Liesiö, Juuso; Andelmin, Juho; Salo, Ahti

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Efficient Allocation of Resources to a Portfolio of Decision Making Units

Juuso Liesiö[a,*], Juho Andelmin[b], Ahti Salo[b]

[a]*Department of Information and Service Management, Aalto University School of Business, P.O. Box 21220, FI-00076 Aalto, Finland*

[b]*Department of Mathematics and Systems Analysis, Aalto University School of Science, P.O. Box 11100, FI-00076 Aalto, Finland*

Abstract

Efficiency analysis is widely employed to evaluate decision making units (DMUs) which convert input resources into outputs. In this paper, we develop models for allocating these resources to DMUs in order to maximize the overall efficiency of the portfolio formed by these DMUs. Our models do not require complete preference information about how valuable the inputs and outputs are relative to each other. Rather, based on incomplete preference information and explicit assumptions about the DMUs' production possibilities, they determine all efficient DMUs portfolios which are then analyzed to provide robust decision recommendations on how much resources should be allocated to each DMU. We illustrate our models by revisiting earlier case studies and show that the use of conventional efficiency analysis in guiding resource allocation decisions can cause inefficiencies.

Keywords: Decision Support Systems, Portfolio Decision Analysis, Multiple Criteria, Resource Allocation, Efficiency Analysis

1. Introduction

Methods of efficiency analysis help compare the performance of decision making units (DMUs) such as hospitals, universities, and firms which consume many kinds of inputs in order to produce many kinds of outputs (see, e.g., Cooper et al. 2000, Podinovski 2005, 2011, Swink et al. 2006, Salo & Punkka 2011, Lim et al. 2014, Eskelinen 2017). These methods estimate the efficient frontier using data on the DMUs' observed input-output mixes. Moreover, some of them make ex-ante assumptions about the functional form of the frontier (see, e.g., Aigner et al. 1977), while others are based on axioms that govern how inputs can be transformed to outputs (e.g. free disposability or decreasing returns to scale; see, e.g., Charnes et al. 1994, Green & Cook 2004, Kuosmanen & Kortelainen 2012).

Typically, the efficiency of each DMU is measured by the distance between its observed input-output mix and the efficient frontier. The resulting efficiency scores offer a *non-parametric* efficiency measure which does not require that inputs and outputs are made commensurable by specifying the exact unit values of each input/output. Thus, these scores help identify which DMUs may not fully use their input resources, allowing managerial actions to be focused on improving the efficiencies of inefficient DMUs (Charnes et al. 1994, see also Cook & Seiford 2009).

[*]Corresponding author

Email addresses: juuso.liesio@aalto.fi (Juuso Liesiö), juho.andelmin@aalto.fi (Juho Andelmin), ahti.salo@aalto.fi (Ahti Salo)

There has also been considerable interest in using efficiency analysis for allocating resources among, for instance, units of a business organization (Golany et al. 1993), fire departments (Athanassopoulos 1998), supermarkets (Korhonen & Syrjänen 2004), ports (Lozano et al. 2011; Fang 2016; Wu et al. 2016), and bank branches (Ray 2016). The resulting models are often referred to as 'centralized' (Lozano & Villa 2004) to distinguish them from traditional models which consider the efficiency of each DMU somewhat independently. Indeed, even a group of individually efficient DMUs may be able to improve its overall performance through optimizing the allocation of the input resources (see, e.g., Nesterenko & Zelenyuk 2007).

Commonly these centralized resource allocation models utilize efficiency analysis in establishing the set of possible input-output mixes that each DMU can attain while enforcing constraints on the total consumption of input resources and/or on the total production of outputs across the group of DMUs. Some models account for the DMUs' efficiency differences through additional constraints that ensure that the DMUs' efficiency scores remain constant even if their input resources change (Korhonen & Syrjänen 2004, Nasrabadi et al. 2012), while others are specifically designed to identify efficiency improving targets for the DMUs to aim at when resource allocations are changed. In particular, the seminal model by Lozano & Villa (2004) identifies target levels for the input resource reductions for all DMUs such that the total output of all the DMUs does not decrease. Several recent articles have extended these models. For instance, Lozano & Villa (2005) introduce the possibility of closing down some of the DMUs by modifying the feasible input-output mixes: their model allows DMUs to be projected onto a virtual point that consumes no input resources and produces no outputs.In turn, Asmild et al. (2009) modify the model so that only resources allocated to inefficient DMUs can be changed. They argue that, in practice, convincing the efficient DMUs to change their mode of operations, which the analysis has identified as a benchmark for the inefficient units, can result in organizational pushback. The model by Fang (2015) produces a sequence of intermediate targets which provides a path of efficiency improvements for each of the DMUs to follow. Fang & Li (2015) consider the costs associated with reallocating resources among DMUs and develop a model to identify revenue maximizing allocation that can be achieved with minimal reallocation costs. Sadeghi & Dehnokhalaji (2019) extend the model of Lozano & Villa (2004) to situations where additional resources need to be efficiently allocated among the DMUs.

Because centralized resource allocation models are motivated by applications involving multiple inputs and/or outputs, they depend on the decision maker's (DM's) preferences that determine how different inputs and outputs are valued. Many of these models do not consider preferences explicitly but rather apply some standard efficiency measures to evaluate the quality of different allocations without any interaction with the DM. However, several authors propose models that explicitly capture the DM's preferences through dedicated parameters whose values can be specified through rigorous preference elicitation process involving the DM. For instance, the model by Lozano & Villa (2004) includes weight coefficients that capture the DM's preferences among reductions in the consumption of different inputs. Korhonen & Syrjänen (2004) propose the Pareto race -method to search for the most preferred trade-off between the different outputs. Fang & Zhang (2008) develop a bi-objective model for resource allocation where the DM can search for the most preferred trade-off between maximizing the average efficiency of all DMUs and maximizing the efficiency of the least efficient DMU. Yang et al. (2018) propose a goal programming approach to capture DM's priorities between maximizing relative improvements in total inputs and outputs. The third objective of their model takes into account the difficulties DMUs can have in changing their input-output mixes by introducing coefficients that the DM can specify using the analytic hierarchy process.

A limitation of these approaches is that the DM must specify preferences among the inputs and outputs until a single optimal resource allocation is obtained. This requirement of complete preference specification is not aligned with the non-parametric tradition of efficiency analysis which typically avoids exact trade-offs among inputs and outputs. That is, even though methods of efficiency analysis involve minimal preference assumptions, the resource allocation models built on top of them require complete preference information in determining the allocation of resources. Yet, in practice, the elicitation of such exact preferences for multiple inputs and outputs may be difficult, time-consuming, or even impossible (White et al. 1982, Hazen 1986, Weber 1987). Furthermore, if there are several DMs with differing viewpoints, the diversity of their preferences cannot be represented by a single preference statement.

In this paper we address these limitations by developing efficiency-based resource allocation models which – in the spirit of efficiency analysis – do not require completely specified preferences for inputs and outputs. Instead, we use approaches of portfolio decision analysis (PDA; Salo et al. 2011; Liesiö et al. 2007, 2008; Liesiö & Salo 2012; Lourenço et al. 2012; Fliedner & Liesiö 2016; Vilkkumaa et al. 2018) to (i) admit incomplete preference information, and to (ii) identify all the alternative resource allocations such that the resulting *portfolio* of DMUs is efficient both in its aggregate outputs and inputs. The analysis of all these efficient portfolios provides answers to the following questions:

- Is the current allocation of resources efficient? Or can more of all outputs be produced by reallocating input resources among the DMUs?
- For each DMU, what are the minimum and maximum amounts of inputs such that the DMU portfolio can be efficient?
- Which DMUs merit incontestably more (or less) resources in the light of stated preference information about the relative values of inputs and outputs?
- Can conventional efficiency scores suggest resource allocations which lead to inefficiencies?

The identification of all efficient DMU portfolios for the available preference information is useful for many reasons. First, the DM need not provide detailed information about how valuable units of each input and output are relative to each other: rather, resource allocation recommendations can be derived from an incomplete preference specification. If no preference statements are given, the model identifies all portfolios with a Pareto optimal input-output mix while, at the other end of the spectrum, exact estimates about the unit values for inputs and outputs generally lead to a unique optimal portfolio. Second, in a group decision setting it is possible to admit preference information which spans the unit values stated by each group member so that the resulting efficient portfolios include the optimal portfolios for all of them. This makes it possible, for instance, to check if there are DMUs which merit incontestably more (or less) input resources, despite the group members' different views on the relative values of inputs and outputs. Finally, even conventional efficiency analyses which are focused on the identification of efficient and inefficient DMUs can be enhanced by analyzing the ranges of input resources that the DMUs receive across all efficient portfolios.

Applying these models to real data sets suggests practical implications on using conventional efficiency analysis methods for resource allocation. Importantly, our results show that, in some cases, reallocating input resources from the inefficient DMUs to the efficient ones can decrease all aggregate outputs simultaneously. Such a reallocation would inevitably lead to an inefficient portfolio which could not be justified by any plausible input/output preferences. Hence, one should be extremely cautious when using conventional efficiency scores and their efficient/inefficient-classifications to motivate resource allocations decisions. We argue that the models developed in this paper provide more rigorous support for resource allocation decisions by identifying the entire set of efficient allocations.

The proposed resource allocation models build on efficiency analysis and hence their recommendations depend on assumptions about how the DMUs' production possibilities are affected by the reallocation of resources, possibly due to events such as the introduction of new production technologies. In general, the DMUs ability to maintain or depart from their current efficiencies may vary from one application to another. This has been recognised by developing resource allocation models of which some assume that the DMUs cannot improve their efficiency (see, e.g., Korhonen & Syrjänen 2004), while others help identify, for example, those input-output mixes that the DMUs should attain as a result of the reallocation in order to improve their efficiency (see., e.g., Lozano et al. 2011).

Because it may be unrealistic to assume that the impacts of resource reallocations on DMU's efficiencies can be predicted very accurately, it can be instructive to examine the implications of alternative assumptions about such impacts. Here, our resource allocation models can be useful, because they do not require that the sets of feasible input-output mixes satisfy specific assumptions, such as maintaining DMUs' efficiencies at their current level. Still, if it reasonable to believe that the efficiencies will change, for example due to the introduction of new production technologies, then approaches such as expert judgement elicitation can be employed to characterize plausible changes in DMUs' efficiencies. Moreover, statistical techniques such as those proposed by Banker & Natarajan (2011) can be employed to formulate and test reasonable hypotheses as to what, if any, efficiency changes can be anticipated. In this paper, the numerical examples are premised on the assumption of non-increasing efficiencies. This assumption is conservative, as it highlights what overall efficiency improvements can be achieved solely through the optimal allocation of resources to the DMU portfolio. If the DMUs' efficiencies were to improve, then the portfolio-level improvements in efficiency could be even greater. Furthermore, it can be instructive to examine how the recommended resource allocations depend on assumptions about the DMUs' efficiencies, because this helps identify (re-)allocations which are less sensitive to such assumptions.

The rest of this paper is structured as follows. Section 2 develops the model for efficiency analysis based resource allocation under incomplete preference information. Section 3 develops computational algorithms for solving the efficient DMU portfolios. Section 4 discusses implications of the model for decision support. Section

5 examines real data from two reported applications. Section 6 extends the model to non-linear preferences and discusses alternative approaches for estimating the DMUs' feasible input-output mixes. Section 7 concludes.

2. Resource Allocation Based on Efficiency Analysis

Let there be m DMUs which consume s different resources as inputs and produce n different outputs. Let $x^j = (x_1^j, \ldots, x_s^j)^T \in \mathbb{R}_+^s$ denote the inputs and $y^j = (y_1^j, \ldots, y_n^j)^T \in \mathbb{R}_+^n$ the outputs of DMU $j = 1, ..., m$. The capability of DMU j to transform inputs into outputs is determined by a polyhedral set of feasible input-output mixes $M^j \subset \mathbb{R}_+^{s+n}$. This definition does not rule out the possibility that the set of feasible input-output mixes can be the same for all DMUs. However, as will become evident later, these sets are often not the same for all DMUs. For instance, if the efficiencies of DMUs cannot be improved, the inefficient DMUs will have smaller sets of feasible input-output mixes than the efficient ones.

By selecting an input-output mix $(x^j, y^j) \in M^j$ for each DMU j, the DM can control the total amount of resources consumed by the portfolio of DMUs ($\sum_{j=1}^m x^j$) as well as the aggregate outputs produced by the portfolio ($\sum_{j=1}^m y^j$). Identifying the preferred input-output mixes for the DMUs requires, among other things, information about which mixes can be attained. Hence, before developing our formal portfolio model in Section 2.2, we show how efficiency analysis can be used to estimate the sets of feasible input-output mixes M^j.

2.1. *Preliminaries on Efficiency Analysis*

The sets of feasible input-output mixes M^j, $j = 1, ..., m$, can be built with many methods. Here, we focus on methods that fall under the general term Data Envelopment Analysis (DEA). In Section 6 we discuss how other methods of efficiency analysis or even expert judgements could be used.

DEA is a family of non-parametric methods in which a production possibility set (PPS) is estimated based on the DMUs' observed input-output mixes and each DMU is given an efficiency score based on its location relative to the efficient frontier of the PPS. In particular, let vectors $\hat{x}^j \in \mathbb{R}^s$ and $\hat{y}^j \in \mathbb{R}^n$ denote the *observed* inputs and outputs of DMU j, respectively. Moreover, let $\hat{X} = (\hat{x}^1, \ldots, \hat{x}^m) \in \mathbb{R}^{s \times m}$ and $\hat{Y} = (\hat{y}^1, \ldots, \hat{y}^m) \in \mathbb{R}^{n \times m}$ denote the matrices that contain the observed inputs and outputs of all the DMUs. Then, the PPS is the polyhedral set

$$T = \{(x, y)^T \in \mathbb{R}_+^{s+n} \mid \hat{X}\lambda \leq x, \hat{Y}\lambda \geq y, \text{ for some } \lambda \in \Lambda\}, \tag{1}$$

where $\Lambda \subset \mathbb{R}_+^m$ is the set of allowed DMU weights λ. Intuitively, the input-output mix (x, y) belongs to T if it is possible to construct a hypothetical DMU as a λ-weighted combination of the observed input-output mixes which consumes no more of the inputs and produces no less of the outputs. Hence, the set of allowed weights $\Lambda \subset \mathbb{R}_+^m$ encodes assumptions on how outputs can be produced from the inputs. For instance, in the CCR-DEA model by Charnes et al. (1978) the set of allowed weights is $\Lambda = \{\lambda \in \mathbb{R}_+^m\}$, which yields a PPS with constant returns to scale. In the BCC-DEA by Banker et al. (1984) the set of allowed weights is the simplex $\Lambda = \Delta^m = \{\lambda \in \mathbb{R}_+^m \mid \sum_{i=1}^m \lambda_i = 1\}$, which implies variable returns to scale. Other models are also possible: for instance, Koopmans (1977) considers models in which the PPS also includes the aggregate mixes of the observed DMUs so that $\Lambda = \{\lambda \in [0, 1]^m\}$.

In computing efficiency scores for the DMUs, DEA models use different approaches for measuring the distance between the observed input-output mix and the efficient frontier of the PPS (see, e.g., Cook & Seiford 2009, Korhonen et al. 2018). In output-oriented DEA models, the DMU's efficiency is the relative increase in outputs that will move the DMU to the (weakly) efficient frontier of the PPS. This score $\sigma^j \in [1, \infty)$ of DMU j is the solution to the LP problem

$$\sigma^j = \max_{\sigma \in \mathbb{R}} \{\sigma \mid (\hat{x}^j, \sigma \hat{y}^j)^T \in T\}. \tag{2}$$

In particular, the efficiency score of efficient DMUs is one, because they are on the efficient border of the PPS; in contrast, inefficient DMUs have a score strictly greater than one. Assuming that changing the allocation of resources to DMU j does not improve its efficiency (cf. Golany et al. 1993, Korhonen & Syrjänen 2004, Nasrabadi et al. 2012) the set of feasible input-output mixes for DMU j is

$$M^j = \{(x^j, y^j)^T \in \mathbb{R}_+^{s+n} \mid (x^j, \sigma^j y^j)^T \in T\}. \tag{3}$$

Inserting (1) into (3) gives

$$M^j = \{(x^j, y^j)^T \in \mathbb{R}_+^{s+n} \mid \hat{X}\lambda^j \le x^j, \hat{Y}\lambda^j \ge \sigma^j y^j, \text{ for some } \lambda^j \in \Lambda\}. \tag{4}$$

Thus, the sets M^j can be modeled as a system of linear constraints on variables (x^j, y^j) and λ^j. In (4), the observed input-output data points for all DMUs are in matrices $\hat{X}$ and $\hat{Y}$, while $(x^j, y^j)^T \in M^j$ corresponds to the feasible input-output mixes of DMU j.

It is worth highlighting that the reallocation of resources will not decrease the efficiencies of the DMUs even though the sets M^j (4) do include input-output mixes with a lower efficiency than that of the observed mix. This is since the objectives of maximizing aggregate outputs and minimizing aggregate input resources of the entire DMU portfolio will force each DMU to choose one of the efficient input-output mixes in sets M^j.

When changes in the resource allocation are relatively small, it is reasonable to assume that DMUs efficiencies remain nearly constant, provided that there are no events that could significantly alter the DMUs' sets of production possibilities (such as the introduction of new production technologies). If such events are foreseen, then expert judgements or statistical tests based on relevant data can be relied on to assess how these events are expected to impact the DMUs' efficiencies. Technically, relaxing the assumption of constant efficiencies leads to a model with a different set of feasible input-output mixes than the set (4). Changing the values of parameters σ^j can be used to control the efficiency scores that the DMUs are expected to achieve due to changes in the resource allocation (cf. Fang 2015). Because different sets of feasible input-output mixes can be used for each DMU, it is also possible to specify different expectations concerning the efficiency for each DMU. Different approaches for establishing the sets of feasible input-output mixes are discussed in Section 6.

Figure 1 illustrates the sets of feasible mixes M^j estimated using output-oriented BCC and CCR models in a problem with a single input and a single output. There are $m = 5$ DMUs whose observed inputs are $\hat{X} = (2, 6, 8, 5, 4)$ and outputs are $\hat{Y} = (1, 6, 7, 4, 2)$. In the CCR model, only DMU $j = 2$ is on the efficient frontier, and hence it is the only DMU whose efficiency score is one. In turn, DMU $j = 4$ is not efficient and

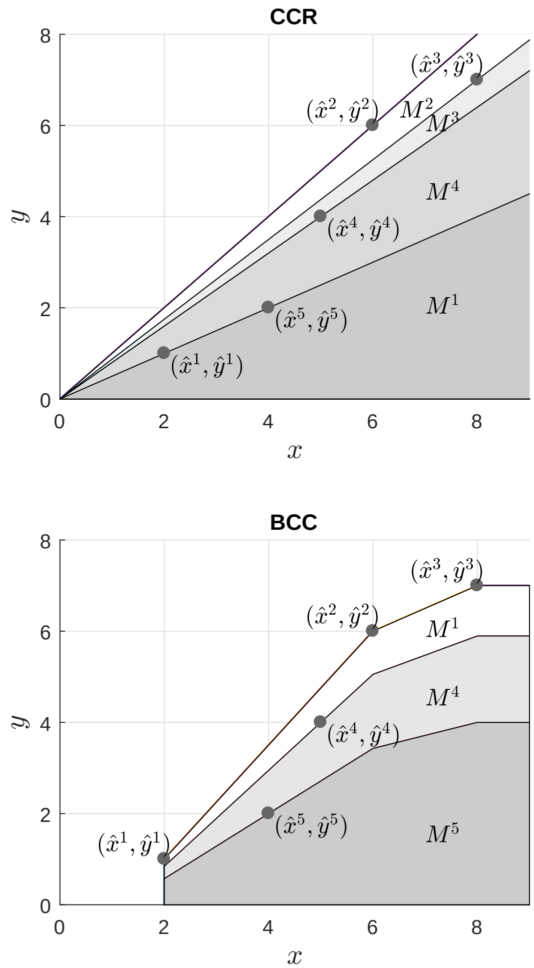

Figure 1: The sets of feasible input-output mixes M^j for $m = 5$ DMUs for the CCR and BCC models. These sets overlap: in the CCR model $M^2 \supset M^3 \supset M^4 \supset M^5 = M^1$ and in the BCC model $M^1 = M^2 = M^3 \supset M^4 \supset M^5$. The observed input-output data points are marked with gray circles.

its efficiency score is $\sigma^4 = 1.25$. Indeed, increasing its output by 25% would result in the input-output mix of $(5, 1.25 \cdot 4) = (5, 5)$ on the efficient frontier. Similarly, in the BCC model the efficiency score of DMU $j = 5$ is $\sigma^5 = 1.75$, because increasing its output by 75% would result in the efficient mix $(4, 1.75 \cdot 2) = (4, 3.5)$.

In input-oriented DEA models, the efficiency score $\theta^j \in (0, 1]$ indicates how much the observed inputs of DMU j should be reduced to achieve a weakly efficient input-output mix. Efficient DMUs thus have an efficiency score of one while that of inefficient ones is strictly less than one. The input-oriented efficient score of DMU j can be obtained from the LP problem

$$\theta^j = \min_{\theta \in \mathbb{R}} \{(\theta \hat{x}^j, \hat{y}^j)^T \in T\}, \tag{5}$$

and the set of feasible input-output mixes for DMU j is

$$M^j = \{(x^j, y^j)^T \in \mathbb{R}_+^{s+n} \mid (\theta^j x^j, y^j)^T \in T\}. \tag{6}$$

It should be noted that the output- and input-oriented DEA models do not generally yield the same sets of feasible input-output mixes. However, in the case of constant returns to scale (i.e., defining $\Lambda = \mathbb{R}^m_+$ in (1)), both orientations give the same sets of feasible input-output mixes M^j.

2.2. Portfolio Model of DMUs

The problem of maximizing the vector of aggregate outputs $\sum_j y^j \in \mathbb{R}^n$ and minimizing the aggregate inputs $\sum_j x^j \in \mathbb{R}^s$ can be viewed as a portfolio problem in which the DM selects an input-output mix $(x^j, y^j) \in M^j$ for each DMU j. A portfolio is thus formally defined as $p = (X, Y)$, where the columns of matrices $X = (x^1, \dots, x^m) \in \mathbb{R}^{s\times m}$ and $Y = (y^1, \dots, y^m) \in \mathbb{R}^{n\times m}$ contain DMUs' inputs and outputs, respectively.

As a decision alternative, the portfolio is feasible only if the input-output mix of each DMU is feasible so that $(x^j, y^j) \in M^j, j = 1, \dots, m$. There may be other constraints as well. For instance, limits on the relative changes in the DMUs' resource allocations lead to linear constraints $(1-\delta)\hat{x}^j \le x^j \le (1+\delta)\hat{x}^j$, thus limiting the possible input changes of DMU j to $\pm\delta * 100\%$ from its observed value $\hat{x}^j$. Furthermore, input resources are usually limited. This can be modeled through the linear constraint $\sum_j x^j \le b$, where the vector $b = [b_1, ..., b_s]^T$ denotes the availability of each resource type. Similarly, the constraint $\sum_j y^j \ge a$ ensures that feasible portfolios meet the production level requirements $a = [a_1, \dots, a_n]^T$.

For the sake of generality, we assume that the *set of feasible portfolios* P is bounded and defined by a system of linear constraints so that

$$P \subset \left\{ p = (X, Y) \;\middle|\; X = (x^1, \dots, x^m) \in \mathbb{R}^{s\times m}, Y = (y^1, \dots, y^m) \in \mathbb{R}^{n\times m}, \right.$$
$$\left. (x^j, y^j) \in M^j \;\forall\; j \in \{1, ..., m\} \right\}, \tag{7}$$

where the sets of feasible input-output mixes M^j have been estimated with a suitable method of efficiency analysis (e.g., equations (2)–(4) for output-oriented DEA).

2.3. Preference Modeling and Decision Recommendations

A DM who seeks to maximize the aggregate portfolio outputs $\sum_{j=1}^m y^j = \sum_{j=1}^m (y^j_1, \dots, y^n_j)^T$ and to minimize the aggregate inputs $\sum_{j=1}^m x^j = \sum_{j=1}^m (x^j_1, \dots, x^j_s)^T$ should select a feasible portfolio $p = (X, Y) \in P$ such that none of the aggregate inputs and outputs can be improved without violating the requirement $p \in P$ or become worse for some aggregate input or output. Hence, the DM can focus on *efficient portfolios*.

Definition 1. Portfolio $p = (X, Y) \in P$ is **efficient** if there does not exist another portfolio $\tilde{p} = (\tilde{X}, \tilde{Y}) \in P$ such that

$$\begin{bmatrix} -\sum_{j=1}^m \tilde{x}^j \\ \sum_{j=1}^m \tilde{y}^j \end{bmatrix} \gneq \begin{bmatrix} -\sum_{j=1}^m x^j \\ \sum_{j=1}^m y^j \end{bmatrix},$$

where $\gneq$ denotes that $\ge$ holds element-wise and at least one of these $s+n$ inequalities is strict. The **set of efficient portfolios** is denoted by $P_E \subseteq P$.

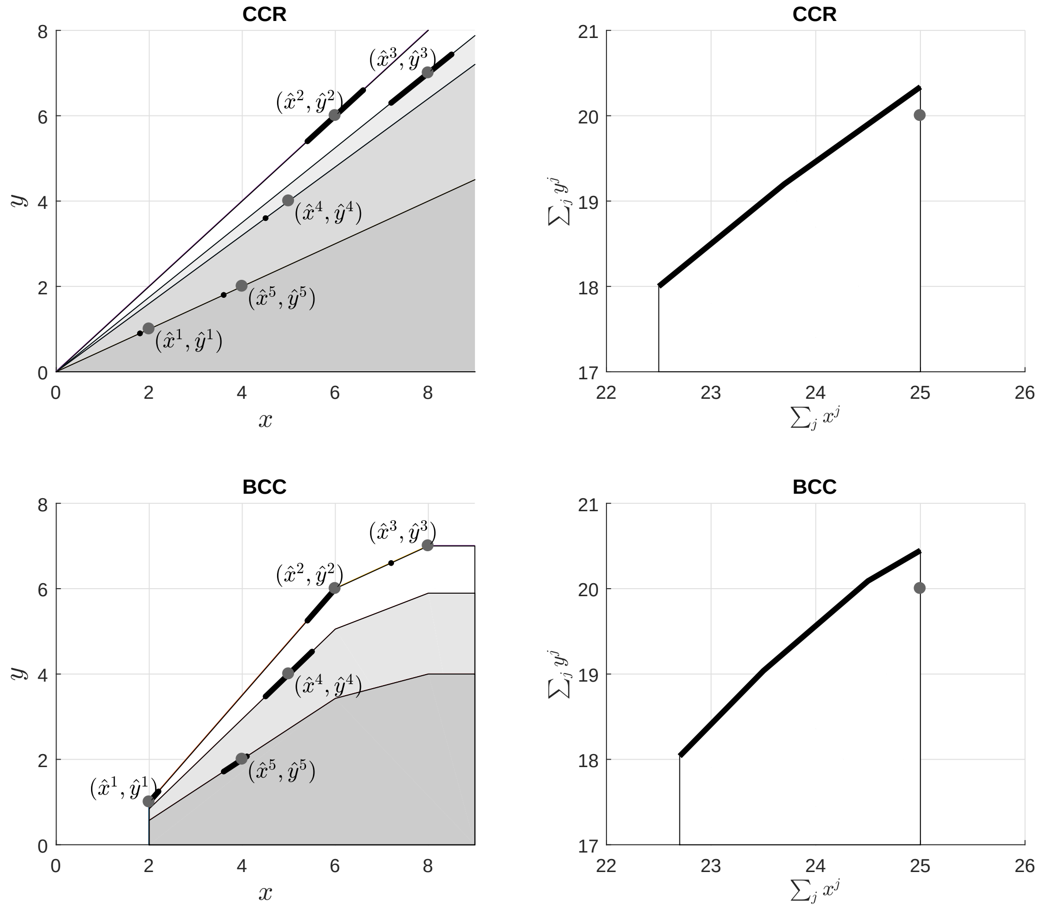

Figure 2: The left panel shows the input-output mixes of the five DMUs and the right panel shows the input-output mixes of the DMU portfolio. The gray circles show the observed inputs and outputs of the DMUs (left) and the DMU portfolio (right). The thick black line in the right panel shows the set of efficient portfolios when the input resources allocated to each DMU can be changed by $\pm 10\%$ from the observed value, but the total resource usage cannot be increased. The set of input-output mixes each DMU obtains across these efficient portfolios is marked with a thick black line in the left panel.

To illustrate the concept of efficient portfolios, we revisit the example from Section 2.1. Suppose that the input resources allocated to each of the five DMUs can be changed $\pm 10\%$ from their observed values and that there are no additional resources available, i.e., the total resource usage cannot be increased from its current value of 25. Figure 2 shows the resulting sets of efficient portfolios, which have been obtained using computational techniques developed in Section 3. In both the CCR- and the BCC-based models, the current allocation of input resources is inefficient: a higher aggregate output can be attained by reallocating resources, while it is also possible to attain the same level of aggregate output with less resources. Even though each point $(\sum_j x^j, \sum_j y^j)$ in the frontier of efficient portfolios produces different aggregate outputs $\sum_j y^j$ for different total use of resources $\sum_j x^j$, it is still possible to derive conclusive decision recommendation for individual DMUs. For instance, in the CCR model the resources of inefficient DMUs $j = 1, 4, 5$ are reduced by 10% in all efficient portfolios, which is the maximum allowed reduction. Moreover, in the BCC model the resources of DMU $j = 3$ are reduced by 10% in all efficient portfolios, while none of these portfolios allocate additional resources to

DMU $j = 2$, although both DMUs $j = 2$ and $j = 3$ are efficient. In turn, additional resources are allocated to the inefficient DMUs $j = 4$ and $j = 5$ in some efficient portfolios. An intuitive explanation for this is that DMUs $j = 4$ and $j = 5$, although inefficient, provide a larger marginal output for additional inputs than the efficient DMUs $j = 2$ and $j = 3$.

Using efficient portfolios to derive decision recommendations does not account for the fact that some inputs and outputs may be more valuable than others. Such preference information can be modeled by defining the virtual input (virtual output) of a portfolio as the weighted sum of its inputs (outputs) and then restricting the values that these weights can attain (cf. Podinovski 2005, 2011, Salo & Punkka 2011, Almeida & Dias 2012, Hinojosa and Marmol 2011, Podinovski & Bouzdine-Chameeva 2013, Kadziński et al. 2017). More precisely, the total virtual input and the virtual output of portfolio $p = (X, Y)$ are

$$\begin{aligned} I_u(X) &= u^T \sum_{j=1}^{m} x^j = \sum_{i=1}^{n} u_i \sum_{j=1}^{m} x_i^j \\ O_v(Y) &= v^T \sum_{j=1}^{m} y^j = \sum_{i=1}^{s} v_i \sum_{j=1}^{m} y_i^j, \end{aligned}$$

respectively, where $u = (u_1, ..., u_s)^T \in \Delta^s$ and $v = (v_1, ..., v_n)^T \in \Delta^n$ are the weights of the different inputs and outputs, respectively (recall that Δ^s denotes the simplex $\Delta^s = \{u \in \mathbb{R}^s_+ | \sum_{i=1}^{s} u_i = 1\}$). Preference statements about the relative values of inputs and outputs correspond to constraints on these weights. For instance, stating that a unit of output $i = 1$ is more valuable than a unit of output $i = 2$ corresponds to the constraint $v_1 \geq v_2$. Similarly, the statement that "one unit of input $i = 2$ is at least as valuable as 2 units of input $i = 1$, but not more valuable than 3 units" leads to the constraint $2u_1 \leq u_2 \leq 3u_1$. The sets of weights which satisfy such constraints, denoted by $S_u \subseteq \Delta^s$ and $S_v \subseteq \Delta^n$, define the *preference information* set $S = S_u \times S_v$.

In general, the virtual input $I_u(X)$ and virtual output $O_v(Y)$ of the portfolio $p = (X, Y)$ depend on what weights are selected from S_u and S_v. Nevertheless, it is possible to establish dominance relations for portfolios by comparing the virtual inputs and outputs for all feasible weight vectors in the sets S_u and S_v.

Definition 2. Let $S = S_u \times S_v$. Portfolio $p = (X, Y)$ S-**dominates** portfolio $\tilde{p} = (\tilde{X}, \tilde{Y})$, denoted by $p \succ_S \tilde{p}$, if

$$\begin{aligned} I_u(X) &\leq I_u(\tilde{X}) \text{ for all } u \in S_u, \\ O_v(Y) &\geq O_v(\tilde{Y}) \text{ for all } v \in S_v, \end{aligned}$$

with at least one strict inequality for some $u \in S_u$ or $v \in S_v$. The **set of** S**-efficient portfolios** is

$$P_E(S) = \big\{p \in P \mid \nexists \tilde{p} \in P \text{ s.t. } \tilde{p} \succ_S p\big\}.$$

The set of S-efficient portfolios contains all the portfolios a rational decision maker should consider in light of preference information S. That is, if the DM selects a dominated portfolio $p' \in P \setminus P_E(S)$, there exists an S-efficient portfolio $p \in P_E(S)$ such that, for any weights in S, the virtual output of p is higher than or equal to that of p' while its virtual input is lower than or equal to that of p'. If there is no preference information so that $S_u = \Delta^s$ and $S_v = \Delta^n$, the set of S-efficient portfolios $P_E(S)$ (Definition 2) is the same as the set of efficient portfolios P_E (Definition 1), as stated in Theorem 1. All proofs are in Appendix A.

Theorem 1. *Assume preference information $S = \Delta^s \times \Delta^n$. Then $P_E(S) = P_E$.*

Additional preference statements about the unit values of inputs and outputs imply additional constraints on weights in S. These constraints reduce the size of the initial information set $S = S_u \times S_v$ which thus becomes $\tilde{S} \subseteq S$ and therefore the corresponding set of S-efficient portfolios $P_E(\tilde{S})$ is usually a subset of $P_E(S)$. However, there exists a special case in which this result does not hold and which needs to be ruled out by introducing an additional assumption. In particular, as long as the revised set $\tilde{S}$ contains at least some of the interior points of S, the corresponding set of S-efficient portfolios $P_E(\tilde{S})$ is always a subset of $P_E(S)$. This result is formally stated in the following theorem where $\mathrm{int}(S)$ denotes the interior of set S.

Theorem 2. *Let $S = S_u \times S_v$ and $\tilde{S} = \tilde{S}_u \times \tilde{S}_v$ be information sets such that $\tilde{S} \subseteq S$ and $\mathrm{int}(S) \cap \tilde{S} \neq \emptyset$. Then $P_E(\tilde{S}) \subseteq P_E(S)$.*

The input resources to be allocated to each DMU vary across the S-efficient portfolios. Still, there may be DMUs whose resource allocation x_i^j either increases or decreases from the observed level $\hat{x}_i^j$ across all S-efficient portfolios, although the exact change in allocation may differ from one portfolio to another. In order to identify such DMUs, we define the *resource range* as the interval between the minimum and maximum amount of resources allocated to a specific DMU across the S-efficient portfolios.

Definition 3. Let $S = S_u \times S_v$. The **resource range** of DMU j with regard to input i is

$$[\underline{r}_i^j(S), \overline{r}_i^j(S)] = \left[\min_{(X,Y) \in P_E(S)} x_i^j \, , \max_{(X,Y) \in P_E(S)} x_i^j \right].$$

For computing these intervals, it is necessary to first determine all S-efficient portfolios which is not trivial. We therefore develop computational approaches in Section 3 for this purpose. But even without using these approaches, the interval notation for the resource ranges (cf. Definition 3) seems justified. In particular, plotting the input values x_i^j of each S-efficient portfolio $p = (X, Y) \in P_E(S)$ gives the real-valued closed interval $[\underline{r}_i^j(S), \overline{r}_i^j(S)]$, even though the set $P_E(S)$ is not necessarily convex. This property is stated in the following corollary.

Corollary 1. *Let $S = S_u \times S_v$. For any point $\alpha \in [\underline{r}_i^j(S), \overline{r}_i^j(S)]$, there exists an S-efficient portfolio $(X, Y) \in P_E(S)$ such that $x_i^j = \alpha$.*

Another intuitive property is that the introduction of additional preference statements cannot make the resource range wider. This implication of Theorem 2 is stated in Corollary 2.

Corollary 2. *Let $S = S_u \times S_v$ and $\tilde{S} = \tilde{S}_v \times \tilde{S}_u$ be information sets such that $\tilde{S} \subset S$ and $\mathrm{int}(S) \cap \tilde{S} \neq \emptyset$. Then*

$$[\underline{r}_i^j(\tilde{S}), \overline{r}_i^j(\tilde{S})] \subseteq [\underline{r}_i^j(S), \overline{r}_i^j(S)].$$

Output ranges can be defined similarly to resource ranges to indicate how the outputs of each DMU vary across the S-efficient portfolios. We focus on the input ranges as they are more directly linked to the actual decisions of a resource allocation problem, i.e., how much resources to allocate to each DMU. In turn, the link between the decisions and output ranges is more indirect, as the latter show the range of outputs each DMU can be expected to produce if different amounts of resources are allocated to it. Nevertheless, ranges of the aggregate inputs/outputs that the efficient portfolios produce can indeed be very useful in conveying to the DM the magnitudes of the tradeoffs between the different input/output types.

3. Computation of Efficient Portfolios and Resource Ranges

Because I_u and O_v are linear in u and v, respectively, possible dominance between any two portfolios can be checked by comparing their virtual inputs and virtual outputs at the extreme points of the sets S_u and S_v. These extreme points $\text{ext}(S_u) = \{u^1, \ldots, u^{t_u}\}$ and $\text{ext}(S_v) = \{v^1, \ldots, v^{t_v}\}$ can be solved using standard LP techniques and are represented on the rows of the matrices

$$U = \begin{pmatrix} u_1^1 & u_2^1 & \cdots & u_s^1 \\ u_1^2 & u_2^2 & \cdots & u_s^2 \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{t_u} & u_2^{t_u} & \cdots & u_s^{t_u} \end{pmatrix} \in \mathbb{R}^{t_u \times s},$$

$$V = \begin{pmatrix} v_1^1 & v_2^1 & \cdots & v_n^1 \\ v_1^2 & v_2^2 & \cdots & v_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{t_v} & v_2^{t_v} & \cdots & v_n^{t_v} \end{pmatrix} \in \mathbb{R}^{t_v \times n}.$$

The following theorem provides sufficient and necessary conditions for dominance based on these matrices.

Theorem 3. *Consider portfolios $p = (X, Y) \in P$, $\tilde{p} = (\tilde{X}, \tilde{Y}) \in P$, and the information set $S = S_u \times S_v$. Then*

$$\tilde{p} \succ_S p \;\Leftrightarrow\; \begin{bmatrix} -U[\sum_{j=1}^m \tilde{x}^j] \\ V[\sum_{j=1}^m \tilde{y}^j] \end{bmatrix} \gneq \begin{bmatrix} -U[\sum_{j=1}^m x^j] \\ V[\sum_{j=1}^m y^j] \end{bmatrix},$$

where rows of the matrices U and V contain the extreme points of S_u and S_v, respectively.

The condition of Theorem 3 states that portfolio $p \in P$ dominates portfolio $p' \in P$ if (i) the virtual output of p is higher than or equal to that of p' and its virtual input is lower than or equal to that of p' at every extreme point, and (ii) p has a higher virtual output or lower virtual input at some extreme point. Hence, the set of S-efficient portfolios $P_E(S)$ for the information set $S = S_u \times S_v$ can be obtained by solving the Pareto optimal solutions to the MOLP problem[1]

$$\underset{(X,Y)\in P}{\text{v–max}} \begin{bmatrix} -U[\sum_{j=1}^m x^j] \\ V[\sum_{j=1}^m y^j] \end{bmatrix}, \tag{8}$$

which has $t_u + t_v$ objective functions (one for each extreme point).

If no preference information has been provided, the feasible weight sets are simplices $S_u = \Delta^s$ and $S_v = \Delta^n$ whose extreme points are the unit vectors $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$ ... $(0, \ldots, 0, 1)$. Thus, U and V are both identity matrices, in which case Problem (8) becomes the $(s + n)$–objective MOLP problem

$$\underset{(X,Y)\in P}{\text{v–max}} \begin{bmatrix} -\sum_{j=1}^m x^j \\ \sum_{j=1}^m y^j \end{bmatrix}, \tag{9}$$

whose Pareto optimal solutions are the efficient portfolios P_E (cf. Definition 1 and Theorem 1).

Formulating the MOLP problem (8) requires enumerating the extreme points of the information set S in order to construct the matrices U and V. When the information set is defined through arbitrary linear constraints on the input and output weights, extreme points can be found using standard vertex enumeration methods (see, e.g., Matheiss & Rubin 1980, Taha 2003). Moreover, special types of preference information, e.g., a ranking of the input/output weights, lead to an information set whose extreme points can be enumerated efficiently with specialized methods (see, e.g., Marmol et al. 1998, Salo & Punkka 2005).

Among the algorithms for MOLP problems, the Multi-Objective Simplex (MO-Simplex, see, e.g., Ehrgott 2005) produces the efficient extreme points of the decision variable space. When applied to Problem (8), it identifies the extreme points of the set $P_E(S)$ which allow the DMUs' resource ranges to be determined. By Definition 3, the bounds of these ranges are obtained by minimizing/maximizing each x_i^j over the set $P_E(S)$. Although this set is not convex, the optimal value of a linear objective function across the efficient set of a MOLP problem is achieved at an extreme point (see, e.g., Benson 1984, Theorem 4.5). Hence, it suffices to check the values x_i^j at each extreme point of $P_E(S)$ generated by the MO-Simplex algorithm.

Unfortunately, the MO-Simplex and other MOLP algorithms which iterate in the decision variable space are impractical in large problems. In general, the number of efficient extreme points in the decision variable space grows exponentially with the size of the problem so that, for example, with 60 decision variables the number of efficient extreme points can exceed 200,000 (Benson 1998). In our model, the total number of decision variables is $mn + ms + m^2$, consisting of the DMUs' inputs and outputs (cf. Equation (7)), and the elements of the vectors $\lambda^1, \ldots, \lambda^m \in \mathbb{R}^m$ (cf. Equation (4)). Hence, solving a small problem with two inputs, two outputs and six DMUs leads to a MOLP model with 60 decision variables. Indeed, our computational experiments suggest that the MO-Simplex algorithms cannot solve instances of Problem (8) of practically relevant size.

The observation that in MOLP problems the number of objective functions is usually far smaller than the number of decision variables has lead to the development of algorithms for solving the problem in the objective

[1]We use v–max[·] to denote an optimization problem with a vector valued objective function. The 'optimal solution' of such a problem is the set of all Pareto optimal solutions.

space (Benson 1998, Löhne 2011). When applied to Problem (8), these algorithms identify the image of the set $P_E(S)$ in the objective space, i.e., the set

$$\Omega = \left\{ \omega = \begin{bmatrix} -U[\sum_{j=1}^m x^j] \\ V[\sum_{j=1}^m y^j] \end{bmatrix} \;\middle|\; (X,Y) \in P_E(S) \right\} \subset \mathbb{R}^{t_v+t_u}. \tag{10}$$

The numerical results presented in this paper, including the illustrative example in Section 2.2 as well as the applications to real data sets in Sections 5 and 6, have been obtained using the algorithm of Löhne (2011). An implementation of this algorithm is available online (Löhne & Weißing 2014). This algorithm produces a set of hyperplane coefficients $(c_f, d_f) \in \mathbb{R}^{t_v+t_u} \times \mathbb{R}$, $f = 1, ..., F$, such that each hyperplane $\{\omega \in \Omega \mid c_f^T \omega = d_f\}$ contains one of the F facets of the set Ω. With some additional computational effort, this information can be used to compute the DMUs' resource ranges. Specifically, the lower (upper) bound of the resource range of DMU j with regard to input i is the minimum (maximum) of F LP problems:

$$\underline{r}_i^j(S) = \min_{f\in\{1,...,F\}} \left[\min_{(X,Y)\in P} \left\{ x_i^j \;\middle|\; c_f^T \begin{bmatrix} -U[\sum_{j=1}^m x^j] \\ V[\sum_{j=1}^m y^j] \end{bmatrix} = d_f \right\} \right] \tag{11}$$

$$\overline{r}_i^j(S) = \max_{f\in\{1,...,F\}} \left[\max_{(X,Y)\in P} \left\{ x_i^j \;\middle|\; c_f^T \begin{bmatrix} -U[\sum_{j=1}^m x^j] \\ V[\sum_{j=1}^m y^j] \end{bmatrix} = d_f \right\} \right]. \tag{12}$$

Essentially, each inner LP problem optimizes over feasible portfolios P with an additional constraint which ensures that the vector of the portfolio's virtual inputs and outputs – evaluated at the extreme points of S_u and S_v – is on a specific facet of Ω.

4. Implications for Decision Support

The above results enable several approaches for supporting resource allocation decisions. The key output of the model is the entire set of S-efficient (Definition 1) or efficient (Definition 2) portfolios. This output makes it possible to show to the decision maker the range of portfolios that are defensible with the incomplete preference information that the decision maker can confidently provide, rather than settling for a single 'optimal' portfolio based on possibly unrealistic assumptions of point-estimate values for input/output weights. Moreover, in group decision setting the set of feasible input/output weights can be constructed so that it includes the weight vectors corresponding to preferences of each individual group member. In this case, computing the set of S-efficient portfolios enables illustrating the range of decisions that align with the groups' preferences, and to identify those DMUs for which the resource allocation recommendations differ among the group members, along with those DMUs for which these recommendations agree.

Once the set of efficient or S-efficient portfolios has been computed, these sets can be readily visualized both in the objective function space and in the decision variable space. Objective function space visualizations include the ranges of outputs (inputs) that S-efficient portfolios produce (consume), projections of the multi-dimensional efficient frontier into a series of two-dimensional plots, each of which has a specific pair of inputs and outputs as its axes, and spider charts. Perhaps the most practical approach for illustrating the efficient portfolios in the decision variable space is the use of the resource ranges (Definition 3). It is also possible to illustrate with two-dimensional graphs how resources are allocated to any pair of DMUs across S-efficient portfolios.

These visualizations can be used in interactive decision support processes which start with relatively loose preference statements about the relative values of inputs and outputs and iteratively ask the DM to provide additional preference statements. At each iteration, the set of S-efficient portfolios is updated to reflect the updated preference information (cf. Theorem 2) and these portfolios are illustrated to the DM, for instance, by showing the ranges of their aggregate inputs and outputs as well as the updated ranges of input resources they allocate to each DMU (resource ranges; Definition 3). If the resource allocation recommendations suggested by the updated resource ranges are not specific enough – e.g., the ranges are too wide or contain both increased and decreased resource values for many DMUs – the DM can be asked to provide additional preference statements which lead to further constraints on the weights, resulting in narrower ranges for the portfolios' aggregate inputs and outputs, and for the DMUs' resource ranges (Corollary 2).

In the interactive process, additional preference information can be elicited with many approaches. For instance, the DM can be asked to rank inputs and outputs based on their unit values (cf. Kirkwood & Sarin 1985), resulting in constraints on the set of weights S. Another possibility is to use the Assurance Region procedure to specify lower and upper bounds on the relative unit values (see, e.g., Thompson et al. 1990). If the values of input and output units are assessed by several experts, a group preference representation can be obtained as the convex hull of the weights implied by each expert's estimates (see, e.g., Salo & Punkka 2011).

Another approach to screen the set of S-efficient portfolios is that the DM fixes the amount of input resources within the resource range he/she prefers to allocate to a specific DMU. Since the entire set of S-efficient portfolios has been solved, this set can be readily updated by discarding those portfolios in which the resources allocated to the selected DMU differ from the fixed value. Thereafter, the visualizations can be readily updated to show, for instance, the new resource ranges for all DMUs computed across the updated set of S-efficient portfolios. This screening process can be continued so that the DM fixes the input level of one DMU at a time by choosing a preferred level from the updated resource range until a single portfolio remains. Since this process is started with the set of S-efficient portfolios, the final recommended portfolio will always be S-efficient.

5. Applications to Real Data

In this section, we apply our models to real data sets. In particular, Section 5.1 presents an application from retail utilizing both constant and variable returns to scale assumptions with an output oriented DEA model. Section 5.2, in turn, presents an application of fast-food chain management that utilizes an input oriented model with variable returns to scale. In the latter application, we also relax the assumption that the set of feasible input-output mixes of each DMU is based on the same production possibility set.

5.1. Allocating Resources to Supermarkets

In the application reported by Korhonen & Syrjänen (2004), the management of a retail chain uses efficiency analysis to support allocation of resources among 25 supermarkets. The data from this application has been examined with different models which rely on complete specification of preferences (see, e.g., Nasrabadi et al., 2012, Fang, 2013). In contrast, our results below are based on incomplete preference information and the corresponding set of efficient and S-efficient portfolios (Definitions 1 and 2).

In this application, outputs to be maximized consist of supermarkets' sales and profits (i.e., $\sum_{j=1}^{25}(y_1^j, y_2^j)^T$) while the aggregate inputs (man-hours and size) are allowed to increase by no more than 1% from their observed values (i.e., $[b_1, b_2]^T = 1.01 \sum_{j=1}^{25} \hat{x}^j$). The observed input and output values as well as the efficiency scores are in Table 1. The input resources of each supermarket can decrease at most by 10% and increase at most by 30% from their observed values (i.e., $x^j \in [0.9\hat{x}^j, 1.3\hat{x}^j],\ j = 1, \ldots, 25$). Korhonen & Syrjänen (2004) consider both variable and constant returns to scale assumptions. Below, we consider models for both assumptions.

Table 1: Supermarkets' observed input/output values and efficiency scores

	Outputs		Inputs		Efficiency scores $\sigma^j_{(\cdot)}$	
Supermarket	Sales	Profit	Man-hours	Size	CCR	BCC
S1	115.3	1.71	79.1	4.99	1.248	1.219
S2	75.2	1.81	60.1	3.30	1.454	1.294
S3	225.5	10.39	126.7	8.12	1.000	1.000
S4	185.6	10.42	153.9	6.70	1.088	1.000
S5	84.5	2.36	65.7	4.74	1.415	1.300
S6	103.3	4.35	76.8	4.08	1.285	1.241
S7	78.8	0.16	50.2	2.53	1.109	1.000
S8	59.3	1.30	44.8	2.47	1.375	1.000
S9	65.7	1.49	48.1	2.32	1.246	1.000
S10	163.2	6.26	89.7	4.91	1.000	1.000
S11	70.7	2.80	56.9	2.24	1.086	1.000
S12	142.6	2.75	112.6	5.42	1.343	1.214
S13	127.8	2.70	106.9	6.28	1.522	1.485
S14	62.4	1.42	54.9	3.14	1.601	1.357
S15	55.2	1.38	48.8	4.43	1.609	1.244
S16	95.9	0.74	59.2	3.98	1.123	1.022
S17	121.6	3.06	74.5	5.32	1.115	1.075
S18	107.0	2.98	94.6	3.69	1.228	1.223
S19	65.4	0.62	47.0	3.00	1.308	1.028
S20	71.0	0.01	54.6	3.87	1.399	1.242
S21	81.2	5.12	90.1	3.31	1.170	1.165
S22	128.3	3.89	95.2	4.25	1.180	1.143
S23	135.0	4.73	80.1	3.79	1.000	1.000
S24	98.9	1.86	68.7	2.99	1.077	1.030
S25	66.7	7.41	62.3	3.10	1.000	1.000
Total	2586.1	81.72	1901.5	102.97		

5.1.1. Constant Returns to Scale Model

Korhonen & Syrjänen (2004) first consider the CCR model in which the supermarkets are able to change their input and output values proportionally in the neighborhood of the observed input-output mix. Specifically, for each supermarket $j = 1, ..., 25$, the set of feasible input-output mixes is

$$M^j = \{(x^j, y^j)^T \in \mathbb{R}^4_+ \mid \hat{X}\lambda^j \le x^j,\, \hat{Y}\lambda^j \ge \sigma^j_{CCR}\, y^j,\, \lambda^j \in \mathbb{R}^{25}_+\}$$

with the additional constraints $x^j \ge \delta_j \hat{x}^j$ and $y^j \le \delta_j \hat{y}^j$. Korhonen & Syrjänen (2004) use these additional constraints to ensure that for each DMU the maximum proportional increase in its outputs is the same as the minimum increase in its inputs. Thus, the constraints are not necessary for the CCR model, but we use them to ensure that our results are comparable to those of Korhonen & Syrjänen (2004).

Figure 3 presents the set of efficient portfolios $P_E = P_E(\Delta^2)$ while Figure 4 shows the corresponding resource ranges. These results were computed in less than five seconds on a standard laptop (see Appendix B for details on the computation environment). The figures also show the three efficient solutions that Korhonen & Syrjänen (2004) identified with the Pareto Race –method (Korhonen & Wallenius, 1988). Even without any preference information, it becomes clear that there are 13 supermarkets in which the man-hours should be reduced by 10% (the maximum allowed decrease). Furthermore, there are three supermarkets (S3, S10, S23) in which the number of man-hours should be increased by 30%, no matter how valuable the outputs are relative to each other.

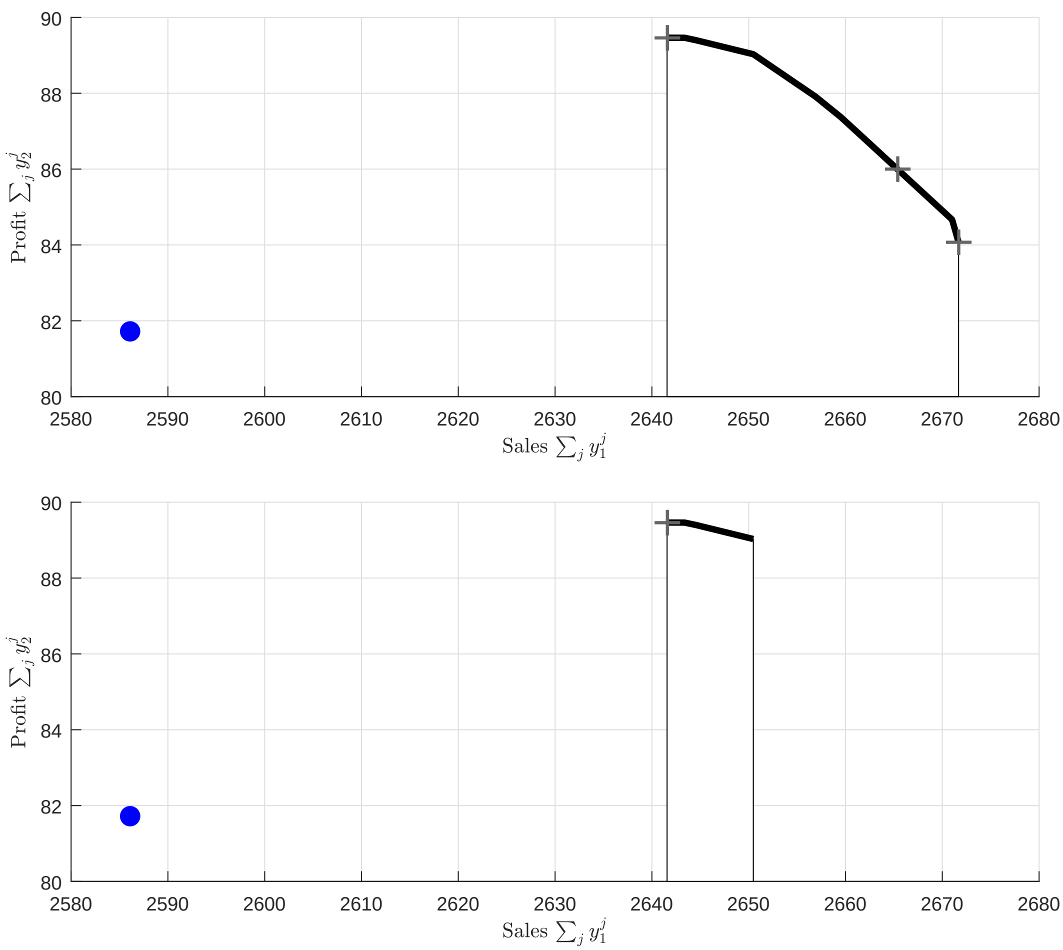

Figure 3: Aggregate outputs of efficient portfolios (top figure) and S-efficient portfolios with weight constraint $v_2 \geq 10v_1$ (bottom figure). The observed input and output values are marked with a circle. The three solutions found by Korhonen & Syrjänen (2004) are marked with crosses.

To illustrate the effects of preference information, suppose that the management states that a one-unit increase in profits is at least as valuable as a 10-unit increase in sales. This preference statement corresponds to the preference information set

$$S_v = \{v = (v_1, v_2)^T \in \Delta^2 \mid v_2 \geq 10v_1\}. \tag{13}$$

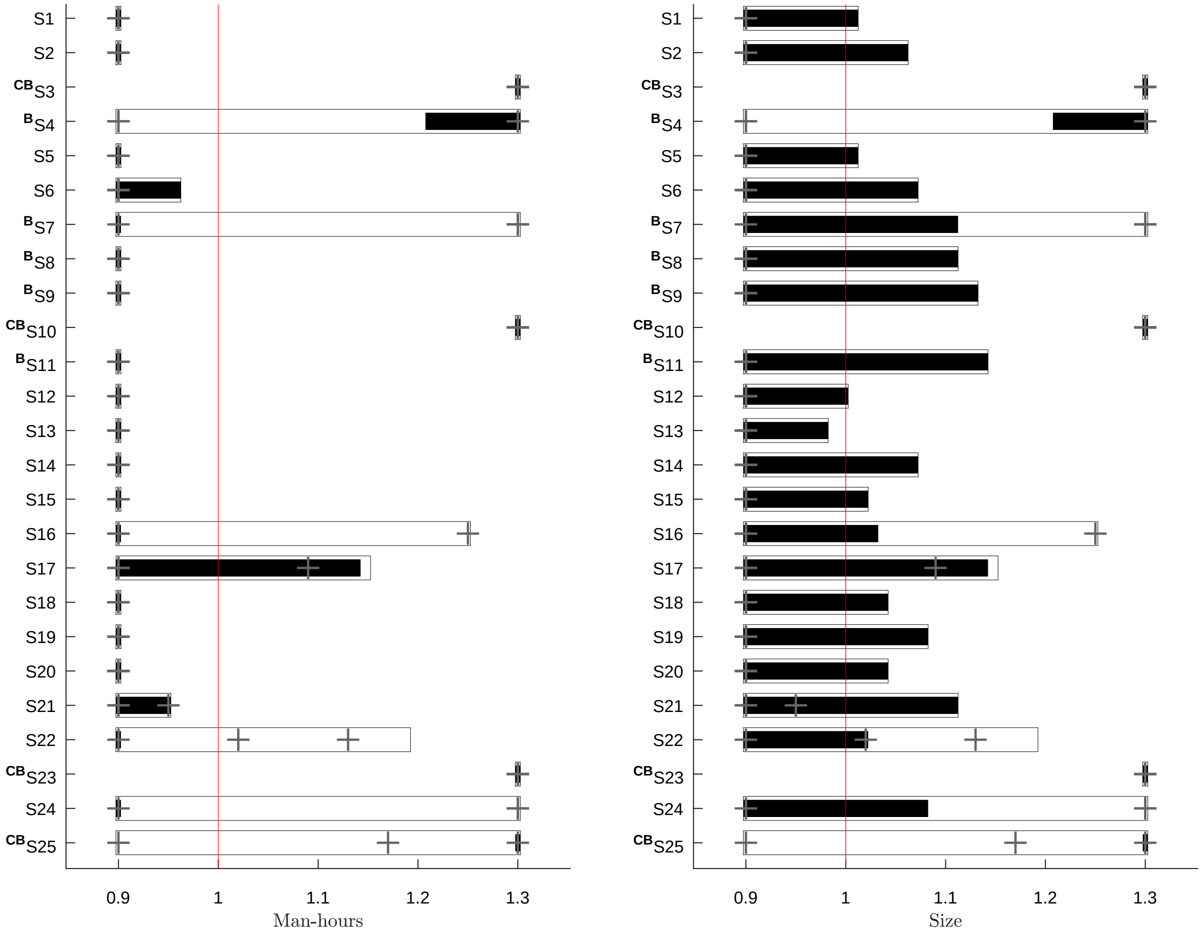

Figure 4: Relative resource ranges over efficient portfolios (white bars) and S-efficient portfolios with weight constraint $v_2 \geq 10v_1$ (black bars). Superscripts $^{\mathbf{C}}$ and $^{\mathbf{B}}$ denote CCR-efficient and BCC-efficient supermarkets, respectively. The three solutions found by Korhonen & Syrjänen (2004) are marked with crosses.

Figure 3 shows how this preference information leads to a smaller set of S-efficient portfolios $P_E(S_v) \subset P_E$ (cf. Theorem 2). Specifically, because profits are now more important than sales, many initially efficient portfolios with high sales but low profits become dominated.

Figure 4 further illustrates how this preference information also leads to narrower resource ranges (cf. Corollary 2). For instance, supermarket S25 now receives the maximum amount of both input resources, whereas the man-hours allocated to supermarkets S7, S16, S22 and S24 are reduced by 10% in each S-efficient portfolio. Also, supermarket S4 receives more of both input resources, but the exact amounts vary across the S-efficient portfolios. The resource range of supermarket S17 remains almost unaffected by the additional preference information.

5.1.2. Variable Returns to Scale Model

In their second example, Korhonen & Syrjänen (2004) use a BCC model in which the set of feasible input-output mixes for each supermarket $j = 1, \ldots, 25$ is given by

$$M^j = \{(x^j, y^j)^T \in \mathbb{R}^4_+ \mid \hat{X}\lambda^j \leq x,\, \hat{Y}\lambda^j \geq \sigma^j_{BCC} y^j,\, \lambda^j \in \Lambda^{BCC}\},$$

where $\Lambda^{BCC} = \{\lambda \in \mathbb{R}^{25}_{+} \mid \sum_{i=1}^{25} \lambda_i = 1\}$. Furthermore, they assume that the supermarkets cannot adjust their floor space, meaning that the linear constraints $x_2^j = \hat{x}_2^j$, $j = 1, \ldots, 25$ must hold.

The computation of efficient portfolios P_E and the corresponding resource ranges took less than 10 seconds (see Appendix B for details on the computation environment). These results are in Figures 5 and 6, which also show the two efficient solutions Korhonen & Syrjänen (2004) found with the reference point method of Wierzbicki (1980). In particular, even though one of these two solutions was generated by emphasizing the sales objective and the other by emphasizing the profits objective, they clearly do not give an exhaustive picture of the full range of trade-offs between the two objectives.

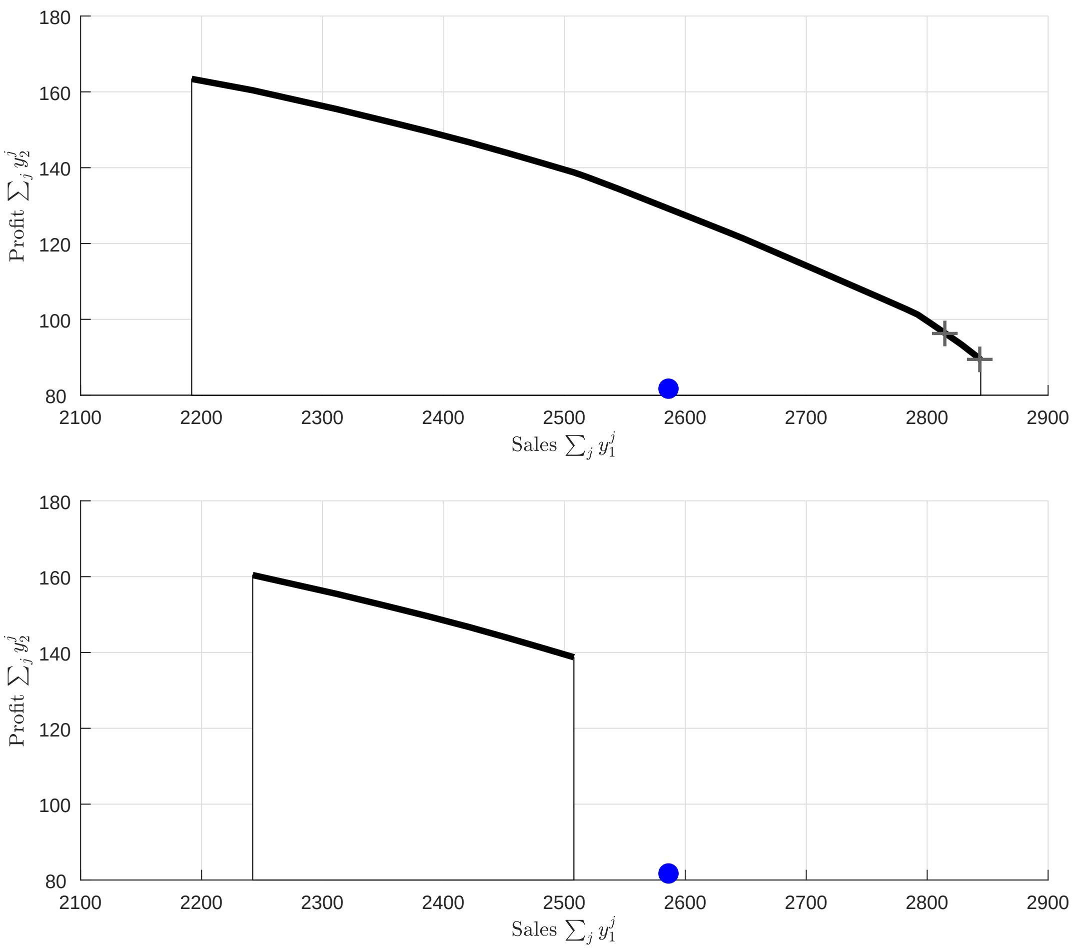

Figure 5: Aggregate outputs of efficient portfolios (top figure) and S-efficient portfolios with weight constraints $10v_1 \leq v_2 \leq 15v_1$ (bottom figure). The observed values are denoted with a circle. The two solutions found by Korhonen & Syrjänen (2004) are marked with crosses.

For 21 supermarkets – which is the large majority – every efficient portfolio suggests consistently either an increase or a decrease in the input resources of these 21 supermarkets, even if the exact magnitude of these changes vary across the portfolios. There are only four supermarkets (S1, S2, S5, S14) which receive more input resources in some portfolios but less in others.

We stress that the conventional efficiency scores do not provide meaningful guidance for allocating resources when seeking to maximize the aggregate outputs. For instance, supermarkets S3, S4, S10, S11, S23 and S25

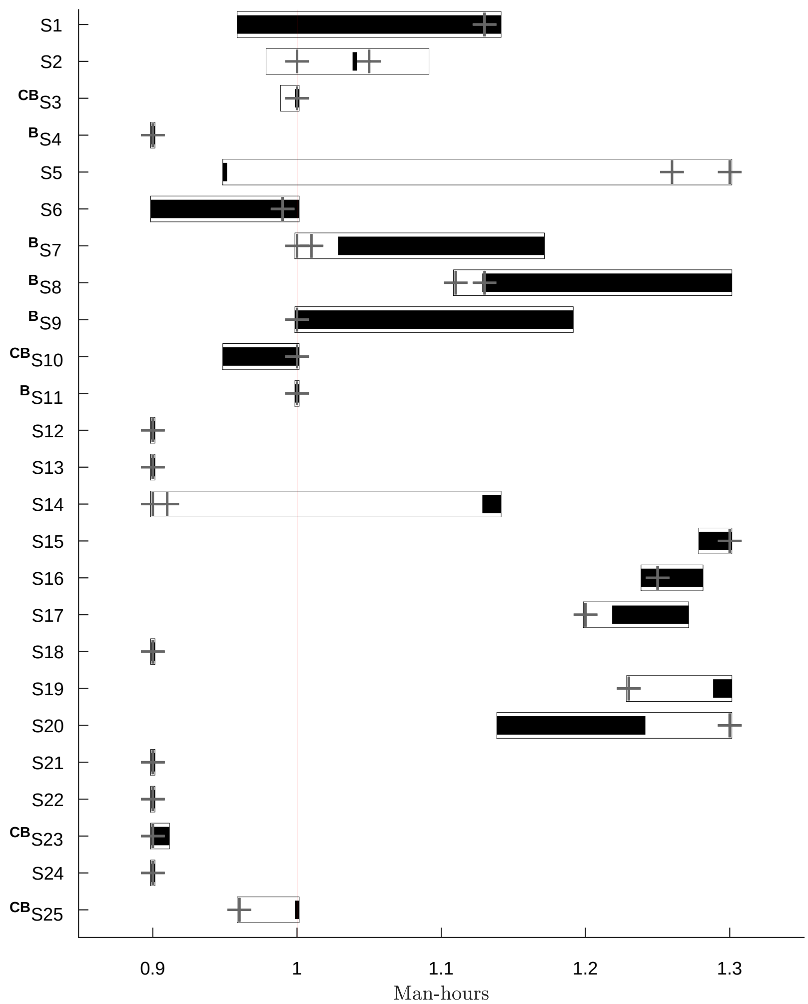

Figure 6: Relative resource ranges over efficient portfolios (white bars) and S-efficient portfolios with weight constraints $10v_1 \leq v_2 \leq 15v_1$ (black bars). Superscripts $^{\mathbf{C}}$ and $^{\mathbf{B}}$ denote CCR-efficient and BCC-efficient supermarkets, respectively. The two solutions found by Korhonen & Syrjänen (2004) are marked with crosses.

have BCC efficiency scores of one and are therefore classified as BCC efficient. Yet, no efficient portfolio suggests giving more resources to these DMUs. In fact, in order to reach portfolio efficiency, the input resources of the BCC efficient DMUs S4 and S23 would have to be reduced by 7-10%. Moreover, the input resources of the BCC inefficient supermarkets S15, S16, S17, S19 and S20 are higher in all efficient portfolios.

To illustrate the effects of preference information, suppose that the management states that a one-unit increase in profits is at least as valuable as a 10-unit increase in sales, but a 15-unit increase in sales is more valuable than a one-unit increase in profits. These statements correspond to the preference information set

$$S_v = \{v = (v_1, v_2)^T \in \Delta^2 \mid 10v_1 \leq v_2 \leq 15v_1\}. \tag{14}$$

As a result of introducing this preference information, the portfolios with the highest profits (i.e., lowest sales) and the lowest profits (i.e., highest sales) now become dominated (see Figure 5). The preference information also produces more specific resource allocation recommendations (Figure 6). Supermarkets S2 and S14 receive more resources for all portfolios in $P_E(S_v)$, whereas the resources of supermarket S5 are reduced by some 5%. Interestingly, the resources of supermarkets S3, S11 and S25 are not affected by this preference information.

5.2. Workforce Allocation in a Fast-Food Chain

We next revisit the application of Lozano (2013) in which the management of a fast-food chain aims to increase the total output production of 20 restaurants (DMUs) while minimizing the required workforce (see Table 2). The outputs consist of monthly servings of meat (y_1), vegetables (y_2), soup (y_3), noodles (y_4), and beverages (y_5). In addition to workforce (x_1), the inputs also include shop size (x_2) which is assumed to be non-adjustable.

Table 2: Input/output data and efficiency scores of the restaurants.

	Inputs		Outputs (10^3 servings)					Efficiency scores
Restaurant	Man-hours (10^3h)	Shop size ($10^2 m^2$)	Meat	Vegetable	Soup	Noodles	Beverage	Input-oriented BCC $\theta^j_{(\cdot)}$
R1/I	3.20	2.00	2.24	2.46	1.22	3.12	0.96	1.000
R2/I	3.40	2.10	2.12	2.52	1.34	3.08	0.88	1.000
R3/I	3.10	1.80	2.08	2.25	1.05	2.85	0.74	1.000
R4/I	3.80	2.20	2.45	2.10	1.30	2.96	0.79	1.000
R5/I	4.20	2.60	2.80	2.78	1.42	3.48	1.05	1.000
R6/I	4.10	2.50	2.65	2.95	1.38	3.25	0.98	1.000
R7/I	3.80	2.30	2.60	2.24	1.15	3.18	0.95	1.000
R8/I	3.80	2.20	2.50	2.15	1.10	3.20	0.82	0.986
R9/I	2.90	1.60	2.10	2.04	0.98	2.88	0.72	1.000
R10/I	4.20	2.80	2.90	2.85	1.52	3.36	1.12	1.000
R11/II	3.40	2.10	2.60	2.45	1.36	3.32	0.82	1.000
R12/II	4.00	2.40	2.78	2.66	1.18	3.15	0.98	1.000
R13/II	3.80	2.60	2.84	2.38	1.25	3.29	0.85	1.000
R14/II	3.40	1.90	2.33	2.20	1.06	2.99	0.82	1.000
R15/II	2.80	1.60	2.00	2.18	1.96	2.84	0.71	1.000
R16/II	3.50	2.20	2.40	2.25	1.26	2.93	0.74	0.914
R17/II	4.20	2.50	2.68	2.50	1.46	3.22	0.92	1.000
R18/II	3.30	1.80	2.05	2.20	1.12	3.02	0.78	1.000
R19/II	3.60	1.90	2.00	2.16	1.02	2.89	0.74	0.886
R20/II	3.10	1.70	2.05	2.12	0.94	2.90	0.68	0.978
Total	71.60	42.80	48.17	47.44	25.07	61.91	17.05	

The restaurants are in one of two 'technology classes' – labeled I and II – with different production possibilities. Following Lozano (2013), we assume that the set of feasible input-output mixes for each restaurant is defined by an input-oriented BCC model, built from the observed input-output mixes of those restaurants that are in the same technology class. Likewise, the efficiency scores are computed relative to the restaurants in the same technology class. Specifically, for class I restaurants $j = 1, ..., 10$, the set of feasible input-output mixes is

$$M^j = \{(x^j, y^j)^T \in \mathbb{R}^7_+ \mid \hat{X}^I \lambda^j \leq \theta^j_I x^j,\, \hat{Y}^I \lambda^j \geq y^j,\, \lambda^j \in \Lambda^{BCC}_I\},$$

where the matrices $\hat{X}^I \in \mathbb{R}^{2\times 10}$ and $\hat{Y}^I \in \mathbb{R}^{5\times 10}$ contain the observed inputs and outputs of class I restaurants, θ^j_I is the efficiency score of restaurant j computed relative to the other restaurants in class I, and $\Lambda^{BCC}_I = \{\lambda \in \mathbb{R}^{10}_+ \mid \sum_{i=1}^{10} \lambda_i = 1\}$. Similarly, for each class II restaurant $j = 11, ..., 20$, the set of feasible input-output mixes is given by

$$M^j = \{(x^j, y^j)^T \in \mathbb{R}^7_+ \mid \hat{X}^{II} \lambda^j \leq \theta^j_{II} x^j,\, \hat{Y}^{II} \lambda^j \geq y^j,\, \lambda^j \in \Lambda^{BCC}_{II}\},$$

where $\hat{X}^{II} \in \mathbb{R}^{2\times 10}$ and $\hat{Y}^{II} \in \mathbb{R}^{5\times 10}$ are the observed inputs and outputs of class II restaurants, θ^j_{II} is the efficiency score of restaurant j computed relative to the other restaurants in class II, and $\Lambda^{BCC}_{II} = \{\lambda \in$

$\mathbb{R}^{10}_{+} \mid \sum_{i=11}^{20} \lambda_i = 1\}$. The assumption of non-adjustable shop sizes is captured by using the constraints $x_2^j = \hat{x}_2^j$, $j = 1, \ldots, 20$. The maximum allowed total workforce is set equal to the observed value of $b_1 = 71.6$.

We consider a setting in which available actions to increase the efficiencies of the DMUs have already been implemented, and the management therefore seeks to increase the total outputs of the DMU portfolio through workforce reallocation. The minimum requirement levels for each aggregate output used by Lozano (2013) cannot be attained unless the restaurants can be assumed to increase their efficiencies without restrictions. Hence, to make the model feasible we decrease the requirement for sold beverages by 0.95 servings which results in the minimum output levels $a = (49, 48, 26, 63, 17.05)^T$.

The intervals of possible value changes in the aggregate inputs and aggregate outputs over all efficient portfolios are in Figure 7. Figure 8 shows the corresponding resource ranges. The total computation time was approximately 650 seconds: it took some 220 seconds to solve the image of the set of efficient portfolios (cf. Equation (10)) and some 430 seconds to determine the resource ranges by solving LP models (11)-(12).

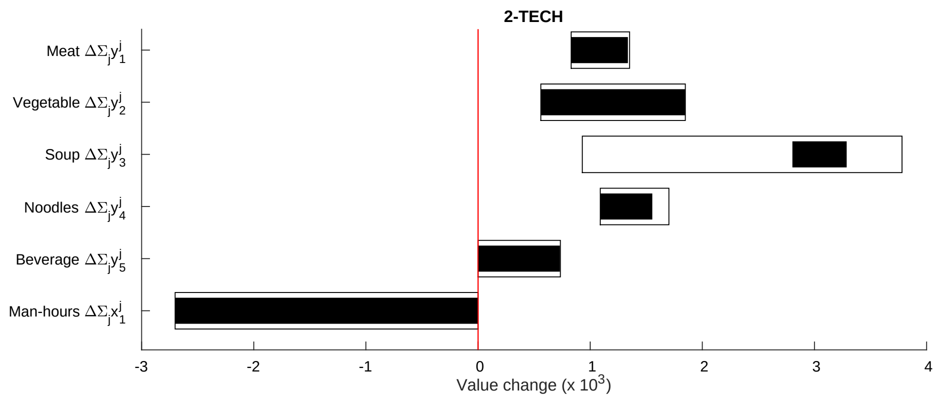

Figure 7: Possible changes in aggregate input and output values over efficient portfolios (white bars) and over S-efficient portfolios with the feasible weight set in Equation (15) (black bars).

Figures 7 and 8 also show the results based on S-efficient portfolios with the set of feasible weights

$$S_v = \{v = (v_1, \ldots, v_5)^T \in \Delta^5 \mid v_1 \geq v_2 \geq v_3 \geq v_4 \geq v_5,\, v_i \geq \frac{1}{10}\ \forall\, i = 1, \ldots, 5\}. \tag{15}$$

This set is based on an ordinal importance ranking of the five outputs. For instance, a unit increase in the number of meat servings is most valuable, while a unit increase in the number of beverage servings is least valuable. Furthermore, the set S_v also guarantees a minimum weight of $1/10$ for each output, which ensures that no serving type can be more than 9 times as valuable as another serving type. With this preference information, it took about 2 seconds to solve the image of the set of efficient portfolios (10), and about 168 seconds to solve the resource ranges from LP models (11)-(12).

Figure 9 shows how the workforce of each restaurant depends on the total workforce available. Specifically, the three sets of relative resource ranges are computed across those efficient/S-efficient portfolios in which the aggregate number of man-hours is equal to 70.6, 69.6, and 69.1. The first two levels correspond to reductions of one and two thousand hours from the current level of 71.6, while the third level is close to the minimum number of man-hours across the efficient portfolios (cf. Figure 7).

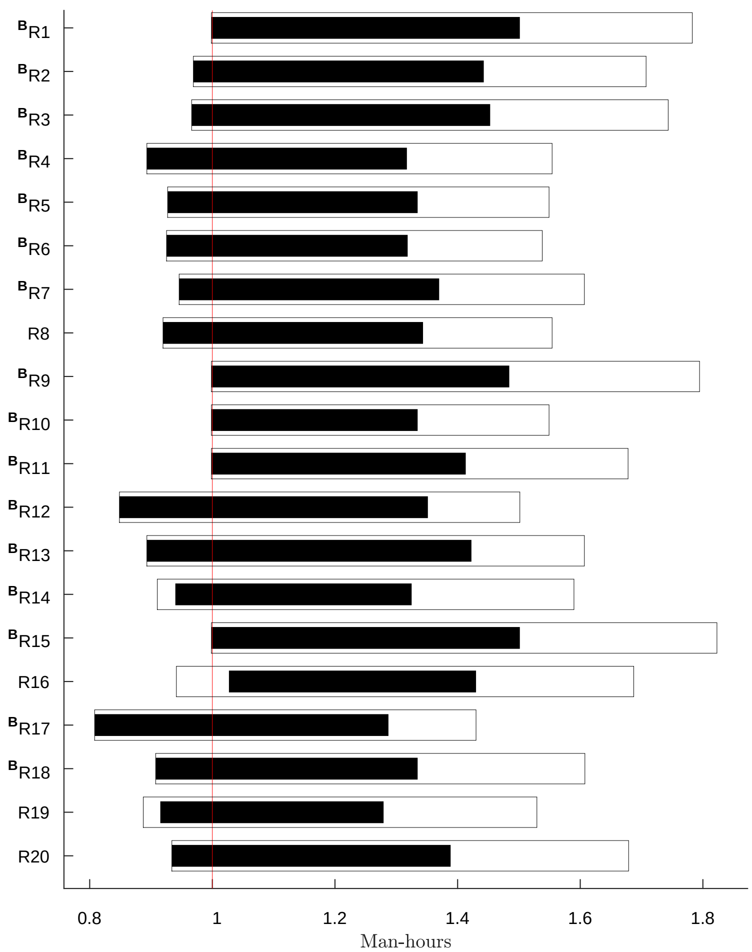

Figure 8: Relative resource ranges over efficient portfolios (white bars) and over S-efficient portfolios with the feasible weight set in Equation (15) (black bars).

Reducing the total number of man-hours tends to yield more specific resource allocation recommendations for individual restaurants. For example, when the total number of man-hours is 70.6, additional resources are allocated to restaurant R12 in every S-efficient portfolio, but when the total number of man-hours is 69.1, its resources are reduced by some 10%.

In this application, too, conventional efficiency scores fail to provide meaningful guidance for allocating resources among the DMUs. For example, when the total number of man-hours is equal to 69.1, restaurants R1 and R16 receive extra resources regardless of preferences, although the former is BCC-efficient and the latter is BCC-inefficient.

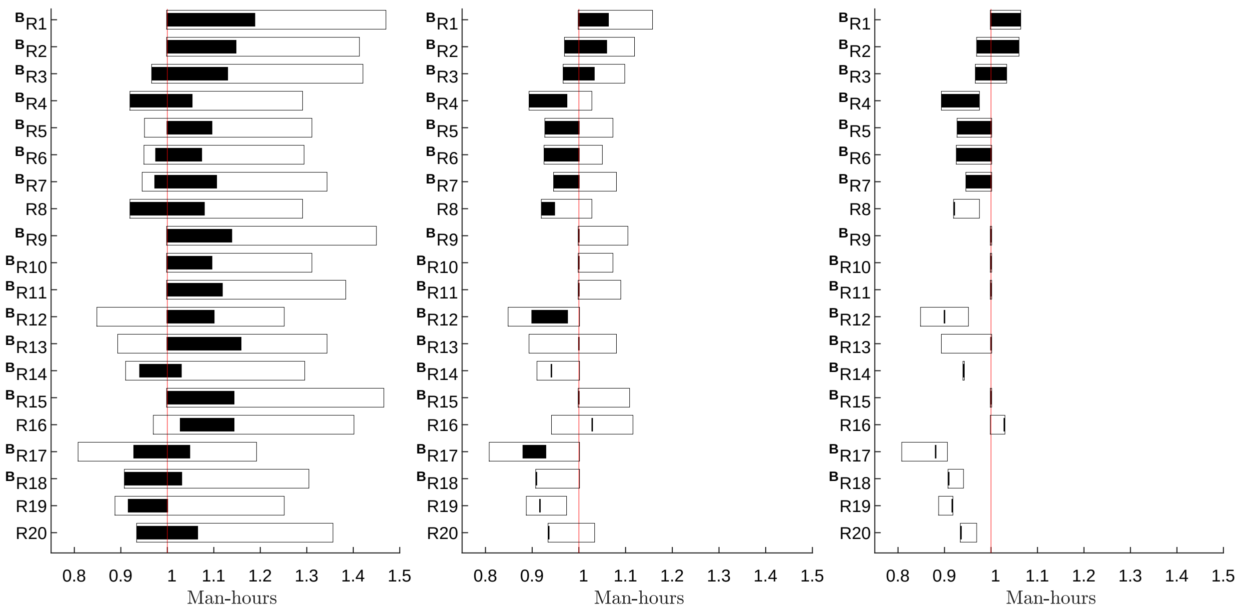

Figure 9: Relative resource ranges over efficient portfolios (white bars) and over S-efficient portfolios for the feasible weight set in Equation (15) (black bars). The total number of man-hours is equal to 70.6 (left figure), 69.6 (middle figure) and 69.1 (right figure). BCC-efficient restaurants (with respect to other restaurants in the same technology class) are marked with the superscript $^{\mathbf{B}}$.

6. Extensions

6.1. Capturing Non-Linear Preferences

Thus far, we have assumed that DMU portfolios are evaluated based on weighted sums of their inputs and outputs. This linear preference model is widely used by standard efficiency analysis methods, as weights can be conveniently interpreted as unit values of the inputs and outputs. Yet, it is based on several assumptions about preferences. First, the inputs and outputs have to be mutually preferentially independent, meaning that each subset of inputs and outputs is preferentially independent (see, e.g., Keeney & Raiffa 1976). In particular, a subset of inputs and outputs is preferentially independent if the preference between two input-output mixes that have equal levels for all inputs and outputs belonging to the complement of this subset does not depend on what these levels are. Second, the inputs and outputs need to exhibit constant marginal values so that a unit increase in any one of the inputs or outputs is of equal value, regardless of the initial values of these inputs and outputs. This makes it impossible to capture diminishing marginal values, even if these can be relevant for modeling output trade-offs, for instance.

Our portfolio model can be readily extended to non-linear preferences for inputs and outputs. The only requirement is that the inputs and outputs are preferentially independent, which is a weaker assumption than mutual preferential independence. In particular, an input is preferentially independent if the DM prefers a decrease in its value no matter what the values of other inputs and outputs are. Similarly, an output is preferentially independent if the DM prefers an increase in its value regardless of the values of the other outputs and inputs. If each input and output is preferentially independent, there exists a value function $\phi : \mathbb{R}^{s+n} \to \mathbb{R}$ which represents the DM's preferences in the sense that portfolio $p = (X, Y)$ is preferred to portfolio $\tilde{p} = (\tilde{X}, \tilde{Y})$ if and only if

$$\phi\Big(\sum_{j=1}^{m} x^j, \sum_{j=1}^{m} y^j\Big) > \phi\Big(\sum_{j=1}^{m} \tilde{x}^j, \sum_{j=1}^{m} \tilde{y}^j\Big).$$

It is reasonable to assume the value function ϕ is strictly decreasing in the first s arguments and strictly increasing in the last n arguments, because the DM seeks to minimize inputs and maximize outputs. The set of efficient portfolios includes an optimal portfolio for any such value function. Moreover, each efficient portfolio is optimal for some value function ϕ. These properties are formally stated by the following theorem.

Theorem 4. *Portfolio $p^* \in P_E(\Delta^s \times \Delta^n)$ if and only if there exists a value function $\phi :\ \mathbb{R}^{s+n} \to \mathbb{R}$ which is strictly decreasing in the first s arguments and strictly increasing in the last n arguments such that*

$$p^* \in \operatorname*{arg\,max}_{p \in (X,Y)^T} \phi\Big(\sum_{j=1}^{m} x^j, \sum_{j=1}^{m} y^j\Big)$$

Importantly, Theorem 4 implies that the decision recommendations produced by the linear preference model without weight restrictions remain valid when the DM's 'true' preferences are represented by any non-linear value function ϕ. In the supermarket example, for instance, there were several DMUs whose input resources were either increased or decreased across all efficient portfolios (cf. Figure 6). These kinds of qualitative results would hold even if the DMUs' resource ranges were to be computed only for those portfolios that maximize some non-linear value function ϕ. Furthermore, Theorem 4 helps identify those portfolios that are optimal for some non-linear value function. In particular, solving the set of efficient portfolios (cf. problem (9)) produces a number of polyhedral sets in $\mathbb{R}^{s+n}$ with known extreme points. Thus, maximizing a non-linear value function ϕ over each one of these sets can be formulated as a continuous optimization problem with $s+n$ linear constraints and one decision variable for each extreme point.

6.1.1. Workforce Allocation Example Revisited

We illustrate the relaxation of the linear preferences assumption by revisiting the fast-food chain application. In particular, we utilize the recently developed CUT method by Argyris et al. (2015), which allows interactive preference elicitation: at each iteration the DM is asked to state her preference between two alternatives (here portfolios), whereafter those alternatives are discarded which do not yield the highest value for any *quasi-concave* value function that is compatible with the given statements. By Theorem 4, the set of efficient portfolios P_E

includes the optimal portfolio for any quasi-concave value function ϕ, which is strictly increasing in outputs and strictly decreasing in inputs. Hence, as the initial set of alternatives for the CUT method we use the extreme points of set of efficient portfolios P_E. This makes it possible to compute the DMUs' resource ranges across portfolios that are optimal for some quasi-concave value function ϕ which is compatible with given preference statements.

As in Section 5.2., the management of the fast-food chain aims to reduce the total amount of workforce by one thousand man-hours (cf. left chart in Figure 9) while satisfying minimum output levels $a = (49, 48, 26, 63, 17.05)^T$. We assume that the management's 'true' preferences among the five portfolio outputs $\sum_{j=1}^m y^j = (\sum_{j=1}^m y_1^j, \ldots, \sum_{j=1}^m y_5^j)$ are captured by the multiplicative value function

$$\phi(\sum_{j=1}^{m} y^j) = \prod_{i=1}^{5} \Big(\sum_{j=1}^{m} y_i^j \Big).$$

Note that this value function is not known to the CUT method, but it is used only to simulate the management's answers to the preference questions presented by the method. The implementation of the CUT method is presented in Appendix C.

Figure 10 illustrates how the workforce allocated to each restaurant changes as the interactive procedure progresses. Specifically, the three sets of resource ranges are computed over those portfolios that have not been discarded after 8, 16, and 28 iterations. The resource ranges shrink considerably at iteration 8, at which point it is already clear that restaurant R16 will gain resources in all the remaining portfolios. Moreover, no restaurant will get the maximum amount of resources suggested by the initial resource ranges. At iteration 16, the resource allocation recommendations become much more conclusive: for every restaurant except R3, R13 and R19, resources are consistently increased or decreased for the remaining portfolios. At iteration 28, this applies to restaurants R3, R13 and R19, too.

As in the previous examples, the standard efficiency scores do not provide meaningful guidance for resource allocation. For example, resources are taken away from restaurants R4 and R8, although R4 is BCC-efficient and R8 is BCC-inefficient. Also, restaurant R16 receives additional resources, whereas resources are taken from restaurant R18, although the former is BCC-inefficient and the latter is BCC-efficient.

6.2. Other Approaches for Estimating DMUs' Feasible Input-Output Mixes

The decision recommendations produced by the model – efficient portfolios and DMUs' resource ranges – depend on the sets of feasible input-output mixes M^j for each DMU. These sets, in turn, depend on (i) the data (DMUs' observed inputs and outputs) and (ii) the efficiency analysis method with which the PPS is estimated and the DMUs' efficiency scores are computed. This dependence on the underlying method is characteristic of many other forms of efficiency analysis. For instance, efficiency rankings provided by two DEA studies may differ because of their different assumptions about the PPS (e.g., constant vs. decreasing returns-to-scale), or because of their different distance measures in computing the efficiency scores (e.g., input-oriented vs. output-oriented).

The selection of an appropriate PPS is thus an important part of modeling process. Many DEA methods are founded on a set of axioms which are assumed to hold for the PPS (e.g., convexity, free-disposability, see,

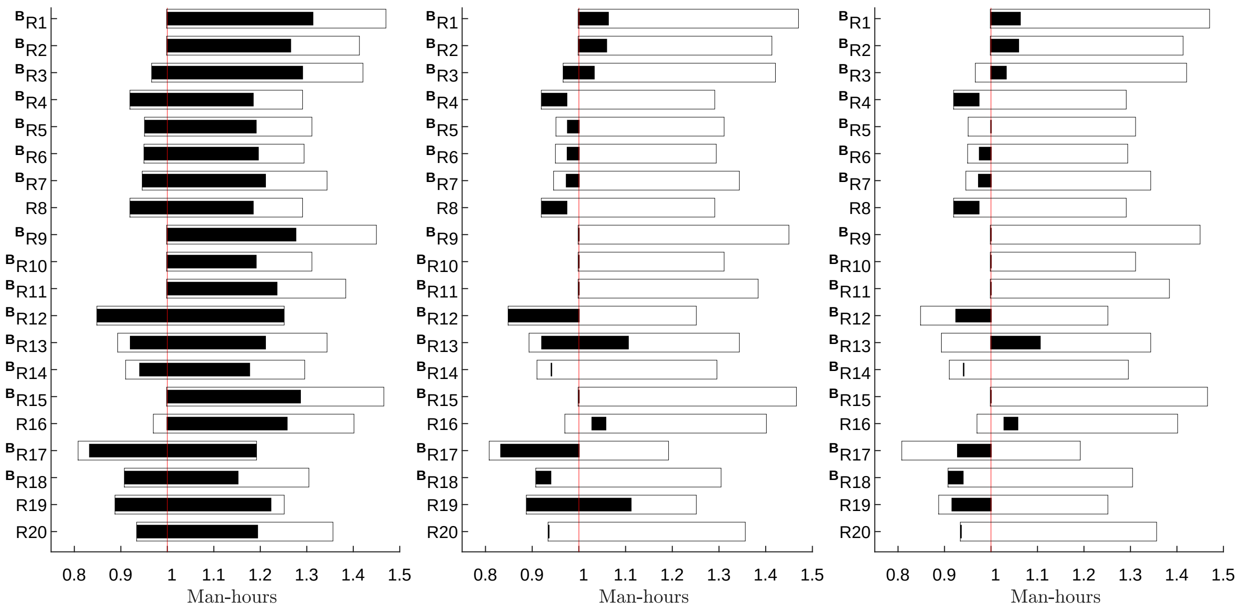

Figure 10: Relative resource ranges computed over efficient portfolios (white bars) and over the portfolios which have not been discarded by the CUT method (black bars) after 8 (left figure), 16 (middle figure), and 28 (right figure) iterations. Restaurants that are BCC-efficient (with respect to other restaurants in the same technology class) are marked with the superscript $^{\mathbf{B}}$.

e.g., Banker et al. 1984) and whose validity needs to be evaluated in specific application contexts. With our model, it is relatively straightforward to compute the set of (S-)efficient portfolios with multiple sets of feasible input-output mixes, each based on a different set of assumptions on the production possibilities and/or utilizing a different orientation (input vs. output). Such an approach provides information on how sensitive the resource allocation recommendations are to the choice of the underlying methods used to construct the PPS.

Apart from the conventional DEA methods that we have considered, there are also other approaches for estimating the sets of feasible input-output mixes M^j. Our model only requires that the sets M^j are bounded and that they can be represented as a system of linear constraints, i.e., requirements which are not very restrictive. In practice, there are finite bounds on the amount of input resources a DMU can consume as well as on the outputs it can produce. Furthermore, production possibility sets can usually be assumed to be convex, so they can be readily approximated with linear constraints.

Alternative approaches for establishing the sets of feasible input-output mixes can be classified into three categories. First, the recently developed *semi-parametric* methods such as StoNED (Kuosmanen & Kortelainen 2012) consider random noise in the observed input and output mixes. Unlike the traditional (non-parametric) DEA models, StoNED can separate efficiency from stochastic noise in the DMUs' observed inputs and outputs, providing an unbiased estimate of the efficient frontier. Furthermore, the resulting PPS is defined through a system of linear constraints and can thus be used as a basis for our model as well. Second, the methods of *parametric efficiency analysis* fit an efficient frontier of a particular functional form to the observed input-output data using regression techniques (Stochastic Frontier Analysis, see, e.g., Aigner et al. 1977). If the functional form is convex, then a piecewise linear approximation of the PPS set could be used in our model. Third, *expert judgement* has been successfully used in some applications to obtain a functional relationship between allocated input resources and produced outputs (see, e.g., Toppila et al. 2011). Such approaches can be combined with our model as long as the resulting functional relationship is convex.

7. Conclusions

Efficiency analysis is often used to measure how well the DMUs perform in producing outputs from the input resources, and also to guide how resources should be allocated among the DMUs. The model developed in this paper builds on top of standard efficiency analysis approaches by addressing these resource allocations explicitly through portfolio decision analysis. The model identifies the entire set of DMU portfolios that are efficient under incomplete information on relative values of different inputs and outputs. This set can be used to visualize and analyze the ranges of inputs and output that can be achieved through efficient allocation of resources. Moreover, identification of the set of efficient DMU portfolios makes it possible to establish for each DMU the range of input resources they receive in the efficient portfolios. Indeed, a major advantage of our model is that it communicates results through these resource ranges, which are directly linked to managerial resource allocations, unlike traditional efficiency measures. Moreover, our analyses of real data show that guiding resource allocations through conventional efficiency scores can result in the loss of efficiency. In particular, allocating more resources to DMUs with high efficiency scores and/or reducing the resources of DMUs with low efficiency scores can lead to a dominated DMU portfolio.

This paper opens up several avenues for future research. First, empirical research on how changes in the allocated resources affect the efficiency scores of individual DMUs could help identify those approaches for estimating the sets of feasible input-output mixes that are most useful in practical applications. Second, the model can be extended to non-convex or even non-continuous sets of feasible input-output mixes by incorporating integer variables in the linear constraint systems that define these sets. In this case, the computation of decision recommendations would call for solving multi-objective mixed integer linear programming problems with suitable algorithms (for a recent survey, see, e.g., Przybylski & Gandibleux 2017). Finally, changes in the outputs of DMUs after changes in their inputs are often uncertain. Thus, modeling the outputs explicitly as random variables while accounting for risk preferences would also call for further methodological advances.

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