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Laplacian-based matrix design for finite-time average consensus in digraphs

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Abstract— **In this paper, we consider the problem of assigning time-varying weights on the links of a time-invariant digraph, such that average consensus is reached in a finite number of steps. More specifically, we derive a finite set of weight matrices that are based on the Laplacian and the Laplacian eigenvalues of the given digraph, such that the product of these weight matrices (in any order) leads to a rank-one matrix. Using the weights associated with this sequence of weight matrices, the nodes run two linear iterations (each with its own initial conditions) and, after a finite number of steps, can calculate the average of the initial values by taking the ratio of the two values they possess at the end of the iteration process. As in the case of undirected graphs, we show that the set of matrices depends on the number of nonzero distinct eigenvalues of the Laplacian matrix. However, unlike the case for undirected graphs, the Laplacian matrix is no longer symmetric, and the number of steps depends not only on the number of distinct eigenvalues but also on their algebraic multiplicities. Illustrative examples demonstrate the validity of the derived results.**

Index Terms— **Finite-time average consensus; Laplacian-based matrix design; digraphs.**

I. Introduction

The advancement and proliferation of smart devices, with impressive sensing, computing and control capabilities, has placed the coordination of multi-agent systems (MAS) at the epicenter of pertinent research topics due to the emergence of a wide range of applications, such as sensor networks [1] (with applications in, e.g., machine-to-machine communication for security, surveillance and environmental management), factory automation networks [2], and autonomous systems [3] such as intelligent transportation systems (ITSs) [4] and robotic systems [5] (with applications in, e.g., remote surgery and space missions). For a thorough discussion on MAS and their applications, the interested reader can refer to, e.g., [6]–[8] as well as references and applications therein.

A MAS consists of a set of agents (nodes) that can exchange information through communication links (edges), forming a (generally) directed communication topology (represented as a directed graph or digraph). Coordination algorithms based on local interactions over several iterations have become very popular due to their simplicity and distributed nature, i.e., nodes are able to coordinate and perform complex tasks using relatively simple rules and local information only. For example, in a typical consensus problem, each node possesses an initial value and the nodes need to follow a strategy to distributively compute the same function of these initial values. When this function is the average of the initial values we say that the nodes reach average consensus (for an extended survey and discussion on the topic, see [9]).

Most existing algorithms for average consensus in digraphs can only guarantee asymptotic convergence. As a result, they cannot be applied to real-world coordination and control applications. Algorithms completing in finite-time are, in general, more desirable, because besides finite-time convergence, it is reported that closed-loop systems under finite-time control usually demonstrate better disturbance rejection properties [10]. The ability to terminate a coordination algorithm in a finite number of steps is also important in applications where, for example, the averaging operation is a first step towards more involved control or coordination tasks.

Finite-time average consensus has recently attracted more attention due to its properties and applicability to practical settings. There have been different strands of research on finite-time consensus in the framework of discrete-time systems. One strand of research aims to take advantage of the minimal polynomial of a matrix [11]–[13]. This method requires nodes to have enough memory to store previous state values and sufficient computing power to check rank conditions on matrices. Another strand of research, which is relevant to the work in this paper, is based on pre-determining the number of steps and update weights that nodes in the network will utilize, so that by the end of these iterations the nodes have values sufficiently close to the average or values that allow them to calculate the exact average [14]–[22]. While this approach is the fastest finite-time approach known in the literature (at the cost of requiring some global knowledge of the communication topology), it is so far limited to undirected graphs only.

In this work, we extend the framework of matrix factorization for finite-time average consensus in undirected graphs to digraphs. More specifically, we first derive a set of matrices (based on the Laplacian and the Laplacian eigenvalues of the digraph) whose product leads to a rank-one matrix. The number of distinct such matrices is equal to the nonzero distinct eigenvalues of the Laplacian matrix. Since the rank-one matrix does not lead to average consensus, as it is the case for undirected graphs, we employ a ratio consensus-inspired approach for obtaining the average after a finite number of steps. Specifically, we run two time-varying

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iterations (each with different initial conditions) using the weight matrices that are based on the graph Laplacian; at the end of the finite window, each node has two values, the ratio of which is the desired average. Finally, we provide illustrative examples that show the validity of the derived results.

Apart from requiring two linear iterations, the method proposed in this paper also has some other differences when compared against the case of undirected graphs in, e.g., [16], [18], [19]. First, the Laplacian matrix of a digraph (defined with respect to the out-degree in this paper) is not a symmetric matrix; this implies that its eigenvalues could be complex numbers and leads to weight matrices that have complex entries in general. In addition, the Laplacian matrix is not guaranteed to be diagonalizable, which implies that some of the eigenvalues could be associated with Jordan blocks of size greater than unity; in such cases, additional uses of the matrix that corresponds to these eigenvalues are required. Regardless, the number of finite steps required is bounded by $n-1$.

The remainder of the paper is organized as follows. In Section II, we provide necessary notation and background. Section III presents the matrix factorization method, and in Section IV, we develop a ratio consensus algorithm that reaches average consensus in a finite number of steps. The results are accompanied with illustrative examples in Section V. Finally, Section VI presents concluding remarks and discusses possible future directions.

II. Notation and Preliminaries

A. Notation

The sets of real (integer) and complex numbers are denoted by $\mathbb{R}$ ($\mathbb{Z}$) and $\mathbb{C}$, respectively. The set of nonnegative numbers (integers) is denoted by $\mathbb{R}_+$ ($\mathbb{Z}_+$). $\mathbb{R}_+^n$ denotes the nonnegative orthant of the n-dimensional real space $\mathbb{R}^n$. Vectors are denoted by small letters whereas matrices are denoted by capital letters. The transpose of a matrix A is denoted by A^{T} and the complex conjugate of a matrix A is denoted by A^*. For $A \in \mathbb{C}^{n\times n}$, A_{ij} denotes the entry in row i and column j, $A_{i\bullet}$ denotes all the entries in row i, and $A_{\bullet j}$ denotes all the entries in column j. The i^{th} component of a vector x is denoted by x_i, and the notation $x \geq y$ implies that $x_i \geq y_i$ for all components i. We denote the all-ones vector by $\mathbb{1}$ and the identity matrix (of appropriate dimensions) by I. We use $\operatorname{diag}(x_i)$ to denote the matrix with elements x_1, x_2, … on the leading diagonal and zeros elsewhere.

In multi-component systems with fixed communication links (edges), the exchange of information between components (nodes) can be conveniently captured by a directed graph (digraph) $\mathcal{G}(\mathcal{N},\mathcal{E})$ of order n ($n \geq 2$), where $\mathcal{N} = \{v_1, v_2, \ldots, v_n\}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{N}\times\mathcal{N}$ is the set of edges. A directed edge from node v_i to node v_j is denoted by $\varepsilon_{ji} = (v_j, v_i) \in \mathcal{E}$ and represents a communication link that allows node v_j to receive information from node v_i. A graph is said to be undirected if and only if $\varepsilon_{ji} \in \mathcal{E}$ implies $\varepsilon_{ij} \in \mathcal{E}$. In this paper, links are not required to be bidirectional, i.e., we deal with digraphs; for this reason, we use the terms "graph" and "digraph" interchangeably. A digraph is called *strongly* connected if there exists a path from each vertex v_i in the graph to each vertex v_j ($v_j \neq v_i$). In other words, for any $v_j, v_i \in \mathcal{N}$, $v_j \neq v_i$, one can find a sequence of nodes $v_i = v_{l_1}$, v_{l_2}, v_{l_3}, …, $v_{l_t} = v_j$ such that link $(v_{l_{m+1}}, v_{l_m}) \in \mathcal{E}$ for all $m = 1, 2, \ldots, t-1$.

All nodes that can transmit information to node v_j directly are said to be in-neighbors of node v_j and belong to the set $\mathcal{N}_j^- = \{v_i \in \mathcal{N} \mid \varepsilon_{ji} \in \mathcal{E}\}$. The cardinality of $\mathcal{N}_j^-$, is called the *in-degree* of v_j and is denoted by $\mathcal{D}_j^- = \left|\mathcal{N}_j^-\right|$. The nodes that receive information from node v_j belong to the set of out-neighbors of node v_j, denoted by $\mathcal{N}_j^+ = \{v_l \in \mathcal{N} \mid \varepsilon_{lj} \in \mathcal{E}\}$. The cardinality of $\mathcal{N}_j^+$, is called the *out-degree* of v_j and is denoted by $\mathcal{D}_j^+ = \left|\mathcal{N}_j^+\right|$.

The adjacency matrix $\mathcal{A}$ is defined as the $n \times n$ matrix with entries

$$\mathcal{A}_{ji} = \begin{cases} 1, & \text{if } (v_j, v_i) \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The Laplacian matrix is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D}$ is the diagonal out-degree matrix, i.e., $\mathcal{D} \triangleq \operatorname{diag}(\mathcal{D}_i^+)$.

In the algorithms we consider, we associate a positive weight p_{ji} with each edge $\varepsilon_{ji} \in \mathcal{E} \cup \{(v_j, v_j) \mid v_j \in \mathcal{N}\}$. The nonnegative matrix $P = [p_{ji}] \in \mathbb{R}_+^{n\times n}$ (with p_{ji} as the entry at its jth row, ith column position) is a weighted adjacency matrix (also referred to as weight matrix) that has zero entries at locations that do not correspond to directed edges (or self-edges) in the graph. In other words, apart from the main diagonal, the zero-nonzero structure of the adjacency matrix P matches exactly the given set of links in the graph.

We use $x_j[k] \in \mathbb{R}$ to denote the information state of node v_j at time t_k. In the synchronous setting we consider, each node v_j updates and sends its information to its neighbors at discrete times $t_0, t_1, t_2, \ldots$. We index nodes' information states and any other information at time t_k by k. Hence, we use $x_j[k]$ to denote the state of node v_j at time t_k.

B. Distributed linear iterations

Each node updates its information state $x_j[k]$ by combining the available information received by its neighbors $x_i[k]$ ($v_i \in \mathcal{N}_j^-$) using the positive weights $p_{ji}[k]$, that capture the weight of the information inflow from agent v_i to agent v_j at time k. In this work, we assume that each node v_j can choose its self-weight $p_{jj}[k]$ and the weights $p_{lj}[k]$ on its outgoing links (v_l, v_j), $v_l \in \mathcal{N}_j^+$ (e.g., by sending $p_{lj}[k]x_j[k]$ to its out-neighbour $v_l \in \mathcal{N}_j^+$. Hence, in its general form, each node updates its information state according to the following relation:

$$x_j[k+1] = p_{jj}[k]x_j[k] + \sum_{v_i \in \mathcal{N}_j^-} p_{ji}[k]x_i[k], \tag{2}$$

for $k \geq 0$, where $x_j[0] \in \mathbb{R}$ is the initial state of node v_j. Let $x[k] = (x_1[k] \;\; x_2[k] \;\; \ldots \;\; x_n[k])^{\mathrm{T}}$ and $P[k] = [p_{ji}[k]] \in \mathbb{R}_+^{n\times n}$. Then (2) can be written in matrix form as

$$x[k+1] = P[k]x[k], \tag{3}$$

where $x[0] = (x_1[0] \;\; x_2[0] \;\; \dots \;\; x_n[0])^{\mathrm{T}} \triangleq x_0$. We say that the nodes asymptotically reach average consensus if

$$\lim_{k\to\infty} x_j[k] = \frac{\sum_{v_i\in\mathcal{N}} x_i[0]}{n}, \quad \forall v_j \in \mathcal{N},$$

or, equivalently

$$\lim_{k\to\infty} x[k] = J_n x[0],$$

where $J_n \triangleq \frac{1}{n}\mathbb{1}\mathbb{1}^{\mathrm{T}}$. Since the above has to hold for any initial set of values $x[0]$, it is equivalent to saying that

$$\lim_{k\to\infty} P[k]P[k-1]\dots P[0] = J_n. \tag{4}$$

If the weights remain unchanged, i.e., if $P[k] = P$ for all k, then (4) becomes $\lim_{k\to\infty} P^k = J_n$. The necessary and sufficient conditions to asymptotically reach average consensus when $P[k] = P$ are the following: (a) P has a simple eigenvalue at one with left eigenvector $\mathbb{1}^{\mathrm{T}}$ and right eigenvector $\mathbb{1}$, and (b) all other eigenvalues of P have magnitude less than 1.

C. Laplacian-based finite-time average consensus in undirected graphs

A related problem is the factorization of the averaging matrix, $\frac{1}{n}\mathbb{1}\mathbb{1}^{\mathrm{T}}$, to orchestrate update matrices at design time, so that average consensus is achieved in a finite number of steps. In other words, one aims for finding a finite number of matrices, $\{P_i\},\ i \in \{1, 2, \dots, m\}$, all adhering to the given structure of the communication topology, such that $\prod_{i=1}^m P_i = J_n$. These matrices define the weights that will be used at each iteration of the form $x[k+1] = P[k]x[k]$ (with $x[0]$ being the initial values of the nodes and $P[k]$ being one of the matrices in $\{P_i\},\ i \in \{1, 2, \dots, m\}$), such that $x[m] = \mu\mathbb{1}$ where μ is the average (i.e., average consensus is reached at iteration m. Note, however, that knowledge of the network topology is required to compute the matrices $\{P_i\},\ i \in \{1, 2, \dots, m\}$.

It was shown in [18] that for an undirected graph there exist Laplacian-based matrices for which this can be achieved. More specifically, for matrices $\{P_i\}$ of the form $P_i \triangleq I - a_i\mathcal{L}$, $a_i > 0$, it was shown that, if a_i are carefully chosen, there exists $m \in \mathbb{N}$, such that

$$\prod_{i=1}^{m} (I - a_i\mathcal{L}) = J_n. \tag{5}$$

The minimum m is achieved if and only if the parameters $a_i, i \in \{1, 2, \dots, m\}$ are the reciprocals of the distinct nonzero Laplacian eigenvalues (i.e., $m \leq n-1$).

III. Extending the Laplacian-based approach to digraphs

While there are several works on the matrix design for finite-time average consensus in undirected graphs (see, e.g., [14]–[22]), there are no results reported on Laplacian-based matrix design for finite-time average consensus in digraphs. One of the main difficulties, as mentioned in the introduction, is that the Laplacian is no longer a symmetric matrix, which makes the extension, from the undirected graphs to directed ones, nontrivial. In this section, we consider a strongly connected directed graph with n nodes and we establish a Laplacian-based matrix design for obtaining a rank-one matrix. More specifically, we first show that a product of matrices $\{P_i\} \in \mathbb{C}^{n\times n}$ of the form $P_i \triangleq I - a_i\mathcal{L}$, $a_i \in \mathbb{C}$, satisfies $\prod_{i=1}^t P_i = \beta c\mathbb{1}^{\mathrm{T}}$, where $t \in \mathbb{N}$, $\beta \in \mathbb{R}_+$, $c \in \mathbb{R}^{n\times 1}$ is the right eigenvector of $\mathcal{L}$ that corresponds to eigenvalue 0 (which is unique since the graph is strongly connected).

Proposition 1: Let $\lambda_1 = 0$ be a simple eigenvalue (i.e., with multiplicity one) and $\lambda_2, \lambda_3, \dots, \lambda_m \in \mathbb{C}$ be the distinct nonzero eigenvalues of the Laplacian matrix $\mathcal{L}$, $m \leq n-1$. Then, a set of products of the matrices $P_i \triangleq I - a_i\mathcal{L}$, with $a_i = 1/\lambda_i$, $i \in \{2, 3, \dots, m-1, m\}$, in any order, converges to a rank-one matrix which is proportional to $c\mathbb{1}^{\mathrm{T}}$, where $c \in \mathbb{R}^{n\times 1}$ is the right eigenvector corresponding to λ_1.

Proof: Since the Laplacian matrix , $\mathcal{L}$, is not symmetric (recall that $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D}$ is the diagonal out-degree matrix), it is not guaranteed that the Laplacian matrix of a digraph is diagonalizable and, therefore, we use Jordan matrix decomposition.

When considering the set of left eigenvectors, the following holds:

$$W^*\mathcal{L} = \Lambda W^*, \tag{6}$$

where W^* is the set of left eigenvectors with $W^*_{1,\bullet} = \beta\mathbb{1}^{\mathrm{T}}$, $\beta \in \mathbb{R}_+$, and Λ is a block-diagonal matrix given by

$$\Lambda = \mathrm{diag}\{B_1, B_2, \dots B_\ell\} \tag{7}$$

with $\ell \in \mathbb{N}$ and $B_i \in \mathbb{C}^{s_i\times s_i}$ being sub-blocks of the form

$$B_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix},$$

where $\lambda_i,\ i \in \{1, 2, \dots, m\}$ ($m \in \mathbb{N}$ possibly different than ℓ) is the ith distinct eigenvalue of $\mathcal{L}$, and $\lambda_1 = 0$. Note that there might be sub-blocks with the same eigenvalue, i.e., $s_2 + \ldots + s_\ell = n-1$ and $\hat{s}_2 + \ldots + \hat{s}_m \leq n-1$, where $\hat{s}_i = \max\{s_{i_1}, \dots, s_{i_r}\}$, s_{i_j} being the jth sub-block with eigenvalue λ_i. For ease of exposition, we assume $s_i = \hat{s}_i$.

When considering the set of right eigenvectors, we get

$$\mathcal{L}Q = Q\Lambda, \tag{8}$$

where Q is the set of right eigenvectors with $Q_{\bullet,1} = c$, where $c \in \mathbb{R}^{n\times 1}$ is the right eigenvector corresponding to λ_1 (the entries of vector c are real due to the Perron-Frobenius Theorem; see, for example, [23]).

The right and left eigenvectors corresponding to a particular sub-block B_i can be defined to have unity inner product. The inner products of a left eigenvector with the right eigenvectors corresponding to different eigenvalues are zero. Therefore, for the sets of right and left eigenvectors of the Laplacian, Q and W^*, respectively, we have that $W^*Q = I$. As a result, $(W^*)^{-1} = Q$.

We can decompose B_i to $B_i = \lambda_i I + F$, where F is the matrix with zero-entries everywhere except in the entries in the super-diagonal which contain ones. For $F \in \mathbb{R}^{s_i \times s_i}$, it is easily deduced that $F^{s_i} = 0$. Suppose that the minimum number of steps to reach average consensus is t. Therefore,

$$\begin{aligned}\prod_{i=1}^{t}(I - a_i\mathcal{L}) &= \prod_{i=1}^{t}(I - a_i(W^*)^{-1}\Lambda W^*) \\ &= \prod_{i=1}^{t}(W^*)^{-1}(I - a_i\Lambda)W^* \\ &= Q\left[\prod_{i=1}^{t}(I - a_i\Lambda)\right]W^* \ . \end{aligned} \tag{9}$$

Note that by choosing $a_i = 1/\lambda_i$ for $i = 2, 3, \ldots, m$, and by taking into account possible multiplicities of the eigenvalues, we have

$$\prod_{i=1}^{m}\left(1 - \frac{B_j}{\lambda_i}\right)^{s_i} = \begin{cases} 1, & \text{if } j = 1, \\ \mathbf{0}_{s_i \times s_i}, & \text{otherwise.} \end{cases} \tag{10}$$

Therefore, $t = \sum_{i=2}^{m} s_i$ and

$$\prod_{i=2}^{m}\left(I - \frac{1}{\lambda_i}\Lambda\right)^{s_i} = \mathrm{diag}\{1, 0, \ldots, 0\}. \tag{11}$$

Substituting (11) into (9) and noting that $Q_{\bullet,1} = c$ and $W^*_{1,\bullet} = \beta\mathbb{1}^{\mathrm{T}}$, we obtain

$$\prod_{i=2}^{m}\left(I - \frac{1}{\lambda_i}\mathcal{L}\right)^{s_i} = \beta c\mathbb{1}^{\mathrm{T}}. \tag{12}$$

It can be easily shown that the order of the matrices appearing in the product does not matter, but it is omitted due to space limitations. The proof is completed. ∎

Remark 1: Note that since $\mathcal{L}$ is not symmetric, some or all of its nonzero eigenvalues are complex numbers. This results to transmissions of two numbers (real and imaginary parts) and multiplications with complex numbers. □

IV. Finite-time average consensus using ratio consensus

In this section, we use the Laplacian-based matrix design from Section III to reach average consensus in a finite number of steps in digraphs by means of ratio consensus.

In [24], an algorithm is suggested that solves the average consensus problem in a directed graph in which each node v_j distributively sets the weights on its self-link and outgoing-links to be $p_{lj} = \frac{1}{1+\mathcal{D}_j^+}$ $\forall (v_l, v_j) \in \mathcal{E}$, so that the resulting weight matrix $P = [p_{lj}]$ is column stochastic, but not necessarily row stochastic. Asymptotic average consensus is reached by using this weight matrix to run two iterations with appropriately chosen initial conditions. The algorithm is stated below for a specific choice of weights on each link that assumes that each node knows its out-degree; however, it works for any set of weights that adhere to the graph structure and forms a primitive column stochastic weight matrix.

Proposition 2 ([24]): Consider a strongly connected digraph $\mathcal{G}(\mathcal{N}, \mathcal{E})$. Let $y_j[k]$ and $x_j[k]$ (for all $v_j \in \mathcal{N}$ and $k = 0, 1, 2, \ldots$) be the result of the iterations

$$y_j[k+1] = p_{jj}y_j[k] + \sum_{v_i \in \mathcal{N}_j^-} p_{ji}y_i[k] \ , \tag{13a}$$

$$x_j[k+1] = p_{jj}x_j[k] + \sum_{v_i \in \mathcal{N}_j^-} p_{ji}x_i[k] \ , \tag{13b}$$

where $p_{lj} = \frac{1}{1+\mathcal{D}_j^+}$ for $v_l \in \mathcal{N}_j^+ \cup \{v_j\}$ (zeros otherwise), and the initial conditions are $y[0] \in \mathbb{R}^n$ and $x[0] = \mathbb{1}$. Then, the solution to the average consensus problem can be asymptotically obtained as

$$\lim_{k\to\infty} \mu_j[k] = \frac{\sum_{v_l \in \mathcal{N}} y_l[0]}{|\mathcal{N}|} \ , \forall v_j \in \mathcal{N} \ ,$$

where $\mu_j[k] = \dfrac{y_j[k]}{x_j[k]}$. □

Remark 2: In Proposition 2, a distributed algorithm is proposed with which the exact average is *asymptotically* reached in a digraph, even if it is not balanced. □

In Theorem 1, we adapt Proposition 2 and ratio consensus for using the properties of the product of matrices for reaching average consensus in a finite number of steps.

Theorem 1: Consider a strongly connected digraph $\mathcal{G}(\mathcal{N}, \mathcal{E})$. Let $y_j[k]$ and $x_j[k]$ (for all $v_j \in \mathcal{N}$ and $k = 0, 1, 2, \ldots, t-1$, $t \leq n-1$) be the result of the iterations

$$y_j[k+1] = \left(1 - \frac{\mathcal{D}_j^+}{\lambda_{i[k]}}\right)y_j[k] + \sum_{v_l \in \mathcal{N}_j^-} \frac{1}{\lambda_{i[k]}}y_l[k] \ , \tag{14a}$$

$$x_j[k+1] = \left(1 - \frac{\mathcal{D}_j^+}{\lambda_{i[k]}}\right)x_j[k] + \sum_{v_l \in \mathcal{N}_j^-} \frac{1}{\lambda_{i[k]}}x_l[k] \ , \tag{14b}$$

where $\lambda_{i[k]}$ is an eigenvalue out of the m distinct eigenvalues of the Laplacian chosen at time step k (i.e., $i[k] \in \{2, 3, \ldots, m\}$ and $i[k_1] \neq i[k_2]$ if $k_1 \neq k_2$), with initial conditions $y[0] \in \mathbb{R}^{n\times 1}$ and $x[0] = \mathbb{1}$. The solution to the average consensus problem can be obtained in $t \leq n-1$ steps, i.e.,

$$\mu_j[t] \triangleq \frac{y_j[t]}{x_j[t]} = \frac{\sum_{v_l \in \mathcal{N}} y_l[0]}{|\mathcal{N}|} \ , \forall v_j \in \mathcal{N} \ .$$

Proof: Using the concept of ratio consensus, of having two iterations running simultaneously, with initial conditions $y[0] \in \mathbb{R}^n$ and $x[0] = \mathbb{1}$, the solution to the average consensus problem can be obtained in t steps as

$$\begin{aligned}\mu_j[t] &= \frac{\left[\prod_{j=2}^{m}\left(I - \frac{1}{\lambda_j}\mathcal{L}\right)^{s_j} y[0]\right]_j}{\left[\prod_{j=2}^{m}\left(I - \frac{1}{\lambda_j}\mathcal{L}\right)^{s_j} x[0]\right]_j} \\ &= \frac{\beta c_j \mathbb{1}^{\mathrm{T}} y[0]}{\beta c_j \mathbb{1}^{\mathrm{T}} x[0]} = \frac{\sum_{v_i \in \mathcal{N}} y_i[0]}{n} \ , \forall v_j \in \mathcal{N} \ .\end{aligned}$$

The proof is completed. ∎

Remark 3: As noted in [18], $t!$ different sequences of matrices can be chosen such that the exact average consensus is reached after t steps. However, the transient behavior could be very different. Selecting the best sequence depends on the criterion and is out of the scope of this paper. However, the effect of the sequence selection is observed in Example 2.

V. Examples

Example 1: We consider a digraph consisting of 6 nodes, as shown in Fig. 1, and with initial conditions

$$y[0] = \begin{bmatrix} -2 & 0 & 2 & 3 & 4 & 5 \end{bmatrix}^{\mathrm{T}}$$

and $x[0] = \mathbb{1}$. Hence, the average is $\mu = 2$.

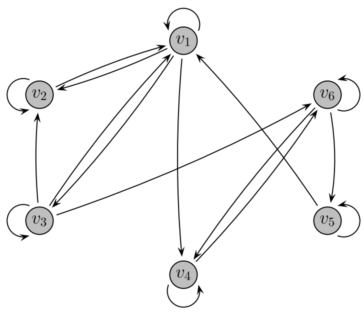

Fig. 1. A digraph consisting of 6 nodes.

The Laplacian matrix is given by

$$\mathcal{L} = \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix}.$$

The eigenvalues of $\mathcal{L}$ are $\lambda_1 = 0$, $\lambda_2 = 4.1322$, $\lambda_3 = 0.8678$, $\lambda_4 = 2.5000 + 0.4052j$, $\lambda_5 = 2.5000 - 0.4052j$ and $\lambda_6 = 1$. The evolution of the states (choosing $a_i = 1/\lambda_{i+2}$ for $i = 0, 1, 2, 3, 4$) is shown in Fig. 2.

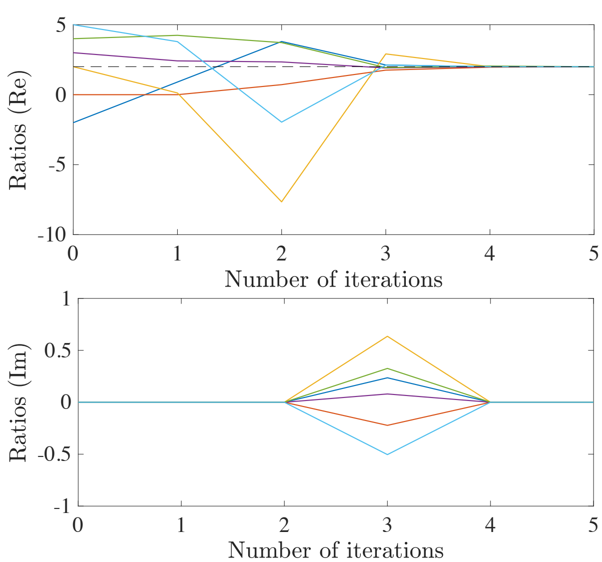

Fig. 2. Average consensus is reached for the ratio $y_j[k]/x_j[k]$ of each node v_j for a digraph of size 6 in 5 steps. The *top figure* shows the evolution of the real part of the state of each node v_j and the *bottom figure* shows the evolution of the imaginary part of the state of each node v_j.

For a different order of the matrices the evolution is different, as shown in Fig. 3.

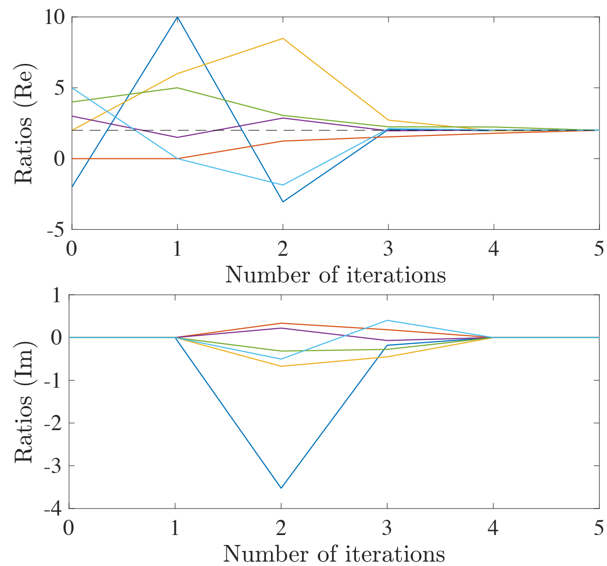

Fig. 3. Average consensus is reached for the ratio $y_j[k]/x_j[k]$ of each node v_j for a digraph of size 6 in 5 steps, but in different order.

Example 2: Next, we consider an example of a larger network, for which asymptotic convergence to the average takes a considerable amount of time. More specifically, we consider a Leslie (a discrete, age-structured model of population growth) matrix consisting of 20 nodes. Due to the structure of the network, the asymptotic convergence time is large. By running the ratio consensus algorithm as in Proposition 2, the evolution of the states is depicted in Fig. 4.

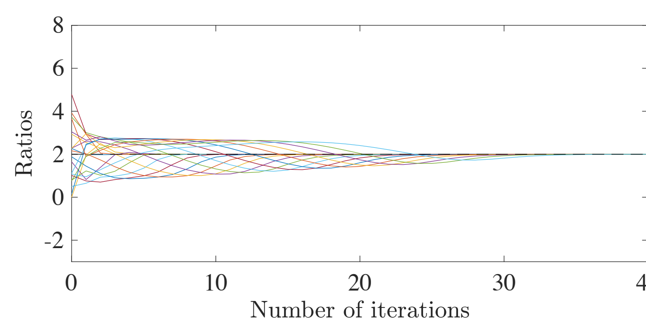

Fig. 4. Asymptotic average consensus for the ratio $y_j[k]/x_j[k]$ of each node v_j for a Leslie digraph of size 20.

As it can be observed, it requires about 40 steps for the error to be considerably small. By comparing with the finite-time average consensus algorithm for directed networks introduced in [13], the minimum number of steps required to compute the average of a node was 28 steps and the maximum 36 time steps. Note that for the algorithm in [13] each node does not require any offline design and knowledge of the network, but it requires large computation and storage capabilities.

For the proposed methodology in this paper, the evolution of the states is shown in Fig. 5. All nodes converge to the average in 19 steps only.

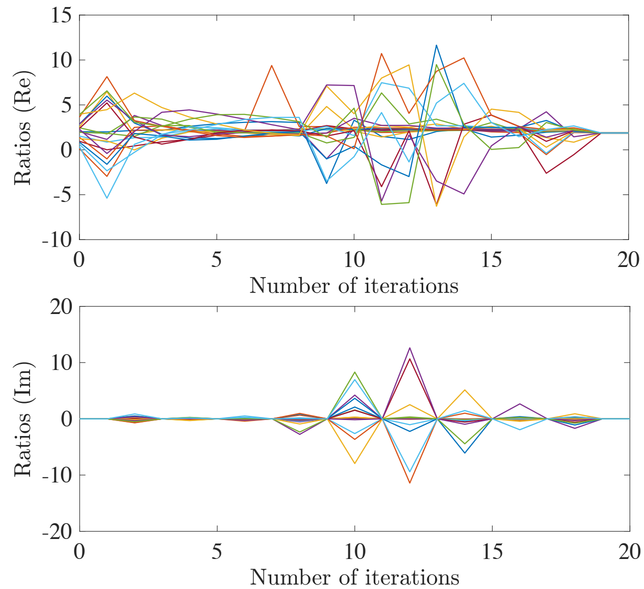

Fig. 5. Average consensus is reached for the ratio $y_j[k]/x_j[k]$ of each node v_j for a digraph of size 20 in 19 steps.

VI. Conclusions and Future Directions

In this paper, we considered the problem of Laplacian-based matrix design for finite-time average consensus in digraphs, i.e., finding the weights on the links of a time-invariant digraph at each time instant such that average consensus is reached in a finite number of steps. More specifically, we derived a set of matrices that are based on the Laplacian and the Laplacian eigenvalues of the digraph whose product leads to average consensus in a finite number of steps. We proved that the number of such weight matrices is equal to the number of nonzero distinct eigenvalues of the Laplacian matrix, as in the case for undirected graphs. However, unlike the case of undirected graph, each weight matrix could have complex-valued entries, and the number of times it is used depends on the algebraic multiplicity of the corresponding eigenvalue. The validity of our results was demonstrated via illustrative examples.

Ongoing research focuses on the analysis of non-ideal cases, e.g., in cases there are estimation errors in computing the eigenvalues or when the measurements are noisy.

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