		$t_s / t = 1/16$		$t_s / t = 1/32$	
E_s / E_c	$\tilde{u}_z^c / \tilde{u}_z^e$	$\tilde{u}_z^s / \tilde{u}_z^e$	$\tilde{u}_z^c / \tilde{u}_z^e$	$\tilde{u}_z^s / \tilde{u}_z^e$	$\tilde{u}_z^c / \tilde{u}_z^e$	$\tilde{u}_z^s / \tilde{u}_z^e$
1	1.00	1.00	1.00	1.00	1.00	1.00
10	0.84	1.04	0.91	1.03	0.91	1.03
10^2	0.30	1.16	0.43	1.13	0.59	1.10
10^3	0.05	1.41	0.07	1.22	0.12	1.18
10^4	0.01	3.44	0.01	1.49	0.01	1.24
10^5	<0.01	23.6	<0.01	4.09	<0.01	1.56

Table 2. Effects of the relative skin layer thickness t_s / t and rigidity ratio E_s / E_c on the transverse displacement modeling error at the mid-surface $z = 0$ ($\tilde{u}_z^c$ ~ classical model, $\tilde{u}_z^s$ ~shear-corrected model, $\tilde{u}_z^e$ ~ exact). Thin plate with $t / L = 1/100$.

	$t_s / t = 1/8$		$t_s / t = 1/16$		$t_s / t = 1/32$	
E_s / E_c	$\tilde{u}_z^c / \tilde{u}_z^e$	$\tilde{u}_z^s / \tilde{u}_z^e$	$\tilde{u}_z^c / \tilde{u}_z^e$	$\tilde{u}_z^s / \tilde{u}_z^e$	$\tilde{u}_z^c / \tilde{u}_z^e$	$\tilde{u}_z^s / \tilde{u}_z^e$
1	1.00	1.00	1.00	1.00	1.00	1.00
10	1.00	1.00	1.00	1.00	1.00	1.00
10^2	0.97	1.01	0.98	1.00	0.99	1.00
10^3	0.78	1.04	0.86	1.03	0.92	1.02
10^4	0.27	1.16	0.38	1.13	0.54	1.09
10^5	0.04	1.41	0.06	1.22	0.10	1.18