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Nitsche's method for the obstacle problem of clamped Kirchhoff plates

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Abstract. The theory behind Nitsche's method for approximating the obstacle problem of clamped Kirchhoff plates is reviewed. A priori estimates and residual-based a posteriori error estimators are presented for the related conforming stabilised finite element method and the latter are used for adaptive refinement in a numerical experiment.

1 Introduction

Nitsche's method is widely used for the numerical approximation of contact problems, cf. [5,9,10,6] and all the references therein. It was first proposed as a non-standard treatment of boundary conditions [23] and as such is related to later discovered discontinuous Galerkin methods. Over 20 years ago (cf. [25]), using a Lagrange multiplier to impose (weakly) the Dirichlet boundary condition on the Poisson equation, we observed that the Lagrange multiplier can be eliminated, element by element, from the stabilised finite element formulation, leading to an optimally conditioned, symmetric and positive definite system corresponding to a method by Nitsche which was, at that time, largely forgotten.

There are, however, fundamental issues with Nitsche's formulation since its analysis requires an additional smoothness assumption and the a posteriori estimates are based on a so-called *saturation assumption*, cf. [1,20,8]. Recently, by going back to the interpretation of Nitsche's method as a stabilised formulation, recalling our analysis for the Stokes problem [26] and using the techniques from [13,15], we were able to give a complete error analysis (both a priori and a posteriori) for the stabilised/Nitsche's method when applied to the membrane obstacle problem [17]. We emphasise that the stabilised form is only needed for the analysis. For the practical implementation, Nitsche's formulation is preferable.

In this note, we will review our latest results on conforming stabilised finite element methods for the plate obstacle problem, cf. [19]. Our method is based on a saddle point formulation with the contact force appearing as an additional unknown (Lagrange multiplier). We present an a priori estimate with minimal regularity assumptions and introduce the Nitsche's formulation with Lagrange multiplier providing an approximation for the contact force and the unknown contact domain. We will also present an a posteriori error estimator and use it for adaptive refinement in a numerical experiment.

Numerical approximation of fourth-order variational inequalities has been previously studied, e.g., in [24,4,3,16,2] but to our knowledge Nitsche's or stabilised methods using conforming C^1-continuous elements were for the first time proposed and rigorously analysed in [19].

2 Problem statement

Let $\Omega \subset \mathbb{R}^2$ denote a polygonal domain occupied by (the mid-surface of) a thin plate of thickness d whose deformation is governed by the Kirchhoff–Love theory. Assume that the vertical displacement u of the plate, resulting from an applied load $f \in L^2(\Omega)$, is constrained by a rigid obstacle $g \in H^2(\Omega)$ and suppose, for simplicity, that the plate is clamped at all edges.

Letting $\boldsymbol{\varepsilon}(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$ denote the infinitesimal strain tensor and $\boldsymbol{K}(u) = -\boldsymbol{\varepsilon}(\nabla u)$ the curvature, the bending moment $\boldsymbol{M}$ is defined by

$$\boldsymbol{M}(u) = \frac{Ed^3}{12(1+\nu)} \left(\boldsymbol{K}(u) + \frac{\nu}{1-\nu} \mathrm{tr}\,(\boldsymbol{K}(u))\, \boldsymbol{I} \right),$$

where E and ν are the Young's modulus and the Poisson ratio (see, e.g., [12]). Defining the bilinear and linear forms a and l by

$$a(w,v) = \int_\Omega \boldsymbol{M}(w) : \boldsymbol{K}(v)\,\mathrm{d}x, \qquad l(v) = \int_\Omega fv\,\mathrm{d}x,$$

the solution to the clamped plate obstacle problem can be characterised as

$$u = \mathrm{argmin}_{v\in K} \left[\frac{1}{2} a(v,v) - l(v) \right],$$

where $K = \{ v \in H^2_0(\Omega) : v \geq g \text{ in } \Omega \}$ or, equivalently, as the solution to the variational inequality: *Find* $u \in K$ *such that*

$$a(u, v-u) \geq l(v-u) \quad \forall\, v \in K\,. \tag{1}$$

Remark 1. The existence of a unique solution to problem (1) follows from standard theory, see [21]. For smooth data, the solution has been shown to be in $H^3_{\mathrm{loc}}(\Omega) \cap C^2(\Omega)$, in convex domains in $H^3(\Omega)$, cf. [14,7], and the smoothness threshold seems to be $H^{7/2-\varepsilon}(\Omega), \varepsilon > 0$, see Example 1 in [3].

Nitsche's method for approximating the plate obstacle problem can be regarded as a stabilised finite element method for the Lagrange multiplier formulation of problem (1) with the stabilisation term arising from its strong form. Associating a Lagrange multiplier λ to the constraint $v \geq g$, the strong form reads as follows:

$$\left.\begin{aligned} \mathcal{A}(u) - \lambda = f \\ \lambda \geq 0, \quad u - g \geq 0, \quad \lambda(u-g) = 0 \end{aligned}\right\} \text{ in } \Omega, \tag{2}$$

$$u = 0 \text{ and } \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where

$$\mathcal{A}(u) = \frac{Ed^3}{12(1-\nu^2)} \Delta^2 u\,.$$

The Lagrange multiplier λ corresponds to a reaction force exerted on the plate by the obstacle and it belongs to the space

$$\Lambda = \{\mu \in Q : \langle v, \mu \rangle \geq 0 \ \forall v \in V \text{ s.t. } v \geq 0 \text{ a.e. in } \Omega\},$$

where $V = H_0^2(\Omega), Q = H^{-2}(\Omega) = [H_0^2(\Omega)]'$ and $\langle \cdot, \cdot \rangle : V \times Q \to \mathbb{R}$ denotes the duality pairing.

Let us define a bilinear form $\mathcal{B} : (V \times Q) \times (V \times Q) \to \mathbb{R}$ and a linear form $\mathcal{L} : V \times Q \to \mathbb{R}$ through

$$\begin{aligned} \mathcal{B}(w, \xi; v, \mu) &= a(w, v) - \langle v, \xi \rangle - \langle w, \mu \rangle, \\ \mathcal{L}(v, \mu) &= (f, v) - \langle g, \mu \rangle. \end{aligned}$$

Problem (2)-(3) can now be written as the following variational inequality:

Find $(u, \lambda) \in V \times \Lambda$ *such that*

$$\mathcal{B}(u, \lambda; v, \mu - \lambda) \leq \mathcal{L}(v, \mu - \lambda) \quad \forall (v, \mu) \in V \times \Lambda. \tag{4}$$

The bilinear form $\mathcal{B}$ is continuous and stable (cf. [19]) with respect to the norm

$$|||(w, \xi)||| = \left(\|w\|_2^2 + \|\xi\|_{-2}^2\right)^{1/2},$$

where $\|\cdot\|_2$ and $\|\cdot\|_{-2}$ are the usual norms in $H^2(\Omega)$ and $H^{-2}(\Omega)$. The saddle-point formulation (4) and formulation (1) are equivalent, cf. [11].

3 Stabilised finite element method

Let $\mathcal{C}_h$ be a conforming shape-regular triangulation of Ω into triangles K and let $V_h \subset V$ and $Q_h \subset Q$ be finite element subspaces. Defining

$$\Lambda_h = \{\mu_h \in Q_h : \mu_h \geq 0 \text{ in } \Omega\} \subset \Lambda$$

and introducing stabilised bilinear and linear forms $\mathcal{B}_h$ and $\mathcal{L}_h$ by

$$\mathcal{B}_h(w,\xi;v,\mu) = \mathcal{B}(w,\xi;v,\mu) - \alpha \sum_{K\in\mathcal{C}_h} h_K^4(\mathcal{A}(w)-\xi, \mathcal{A}(v)-\mu)_K,$$
$$\mathcal{L}_h(v,\mu) = \mathcal{L}(v,\mu) - \alpha \sum_{K\in\mathcal{C}_h} h_K^4(f, \mathcal{A}(v)-\mu)_K,$$

where $\alpha > 0$ is a stabilisation parameter, the stabilised finite element method becomes: *Find* $(u_h,\lambda_h)\in V_h\times \Lambda_h$ *such that*

$$\mathcal{B}_h(u_h,\lambda_h;v_h,\mu_h-\lambda_h) \leq \mathcal{L}_h(v_h,\mu_h-\lambda_h) \quad \forall (v_h,\mu_h)\in V_h\times\Lambda_h. \tag{5}$$

The stabilised formulation is consistent and, assuming that $\alpha\in(0,C_I)$ where $C_I>0$ is the constant from the inverse inequality

$$C_I \sum_{K\in\mathcal{C}_h} h_K^4 \|\mathcal{A}(w_h)\|_{0,K}^2 \leq a(w_h,w_h) \quad \forall w_h \in V_h,$$

it is stable which leads to a quasi-optimal a priori error estimate (see [19]):

$$|\!|\!|(u-u_h,\lambda-\lambda_h)|\!|\!| \lesssim \inf_{\substack{v_h\in V_h,\\ \mu_h\in\Lambda_h}} \Big(|\!|\!|(u-v_h,\lambda-\mu_h)|\!|\!| + \sqrt{\langle u-g,\mu_h\rangle}\Big) + \mathrm{osc}(f).$$

Above $\mathrm{osc}(f)$ denotes the data oscillation defined by

$$\mathrm{osc}(f)^2 = \sum_{K\in\mathcal{C}_h} h_K^2 \|f-f_h\|_{0,K},$$

with $f_h\in V_h$ standing for the L^2-projection of f.

4 Nitsche's method

Assume that the finite element spaces consist of a C^1-element for the displacement field (in our example, the Argyris element) coupled with a piecewise polynomial and discontinuous approximation of the Lagrange multiplier. The Lagrange multiplier can thus be eliminated elementwise from the stabilised formulation (5) leading to the Nitsche's method:

Find $u_h\in V_h$ *such that*

$$a_h(u_h,v_h;u_h) = l_h(v_h;u_h) \quad \forall v_h\in V_h, \tag{6}$$

where

$$\begin{aligned} a_h(u_h,v_h;w_h) &= a(u_h,v_h) + \big(\tfrac{1}{\alpha\mathcal{H}^4}u_h,v_h\big)_{\Omega_C(w_h)} - (\mathcal{A}(u_h),v_h)_{\Omega_C(w_h)} \\ &\quad - (u_h,\mathcal{A}(v_h))_{\Omega_C(w_h)} - \big(\alpha\mathcal{H}^4\mathcal{A}(u_h),\mathcal{A}(v_h)\big)_{\Omega\setminus\Omega_C(w_h)}, \\ l_h(v_h;w_h) &= (f,v_h) + \big(\tfrac{1}{\alpha\mathcal{H}^4}g,v_h\big)_{\Omega_C(w_h)} - (g,\mathcal{A}(v_h))_{\Omega_C(w_h)} \\ &\quad - (f,v_h)_{\Omega_C(w_h)} - (\alpha\mathcal{H}^4 f,\mathcal{A}(v_h))_{\Omega\setminus\Omega_C(w_h)}. \end{aligned}$$

Above, $\mathcal{H} \in L^2(\Omega)$ is defined as $\mathcal{H}|_K = h_K$, $\forall K \in \mathcal{C}_h$, and the contact set $\Omega_C(w_h) = \{(x,y) \in \Omega : F(w_h) > 0\}$, where

$$F(w_h) = \frac{1}{\alpha \mathcal{H}^4}\left(g - w_h + \alpha \mathcal{H}^4(\mathcal{A}(w_h) - f)\right)_+ , \qquad w_+ = \max(w, 0),$$

is the reaction force (Lagrange multiplier), is approximated iteratively using the previous displacement field to linearise problem (6).

Based on an a posteriori error analysis of the stabilised method, the local error estimator used in an adaptive refinement strategy is defined as

$$\mathcal{E}_K^2 = \eta_K^2 + \frac{1}{2}\sum_{E\subset K} \eta_E^2 + ((u_h - g)_+, \lambda_h)_K + \|(g - u_h)_+\|_{2,K}^2,$$

where E are the (interior) edges of $K \in \mathcal{C}_h$ and

$$\eta_K^2 = h_K^4 \|\mathcal{A}(u_h) - \lambda_h - f\|_{0,K}^2, \quad \eta_E^2 = h_E^3 \|[\![V_n(u_h)]\!]\|_{0,E}^2 + h_E \|[\![M_{nn}(u_h)]\!]\|_{0,E}^2,$$

with $[\![V_n(u_h)]\!]$ and $[\![M_{nn}(u_h)]\!]$ denoting the jumps over interior edges of the Kirchhoff shear force and the normal moment, see [18,19] for more details.

5 Numerical results

We will consider the problem from [2, Example 7.4] where the domain Ω is given by $\Omega = (-0.5, 0.5)^2 \setminus [0, 0.5]^2$ and the obstacle and load function are

$$g(x) = -\sin(2\pi(x + 0.5)(y + 0.5))\sin(4\pi(x - 0.5)(y - 0.5)) - 0.35$$

$$f(x) = \begin{cases} 500\, e^{(x+0.25)^2 + (y+0.25)^2}, & x \leq 0,\, y > 0 \\ 0, & x \leq 0,\, y \leq 0 \\ 1000\left(0.5 + (x - 0.25)^2 + (y + 0.25)^2\right)^{3/2}, & x > 0,\, y \leq 0 \end{cases}$$

The stabilisation parameter is chosen as $\alpha = 10^{-5}$, and the marking and adaptive refinement strategies are as in [19]. For this problem the contact set is one-dimensional. The discrete solution and the discrete contact set are visualised in Fig. 1. The adaptive meshes and the total error are given in Fig. 2. Note that due to the re-entrant corner, the exact solution belongs to $H^{2.54}(\Omega)$ (cf. [22]) which corresponds to the convergence rate $N^{-0.27}$ obtained with uniform refinement, with N denoting the number of degrees of freedom, and that the adaptive meshing strategy recovers the optimal rate of convergence N^{-2} for fifth order elements.

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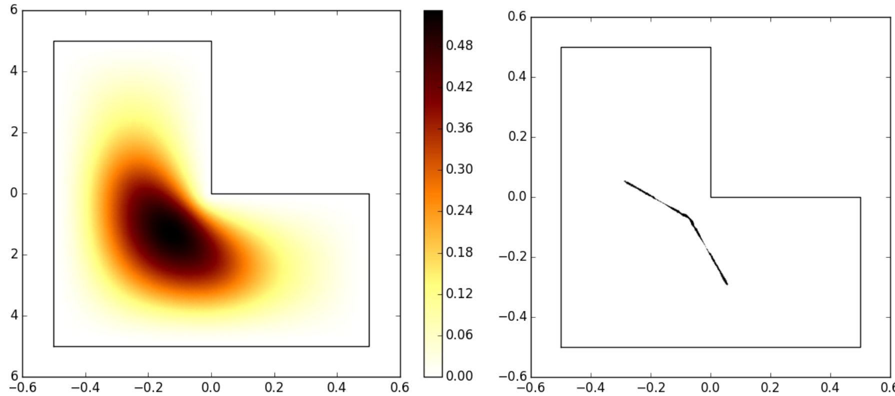

Fig. 1. The discrete solution u_h and the discrete contact set, $\Omega_C(u_h)$, after two adaptive mesh refinements.

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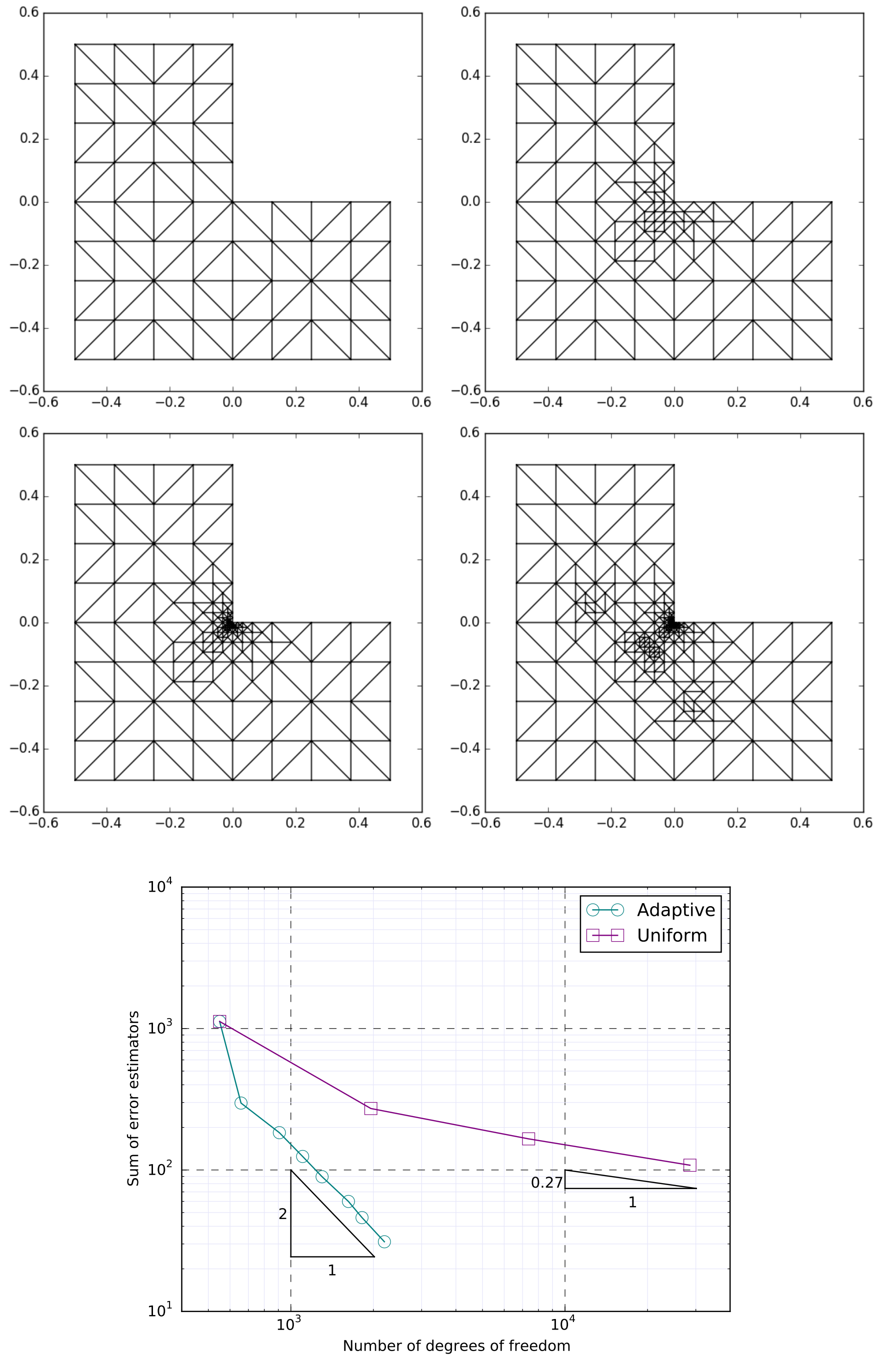

Fig. 2. A sequence of adaptively refined meshes and the total error as a function of the number of degrees of freedom. The upper-left panel depicts the initial mesh.

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