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Abstract In this paper, we derive a general closed-form solution for the static, axisymmetric bending of solid and annular microstructure-dependent Mindlin plates based on the modified couple-stress theory. The solution is developed by employing a suitable change of displacement variables. The couple-stress solution contains modified Bessel functions which do not appear in the classical case. The stress distributions of the annular microstructure-dependent plate are obtained in terms of load resultants by using the general closed-form solution. Analytical and numerical examples considering solid and annular circular plates are presented. The examples include simply-supported and clamped solid and annular plates subjected to a uniformly distributed load and annular plates attached to a point-loaded rigid shaft.

Keywords Mindlin plate · couple-stress · axisymmetric bending · analytical solution

1 Introduction

It is well known that the mechanical response of many micron-scale structures is size-dependent. For example, the normalized bending stiffness of a small epoxy beam increases with decreasing beam thickness in a manner that cannot be explained by the tools provided by classical elasticity [11]. Other representative examples of size-dependency at small-scales include, but are not limited to, the torsion of thin copper wires [4] and the bending of nickel foils and polypropylene cantilever beams [15, 25]. In order to characterize the size-dependent effects of small-scale structures, a variety of higher-order continuum theories have been developed in recent years. A widely accepted framework among these is the modified couple-stress theory by Yang et al. [27]. This theory introduces a single microstructural length scale parameter into the constitutive relations.

The modified couple-stress theory has been used to derive a good number of microstructure-dependent classical, first- and third-order beam and plate models. Park and Gao [16] and Kong et al. [10] developed static and dynamic models for the Euler-Bernoulli beam. Timoshenko beam models based on the modified couple-stress theory were formulated by Ma et al. [12], Reddy [17] and Asghari et al. [1, 2]. A non-classical third-order Reddy–Levinson beam model was first presented by Ma et al. [14]. Unified treatments on different microstructure-dependent higher-order beam models

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can be found in the papers by Şimşek and Reddy [23, 24]. Solid and annular circular Mindlin plate models based on the modified couple-stress theory, which are the main subject of this study, have been derived and investigated by several authors. Reddy and Berry [18] developed nonlinear models for the axisymmetric bending of circular Kirchhoff and Mindlin plates. Ke et al. studied the linear bending, vibration and buckling [6], nonlinear free vibration [7] and post-buckling [8] of axisymmetric microstructure-dependent Mindlin plates by using the differential quadrature (DQ) method. Zhou and Gao [29] solved an axisymmetric bending problem of a clamped circular Mindlin plate by using Fourier–Bessel series and Eshraghi et al. [3] analysed the bending and vibrations of circular microplates under thermal loading by employing the DQ method. Reddy et al. [20] developed finite element models for the microstructure-dependent, geometrically nonlinear axisymmetric bending of circular plates. Modified couple-stress theory based formulations for various rectangular plates can be found in the works of Tsiatas [26], Yin et al. [28], Ma et al. [13], Jomehzadeh et al. [5], and Reddy and Kim [9, 19].

The governing equations of microstructure-dependent Timoshenko beams are typically solved by using Fourier series [12, 17], a closed-form analytical solution [2] or numerical methods, see e.g [22]. While series solutions and numerical approaches are also available for circular Mindlin plates based on the modified couple-stress theory, a general closed-form solution has not been found in the literature to date. Therefore, in this paper we derive such an analytical solution for the static, axisymmetric bending problem of solid and annular microstructure-dependent Mindlin plates. The stretching and dynamic behavior of the plates are not considered. The static closed-form general solution provides an easy way to obtain solutions for many practical linear cases. From a different point of view, it offers reference solutions to validate (to an extent) theoretically more involved models and the numerical and inexact analytical approaches used to study them.

The remainder of this paper is organized as follows. In Section 2, the circular microstructure-dependent Mindlin plate model by Reddy and Berry [18] is reviewed briefly. The general solution to the governing equations of the model is developed by taking use of the same change of variables as utilized in the case of a microstructure-dependent Timoshenko beam by Asghari et al. [2]. The homogeneous part of the general solution consists of logarithmic terms and modified Bessel functions. In Section 3, case studies considering solid and annular plates are presented and the differences between classical and couple-stress solutions are highlighted. Conclusions are finally drawn in Section 4.

2 Circular couple-stress Mindlin plate

2.1 Governing equations

Fig. 1 presents an annular Mindlin plate subjected to a rotationally symmetric transverse load $q(r)$. The thickness of the plate is h and the outer and inner radii of the plate are a and b, respectively. With the exclusion of the stretching part, the displacement field of the plate can be written as [18]

$$U_r(r,z) = z\phi(r), \quad U_z(r,z) = w(r), \tag{1}$$

where $w(r)$ is the transverse deflection of the mid-surface of the plate and $\phi(r)$ is the rotation of the normal of the mid-surface. The non-zero strain components of the plate are

$$\epsilon_r = \frac{\partial U_r}{\partial r} = z\frac{\partial \phi}{\partial r}, \quad \epsilon_\theta = \frac{U_r}{r} = \frac{z}{r}\phi, \quad \gamma_{rz} = \frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r} = \phi + \frac{\partial w}{\partial r} \tag{2}$$

and the rotation and curvature components read

$$\omega_\theta = \frac{1}{2}\left(\frac{\partial U_r}{\partial z} - \frac{\partial U_z}{\partial r}\right) = \frac{1}{2}\left(\phi - \frac{\partial w}{\partial r}\right), \tag{3}$$

$$\chi_{r\theta} = \frac{1}{2}\left(\frac{\partial \omega_\theta}{\partial r} - \frac{\omega_\theta}{r}\right) = \frac{1}{4}\left[\frac{\partial \phi}{\partial r} - \frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\left(\phi - \frac{\partial w}{\partial r}\right)\right], \tag{4}$$

respectively. The constitutive relations of the plate are

$$\begin{aligned} \sigma_r &= \frac{E(z)}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta), & \sigma_\theta &= \frac{E(z)}{1-\nu^2}(\nu\epsilon_r + \epsilon_\theta), \\ \tau_{rz} &= G(z)\gamma_{rz}, & m_{r\theta} &= 2G(z)l^2\chi_{r\theta}, \end{aligned} \tag{5}$$

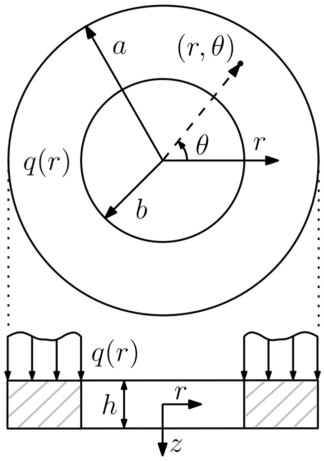

Fig. 1 Annular Mindlin plate under a rotationally symmetric transverse load $q(r)$

where $E(z)$, $G(z)$ and ν are the Young's modulus, shear modulus and Poisson ratio, respectively, and l is the microstructural length scale parameter related to the couple-stress $m_{r\theta}$. On the basis of the above relations, the load resultants per unit length are defined as

$$M_r(r) = \int_{-h/2}^{h/2} \sigma_r z dz = D\left(\frac{\partial \phi}{\partial r} + \frac{\nu}{r}\phi\right), \tag{6}$$

$$M_\theta(r) = \int_{-h/2}^{h/2} \sigma_\theta z dz = D\left(\nu\frac{\partial \phi}{\partial r} + \frac{1}{r}\phi\right), \tag{7}$$

$$Q_r(r) = \int_{-h/2}^{h/2} \tau_{rz} dz = D_Q\left(\phi + \frac{\partial w}{\partial r}\right), \tag{8}$$

$$P_{r\theta}(r) = \int_{-h/2}^{h/2} m_{r\theta} dz = \frac{1}{2}S_{r\theta}\left[\frac{\partial \phi}{\partial r} - \frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\left(\phi - \frac{\partial w}{\partial r}\right)\right], \tag{9}$$

where we have $D_Q = K_s S_{rz}$ with the shear coefficient K_s, and D, S_{rz} and $S_{r\theta}$ are the bending, transverse shear, and couple-stress related stiffness coefficients, respectively. These coefficients may be determined, for example, according to an isotropic material or a two-constituent functionally graded material [20].

The equilibrium equations and boundary conditions of the plate are obtained by employing the principle of virtual displacements, see Ref. [18]. The equilibrium equations can be written in the form

$$(rM_r)' - M_\theta + \frac{1}{2}\left[(rP_{r\theta})' + P_{r\theta}\right] = rQ_r, \tag{10}$$

$$(rQ_r)' + \frac{1}{2}\left[(rP_{r\theta})'' + P_{r\theta}'\right] = -rq(r), \tag{11}$$

where prime denotes differentiation with respect to r. Finally, the boundary conditions are determined by specifying one element in each of the following pairs at $r = b$ and $r = a$

$$r\hat{V} \quad \text{or} \quad w, \tag{12}$$

$$r\hat{M} \quad \text{or} \quad \phi, \tag{13}$$

$$r\hat{P} \quad \text{or} \quad \theta \equiv -w', \tag{14}$$

where $\theta(r)$ is the slope at the midsurface of the plate and

$$\hat{V} = Q_r + \frac{1}{2r}\left[(rP_{r\theta})' + P_{r\theta}\right], \quad \hat{M} = M_r + \frac{1}{2}P_{r\theta}, \quad \hat{P} = -\frac{1}{2}P_{r\theta}\ . \tag{15}$$

2.2 General solution

Like in the case of a couple-stress Timoshenko beam [2], the plate equations can be solved analytically by introducing the variables

$$\gamma \equiv \gamma_{rz} = \phi + w' \quad \text{and} \quad \omega \equiv 2\omega_\theta = \phi - w' \ . \tag{16}$$

Now we can write the equilibrium equations (10) and (11) in the form

$$\frac{D_Q}{r}(r\gamma' + \gamma) + \frac{S_{r\theta}}{4r^3}(r^3\omega''' + 2r^2\omega'' - r\omega' + \omega) + q(r) = 0, \tag{17}$$

$$\frac{1}{2r}\left[Dr(r\gamma'' + \gamma') - (D + 2D_Q r^2)\gamma\right] + \frac{\beta_2}{4r}\left[r(r\omega'' + \omega') - \omega\right] = 0 \ . \tag{18}$$

In the following, we include a uniformly distributed load $q(r) = q$ into the general solution as a special case to complement the homogeneous part which is of main interest here. The outline of the following solution procedure remains the same even if a different distributed load is used. By differentiating Eq. (18) once with respect to r and combining the result with Eq. (17), we obtain

$$r\gamma''' + 2\gamma'' - \left(\frac{1}{r} + 4\alpha^2 r\right)\gamma' + \left(\frac{1}{r^2} - 4\alpha^2\right)\gamma = qr\frac{2\beta_2}{DS_{r\theta}}, \tag{19}$$

where

$$\alpha = \sqrt{\frac{D_Q\beta_1}{DS_{r\theta}}} \quad \text{with} \quad \beta_1 = D + S_{r\theta} \quad \text{and} \quad \beta_2 = 2D + S_{r\theta} \ . \tag{20}$$

We note that differential equation (19) is of such a form that it can be readily solved by a symbolic mathematical tool such as Maple. The solution to Eq. (19) is

$$\gamma = B_1\frac{1}{r} + B_2 I_1(2\alpha r) + B_3 K_1(2\alpha r) - \frac{qr}{4}\frac{\beta_2}{D_Q\beta_1}, \tag{21}$$

where $I_1(2\alpha r)$ and $K_1(2\alpha r)$ are modified Bessel functions of the first and second kind, respectively. It is interesting to note that the very same special functions appear in solutions of the ordinary third-order circular plate theory, where the total differential order of the governing equations is also six [21]. By substituting Eq. (21) into Eq. (18) and solving for variable ω, we obtain

$$\begin{aligned}\omega = {} & B_1\frac{D_Q r}{\beta_2}(2\ln r - 1) + B_4\frac{1}{r} + B_5 r - \frac{qr^3}{8\beta_1} \\ & + \frac{D_Q - 2D\alpha^2}{\beta_2\alpha^2}\left[B_2 I_1(2\alpha r) + B_3 K_1(2\alpha r)\right] \ .\end{aligned} \tag{22}$$

The original kinematic variables are retrieved from

$$w = \int \frac{1}{2}(\gamma - \omega)dr \quad \text{and} \quad \phi = \frac{1}{2}(\gamma + \omega), \tag{23}$$

which yield

$$\begin{aligned} w(r) = {} & C_1 - \frac{1}{4}C_2 r^2 - \frac{1}{2}C_3\ln r + \frac{1}{2}C_4\left[\ln r + \frac{D_Q r^2}{\beta_2}(1 - \ln r)\right] \\ & + \frac{\beta_2}{4\alpha\beta_1}\left\{C_5[I_0(2\alpha r) - 1] + C_6 K_0(2\alpha r)\right\} + \frac{qr^2(D_Q r^2 - 4\beta_2)}{64 D_Q\beta_1},\end{aligned} \tag{24}$$

$$\begin{aligned} \phi(r) = {} & \frac{1}{2}C_2 r + \frac{1}{2r}C_3 + \frac{1}{2}C_4\left[\frac{1}{r} + \frac{D_Q r}{\beta_2}(2\ln r - 1)\right] \\ & + \frac{S_{r\theta}}{2\beta_1}\left[C_5 I_1(2\alpha r) + C_6 K_1(2\alpha r)\right] - \frac{qr(D_Q r^2 + 2\beta_2)}{16 D_Q\beta_1} \ .\end{aligned} \tag{25}$$

In the case of a solid circular plate without a concentrated load at the center, we choose to set $C_3 = C_4 = C_6 = 0$ in order to avoid the cases $\{\ln r, K_{0,1}(2\alpha r)\} \to \infty$ for $r \to 0$. In physical terms our choice means that the displacements are taken to be finite at the centerpoint of the solid plate. Furthermore, it is also worth noting that for $r = 0$ we have $I_0(2\alpha r) - 1 = I_1(2\alpha r) = 0$. Finally, by setting $C_5 = C_6 = S_{r\theta} = 0$ in Eqs. (24) and (25) we obtain the general solution for the corresponding classical, annular Mindlin plate under a uniformly distributed load.

The general solution can be used to obtain the stress distributions in terms of the load resultants given by Eqs. (6)–(9) and $(15)_1$. We calculate the load resultants using Eqs. (24) and (25) and the stresses (5). Then we can express constants C_2, C_3, C_4, C_5 and C_6 in terms of the load resultants and substitute them back into the stresses (5) to obtain the stress distributions

$$\begin{aligned} \sigma_r(r,z) &= \frac{E(z)M_r(r)z}{D(1-\nu^2)}, \qquad \sigma_\theta(r,z) = \frac{E(z)M_\theta(r)z}{D(1-\nu^2)}, \\ \tau_{rz}(r,z) &= \frac{G(z)Q_r(r)}{D_Q}, \qquad m_{xy}(x,z) = \frac{G(z)P_{r\theta}(r)l^2}{S_{r\theta}} \; . \end{aligned} \tag{26}$$

Note that the uniformly distributed load does not appear explicitly in the above relations although it is included in the general solution given by Eqs. (24) and (25).

3 Case studies

In this section, we study solid and annular plates subjected to a uniformly distributed load and annular plates attached to a point-loaded rigid shaft. Both simply-supported and clamped conditions are used for the outer plate edges. The plate material is assumed to be linearly isotropic and homogeneous leading to the stiffness coefficients

$$D = \frac{Eh^3}{12(1-\nu^2)}, \quad D_Q = K_s G h, \quad S_{r\theta} = Ghl^2, \tag{27}$$

where $G = E/[2(1+\nu)]$. For the linear plates at hand, the parameter values for numerical calculations are taken as $E = 10^3$ GPa, $\nu = 0.35$, $h = 1$ mm, $a = 10h$, $b = 2h$ and $K_s = 5/6$. The value of the microstructural length scale parameter l is varied in the calculations.

3.1 Solid circular plates

Let us consider solid circular plates subjected to a uniformly distributed load $q = 10^3$ MPa. The boundary conditions for clamped and simply-supported plates are

$$\begin{aligned} r = a: \;\; & w = \phi = \theta = 0, \\ r = a: \;\; & w = \hat{M} = \hat{P} = 0, \end{aligned} \tag{28}$$

respectively. In both cases the three non-zero constants C_1, C_2 and C_5 in Eqs. (24) and (25) are solved using Eqs. (28). The displacement components of the clamped plate read

$$\begin{aligned} w(r) = &\frac{q(a^2-r^2)}{64 D_Q \beta_1^2}\left[16D\beta_1 + 4S_{r\theta}^2 + D_Q\beta_1(a^2-r^2)\right] \\ &- qa\frac{D S_{r\theta}\alpha\beta_2^2}{16 D_Q^2 \beta_1^3}\frac{I_0(2\alpha a) - I_0(2\alpha r)}{2I_0(\alpha a)}, \end{aligned} \tag{29}$$

$$\phi(r) = \frac{qr}{16 D_Q \beta_1^2}\left[D_Q\beta_1(a^2-r^2) - 2S_{r\theta}\beta_2\right] + qa\frac{S_{r\theta}\beta_2}{8D_Q\beta_1^2}\frac{I_1(2\alpha r)}{I_1(2\alpha a)}, \tag{30}$$

The classical solution is obtained by excluding the Bessel terms and setting $S_{r\theta}$ to zero. When it comes to the the simply-supported plate, it is more convenient to work with some of the load resultants

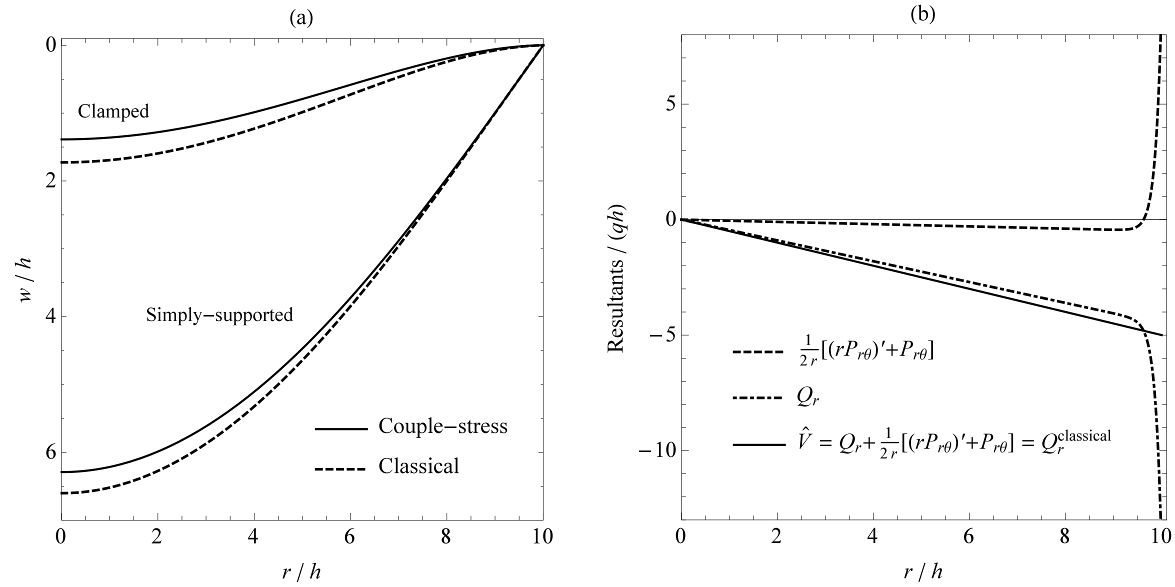

Fig. 2 (a) Simply-supported and clamped solid plates ($l = 0.25h$) under a uniformly distributed load. (b) Shear force terms of the simply-supported microstructure-dependent circular plate. Note that the dashed lines are bounded so that the effective shear force has a finite value at $r/h = 10$

instead of very lengthy explicit expressions for the displacements. The effective shear force $(15)_1$ of the simply-supported plate is the sum of the terms

$$Q_r = -\frac{q}{8}\left[\frac{2\beta_2 r}{\beta_1} + \frac{D_Q a^2}{D\alpha}\frac{I_1(2\alpha r)}{I_2(2\alpha a)}\right], \tag{31}$$

$$\frac{1}{2r}\left[(rP_{r\theta})' + P_{r\theta}\right] = \frac{qS_{r\theta}}{8\beta_1}\frac{\alpha a^2 I_1(2\alpha r) - 2rI_2(2\alpha a)}{I_2(2\alpha a)}\ . \tag{32}$$

Fig. 2(a) displays the transverse deflections of the clamped and simply-supported plates subjected to the uniformly distributed load. The microstructural length scale parameter has a value of $l = 0.25h$. It can be seen that the microstructure-dependent plates are stiffer than their classical counterparts. This is in line with the general trends observed in small-scale structural components. Fig. 2(b) shows that the effective shear force of the simply-supported plate is equal to the shear force $-qr/2$ of the classical solution. This can also be shown analytically by Eqs. $(15)_1$, (31) and (32).

3.2 Annular circular plates attached to a rigid shaft

Next we consider annular plates attached to a rigid shaft which is subjected to a vertical load $P = 10^5$ N at its centerpoint. The boundary conditions for the inner edges read

$$r = b:\ \phi = \theta = 0 \quad \text{and} \quad 2\pi(r\hat{V}) = -P\ . \tag{33}$$

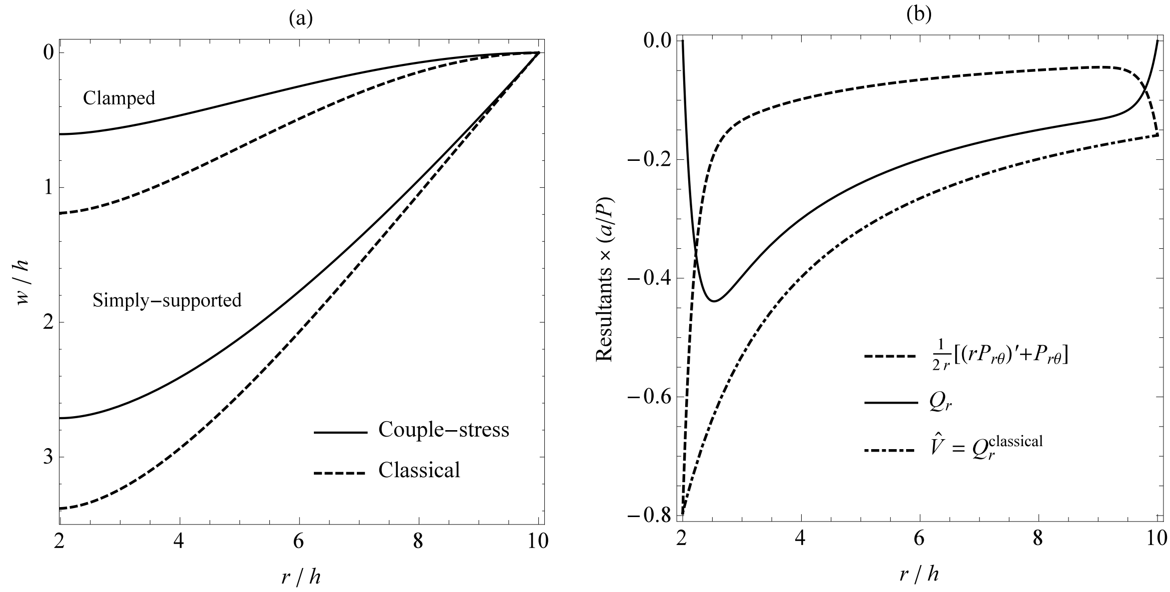

Fig. 3 (a) Simply-supported and clamped annular plates ($l = 0.5h$) attached to a point-loaded rigid shaft. (b) Shear force terms of the clamped microstructure-dependent circular plate

The outer edge boundary conditions for the clamped and simply-supported plates are the given by Eqs. (28). For the clamped plate we obtain

$$\begin{aligned} Q_r = {} & \frac{PD_Q\beta_1\beta_2}{S\eta}\big\{rI_1(2\alpha r)[aK_1(2\alpha a) - bK_1(2\alpha b)] \\ & + aI_1(2\alpha a)[bK_1(2\alpha b) - rK_1(2\alpha r)] + bI_1(2\alpha b)[rK_1(2\alpha r) - aK_1(2\alpha a)]\big\}, \end{aligned} \tag{34}$$

$$\begin{aligned} \frac{1}{2r}\left[(rP_{r\theta})' + P_{r\theta}\right] = {} & \frac{P}{\eta}\big\{aI_1(2\alpha a)\left[D_Q\beta_1 bK_1(2\alpha b) + D\alpha^2\beta_2 rK_1(2\alpha r)\right] \\ & - bI_1(2\alpha b)\left[D_Q\beta_1 aK_1(2\alpha a) + D\alpha^2\beta_2 rK_1(2\alpha r)\right] \\ & + D\alpha^2\beta_2 rI_1(2\alpha r)\left[bK_1(2\alpha b) - aK_1(2\alpha a)\right]\big\}, \end{aligned} \tag{35}$$

where

$$\eta = 2\pi rab\left(D_Q\beta_1 + D\alpha^2\beta_2\right)\left[I_1(2\alpha b)K_1(2\alpha a) - I_1(2\alpha a)K_1(2\alpha b)\right] . \tag{36}$$

We note that the logarithmic terms are not present in Eqs. (34) and (35). In addition, while the sum of Eqs. (34) and (35) gives the effective shear force, it can also be obtained from the equation

$$\hat{V} = \frac{1}{r}\left[(rM_r)' - M_\theta + (rP_{r\theta})' + P_{r\theta}\right] = -\frac{P}{2\pi r} . \tag{37}$$

Fig. 3(a) presents the transverse deflections of the clamped and simply-supported plates attached to the point-loaded rigid shaft. The microstructural length scale parameter has a value of $l = 0.5h$. Similarly to the solid circular plates, the microstructure-dependent plates are stiffer than their classical counterparts. The stiffness increases for larger values of l, which can be observed by comparing Figs. 2(a) and 3(a). In Fig. 3(a), the maximum displacement of the clamped classical plate is double that of the microstructure-dependent plate. It is interesting to note that numerical calculations suggest the transverse deflection to be bounded near the centerpoint of the microstructure-dependent plate in the present case for values higher than $l \approx 0.1h$ when $b \to 0$. The corresponding classical deflection is unbounded, i.e., approaches infinity. Fig. 3(b) shows the effective shear force of the clamped plate.

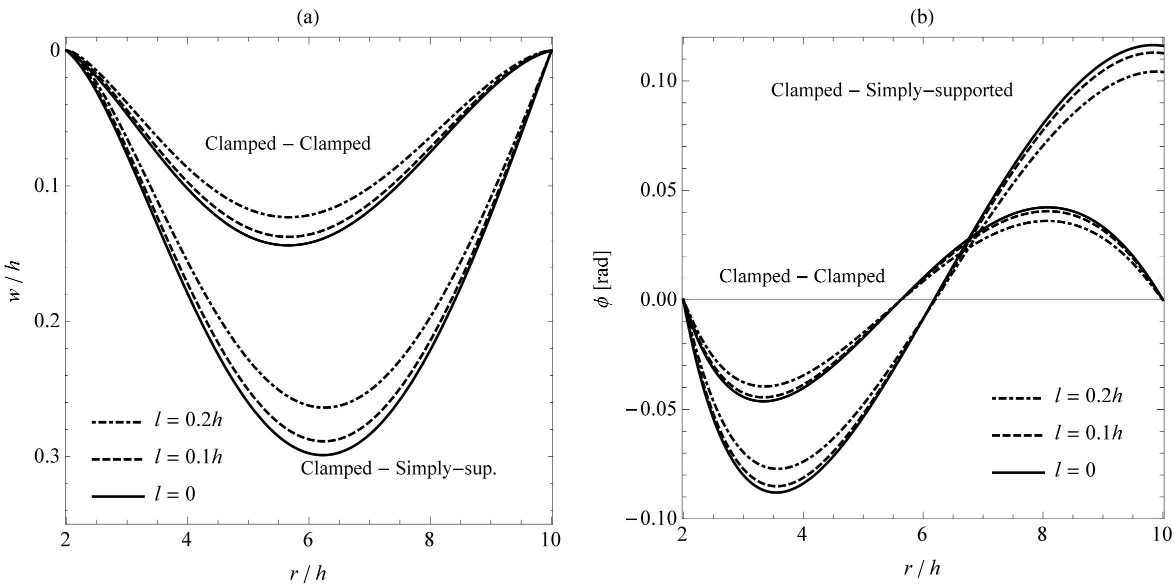

Fig. 4 (a) Deflections and (b) rotations of annular plates under a uniformly distributed load

3.3 Annular circular plates with clamped inner edge

Finally, we study annular plates that have a clamped inner edge and the outer edge is either clamped or simply-supported. The plates are subjected to a uniformly distributed load $q = 10^3$ MPa. The boundary conditions can be adapted from Eqs. (28). Figs. 4(a) and 4(b) show the deflections $w(r)$ and rotations $\phi(r)$ of the plates. It can be seen that a change in the outer edge boundary condition shifts the locations of the maximum deflections and the corresponding zero rotations, but the value of the microstructural length scale parameter has no notable effect on the location of the maximum deflections.

4 Conclusions

In this work, we derived a general analytical solution for the static, axisymmetric bending of solid and annular circular Mindlin plates based on the modified couple-stress theory. The solution provides a straightforward approach to study engineering plate applications where microstructural effects play a pivotal role. In addition, the general solution can be used to develop reference solutions to validate numerical methods which are used to study microstructure-dependent plates.

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