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Abstract

We consider Social Distance Games (SDGs), that is cluster formation games in which the utility of each agent only depends on the composition of the cluster she belongs to, proportionally to her harmonic centrality, i.e., to the average inverse distance from the other agents in the cluster. Under a non-cooperative perspective, we adopt Nash stable outcomes, in which no agent can improve her utility by unilaterally changing her coalition, as the target solution concept. Although a Nash equilibrium for a SDG can always be computed in polynomial time, we obtain a negative result concerning the game convergence and we prove that computing a Nash equilibrium that maximizes the social welfare is NP-hard by a polynomial time reduction from the NP-complete Restricted Exact Cover by 3-Sets problem. We then focus on the performance of Nash equilibria and provide matching upper bound and lower bounds on the price of anarchy of $\Theta(n)$, where n is the number of nodes of the underlying graph. Moreover, we show that there exists a class of SDGs having a lower bound on the price of stability of $\frac{6}{5} - \epsilon$, for any $\epsilon > 0$. Finally, we characterize the price of stability of SDGs for graphs with girth 4 and girth at least 5, the girth being the length of the shortest cycle in the graph.

1. Introduction

Coalition formation is a pervasive aspect of social life and it has been studied extensively in algorithmic game theory using the natural model of *Hedonic Games* (HGs), introduced by Dreze and Greenberg (1980) and further explored in several papers, such as Banerjee, Konishi, and Sönmez (2001); Bogomolnaia and Jackson (2002); and Cechlárová and Romero-Medina (2001). An HG consists of a set of selfish agents (humans, robots, software agents, etc.) having preferences over coalitions that might include them, regardless of which other coalitions may or may not be present. In other words, the utility of each agent only depends on the composition of the coalition she belongs to, without any form of externality, that is, without caring about the structure of the other coalitions. The outcome of an HG is a partition of the agents' set into disjoint coalitions (or clusters), referred to as a *clustering* or *coalition structure*.

Stability is the main criterion that has been used to analyze which coalition structures will arise: an outcome should be resistant to individual/group deviations, with different types of deviations giving rise to different notions of stability (such as core stability, individual stability, Nash stability).

A significant stream of research (Aziz, Brandt, & Harrenstein, 2013; Aziz, Brandt, & Seedig, 2013; Bogomolnaia & Jackson, 2002; Banerjee et al., 2001; Elkind & Wooldridge, 2009; Gairing & Savani, 2010) characterizes the existence and the properties of stable clusterings in hedonic games. For an overview, see the recent survey by Aziz and Savani (2016).

Under a non-cooperative perspective, we focus on Nash stable outcomes in which, unlike the core, every agent cannot coordinate with the others in order to understand if she can improve her utility. This notion of stability is better suited in settings in which it is not possible to assume the ability of coordination, or the cost of coordination is excessively high, like in huge distributed environments characterized by decentralization, autonomy, and general lack of cooperation among the network entities. A clustering is *Nash stable*, or it is a Nash equilibrium, if no agent can improve her utility by unilaterally changing her own coalition. In this setting, the outcome generated by such uncoordinated selfish agents can be inferior to a centrally designed one. Hence, one of the main issues is evaluating the induced degradation of the system performance. In particular, the most popular measure of the inefficiency of equilibria is the price of anarchy (PoA) (Koutsoupias & Papadimitriou, 1999; Papadimitriou, 2001), defined as the worst-case ratio between the social welfare (i.e., the sum of the agents utilities) of a best clustering and the social welfare of a Nash stable clustering. Notice that for this definition to make sense, a game should allow at least one Nash equilibrium. Roughly speaking, a low PoA means that agents' selfish behaviour has a small impact on the system performance; a high PoA, instead, gives evidence that agents can really hurt the system and some sort of coordination is useful. Since a game has an high PoA even if only one of its multiple equilibria is highly inefficient, to differentiate between games in which all equilibria are inefficient and those in which only some equilibria are inefficient, we adopt a related optimistic measure, called the price of stability (PoS) (Anshelevich, Dasgupta, Kleinberg, Tardos, Wexler, & Roughgarden, 2004), that is the best-case ratio between the social welfare of a best clustering and the social welfare of a Nash stable clustering. In other words, the PoS is related to the best possible outcome that selfish agents might accept. The PoS thus quantifies the necessary quality degradation caused by imposing the game-theoretic constraint of stability.

Since the number of possible coalitions and coalition structures is exponential, the mere specification of the input with all the corresponding values is already intractable. Therefore, researchers have focused on suitable subclasses allowing concise descriptions. Along this line, a growing interest in HGs has concerned instances in which agents entertain preferences over the other agents, which are then naturally lifted to preferences over coalitions. *Additively Separable* HGs (Banerjee et al., 2001) are a corresponding natural class of HGs, that can be succinctly represented by means of a preference graph of the agents. In these games, each agent has a value for any other one, and her utility for being in a coalition is simply the sum of the values she ascribes to its members, or analogously her degree centrality in the coalition. A related class of HGs is given by the *Fractional Hedonic Games* (FHGs) (Aziz, Brandt, & Harrenstein, 2014), in which the utility of an agent is still given by her degree centrality in the own coalition, but it is also divided by the size of her cluster. Given that the utility of an agent is not simply defined as her total happiness, but as its ratio with the coalition size, FHGs can model several natural behavioral dynamics in social environments, like the well-known *Bakers and Millers* game.

In this paper we focus on *Social Distance Games* (SDGs), another important subclass of HGs introduced by Brânzei and Larson (2011), still defined around the notion of node centrality. In particular, they are based on the concept of social distance (i.e., the number of hops required to reach one node from another), which has become famous since Milgram's study on six degrees of

separation. In SDGs the utility of an agent is given by the average inverse distance from all the other nodes in her coalition, that is by her harmonic centrality, divided by the size of the coalition. The basic idea is that each agent prefers to maintain stronger ties with closer ones, in such a way that also non neighbor agents contribute to her utility in a decreasing way with respect to their distance. The utility formulation is a variant of the closeness centrality and reflects the principle of homophily, that similarity breeds connection and people tend to form communities with similar others (McPherson, Lovin, & Cook, 2001). Namely, it is related to several other classical measures from network analysis, such as degree, closeness, betweenness, and eigenvector centrality (Gomez, Gonzlez-Arangena, Manuel, Owen, del Pozo, & Tejada, 2003), all of which are used to determine how a node is embedded in the network. Among such notions, the harmonic centrality used in SDGs has been identified as one of the best indexes, as it is the unique one satisfying a set of desirable properties (Boldi & Vigna, 2014).

SDGs not only have the merit of being able to depart from the previous models that consider only the direct social neighborhood, but also provide a suitable abstraction of the utility definition, able to isolate the notion of node centrality in a modular way. Thus, they are capturing recent research attention and will possibly lead to other interesting models related to different suitable measures of social centrality.

1.1 Related Work

In the literature, an extensive research considered HGs from a strategic cooperative point of view, with the aim of characterizing the existence and the properties of coalition structures such as, for instance, the core (Bogomolnaia & Jackson, 2002; Banerjee et al., 2001; Elkind & Wooldridge, 2009; Gairing & Savani, 2010). A clustering is core stable if there is no group of agents who can all be better off by forming a new deviating coalition. Non-cooperative studies on HGs in which self-organized clusterings are obtained from the decisions taken by independent and selfish agents can be found in Bloch and Diamantoudi (2011); Feldman, Lewin-Eytan, and Naor (2015); and Gairing and Savani (2010). More precisely, Banerjee et al. (2001) study properties guaranteeing the existence of core allocations for HGs with additively separable utilities, while Bogomolnaia and Jackson (2002) consider several forms of clustering stability like the core and the Nash. Aziz, Brandt, and Seedig (2011); Ballester (2004); and Olsen (2009) deal with computational complexity issues related to HGs, also considering additively separable utilities. In particular, Olsen (2009) shows that the problem of deciding whether a Nash stable clustering exists in an additively separable HG is NP-complete, as well as the one of deciding whether a Nash stable clustering different from the grand coalition exists in an additively separable HG with non-negative and symmetric preferences. Bloch and Diamantoudi (2011) study non-cooperative games of coalition formation and identify conditions for stable outcomes. In a similar way, Apt and Witzel (2009) study how certain proposed rules can transform clusterings into other ones with specific properties. Feldman et al. (2015) investigate some interesting subclasses of HGs from a non-cooperative point of view, by characterizing Nash equilibria and providing upper and lower bounds on both the price of stability and the price of anarchy.

Fractional Hedonic Games are the class of games most related to the *Social Distance Games* considered in this paper. As already mentioned, they have been traditionally investigated under the additively-separable property. Namely, the utility of an agent is given by the sum of her preferences for each single member of her coalition, i.e. by her degree centrality, divided by the size of the

coalition. FHGs have been investigated in Aziz et al. (2014); Aziz, Gaspers, Gudmundsson, Mestre, and Täubig (2015); and Brandl, Brandt, and Strobel (2015) from a cooperative perspective and in Bilò, Fanelli, Flammini, Monaco, and Moscardelli (2014); Bilò, Fanelli, Flammini, Monaco, and Moscardelli (2015); and Olsen (2012) from a non-cooperative viewpoint. In particular, we point out that Aziz et al. (2014) focus on core stable partitions and present a number of classes of graphs for which the core is non-empty, including graphs with girth at least 5, the girth being the length of the shortest cycle in the graph.

Olsen (2012) introduces a modified version of FHGs further investigated in Kaklamanis, Kanellopoulos, and Papaioannou (2016); and Monaco, Moscardelli, and Velaj (2018), where the utility of an agent is divided by the size of the coalition she belongs to minus 1. Authors study the existence, complexity and performance of solutions that lead to Nash, strong Nash, and core stable outcomes.

Apart from examining standard stability notions, Aziz et al. (2013); and Elkind, Fanelli, and Flammini (2016) investigate Pareto optimality in HGs and FHGs. Pareto optimal solutions have been in fact considered in HGs as outcomes arising from the strategic interaction of the agents, as they are stable under the deviation of the grand coalition. In fact, they do not permit a simultaneous deviation by all the agents making all of them weakly better off and some strictly better off. In such papers the authors give intractability results on the determination of Pareto optimal solutions and suitable bounds on their price of anarchy, called price of Pareto optimality in this setting.

Flammini, Monaco, Moscardelli, Shalom, and Zaks (2018) study HGs where agents come in an online fashion, one at a time. They consider additively separable HGs and FHGs and show tight bounds in the cases where the edge-weights can be constant, positive, or unrestricted.

Flammini, Monaco, and Zhang (2017) study strategyproof mechanisms for additively separable HGs and FHGs. More precisely, they show partitioning methods without payments such that agents do not gain anything, utility wise, by lying about their true preferences.

To the best of our knowledge *Social Distance Games* have been considered in the cooperative context of core stability by Brânzei and Larson (2011). As already observed, they are related to FHGs in the sense that, in the definition of agents utilities, the degree centrality measure is substituted by the harmonic centrality. Brânzei and Larson (2011) claim that finding the best clustering can be shown to be NP-hard on SDGs via a reduction from the NP-complete *Partition into Triangles* problem, and provide an algorithm to approximate the optimal welfare within a factor of 2. They analyze core stable solutions, determining their welfare and stability characteristics, and showing that they possess the small world property.

Balliu, Flammini, and Olivetti (2017b) investigate Pareto stability in SDGs and show that, while computing a Pareto stable solution maximizing the social welfare in bounded degree graphs is NP-hard, a suitably approximating one can be determined in polynomial time. They also provide asymptotically tight bounds on the price of Pareto optimality for several classes of social graphs.

1.2 Our Contribution

In this paper we approach SDGs from the viewpoint of non-cooperative game theory with the aim of understanding the existence, computability and performances of Nash stable clusterings. We first focus on the existence of Nash stable clusterings: even if SDGs always admit a Nash equilibrium, we show that they may not converge to Nash equilibria and then give a polynomial time reduction from the NP-complete RESTRICTED EXACT COVER *by 3*-SETS ($RXC3$) problem (Gonzalez, 1985) to prove that it is NP-hard to find a best Nash equilibrium. We then study the performances of the Nash

equilibria and provide matching upper and lower bounds on the price of anarchy of $\Theta(n)$, where n is the number of nodes of the underlying graph. Also, we show that there exists a class of SDGs having a lower bound on the price of stability of $\frac{6}{5} - \epsilon$, for any arbitrarily small $\epsilon > 0$. Finally, we characterize the price of stability of SDGs for graphs with girth 4 and girth at least 5, the *girth* being the length of the shortest cycle in the graph. Our results are summarized in Tables 1 and 2.

Notice that graphs with limited girth, while generalizing known topologies such as bipartite graphs and being among the classical ones investigated in this setting, seem to identify the SDGs for which good stable solutions can be computed in polynomial time. In fact, for a girth greater than 4 we prove that, by partitioning the nodes into stars, it is possible to efficiently determine a stable solution which has a social welfare at least equal to the optimal one divided by 1.207. On the other hand, as we will show in the sequel, there are graphs with girth at most 4 that do not admit a stable star partition.

We finally remark that, after the preliminary version of this paper appeared in Balliu, Flammini, Melideo, and Olivetti (2017a), an improved lower bound of 2 of the price of stability for SDGs on general graphs has been provided by Kaklamanis, Kanellopoulos, and Patouchas (2018). In that paper, the authors have also proven that the price of stability in trees is 1; moreover, they have studied the price of stability for a modified version of SDGs, where the utility is computed with respect to the cluster size minus 1.

Existence of a Nash equilibrium	Always
Convergence to a Nash equilibrium	Not guaranteed
Finding a best Nash equilibrium	NP-hard

Table 1: Nash equilibria in SDGs.

	Lower Bound	Upper Bound	Graph Topology
	$6/5 - \epsilon$	?	general
PoS	$169/160$	?	girth 4
	?	$1/2 + 1/\sqrt{2}$	girth greater that 4
PoA	$\Omega(n)$	$O(n)$	general

Table 2: Price of Stability and Price of Anarchy in SDGs.

2. Model and Preliminaries

Consider an undirected graph $G = (V, E)$. A *coalition* or *cluster* is a non-empty subset of V. The set of all nodes V is called the *grand coalition*, whereas a coalition of size 1 is said a *singleton coalition* and its node *singleton* or *isolated*. For any integer $n > 0$ denote with $[n]$ the set of integers

$\{1, \ldots, n\}$ and with $[k, n]$, for $k \leq n$ the subset $\{k, \ldots, n\} \subseteq [n]$. A *clustering* or *coalition structure* is a partition of V into $k > 0$ coalitions $\mathcal{C} = \{C_1, \ldots, C_k\}$ such that $C_i \subseteq V$ for each $i \in [k]$, $\bigcup_{i \in [k]} C_i = V$ and $C_i \cap C_j = \emptyset$ for any $i, j \in [k]$ with $i \neq j$. Given a coalition $C \subseteq V$, we denote by $G(C)$ the subgraph induced by C. For brevity, we will often identify $G(C)$ directly with the corresponding coalition C and we say that a coalition C is connected or disconnected if $G(C)$ is so.

Let the *harmonic centrality* of a node $x \in V$ in an undirected graph $G = (V, E)$ be defined as $\mu_x(G) = \sum_{y \in V \setminus \{x\}} \frac{1}{d_G(x,y)}$, where for any pair of nodes $x, y \in V$, $d_G(x, y)$ denotes the length of a shortest path connecting x and y in G. The sum of the inverse of the social distances can be viewed as the similarity of a node in its graph, and it indicates its centrality.

We consider coalition forming games in which each node $x \in V$ is an agent. More precisely, we focus on the following class of games.

Definition 2.1 *(Brânzei & Larson, 2011) A* Social Distance Game *SDG(G), or SDG in short, is represented as an undirected graph $G = (V, E)$ where (i) V is the set of n agents and (ii) the* utility *of an agent $x \in V$ in a given coalition C is a suitable function of her harmonic centrality in the subgraph induced by C and of the size of C, that is:*

$$u_x(C) = \frac{\mu_x(G(C))}{|C|} = \begin{cases} 0 & \text{if } C = \{x\} \\ \frac{1}{|C|} \sum_{y \in C \setminus \{x\}} \frac{1}{d_C(x,y)} & \text{otherwise} \end{cases}$$

If x and y are disconnected in C, then $d_C(x, y) = \infty$.

We denote by $\mathcal{C}(x)$ the coalition of $\mathcal{C}$ including node x. Let us use $u_x(\mathcal{C})$ as a shorthand for the utility $u_x(\mathcal{C}(x))$ of x in a given clustering $\mathcal{C}$. Each agent chooses the coalition to belong to with the aim of maximizing her utility. A *solution* or *outcome* of a SDG is a clustering $\mathcal{C}$. The *social welfare* $SW(\mathcal{C})$ of a clustering $\mathcal{C}$ is the sum of the agents' utilities, i.e., $SW(\mathcal{C}) = \sum_{x \in V} u_x(\mathcal{C})$.

In the rest of this paper, we can assume that the input graph is non-singleton and connected, since disconnected graphs can be analyzed componentwise.

For any pair of agents x and y, we denote with $(\mathcal{C}, x, y)$ the clustering obtained from $\mathcal{C}$ by moving x from $\mathcal{C}(x)$ to $\mathcal{C}(y)$. A clustering $\mathcal{C}' = (\mathcal{C}, x, y)$ is an improving deviation for the agent x in $\mathcal{C}$ if $u_x(\mathcal{C}) < u_x(\mathcal{C}')$. In this case we say that agent x makes an improving move. An improvement path is a sequence of improving deviations. A game has the finite improvement property (FIP) if every improvement path of the game is finite.

An agent x is *Nash stable* in a clustering $\mathcal{C}$ if she cannot perform any improving move, that is if $\forall y \in V$, $u_x(\mathcal{C}) \geq u_x(\mathcal{C}')$, where $\mathcal{C}' = (\mathcal{C}, x, y)$. A clustering $\mathcal{C}$ is *Nash stable*, or a *Nash equilibrium*, if every agent x is Nash stable in $\mathcal{C}$. If an improvement path is finite then its last clustering is a Nash equilibrium.

We are interested in bounding the performances of Nash equilibria with respect to the social optimum OPT, i.e. the social welfare of a best clustering $\mathcal{C}^*$ that maximizes $SW(\mathcal{C}^*)$, that is such that $\text{OPT} = SW(\mathcal{C}^*) = \max_{\mathcal{C}} SW(\mathcal{C})$. Notice that $\mathcal{C}^*$ is not necessarily an equilibrium. If $\mathcal{N}$ denotes the set of the Nash equilibria, a best (worst) Nash equilibrium is a Nash equilibrium $\mathcal{C} \in \mathcal{N}$ that maximizes (minimizes) $SW(\mathcal{C})$. The best (worst) social cost is the social welfare of a best (worst) Nash equilibrium. The *price of anarchy* (PoA) is the ratio between the social optimum and the worst social cost in a game i.e., $\text{PoA} = \max_{\mathcal{C} \in \mathcal{N}} \frac{\text{OPT}}{SW(\mathcal{C})}$; the *price of stability* (PoS) is the ratio between the social optimum and the best social cost, i.e., $\text{PoS} = \min_{\mathcal{C} \in \mathcal{N}} \frac{\text{OPT}}{SW(\mathcal{C})}$.

3. Nash Equilibria: Existence, Convergence and Complexity of their Finding

In order to establish the existence of Nash equilibria for SDGs and to prove that determining a Nash equilibrium can be done in polynomial time, it is sufficient to notice that the grand coalition is Nash stable. In fact, no agent can have any improving move by deciding to form a singleton coalition on her own. Thus, on the basis of these considerations, we can state the following:

Proposition 1 *A SDG always admits a Nash equilibrium and it can be computed in polynomial time.*

3.1 Convergence to Equilibrium in SDGs

In the sequel of this section we investigate if, starting from an arbitrary initial clustering, an improvement path is always guaranteed to converge to a Nash equilibrium, and we provide a negative answer. In fact, Theorem 1 shows that SDGs may admit an infinite sequence of improving deviations (i.e., the FIP property is not guaranteed).

Theorem 1 *SDGs may not converge to Nash equilibria.*

Proof. Consider the instance $SDG(G)$ where G is the bipartite graph depicted in Figure 1 with vertex set $V = X \cup Y \cup Z$, where $X = \{x_i | i \in [20]\}$, $Y = \{y_i | i \in [15]\}$ and $Z = \{z_1, z_2\}$. In the following, for $i \leq j$, we write $X_{i,j}$ and $Y_{i,j}$ as a shorthand for $\{x_i, \ldots, x_j\} \subseteq X$ and $\{y_i, \ldots, y_j\} \subseteq Y$, respectively.
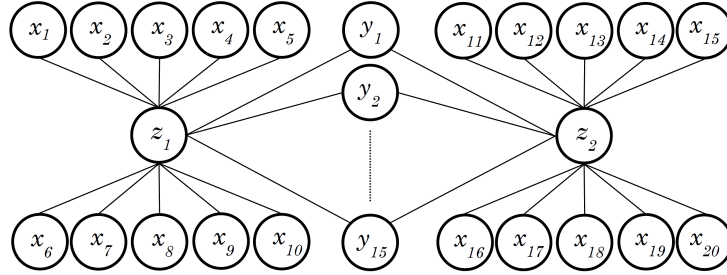

Figure 1: A SDG not converging to a Nash equilibrium.

Starting from the clustering $\mathcal{C} = \{X_{1,10} \cup \{z_1\}, Y, X_{11,20} \cup \{z_2\}\}$ depicted in Figure 2, we prove the existence of a cyclic sequence of improving deviations which shows the claim.

Namely, we consider the following improvement path:

- $\mathcal{C}^{(1)} = (\mathcal{C}, z_1, y_1) = \{X_{1,10}, Y \cup \{z_1\}, X_{11,20} \cup \{z_2\}\}$ (see Figure 3) is the improving deviation obtained from $\mathcal{C}$ by moving z_1 to $\mathcal{C}(y_1)$. Since $u_{z_1}(\mathcal{C}) = \frac{10}{11}$ and $u_{z_1}(\mathcal{C}^{(1)}) = \frac{15}{16} > \frac{10}{11}$, agent z_1 can really make an improving move.

- $\mathcal{C}^{(2)} = (\mathcal{C}^{(1)}, z_2, z_1) = \{X_{1,10}, Y \cup Z, X_{11,20}\}$ (see Figure 4) is the improving deviation resulting from $\mathcal{C}^{(1)}$ by moving agent z_2 in $\mathcal{C}^{(1)}(z_1)$. In fact z_2 can improve her utility from $u_{z_2}(\mathcal{C}^{(1)}) = \frac{10}{11}$ to $u_{z_2}(\mathcal{C}^{(2)}) = u_{z_2}(Y \cup Z) = \frac{15+\frac{1}{2}}{17} = \frac{31}{34}$ by moving to $\mathcal{C}^{(1)}(z_1) = Y \cup \{z_1\}$.
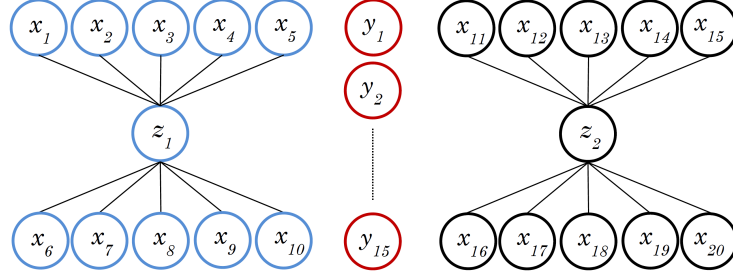
Figure 2: The initial clustering $\mathcal{C}$, where $u_{z_1}(\mathcal{C}) = u_{z_2}(\mathcal{C}) = \frac{10}{11}$.

- Since $\forall x \in X$, $u_x(\mathcal{C}^{(2)}(x)) = 0$, agents $x_1, \ldots, x_5$ and $x_{11}, \ldots, x_{15}$ can increase their utility by moving one after another to the coalition including agents in $Y \cup Z$. Thus starting from $\mathcal{C}^{(2)}$, we consider 10 improving moves inducing the clustering $\mathcal{C}^{(12)} = \{X_{6,10}, Y \cup Z \cup X_{1,5} \cup X_{11,15}, X_{16,20}\}$ depicted in Figure 5.

- Notice that $u_{z_1}(\mathcal{C}^{(12)}) = \frac{20 + \frac{1}{2} + \frac{5}{3}}{27} = \frac{133}{162}$ and thus agent z_1 can increase her utility to $\frac{5}{6}$ by moving back to $X_{6,10}$. Thus $\mathcal{C}^{(13)} = (\mathcal{C}^{(12)}, z_1, x_6) = \{X_{6,10} \cup \{z_1\}, Y \cup \{z_2\} \cup X_{1,5} \cup X_{11,15}, X_{16,20}\}$ (see Figure 6).

- Since $u_{z_2}(\mathcal{C}^{(13)}) = \frac{10}{13}$, agent z_2 induces an improving deviation by moving back to $X_{16,20}$, that is $\mathcal{C}^{(14)} = (\mathcal{C}^{(13)}, z_2, x_{16}) = \{X_{6,10} \cup \{z_1\}, Y \cup X_{1,5} \cup X_{11,15}, X_{16,20} \cup \{z_2\}\}$ and $u_{z_2}(\mathcal{C}^{(14)}) = \frac{5}{6} > \frac{10}{13}$ (see Figure 7).

- Agents $x_1, \ldots, x_5$ and $x_{11}, \ldots, x_{15}$ have now utility zero in $\mathcal{C}^{(14)}$ and thus they can increase the utility by moving back one after another to the cluster containing z_1 and z_2, respectively. Then $\mathcal{C}$ is again reached and therefore the claim follows.
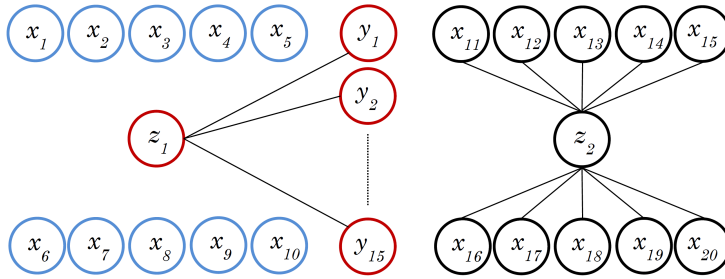

Figure 3: $\mathcal{C}^{(1)} = (\mathcal{C}, z_1, y_1)$, where $u_{z_1}(\mathcal{C}^{(1)}) = \frac{15}{16} > \frac{10}{11}$.

$\square$

3.2 NP-completeness of Finding a Best Nash Equilibrium for SDGs

While determining the existence of a Nash equilibrium can be trivially done in polynomial time, as the grand coalition is stable, in the sequel of this section we show that unfortunately computing
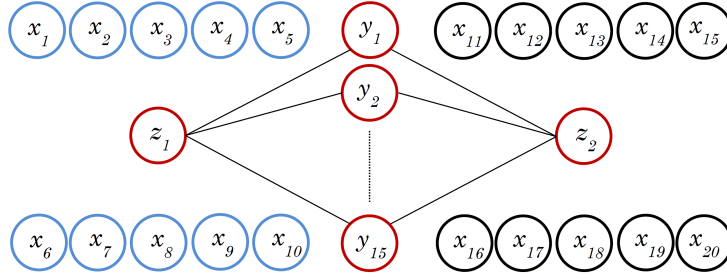
Figure 4: $\mathcal{C}^{(2)} = (\mathcal{C}^{(1)}, z_2, z_1)$, where $u_{z_1}(\mathcal{C}^{(2)}) = u_{z_2}(\mathcal{C}^{(2)}) = \frac{15+\frac{1}{2}}{17} = \frac{31}{34} > \frac{10}{11}$.
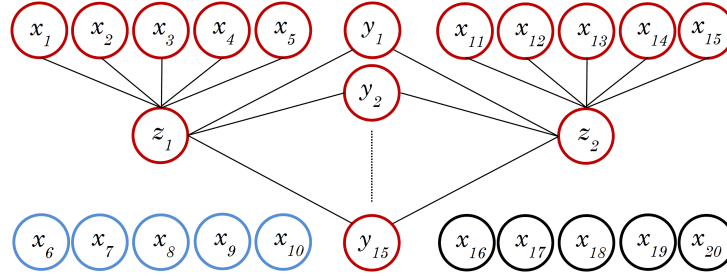

Figure 5: $\mathcal{C}^{(12)} = \{X_{6,10}, Y \cup Z \cup X_{1,5} \cup X_{11,15}, X_{16,20}\}$, where $u_{z_1}(\mathcal{C}^{(12)}) = \frac{133}{162}$.

a best Nash equilibrium for SDGs is NP-hard. To this aim, we provide a polynomial time reduction from the NP-complete RESTRICTED EXACT COVER by 3-SETS $(RXC3)$ problem (Gonzalez, 1985) which is defined as follows.

RESTRICTED EXACT COVER BY 3-SETS $(RXC3)$ problem. Let $m = 3p$ for some integer $p \geq 1$. An instance of the $RXC3$ problem consists of a universe set $U = \{u_1, \ldots, u_m\}$ of m elements and a collection $\mathcal{B} = \{B_1, \ldots, B_m\}$ of 3-elements subsets of U (triples) such that $\bigcup_{j=1}^m B_j = U$ and each element of U appears exactly in three triples. The problem is to decide if $\mathcal{B}$ contains an exact cover for U, i.e. a subcollection $\mathcal{F} \subseteq \mathcal{B}$ such that every element of U occurs in exactly one triple of $\mathcal{F}$.

In the following we show how to transform an instance of $RXC3$ to a suitable instance of SDG with the technique of local replacement. Roughly speaking, we use a subgraph to represent every triple in $\mathcal{B}$ (see Figure 8) and we show that the composition of these subgraphs is a graph with certain properties that has a Nash stable clustering with social welfare at least $\frac{257}{20}p$ if and only if $\mathcal{B}$ contains an exact cover.

NP-hardness Reduction from $RXC3$ to SDG. Suppose we are given an instance $\mathcal{I} = (U, \mathcal{B})$ of $RXC3$. We can define an instance of SDG by specifying the underlying undirected graph $G_{\mathcal{I}} = (V, E)$ as follows:

- For each of the m elements $u \in U$, we have a node $y \in V$. For convenience, for all $i \in [m]$ we denote by $Y_i = \{y_{i,1}, y_{i,2}, y_{i,3}\}$ the set of nodes associated with the triple $B_i = (u_{i,1}, u_{i,2}, u_{i,3}) \in \mathcal{B}$. Thus, let $Y = \cup_{i=1}^m Y_i$ be the set of the m nodes associated with U. For each triple $B_i \in \mathcal{B}$, with $i \in [m]$, we consider a set of four nodes $K_i = \{z_{i,1}, z_{i,2}, z_{i,3}, z_{i,4}\}$
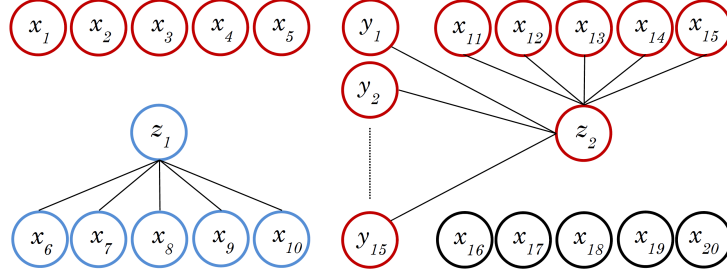
Figure 6: $\mathcal{C}^{(13)} = (\mathcal{C}^{(12)}, z_1, x_6) = \{X_{6,10} \cup \{z_1\}, Y \cup \{z_2\} \cup X_{1,5} \cup X_{11,15}, X_{16,20}\}$ where $u_{z_1}(\mathcal{C}^{(13)}) = \frac{5}{6} > \frac{133}{162}$.
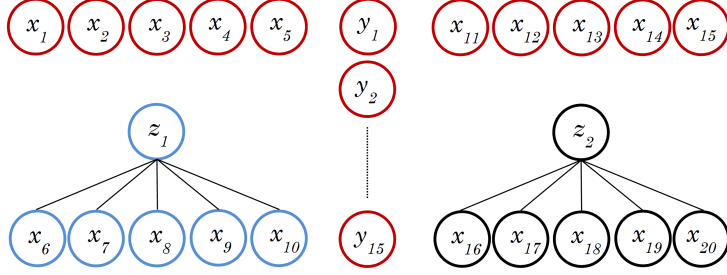

Figure 7: $\mathcal{C}^{(14)} = (\mathcal{C}^{(13)}, z_2, x_{16}) = \{X_{6,10} \cup \{z_1\}, Y \cup X_{1,5} \cup X_{11,15}, X_{16,20} \cup \{z_2\}\}$ where $u_{z_2}(\mathcal{C}^{(14)}) = \frac{5}{6} > \frac{10}{13}$.

and one additional node x_i. Therefore, if $X = \{x_1, \ldots, x_m\}$ and $K = \cup_{i=1}^m K_i$, we have that $V = K \cup X \cup Y$, with $|V| = 6m$.

- For what concerns the set of edges E, for each $i \in [m]$, $G_{\mathcal{I}}$ contains one 4-clique $K_4(K_i)$ which includes all vertices in K_i (i.e., six edges), a set of three edges $\{(x_i, z_{i,2}), (x_i, z_{i,3}),$- $(x_i, z_{i,4})\}$ and a set of further three edges $\{(x_i, y_{i,1}), (x_i, y_{i,2}), (x_i, y_{i,3})\}$. Consequently, it must be $|E| = 12m$.

  As an example, a triple $B_i = (u_{i,1}, u_{i,2}, u_{i,3}) \in \mathcal{B}$ of $RXC3$ corresponds to the subgraph of $G_{\mathcal{I}}$ depicted in Figure 8. We notice that, since each element of U appears exactly in three triples, each $y \in Y$ has exactly three incident edges.

Clearly such a reduction can be done in polynomial time. In order to complete the proof and prove the hardness, we show that the constructed SDG instance $G_{\mathcal{I}}$ has a Nash stable clustering with social welfare at least $\frac{257}{20}p$ if and only if the $RXC3$ instance $\mathcal{I}$ has an exact cover.

Before proving the main theorem, let us analyze some fundamental properties of Nash stable clusterings for $G_{\mathcal{I}}$.
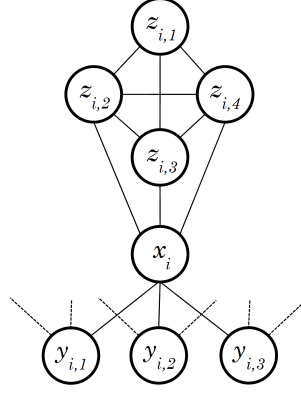
Figure 8: Subgraph of $G_{\mathcal{I}}$ associated with a triple $B_i = (u_{i,1}, u_{i,2}, u_{i,3}) \in \mathcal{B}$ of $RXC3$.

3.2.1 STRUCTURAL PROPERTIES OF STABLE CLUSTERINGS FOR ANY INSTANCE $G_{\mathcal{I}}$ OF SDGS.

In the following we discuss some structural properties of Nash stable clusterings for $G_{\mathcal{I}}$. In particular, Lemma 1 stresses that agents of K_i (for $i \in [m]$) must always belong to the same coalition, whereas Lemmas 2 and 3 deal with the cases in which agents of K_i must belong to the same coalition as x_i.

Lemma 1 *Let $\mathcal{C}$ be any Nash stable clustering for an instance $G_{\mathcal{I}}$ of SDG. For all $i \in [m]$, agents in K_i are always included in the same coalition.*

Proof. For any $i \in [m]$, denote by $C_1 = \mathcal{C}(z_{i,1})$ the coalition containing $z_{i,1}$ in the stable clustering $\mathcal{C}$. We show that, in order to guarantee stability, it must be $K_i \subseteq C_1$. Arguing by contradiction, assume that there exists an agent in K_i which is not included in C_1. Without loss of generality, let $z_{i,4} \notin C_1$ such an agent and let $C_4 = \mathcal{C}(z_{i,4})$.

A case analysis shows that such an assumption leads to a contradiction. We can distinguish different cases based on the cardinality of C_1 and C_4. Since in a stable clustering an agent cannot be isolated, C_1 and C_4 must contain at least an agent adjacent to $z_{i,1}$ and $z_{i,4}$, respectively, and therefore $|C_1| \geq 2$ and $|C_4| \geq 2$.

- First of all, assume that $|C_1| = 2$, and let $C_1 = \{z_{i,1}, z_{i,2}\}$. We can distinguish three cases depending on the cardinality of C_4:

  – $|C_4| = 2$. If $C_4 = \{x_i, z_{i,4}\}$, $z_{i,3}$ cannot be Nash stable as it is disconnected, i.e., $u_{z_{i,3}}(\mathcal{C}) = 0$, and therefore she can improve her utility by moving to C_1 or C_4. If $C_4 = \{z_{i,3}, z_{i,4}\}$, all agents in K_i are not Nash stable, as each of them can make an improving move to the other coalition and achieve a utility of $\frac{2}{3} > \frac{1}{2}$. Both cases lead to a contradiction. Clearly, $z_{i,4}$ would be disconnected and not stable in any other coalition.

  – $|C_4| = 3$. The only coalition without disconnected agents is $C_4 = \{z_{i,3}, z_{i,4}, x_i\}$. Nevertheless, agents $z_{i,1}$ and $z_{i,2}$, with $u_{z_{i,1}}(C_1) = u_{z_{i,2}}(C_1) = \frac{1}{2}$, can improve by moving to C_4, as they would achieve utilities $u_{z_{i,1}}((C_1, z_{i,1}, z_{i,4})) = \frac{2 + \frac{1}{2}}{4} = \frac{5}{8}$ and $u_{z_{i,2}}((C_1, z_{i,2}, z_{i,4})) = \frac{3}{4}$: a contradiction.

- $|C_4| > 3$. In such a case it must necessarily be $\{z_{i,3}, z_{i,4}, x_i\} \subset C_4$. Since $u_{z_{i,4}} = u_{z_{i,3}} \leq \frac{2 + \frac{|C_4|-3}{2}}{|C_4|} = \frac{|C_4|+1}{2|C_4|} = \frac{1}{2} + \frac{1}{2|C_4|} < \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, agents $z_{i,3}$ and $z_{i,4}$ can make an improving move to C_1 achieving a utility of $\frac{2}{3}$: a contradiction.

- Assume now that $|C_1| = 3$. If $C_1 = \{z_{i,1}, z_{i,2}, z_{i,3}\}$, any possible coalition C_4 without disconnected agents must include x_i. In such a case, since $u_{z_{i,4}}(C_4) < \frac{1 + \frac{|C_4|-2}{2}}{|C_4|} = \frac{1}{2}$, $z_{i,4}$ can improve her utility by moving to C_1. On the other hand, if $C_1 = \{z_{i,1}, z_{i,2}, x_i\}$, it must be $C_4 = \{z_{i,3}, z_{i,4}\}$. Agents $z_{i,3}$ and $z_{i,4}$ can move to C_1, whereas $z_{i,1}$ and x_i can move to C_4, improving their utilities from $\frac{1}{2}$ to $\frac{3}{4}$ and $\frac{2}{3}$, respectively. Both cases lead to a contradiction.

- Finally, in order to conclude the proof, let us consider the case $|C_1| \geq 4$. Since by assumption $C_1 \neq K_i$, we have that $x_i \in C_1$. Thus, it must be $\{z_{i,1}, z_{i,2}, x_i\} \subseteq C_1$ and $C_4 = \{z_{i,3}, z_{i,4}\}$. Since $u_{z_{i,1}}(C_1) \leq \frac{1 + \frac{1}{2} + \frac{|C_1|-3}{3}}{|C_1|} = \frac{6 + 3 + 2|C_1| - 6}{6|C_1|} = \frac{3 + 2|C_1|}{6|C_1|} = \frac{1}{2|C_1|} + \frac{1}{3} < \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$, agent $z_{i,1}$ can make an improving move to C_4, by achieving a utility of $\frac{2}{3}$: a contradiction.

$\square$

Lemma 1 guarantees that in any Nash stable clustering for an instance $G_{\mathcal{I}}$ of SDG, all the four agents in K_i are always included in the same coalition, for all $i \in [m]$. However, a 4-clique K_i may be together with some other K_j in a coalition C. The remaining structural properties, aiming at characterizing conditions under which a stable coalition can include one or more K_i, depend on how many agents $x_i \in X$ are included in a same *connected* coalition. Namely, Lemmas 2 and 3 distinguish the two cases where a Nash stable coalition C contains one or more agents of X. We will refer to the coalitions containing at most one agent $x_i \in X$ as *basic* coalitions or (basic) x_i-coalitions to emphasize that they just contain x_i. On the other hand, coalitions including at least two agents of X are called *composed* coalitions.

Lemma 2 *Let $\mathcal{C}$ be any Nash stable clustering for an instance $G_{\mathcal{I}}$ of SDG. For any basic x_i-coalition $C \in \mathcal{C}$ containing at most two agents of Y_i, agents in K_i must belong to C as well, i.e., $K_i \subset C$. On the other hand, if C contains exactly three agents of Y_i, then K_i may not be included in C.*

Proof. Let C be a Nash stable basic x_i-coalition in $\mathcal{C}$ including at most two agents of Y_i (just like coalitions t_0, t_1, t_2 depicted in Figure 9). By Lemma 1, since in $\mathcal{C}$ all the agents in K_i are within the same coalition, the cases are two: either $K_i \subseteq C$ (as we want to prove) or $K_i \not\subset C$. Assume by contradiction that $K_i \not\subset C$. In such a case $|C| \leq 3$, and thus $u_{x_i}(C) \leq \frac{2}{3}$. Therefore, agent x_i could improve her utility by moving to K_i, as she would achieve utility $u_{x_i}(K_i \cup \{x_i\}) = \frac{7}{10} > \frac{2}{3}$: a contradiction. Thus, in order to guarantee stability, it must be $K_i \subset C$.

On the other hand, we can notice that if $C = \{x_i, y_{i,1}, y_{i,2}, y_{i,3}\}$, coalition K_i is actually stable even if $K_i \not\subset C$ (see the two coalitions corresponding to type t_4 in Figure 9). In fact, no agent in K_i can make an improving move by moving to C, because agent $z_{i,1}$ would obtain a null utility, whereas agents $z_{i,2}, z_{i,3}, z_{i,4}$ would decrease their utility from $\frac{3}{4}$ to $\frac{1}{2}$. Moreover, by moving to any other coalition in $\mathcal{C}$, all agents in K_i would be disconnected and would obtain a utility zero. We can also stress that, by moving from C to K_i, agent x_i would decrease her utility from $\frac{3}{4}$ to $\frac{7}{10}$, and agents $y_{i,1}, y_{i,2}, y_{i,3}$ would get utility zero. $\square$

It is worthwhile underlining that by Lemmas 1 and 2, *Nash stable basic coalitions* can take only the forms depicted in Figure 9, where the subfigure t_4 actually shows a pair of coalitions, i.e., K_i and $Y_i \cup \{x_i\}$.
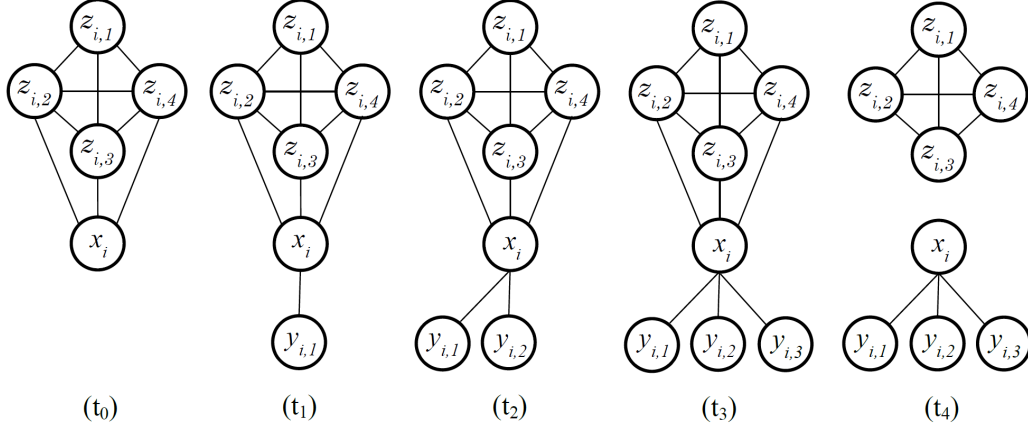

Figure 9: Nash stable basic coalitions for any instance $G_{\mathcal{I}}$ of SDG.

The following structural result holds for composed coalitions.

Lemma 3 *Let $\mathcal{C}$ be any Nash stable clustering for an instance $G_{\mathcal{I}}$ of SDG. If two agents x_i and x_j of X belong to a same connected composed coalition $C = \mathcal{C}(x_i) = \mathcal{C}(x_j)$, then agents in K_i and K_j must belong to the same coalition as well, i.e., $K_i \cup K_j \subset C$.*

Proof. First of all we notice that if x_i and x_j belong to the same Nash stable connected coalition C, there must exist a path between them. Thus, there exists at least a node $y \in Y_i \cap Y_j$ such that $\{x_i, x_j, y\} \subseteq C$ (see Figure 10).
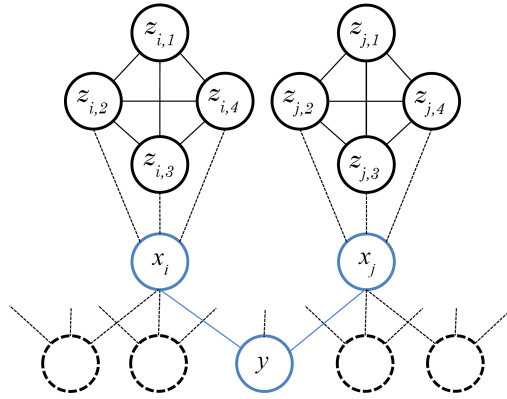

Figure 10: A stable connected composed coalition containing x_i and x_j.

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It is worth recalling that by Lemma 1 all the four agents in K_i (resp. K_j) must be together in a same coalition. Arguing by contradiction, assume that $K_i \not\subset C$. A case analysis shows that, in order to guarantee stability, agents in K_i must be in C.

Let $\nu_i = |Y_i \cap C| \geq 1$ and $\nu_j = |Y_j \cap C| \geq 1$ denote the number of agents adjacent to x_i and x_j which are in Y_i and Y_j, respectively. We stress that, if $K_i \not\subset C$, agent x_i has $\nu_i \leq 3$ agents at distance 1, $1 + \alpha$ agents at distance 2 (included x_j), for $\alpha \geq 0$, and all the other $\beta \geq 0$ agents at distance at least 3. Namely:

$$u_{x_i}(C) \leq \frac{\nu_i + \frac{1+\alpha}{2} + \frac{\beta}{3}}{\nu_i + 2 + \alpha + \beta} = \frac{6\nu_i + 3 + 3\alpha + 2\beta}{6(\nu_i + 2 + \alpha + \beta)}. \tag{1}$$

Assume first that $\nu_i \leq \nu_j$. We notice that when $\nu_i = \nu_j = 3$ there must be at least one agent in $Y_j \cap C$ adjacent to x_j at distance 3, and then $\beta \geq 1$. Thus, Equation (1) implies that:

$$u_{x_i}(C) \leq \begin{cases} \frac{9+3\alpha+2\beta}{6(3+\alpha+\beta)} \leq \frac{3(3+\alpha+\beta)}{6(3+\alpha+\beta)} = \frac{1}{2} & \text{if } \nu_i = 1 \\[2mm] \frac{15+3\alpha+2\beta}{6(4+\alpha+\beta)} < \frac{4(4+\alpha+\beta)}{6(4+\alpha+\beta)} = \frac{2}{3} & \text{if } \nu_i = 2 \\[2mm] \frac{21+3\alpha+2\beta}{6(5+\alpha+\beta)} \leq \frac{4(5+\alpha+\beta)+1}{6(5+\alpha+\beta)} = \frac{2}{3} + \frac{1}{6(5+\alpha+\beta)} \leq \frac{2}{3} + \frac{1}{36} = \frac{25}{36} < \frac{7}{10} & \text{if } \nu_i = 3 \end{cases} \tag{2}$$

All these cases lead to a contradiction, since agent x_i could always improve her utility by moving from C to K_i. In fact, $u_{x_i}(K_i \cup \{x_i\}) = \frac{3+\frac{1}{2}}{5} = \frac{7}{10} > u_{x_i}(C)$ (see Equation (2)) and therefore, if $\nu_i \leq \nu_j$, agents in K_i must be included in C.

Assume now that $\nu_i > \nu_j$. By applying the previous arguments to x_j, agents in K_j must be in C. If by contradiction $K_i \not\subset C$ but $K_j \subset C$, agent x_i has $\nu_i \leq 3$ agents at distance 1 (notice that $\nu_i \geq 2$ since $\nu_j \geq 1$), $1 + \alpha$ agents at distance 2 (included x_j), for $\alpha \geq 0$, 4 agents at distance at least 3 ($z_{i,1}, z_{i,2}, z_{i,3}, z_{i,4}$) and further $\beta \geq 0$ agents at distance at least 3. That is:

$$u_{x_i}(C) \leq \frac{\nu_i + \frac{1+\alpha}{2} + \frac{4+\beta}{3}}{\nu_i + 6 + \alpha + \beta} = \frac{6\nu_i + 11 + 3\alpha + 2\beta}{6(\nu_i + 6 + \alpha + \beta)} < \frac{3(\nu_i + 6 + \alpha + \beta) + 3\nu_i}{6(\nu_i + 6 + \alpha + \beta)}$$

$$= \frac{1}{2} + \frac{\nu_i}{2(\nu_i + 6 + \alpha + \beta)} \leq \frac{1}{2} + \frac{3}{2(2+6)} = \frac{1}{2} + \frac{3}{16} = \frac{11}{16} < \frac{7}{10},$$

which is a contradiction since x_i could improve her utility to $\frac{7}{10}$ by moving from C to K_i. Therefore, if $\nu_i > \nu_j$, agents in K_i must be in C as well.

The claim follows by applying symmetric arguments to x_j. $\qquad \square$

3.2.2 STABILITY PRESERVING TRANSFORMATIONS OF INSTANCES $G_\mathcal{I}$ OF SDGS.

In the following we provide some transformation rules for coalitions in Nash stable clusterings, which obtain a strictly higher social welfare. Indeed, we first prove that every composed coalition in a stable clustering for $G_\mathcal{I}$ can be split in basic coalitions leading to a new clustering with a strictly higher social welfare (Proposition 4). Then, in Lemma 4, we prove that starting from such a (not necessarily stable) clustering, the game tends to converge to a stable clustering with at least the same social welfare, still composed only of basic coalitions. Splitting every resulting coalition of type t_3 in two subcoalitions corresponding to type t_4 preserves stability and further increases the

social welfare (Corollary 1). Therefore, we can finally state that a best Nash equilibrium for an instance $G_{\mathcal{I}}$ of SDG can always be found among clusterings including only basic coalitions of type t_0, t_1, t_2, t_4 (Corollary 2).

Let us preliminarly compute a lower bound on the average utility of the agents in any basic coalition of type t_0, t_1, t_2, t_4 (Proposition 2) and an upper bound on the agents' utilities in any stable composed coalition (Proposition 3).

Proposition 2 *In any basic coalition of type t_0, t_1, t_2, t_4 the average utility of the agents in the coalition is at least 0.639.*

Proof. With reference to the coalitions depicted in Figure 9, we can compute the social welfare as follows:

$$SW(t_0) = \frac{3 \cdot 4 + 2(3 + \frac{1}{2})}{5} = \frac{19}{5} = 3.8; \tag{3}$$

$$SW(t_1) = \frac{(3 + \frac{1}{2} + \frac{1}{3}) + 4(4 + \frac{1}{2}) + (1 + \frac{3}{2} + \frac{1}{3})}{6} = \frac{\frac{23}{6} + 18 + \frac{17}{6}}{6} = \frac{37}{9} \approx 4.1; \tag{4}$$

$$SW(t_2) = \frac{(3 + \frac{1}{2} + \frac{2}{3}) + 3 \cdot 5 + (5 + \frac{1}{2}) + 2(1 + \frac{4}{2} + \frac{1}{3})}{7} = \frac{\frac{25}{6} + 15 + \frac{11}{2} + \frac{20}{3}}{7}$$
$$= \frac{94}{21} \approx 4.47; \tag{5}$$

$$SW(t_4) = (4 \cdot \frac{3}{4}) + (\frac{3}{4} + 3 \cdot \frac{1 + \frac{2}{2}}{4}) = 3 + \frac{9}{4} = \frac{21}{4} = 5.25. \tag{6}$$

The claims follows by noting that by Equations (3-6), the average utility $\bar{u}$ of agents in basic coalitions of type t_0, t_1, t_2, t_4 is: $\bar{u}(t_0) = \frac{\frac{19}{5}}{5} = \frac{19}{25} = 0.76$, $\bar{u}(t_1) = \frac{\frac{37}{9}}{6} = \frac{37}{54} > 0.68$, $\bar{u}(t_2) = \frac{\frac{94}{21}}{7} = \frac{94}{147} > 0.639$, $\bar{u}(t_4) = \frac{\frac{21}{4}}{8} = \frac{21}{32} > 0.656$. $\qquad\square$

Proposition 3 *Let $\mathcal{C}$ be any Nash stable clustering for an instance $G_{\mathcal{I}}$ of SDG. In any connected composed coalition the utility of all the agents in the coalition is strictly less than 0.635.*

Proof. Let $C = \mathcal{C}(x_i) = \mathcal{C}(x_j)$ be a connected composed coalition including agents $x_i, x_j \in X$. By Lemma 3, we know that $K_i \cup K_j \subseteq C$ and that there must be a path between x_i and x_j. Thus, we can give an upper bound to the utility of all agents in C as follows (see Figure 10).

- Each $z_{i,1}$ has at least 3 agents at distance 1 ($z_{i,2}, z_{i,3}, z_{i,4}$), one agent at distance 2 (x_i), one agent at distance 4 (x_j), 3 agents at distance 5 ($z_{j,2}, z_{j,3}, z_{j,4}$), and one agent at distance 6 ($z_{j,1}$). All the other α agents (including the ones in $Y_i \cup Y_j$) have distance at least 3, so that $u_{z_{i,1}}(C) \leq \frac{3 + \frac{1}{2} + \frac{\alpha}{3} + \frac{1}{4} + \frac{3}{5} + \frac{1}{6}}{10 + \alpha} < 0.46$.

- Each $z_{i,2}$ (and symmetrically $z_{i,3}$ and $z_{i,4}$) has at least 4 agents at distance 1 ($z_{i,1}, z_{i,3}, z_{i,4}, x_i$), one agent at distance 3 (x_j), 3 agents at distance 4 ($z_{j,2}, z_{j,3}, z_{j,4}$), and one agent at distance 5 ($z_{j,1}$). All the other α agents have distance at least 2, and thus $u_{z_{i,2}}(C) \leq \frac{4 + \frac{\alpha}{2} + \frac{1}{3} + \frac{3}{4} + \frac{1}{5}}{10 + \alpha} \leq \frac{317}{600} < 0.53$.

- Each x_i can have at most 6 agents at distance 1 ($z_{i,2}, z_{i,3}, z_{i,4}, y_{i_1}, y_{i_2}, y_{i_3}$), 2 agents at distance 2 ($z_{i,1}, x_j$), 3 agents at distance 3 ($z_{j,2}, z_{j,3}, z_{j,4}$), and one agent at distance 4 ($z_{j,1}$). All the other α nodes have distance at least 2, so that $u_{x_i}(C) \leq \frac{6+\frac{\alpha+2}{2}+\frac{3}{3}+\frac{1}{4}}{13+\alpha} \leq \frac{33}{52} < 0.635$.

- Each $y \in Y \cap C$ has at most 3 agents at distance 1 (x_i, x_j, x_k), 6 agents at distance 2 ($z_{i,2}, z_{i,3}, z_{i,4}, z_{j,2}, z_{j,3}, z_{j,4}$), and 2 agents at distance 3 ($z_{i,1}, z_{j,1}$). All the other α nodes have distance at least 2. This gives an upper bound to the utility of y of $\frac{3+\frac{6+\alpha}{2}+\frac{2}{3}}{12+\alpha} \leq \frac{5}{9} < 0.56$.

$\square$

Based on the above results, Proposition 4 analyzes how the social welfare of a stable clustering $\mathcal{C}$ changes when partitioning every disconnected coalition into connected subcoalitions, and then splitting every resulting connected composed coalition in basic coalitions of type t_0, t_1, t_2, t_4.

Proposition 4 *Let $\mathcal{C}$ be any Nash stable clustering for an instance $G_{\mathcal{I}}$ of SDG. All the composed coalitions $C \in \mathcal{C}$ can be split in basic coalitions of type t_0, t_1, t_2, t_4, obtaining a new (non necessarily stable) clustering $\bar{\mathcal{C}}$ with a strictly higher social welfare.*

Proof. Consider any clustering $\bar{\mathcal{C}}$ obtained from $\mathcal{C}$ by first partitioning every disconnected coalition $C \in \mathcal{C}$ into connected subcoalitions, and then later splitting every connected composed coalition in subcoalitions of type t_0, t_1, t_2, t_4.

First of all, let $C \in \mathcal{C}$ be a disconnected coalition and let $k > 1$ denote the number of connected subcoalitions $C_1, \ldots, C_k$ in C, such that $C_1 \cap \ldots \cap C_k = \emptyset$ and $C = C_1 \cup \ldots \cup C_k$. Moreover, let $\mathcal{C}'$ be the clustering obtained from $\mathcal{C}$ by partitioning C into the k connected subcoalitions $C_1, \ldots, C_k$, i.e., $\mathcal{C}' = \mathcal{C} \cup \{C_1, \ldots, C_k\} \setminus \{C\}$.

Clearly, $u_x(\mathcal{C}') = u_x(\mathcal{C})$ for all $x \in V \setminus C$. On the other hand, since $\mu_x(G(C)) = \mu_x(G(C_i))$ for every $x \in C_i$, it must be $u_x(\mathcal{C}') = \frac{\mu_x(G(C_i))}{|C_i|} > \frac{\mu_x(G(C))}{|C|} = u_x(\mathcal{C}), \forall x \in C$. This proves that the clustering obtained from $\mathcal{C}$ by partitioning every disconnected coalition C into connected subcoalitions has a strictly higher social welfare.

Now, let $\bar{\mathcal{C}}$ be any clustering obtained by further splitting every resulting connected composed coalition in basic coalitions of type t_0, t_1, t_2, t_4. Therefore, since the worst average utility 0.639 of the agents in any basic coalition of type t_0, t_1, t_2, t_4 (see Proposition 2) is greater than the best utility 0.635 of all the agents in the composed coalitions (see Proposition 3), the social welfare actually strictly increases in any $\bar{\mathcal{C}}$ as well. This proves the claim. $\square$

Notice that all the coalitions of type t_3 possibly contained within the Nash stable clustering $\mathcal{C}$ remain unchanged in $\bar{\mathcal{C}}$ after the abovementioned transformations. Therefore, such clustering $\bar{\mathcal{C}}$ can actually include only coalitions of the five types $t_0, \ldots, t_4$ depicted in Figure 9.

In the following, with a little abuse of language, sometimes we will refer to coalitions of type t_i directly as coalitions t_i, for $i \in \{0, \ldots, 4\}$.

Lemma 4 *Let $\bar{\mathcal{C}}$ be any clustering for an instance $G_{\mathcal{I}}$ of SDG containing only basic coalitions of the five types $t_0, \ldots, t_4$. Starting from $\bar{\mathcal{C}}$, the game admits a Nash equilibrium with at least the same social welfare.*

Proof. In order to prove that the game admits a Nash equilibrium starting from any $\bar{\mathcal{C}}$, we can define a suitable potential function which strictly increases each time an agent makes an improving move.

Therefore, we argue that the game has the finite improvement path property, and thus it admits a Nash equilibrium.

Given any clustering $\bar{\mathcal{C}}$ including only coalitions of the types $t_0, \ldots, t_4$, and denoted by α_i the number of coalitions of type t_i included in $\bar{\mathcal{C}}$, we consider the following potential function, coincident with the social welfare:

$$\phi(\bar{\mathcal{C}}) = SW(\bar{\mathcal{C}}) = \sum_{i=0}^{4} \alpha_i SW(t_i). \tag{7}$$

First of all we can notice that agents in $K_i \cup \{x_i\}$ are stable in every coalition in $\bar{\mathcal{C}}$:

- Any agent $z \in K_i$ would be disconnected in any other coalition $t_j \cup \{z\}$, and thus by moving her utility would decrease to zero.

- Any agent x_i would obtain a utility zero by moving from t_i to t_j whenever the new formed coalition $t_j \cup \{x_i\}$ is disconnected. This happens for instance when x_i moves from t_i, for $i \in \{3, 4\}$, or when x_i moves to $t_j = t_0$. On the other hand, for $i \in \{0, 1, 2\}$, agent x_i has utility $u_{x_i}(t_i) = \frac{3+i+\frac{1}{2}}{5+i} \geq \frac{7}{10}$, and she would decrease her utility by moving to any other basic coalition t_j with $j \geq 1$ even if $t_j \cup \{x_i\}$ is connected. In fact:

  - $u_{x_i}(t_1 \cup \{x_i\}) = \frac{1+\frac{1}{2}+\frac{3}{3}+\frac{1}{4}}{7} = \frac{11}{28} < \frac{7}{10} \leq u_{x_i}(t_i)$;

  - $u_{x_i}(t_j \cup \{x_i\}) \leq \frac{2+\frac{1}{2}+\frac{3+(j-2)}{3}+\frac{1}{4}}{7+(j-1)} \leq \frac{15}{32} < \frac{7}{10} \leq u_{x_i}(t_i)$, for $j \in \{2, 3\}$;

  - $u_{x_i}(t_4 \cup \{x_i\}) \leq \frac{2+\frac{1}{2}+\frac{1}{3}}{5} = \frac{17}{30} < \frac{7}{10} \leq u_{x_i}(t_i)$.

It remains to examine agents $y \in Y$ included in any t_i. In every basic coalition we can compute the following utilities $u_y(t_i)$:

$$u_y(t_i) = \frac{1+\frac{3+(i-1)}{2}+\frac{1}{3}}{6+(i-1)} = \begin{cases} \frac{17}{36} \approx 0.472 & \text{if } i = 1 \\ \frac{10}{21} \approx 0.476 & \text{if } i = 2 \\ \frac{23}{48} \approx 0.479 & \text{if } i = 3 \end{cases} \tag{8}$$

$$u_y(t_4) = \frac{1+\frac{2}{2}}{4} = \frac{1}{2} = 0.5.$$

Since no agent y can move to t_3 or t_4 without decreasing her utility to zero, we can focus only on possible moves from t_i to t_j with $j \leq 2$. Moreover, since if y moves from t_i to t_j we either obtain t_{j+1} or a coalition where y is disconnected, by Equation (8) it is evident that t_3 and t_4 are already stable, because every y in t_3 and t_4 would not increase her utility by moving to t_j with $j \leq 2$, and that every $y \in t_i$ for $i \in \{1, 2\}$ can improve her utility only by moving to a coalition of type t_j for $i \leq j \leq 2$.

Therefore, in a clustering $\bar{\mathcal{C}}$ all the coalitions of type t_0, t_3 and t_4 are already Nash stable, whereas if $\bar{\mathcal{C}}$ is not Nash stable it necessarily depends on the instability of some agent y in t_1 or t_2. In the following we prove that when such an agent y performs an improving move to t_j, for $i \leq j \leq 2$, the social welfare strictly increases, and the new coalitions are still of types $t_0, \ldots, t_4$.

For the sake of simplicity, let us denote by $(\bar{\mathcal{C}}, t_i, t_j)$ the improving deviation of an agent y from a cluster of type t_i to a cluster of type t_j, in $\bar{\mathcal{C}}$. Since we know that y can move only from t_i to t_j, for $i \in \{1, 2\}$ and $i \leq j \leq 2$, we can distinguish three cases:

1. if $t_i = t_j = t_1$, i.e., y moves from a coalition of type t_1 to another coalition of type t_1, from two coalitions of type t_1 we obtain two coalitions respectively of type t_0 (stable) and t_2, and therefore: $\phi((\bar{\mathcal{C}}, t_1, t_1)) - \phi(\bar{\mathcal{C}}) = SW(t_0) + SW(t_2) - 2SW(t_1) = \frac{19}{5} + \frac{94}{21} - 2 \cdot \frac{37}{9} = \frac{869}{105} - \frac{74}{9} \approx 8.27 - 8.22 = 0.05$;

2. if $t_i = t_j = t_2$, i.e., y moves from a coalition of type t_2 to another coalition of type t_2, from two coalitions of type t_2 we obtain two coalitions of type t_1 and t_3 (stable), and therefore: $\phi((\bar{\mathcal{C}}, t_2, t_2)) - \phi(\bar{\mathcal{C}}) = SW(t_1) + SW(t_3) - 2SW(t_2) = \frac{37}{9} + \frac{39}{8} - 2 \cdot \frac{94}{21} = \frac{647}{72} - \frac{188}{21} \approx 8.98 - 8.95 = 0.03$;

3. if $t_i = t_1$, $t_j = t_2$ i.e., y moves from a coalition of type t_1 to a coalition of type t_2, from two coalitions of type t_1 and t_2 we obtain two coalitions of type t_0 and t_3 (both stable), and therefore: $\phi((\bar{\mathcal{C}}, t_1, t_2)) - \phi(\bar{\mathcal{C}}) = SW(t_0) + SW(t_3) - (SW(t_1) + SW(t_2)) = \frac{19}{5} + \frac{39}{8} - (\frac{37}{9} + \frac{94}{21}) = \frac{347}{40} - \frac{541}{63} \approx 8.675 - 8.587 = 0.088$.

In every case the social welfare strictly increases. Since the number of possible clusterings is finite, this implies that starting from any clustering $\bar{\mathcal{C}}$ consisting of coalitions of type $t_0, \ldots, t_4$, there exists a finite improvement path converging to a Nash stable clustering (still containing coalitions of type $t_0, \ldots, t_4$) with a strictly higher social welfare. $\qquad\square$

We can now prove the following corollary.

Corollary 1 *Let $\mathcal{C}$ be any Nash stable clustering for an instance $G_{\mathcal{I}}$ of SDG including only basic coalitions. Every x_i-coalition $C = K_i \cup \{x_i\} \cup Y_i$ of type t_3 can be split in two sub-coalitions $C' = K_i$ and $C'' = \{x_i\} \cup Y_i$, obtaining a new stable clustering $\mathcal{C}' = \mathcal{C} \setminus \{C\} \cup \{C', C''\}$ with a strictly higher social welfare.*

Proof. Consider the clustering $\mathcal{C}'$ obtained from $\mathcal{C}$ by splitting a coalition C of type t_3 (see Figure 9) in two sub-coalitions $C' = K_i$ and $C'' = \{x_i\} \cup Y_i$ corresponding to type t_4. The stability of $\mathcal{C}'$ directly arises from the proofs of Lemmas 2 and 4. Moreover, we can easily verify that $\mathcal{C}'$ has a strictly higher social welfare. In fact:

$$SW(t_3) = \frac{(3 + \frac{1}{2} + \frac{3}{3}) + 3(4 + \frac{3}{2}) + (6 + \frac{1}{2}) + 3(1 + \frac{5}{2} + \frac{1}{3})}{8} = \frac{\frac{9}{2} + \frac{33}{2} + \frac{13}{2} + \frac{23}{2}}{8} \tag{9}$$
$$= \frac{39}{8} = 4.875$$

and therefore, by Equations (6) and (9), $SW(t_4) - SW(t_3) = \frac{21}{4} - \frac{39}{8} = \frac{3}{8} = 0.375$. $\qquad\square$

On the basis of these considerations, we can state the following result.

Corollary 2 *Any best Nash equilibrium for an instance $G_{\mathcal{I}}$ of SDG must include only basic coalitions of the 4 types t_0, t_1, t_2, t_4.*

We are now ready to claim the following main theorem, whose proof comes directly from the following Lemmas 5 and 6 below.

Theorem 2 *Computing a best Nash equilibrium for SDGs is NP-hard.*

Lemma 5 *If there is an exact cover for the input instance $\mathcal{I}$ of $RXC3$, then there exists a Nash equilibrium for the instance $G_\mathcal{I}$ of SDG with social welfare at least $\frac{257}{20}p$.*

Proof. If there is an exact cover, then there is a clustering consisting of exactly p coalitions of type t_4 associated to the triples of the cover, and $(m - p) = 2p$ remaining coalitions of type t_0. By Lemma 4 such a clustering is stable and the claim follows by simply summing up the utilities of all the agents, i.e., $\frac{21}{4}p + \frac{19}{5}(2p)$. □

Lemma 6 *If there is not an exact cover for the input instance $\mathcal{I}$ of $RXC3$, then every Nash equilibrium for the instance $G_\mathcal{I}$ of SDG has social welfare strictly less than $\frac{257}{20}p$.*

Proof. Assume then that there is not an exact cover. By Corollary 2, the stable clusterings achieving the best social welfare include only basic coalitions of the 4 types t_0, t_1, t_2, t_4. Consider any such stable clustering $\mathcal{C}$. For each $i \in [m]$, we rearrange the social welfare of each coalition of $\mathcal{C}$ in such a way that $\frac{19}{5}$ is always accounted to the nodes $z_{i,1}, z_{i,2}, z_{i,3}, z_{i,4}, x_i$, and the surplus is equally shared among the agents $y \in Y_i$ belonging to the coalition. We have three cases for charging such y agents, depending on how many of them are contained in each coalition: in case of t_1 there is a single y agent and we account to her all the remaining social welfare, that is $\frac{14}{45} \approx 0.31$; in the case t_2 there are two y agents, and we account $\frac{71}{210} \approx 0.338$ to each of them; in the case t_4 there are three y agents, and we account to each $\frac{29}{60} \approx 0.483$. Notice that, since $\frac{29}{60} > \frac{71}{210} > \frac{14}{45}$, a clustering with social welfare equal to $\frac{29}{60}m + \frac{19}{5}m = \frac{257}{60}m = \frac{257}{20}p$ can be obtained if and only if $\frac{29}{60}$ is accounted to all the y nodes, which would imply the existence of an exact cover: a contradiction. □

4. Price of Anarchy and Price of Stability

In this section we first provide matching upper and lower bounds on the price of anarchy of SDGs. We then focus on the price of stability, and prove that there exists an instance of SDGs for which the lower bound on the PoS is $\frac{6}{5} - \epsilon$, for any $\epsilon > 0$. Finally, we study the PoS for graphs with girth 4 and girth at least 5, the *girth* being the length of the shortest cycle in the graph. In the sequel we call *star* a tree consisting of one vertex (the center) adjacent to all the other vertices (the leaves).

Theorem 3 *There exists a class of SDGs having price of anarchy $\Theta(n)$.*

Proof. We recall that, by the remark made in Section 2, we can focus on SDGs for non-singleton and connected graphs. By the definition of the game, the social welfare of any clustering is upper bounded by $n - 1$ (and therefore $\text{OPT} \leq n - 1$), as every agent has utility at most $\frac{n-1}{n}$. Such an upper bound can be attained only by the grand coalition in complete connected graphs. On the other hand, since in any equilibrium $\mathcal{C}$ every agent has at least one neighbor in her coalition, thus having utility at least $\frac{1}{n}$, it is $SW(\mathcal{C}) \geq 1$.

It remains to show that there exists a class of SDGs having price of anarchy $\Omega(n)$. To this end, for any positive integer n', consider the graph of $n = 8n'$ nodes depicted in Figure 11. In the Nash stable solution $\mathcal{C}$ illustrated in Figure 12,

$$SW(\mathcal{C}) = \frac{n}{4} \cdot \frac{2(1 + \frac{1}{2} + \frac{1}{3}) + 2(2 + \frac{1}{2})}{4} = \frac{n}{4} \cdot \frac{\frac{26}{3}}{4} = \frac{13}{3}n'.$$

Another Nash stable solution $\mathcal{C}'$ is shown in Figure 13. In this case, the social welfare is

$$SW(\mathcal{C}') = 2 \cdot \frac{[2(1 + \frac{1}{2} + \frac{1}{3}) + 2(2 + \frac{1}{2})] \cdot \frac{n}{2}}{n/2} \cdot \frac{\frac{n}{2}}{4} = 2 \cdot \frac{\frac{26}{3}n'}{4n'} = \frac{13}{3}$$

and therefore

$$\text{PoA} \geq \frac{\frac{13n'}{3}}{\frac{13}{3}} = n' = \Omega(n).$$
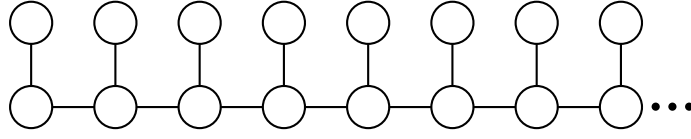
Thus, the claim follows. $\qquad\square$

Figure 11: A SDG with n agents having PoA $= \Omega(n)$.
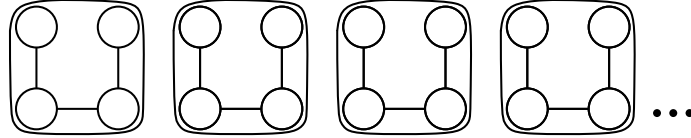

Figure 12: A Nash stable solution with $SW = \frac{13n}{24}$ for the SDG depicted in Figure 11.
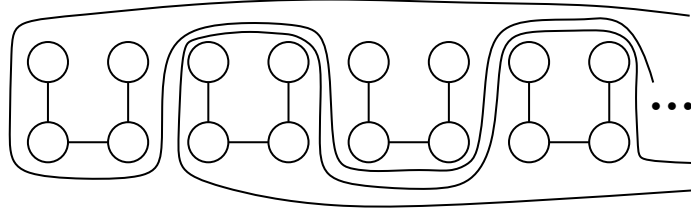

Figure 13: A Nash stable solution with $SW = \frac{13}{3}$ for the SDG depicted in Figure 11.

Let us now turn our attention to the PoS. A first general lower bound is established in the following theorem.

Theorem 4 *There exists a class of SDGs having* PoS *at least* $\frac{6}{5} - \epsilon$, *for any arbitrarily small* $\epsilon > 0$.

Proof. For any integer $\nu \geq 3$, consider a graph of $2\nu+2$ nodes consisting of a clique of $\nu+1$ agents $K_{\nu+1} = \{z_i | i \in [\nu + 1]\}$, each connected to the center x of a star of ν leaves $Y = \{y_i | i \in [\nu]\}$. An example of the graph for $\nu = 3$ is depicted in Figure 14(a).

Let $S = \{x\} \cup Y$ be the set of the agents in the star. Consider the clustering $\mathcal{C} = \{K_{\nu+1}, S\}$ (see Figure 14(b)), with social welfare

$$SW(\mathcal{C}) = \nu + \frac{\nu}{(\nu + 1)} \frac{(\nu + 3)}{2} = \frac{\nu(3\nu + 5)}{2(\nu + 1)}.$$
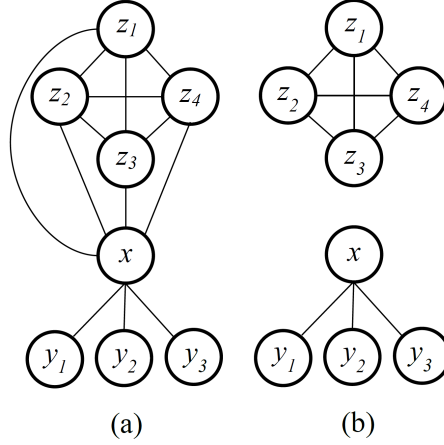
Figure 14: A SDG (a) and a corresponding clustering which is not Nash stable (b).

$\mathcal{C}$ is not Nash stable, since agent x can increase her utility from $\frac{\nu}{\nu+1}$ to $\frac{\nu+1}{\nu+2}$ by moving to $K_{\nu+1}$.

We now prove that the grand coalition (Figure 14(a)) is the only (and consequently, the best) Nash stable solution.

First of all, in any stable clustering, each agent $y_i \in Y$ cannot be isolated, and thus must be connected to x, otherwise she would have utility 0. Therefore, all agents in S must be in the same coalition. Moreover, there can be no pair of coalitions containing only agents of $K_{\nu+1}$, otherwise all agents in one coalition of minimum cardinality would move one after the other to the other coalition to increase their utility (even starting from two subsets of $K_{\nu+1}$ of the same size). As a consequence, any stable clustering cannot contain more than two coalitions, as being all the agents of S in a same coalition, the remaining two coalitions should be contained in $K_{\nu+1}$. Thus, in order to prove that also all agents in $K_{\nu+1}$ must be in the same coalition of x, it remains to show that any clustering consisting of two different coalitions $C_1 \subset K_{\nu+1}$ and $C_2 = V \setminus C_1$ (i.e., C_2 includes S and the remaining agents of $K_{\nu+1}$) is not stable. Arguing by contradiction, assume that there exists such a stable clustering $\mathcal{C} = \{C_1, C_2\}$. Let $\nu_1 = |C_1| > 1$ (C_1 cannot be singleton) and $\nu_2 = |K_{\nu+1} \cap C_2| \geq 1$ (as mentioned above, if $\nu_2 = 0$ the clustering is not stable) be the number of agents of $K_{\nu+1}$ belonging to C_1 and C_2 respectively, with $\nu_1 + \nu_2 = \nu + 1$.

Let z_1 be an agent in C_1 and z_2 be an agent in $K_{\nu+1} \cap C_2$. The utilities of z_1 and z_2 are the following:

- $u_{z_1}(C_1) = \frac{\nu_1 - 1}{\nu_1}$: if z_1 moves, her utility becomes $u_{z_1}(C_2 \cup \{z_1\}) = \frac{\nu_2 + \frac{\nu}{2} + 1}{\nu + \nu_2 + 2}$;

- $u_{z_2}(C_2) = \frac{\nu_2 + \frac{\nu}{2}}{\nu + \nu_2 + 1}$: if z_2 moves, her utility becomes $u_{z_2}(C_1 \cup \{z_2\}) = \frac{\nu_1}{\nu_1 + 1}$.

We want to prove that (i) if agent z_1 does not want to move, then z_2 prefers to move, and (ii) if agent z_2 does not want to move, then z_1 prefers to move.

It is easy to see that, for any choice of ν_1 and ν_2:

- $u_{z_1}(C_1) < u_{z_2}(C_1 \cup \{z_2\})$, and

- $u_{z_2}(C_2) < u_{z_1}(C_2 \cup \{z_1\})$.

Now, z_1 does not want to move when $u_{z_1}(C_1) \geq u_{z_1}(C_2 \cup \{z_1\})$. This implies that $u_{z_2}(C_2) < u_{z_1}(C_2 \cup \{z_1\}) \leq u_{z_1}(C_1) < u_{z_2}(C_1 \cup \{z_2\})$, that is z_2 wants to move. Similarly, when z_2 does not want to move, that is when $u_{z_2}(C_2) \geq u_{z_2}(C_1 \cup \{z_2\})$, we obtain $u_{z_1}(C_1) < u_{z_2}(C_1 \cup \{z_2\}) \leq u_{z_2}(C_2) < u_{z_1}(C_2 \cup \{z_1\})$, that is z_1 wants to move. In both cases we can conclude that the clustering would not be stable. Therefore, the grand coalition is the only Nash stable solution. Its social welfare is $SW(V) = (\nu + 1) \cdot u_{z_i}(V) + u_x(V) + \nu \cdot u_{y_i}(V) = \frac{\nu(5\nu+11)+4}{4\nu+4}$, where $u_{z_i}(V) = \frac{\nu+1+\frac{\nu}{2}}{2\nu+2}$ for all $i \in [\nu + 1]$, $u_x(V) = \frac{2\nu+1}{2\nu+2}$ and $u_{y_i}(V) = \frac{1+\nu}{2\nu+2}$ for all $i \in [\nu]$.

Hence, the ratio $\frac{SW(\mathcal{C})}{SW(V)} = \frac{2\nu(3\nu+5)}{\nu(5\nu+11)+4}$ tends to $\frac{6}{5}$ as ν tends to infinity, proving the theorem. $\square$

The following lower bound on the PoS for graphs of girth 4 holds.

Theorem 5 *There exists an instance of SDGs, in which the underlying graph has girth 4, that has* PoS *at least* $169/160 = 1.05625$.
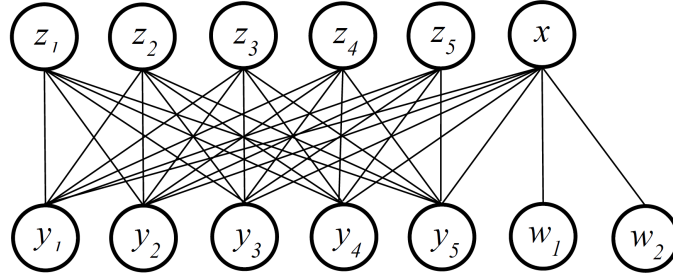

Figure 15: A SDG with PoS $\geq \frac{169}{160}$.

Proof. Consider the graph in Figure 15 and let us denote $Z = \{z_1, \ldots, z_5\}$, $Y = \{y_1, \ldots, y_5\}$ and $S = \{x, w_1, w_2\}$. We first show that there exists an unstable clustering that achieves high social welfare, and then that any stable solution achieves quite less.

More precisely, consider the clustering $\bar{\mathcal{C}} = \{Y \cup Z, S\}$. Since $u_{z_i}(\bar{\mathcal{C}}) = u_{y_i}(\bar{\mathcal{C}}) = \frac{7}{10}$ for all $i \in [5]$, $u_x(\bar{\mathcal{C}}) = \frac{2}{3}$ and $u_{w_1}(\bar{\mathcal{C}}) = u_{w_2}(\bar{\mathcal{C}}) = \frac{1}{2}$, it has social welfare $SW(\bar{\mathcal{C}}) = 10 \cdot \frac{7}{10} + \frac{2}{3} + 1 = \frac{26}{3}$, but it is not stable, as x can move and increase her utility from $u_x(S) = \frac{2}{3}$ to $u_x(Y \cup Z \cup \{x\}) = \frac{5+\frac{5}{2}}{11} = \frac{15}{22}$ by switching coalition.

We now prove that the best stable solution is the grand coalition, that has $SW(V) = 5u_{z_i}(V) + 5u_{y_i}(V) + u_x(V) + 2u_{w_i}(V) = \frac{320}{39}$, where $u_{z_i}(V) = \frac{5+\frac{5}{2}+\frac{2}{3}}{13} = \frac{49}{78}$ and $u_{y_i}(V) = \frac{6+\frac{6}{2}}{13} = \frac{9}{13}$ for $i \in [5]$, $u_x(V) = \frac{7+\frac{5}{2}}{13} = \frac{19}{26}$ and $u_{w_i} = \frac{1+\frac{6}{2}+\frac{5}{3}}{13} = \frac{17}{39}$ for $i \in \{1, 2\}$. The result then derives by the ratio between the above two social welfares.

Notice first that, in any stable clustering, w_1 and w_2 must be in the same coalition of x, otherwise they would have utility 0 and could improve their utility by moving.

Let $\mathcal{C}$ be a stable clustering not being the grand coalition. In the remainder of the proof we will make use of the following properties of $\mathcal{C}$:

- *connection-property*: every coalition of $\mathcal{C}$ containing an agent $z \in Z$ (resp. $y \in Y$) must contain at least one agent $y \in Y$ (resp. $z \in Z \cup \{x\}$). In fact, otherwise z (resp. y) would have utility 0 and could improve her utility by moving.

- *half-property*: an agent $z \in Z$ (resp. $y \in Y$) has utility strictly greater than $1/2$ if and only if it is in a coalition with at least two agents of Y (resp. of $Z \cup \{x\}$). In fact, recalling the definition of the utilities of SDGs, if $z \in Z$ (resp. $y \in Y$) is connected with one agent of Y (resp. $Z \cup \{x\}$) (by the connection-property at least one must be in her coalition), she would have only one agent at distance 1 and eventual other ones at distance at least 2, so her utility would be at most $1/2$. On the other hand, if z (resp. y) is connected with two agents of Y (resp. of $Z \cup \{x\}$), then she has at least 2 nodes at distance 1, eventual nodes at distance 2 and at most 2 nodes at distance 3, so her utility would be strictly greater than $1/2$.

Having in mind the above properties, we distinguish two cases depending on whether $\mathcal{C}$ does not contain or contains a coalition C with at least two agents of $Z \cup \{x\}$.

In the former case, by the connection-property and the fact that agents w_1 and w_2 must always be with x, up to symmetries the only possibility for $\mathcal{C}$ is that $\mathcal{C} = \{\{z_1, y_1\}, \ldots, \{z_5, y_5\}, S\}$. Such a clustering has social welfare $SW(\mathcal{C}) = 5 + \frac{5}{3} = \frac{20}{3}$, which is strictly less than the $\frac{320}{39}$ one obtained by the grand coalition.

So let us assume then that $\mathcal{C}$ contains a coalition C with at least two agents of $Z \cup \{x\}$. In the remaining part of the proof we show that under this assumption $\mathcal{C}$ cannot be stable.

Notice first that in $\mathcal{C}$ there cannot be any coalition C_1 with only one agent in $Z \cup \{x\}$. In fact, by the connection-property, C_1 should have at least one agent $y \in Y$, that by the half-property would be interested in moving to C. This also implies that $\mathcal{C}$ can contain at most 3 coalitions, so let us assume there is also a coalition C_2 with at least two agents of Y. Therefore, in a similar way, there cannot be a coalition containing a single agent $y \in Y$ and at least one agent $z \in Z$, because by the half-property z would be interested in moving to C_2.

As a consequence of the above observations, every coalition not containing x must have at least two agents of Z and at least two agents of Y.

If $\mathcal{C}$ has two coalitions C_1 and C_2 not containing x, then recalling that $\mathcal{C}$ has at most 3 coalitions and that by the connection-property the remaining coalition C_3 containing x either has both an agent of Z and Y or none of them, up to symmetries the following cases can hold:

   i. $C_1 = \{z_1, z_2, y_1, y_2\}, C_2 = \{z_3, z_4, y_3, y_4\}, C_3 = \{z_5, y_5, x, w_1, w_2\}$,

   ii. $C_1 = \{z_1, z_2, z_3, y_1, y_2\}, C_2 = \{z_4, z_5, y_3, y_4, y_5\}, C_3 = \{x, w_1, w_2\}$,

   iii. $C_1 = \{z_1, z_2, y_1, y_2\}, C_2 = \{z_3, z_4, z_5, y_3, y_4, y_5\}, C_3 = \{x, w_1, w_2\}$.

In case i. by the half-property z_5 is interested to move to C_1 or C_2. In case ii. z_3 can improve her utility from $u_{z_3}(C_1) = \frac{1+1+\frac{1}{2}+\frac{1}{2}}{5} = \frac{3}{5}$ to $u_{z_3}(C_2 \cup \{z_3\}) = \frac{1+1+1+\frac{1}{2}+\frac{1}{2}}{6} = \frac{2}{3} > \frac{3}{5}$. Finally in case iii. z_2 can improve her utility from $u_{z_2}(C_1) = \frac{1+1+\frac{1}{2}}{4} = \frac{5}{8}$ to $u_{z_2}(C_2 \cup \{z_2\}) = \frac{1+1+1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}}{7} = \frac{9}{14} > \frac{5}{8}$.

Therefore, since $\mathcal{C}$ cannot have two coalitions C_1 and C_2 not containing x, the only possibility left is that $\mathcal{C}$ consists of just two coalitions (recall that we are under the assumption that $\mathcal{C}$ is not the grand coalition).

By exploiting symmetries and the above constraints, we can describe all possible clusterings containing exactly two coalitions using just two parameters, α and β. All such clusterings (up to symmetries) are:

$$\{\{\{z_1, \ldots, z_\alpha, y_1, \ldots, y_\beta\}, \{z_{\alpha+1}, \ldots, z_5, y_{\beta+1}, \ldots, y_5, x, w_1, w_2\}\} \mid 2 \le \alpha \le 4, \ 2 \le \beta \le 3\}.$$

By means of a case analysis we show that they are all unstable, as there is always at least one agent that has an improving move.

Let $C_1 = \{z_1, \ldots, z_\alpha, y_1, \ldots, y_\beta\}$ and $C_2 = \{z_{\alpha+1}, \ldots, z_5, y_{\beta+1}, \ldots, y_5, x, w_1, w_2\}$. Then:

- If $\alpha = 2$ agent y_1 moves. In fact, her original utility is $u_{y_1}(C_1) = \frac{2 + \frac{\beta-1}{2}}{2+\beta}$, while her new one is $u_{y_1}(C_2 \cup \{y_1\}) = \frac{4 + \frac{7-\beta}{2}}{12-\beta}$, and for any β in our domain, $u_{y_1}(C_1) < u_{y_1}(C_2 \cup \{y_1\})$.

- If $\beta = 3$ agent z_5 moves. In fact, her original utility is $u_{z_5}(C_2) = \frac{2 + \frac{5-\alpha}{2} + \frac{2}{3}}{10-\alpha}$, while her new one is $u_{z_5}(C_1 \cup \{z_5\}) = \frac{3 + \frac{\alpha}{2}}{4+\alpha}$, and $u_{z_5}(C_2) < u_{z_5}(C_1 \cup \{z_5\})$ for any α such that $3 \leq \alpha \leq 4$.

- The remaining case is $3 \leq \alpha \leq 4$ and $\beta = 2$. Under this assumption y_5 moves. In fact, her original utility is $\frac{8-\alpha}{11-\alpha}$, while her new one is $\frac{\alpha+1}{\alpha+3}$, which is strictly greater for any α such that $3 \leq \alpha \leq 4$.

In conclusion, there cannot neither be any stable clustering consisting of two coalitions, hence the claim. □

Finding an upper bound on the price of stability in SDGs seems to be a challenging task. However, we can make progress in this direction by considering more restrictive settings. More precisely, we provide an upper bound of $\frac{5}{4}$ on the price of stability on graphs with girth at least 5. Notice that this result is not comparable with the lower bound on PoS obtained in Theorem 4, since the lower bound is given by a construction having girth less than 5. Before proving such a lower bound, we also show that if we have the promise that the graph has girth 4, then there exists an instance of SDGs in which the only stable solution is the grand coalition.

Lemma 7 *There exists an instance of SDGs in which the underlying graph has girth 4 and the only stable solution is the grand coalition.*

Proof. Consider the graph in Figure 16 and let us analyze all possible ways for partitioning the nodes in a stable way.
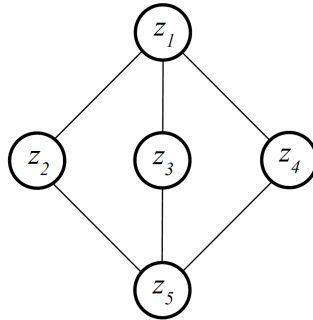

Figure 16: A graph with girth 4.

First of all, notice that an agent cannot be isolated and that, if a coalition is not connected, an agent will move because her utility would be zero. So, besides the grand coalition, we can only partition V in two connected coalitions of size 2 and 3. We only consider the case $\mathcal{C} =$

$\{\{z_1, z_2\}, \{z_3, z_4, z_5\}\}$, as all the others are symmetric. $\mathcal{C}$ is not stable, since z_1 can increase her utility from $\frac{1}{2}$ to $\frac{5}{8}$. This implies the theorem. $\qquad\square$

For a greater girth the following result holds.

Lemma 8 *Given any graph with girth greater than* 4*, it is possible to obtain a stable solution* $\mathcal{C}$ *with* $SW(\mathcal{C}) \geq \frac{n}{2}$ *by partitioning the graph into stars.*

Proof. The following simple procedure allows to determine a partition of the graph into stars, each of at least 2 nodes, together with the corresponding stable clustering. Once constructed a rooted spanning tree T of the graph G, let x be one of the deepest leaves and y be her parent in T. Put y and all her children (x included) in the same coalition and iterate the procedure on the residual tree, until either the tree becomes empty or only the root is left. In this last case insert the root in the last formed coalition. Notice that every coalition induces exactly a star subgraph, as such a subgraph has all the leaves connected to a center and, by the assumption on the girth, there can not be edges between leaves.

It is easy to check that in a star all the leaves have utility $\frac{1}{2}$, while the center has utility at least $\frac{1}{2}$. For what concerns the stability, consider any agent in a star. Since the girth is greater than 4, she can have only one neighbor in a different star, hence she cannot achieve utility greater than $\frac{1}{2}$ by moving, hence the claim. $\qquad\square$

Hence, we can finally state the following upper bound on the PoS for graphs of girth greater than 4.

Theorem 6 *The upper bound on the* PoS *of SDGs in which the underlying graph has girth strictly greater than* 4 *is* $\frac{1}{2} + \frac{1}{\sqrt{2}} \approx 1.207$.

Proof. Consider any clustering $\mathcal{C}$ and let $C_1, \ldots, C_k$ be the non-singleton coalitions in $\mathcal{C}$, that is such that $|C_i| \geq 2$ for $i \in [k]$.

Let $n_i = |C_i|$, E_i be the subset of the edges induced by C_i, and δ_x be the degree of agent x in the subgraph induced by her coalition. We can give an upper bound to the utility of x by considering her neighbors in her coalition C_i at distance 1 and all the other agents in C_i as if they were at distance 2. Hence, the contribute SW_i of any coalition C_i to the social welfare for $i \in [k]$, is

$$
\begin{aligned}
SW_i &\leq \sum_{x \in C_i} \frac{\delta_x + \frac{n_i - \delta_x - 1}{2}}{n_i} = \sum_{x \in C_i} \frac{\frac{n_i - 1}{2} + \frac{\delta_x}{2}}{n_i} = \\
&= \frac{n_i - 1}{2} + \sum_{x \in C_i} \frac{\delta_x}{2n_i} = \frac{n_i - 1}{2} + \frac{2|E_i|}{2n_i} = \\
&= \frac{n_i - 1}{2} + \frac{|E_i|}{n_i}.
\end{aligned}
$$

Since the subgraph induced by C_i cannot have a girth lower than the one of G, and thus has girth greater than 4, from Dutton and Brigham (1991) we know that $|E_i| \leq \frac{n_i \sqrt{n_i - 1}}{2}$. Thus,

$$
SW_i \leq \frac{n_i - 1}{2} + \frac{\sqrt{n_i - 1}}{2},
$$

and

$$SW(\mathcal{C}) = \sum_{i \in [k]} SW_i \leq \frac{\sum_{i \in [k]}(n_i - 1) + \sum_{i \in [k]} \sqrt{n_i - 1}}{2} =$$

$$= \frac{n - k + \sum_{i \in [k]} \sqrt{n_i - 1}}{2} \leq \frac{n - k + k\sqrt{\frac{n}{k} - 1}}{2}.$$

From Lemma 8, we can guarantee a social welfare of at least $\frac{n}{2}$, hence

$$\text{PoS} \leq \frac{\frac{n - k + k\sqrt{\frac{n}{k} - 1}}{2}}{\frac{n}{2}} \leq \frac{1}{2} + \frac{1}{\sqrt{2}},$$

where the last inequality comes from standard maximization arguments, thus proving the theorem. $\square$

5. Conclusions

We investigated Nash stability in SDGs. Several issues remain open. First of all, we have shown that there is a class of SDGs for which a $\frac{6}{5}$ lower bound on the price of stability holds; it would be nice to provide a corresponding upper bound. Another relevant question that naturally emerges is whether there exists a polynomial time algorithm for determining the existence of a Nash stable clustering for SDGs different from the grand coalition. As we have seen, this is not guaranteed for graphs with girth less or equal to 4. We notice that, besides this computational issue, a deeper understanding of the conditions for the existence of such non trivial equilibria would be particularly important for providing better bounds on the price of stability. A related open question is that of identifying special graphs in which a best equilibrium or a best non stable clustering can be computed in polynomial time. Finally, it would be interesting to generalize our results to weighted graphs and to consider classes of hedonic and fractional hedonic games induced by other classical centrality measures, like the ones presented by Gomez et al. (2003). On this respect, it would be particularly worth to consider models in which being a singleton is not the worst choice.

Acknowledgements

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