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Instantaneous radiation from time-varying electric and magnetic dipoles

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Radiation from magnetic and electric dipole moments is a key subject in the theory of electrodynamics. Although people treat the problem thoroughly in the context of the frequency domain, the problem is still not well understood in the context of the time domain especially if dipole moments arbitrarily vary in time under the action of external forces. Here, we scrutinize the *instantaneous* power radiated by magnetic and electric dipole moments and report findings that are different from the conventional understanding of their instantaneous radiation found in textbooks. In contrast to the traditional far-field approach based on the Poynting vector, our analysis employs a near-field method based on the induced electromotive force, leading to corrective terms that are found to be consistent with time-domain numerical simulations, unlike previously reported expressions. Beyond its theoretical value, this paper may also have significant impact in the field of time-varying metamaterials especially in the study of radiation from subwavelength meta-atoms, scatterers, and emitters that are temporally modulated.

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I. INTRODUCTION

Electromagnetic radiation is conventionally identified and studied by looking at the Poynting vector [1] at long distances from the source. This perspective towards radiation is supported by the use of the Sommerfeld radiation condition [2,3] which is applied at those distances. As a consequence, one can say that the total energy per unit time radiated from a dipole moment can be obtained by using the following expression [4]:

$$P = \oint_S [\mathbf{E} \times \mathbf{H}] \cdot d\mathbf{S}, \tag{1}$$

in which $\mathbf{E}$ and $\mathbf{H}$ represent the "far-field" components of the time-varying electric and magnetic fields, respectively, generated by the dipole moment. As is known, these far-field components attenuate as $1/r$ in which r is the distance from the origin where the dipole is located. In the literature, the electric current density inducing the dipole moment is usually assumed to be a time-harmonic function. Therefore, applying the method of phasors, the above equation is simplified, and people are used to write the averaged energy per unit time over one cycle. We have

$$P_{\rm av} = \oint_S \frac{1}{2}\mathrm{Re}[\overline{\mathbf{E}} \times \overline{\mathbf{H}}^*] \cdot d\mathbf{S}, \tag{2}$$

in which * denotes the complex conjugate and Re[ ] gives the real part. Here, $\overline{\mathbf{E}}$ and $\overline{\mathbf{H}}$ are the far-field components of the complex phasors of the electric and magnetic fields in the frequency domain. If the background medium is lossless, the surface S (the spherical surface enclosing the dipole moment) in Eq. (2) can have a finite radius (e.g., Ref. [5]). In this case, one may think that the "near-field" components, attenuating as $1/r^3$ and $1/r^2$, must be also taken into account. However, these components of the complex phasors of the electric and magnetic fields are out of phase. As a result, the real part of the corresponding vector product in Eq. (2) for those components is zero (although this is not true for far-field components since they are in phase and the real part does not vanish). Consequently, as mentioned above, in the approach based on Eq. (2), we are conventionally interested in the far-field components. It is worth noting that, here, the background medium in which the dipole moment is located is considered to be a vacuum (for a lossy or dispersive background, the interpretation of Eq. (2) needs to be revisited, see, e.g., Ref. [6]).

In this paper, we introduce an alternative approach to study radiation from a general time-varying dipole moment, that uses the fields at the vicinity of the dipole (note that the fields have a singularity at the origin $r = 0$). In the context of the frequency domain, this is similar to Green's tensor approach which uses the fields at the dipole's origin as well, see e.g., Chap. 8 in Ref. [7]. However, our proposed approach provides us with a possibility to understand radiation also in the time domain (instantaneous radiation). It appears that this point of view has not been explicitly explored in the literature, and it leads to interesting new results and insights. Counterintuitively, we demonstrate that the nonsingular component for the fields at the location of the dipole determines not only the time-averaged power given by (2), but also additional instantaneous exchange of power between the dipole and the fields. This nonsingular component is proportional to the second and the third derivatives of the electric current of the electric and magnetic dipole moments, respectively. Interestingly, this nonsingular component is contributing to the induced electromotive force which results in radiation. We compare the time-varying induced electromotive force with the Lorentz-friction force exerted on one single accelerated electron and explicitly show the fundamental resemblance between these two classical concepts. In the end, we show some simulated results confirming our theoretical expectations about the instantaneous radiation. This paper may have an impact on the study of the transient description of antennas [8–13], the transient description of the reactive power around the dipole [14–18], the time-modulated scatterers [19,20], and the time-modulated metamaterials (see, e.g., Refs. [21–23]).

The paper is organized as follows: In Secs. II and III, we show the corresponding analytical derivations regarding magnetic and electric dipole moments, respectively. In Sec. IV, we demonstrate the full-wave simulation results confirming the analytical results achieved in the previous sections. In Sec. V, we repeat our derivations assuming that the electric and magnetic dipole moments have arbitrary temporal variation (not only time harmonic), and, finally, in Sec. VI, we conclude the paper.

II. MAGNETIC DIPOLE

Let us consider a radiating loop whose radius a is extremely small $a \rightarrow 0$. We assume that the loop is centered at the origin on the xy plane and that it carries uniform electric current I_0. Under these assumptions, such an electric-current loop is equivalent to a point magnetic dipole that is directed along the z axis and located at the origin of the Cartesian coordinate system. Therefore, in the frequency domain, the θ component of the generated complex magnetic field can be given by [24,25]

$$\overline{H}_\theta = -\frac{(ka)^2 I_0}{4r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]\sin\theta\,\exp[-jkr], \quad (3)$$

where k is the free-space wave number. Here, we choose the electrical engineering convention: $\exp(+j\omega t)$ for representing harmonic time variation (one can simply convert to the physics convention by replacing the imaginary unit j by $-i$). In order to find the electromotive force induced by the fields into the antenna, we derive the magnetic flux crossing the loop area and apply the Faraday law. Therefore, we first need to contemplate the magnetic field at the location of the loop, set the elevation angle θ to $\pi/2$, and calculate the corresponding magnetic flux density. To do that, in the above equation, we employ the Taylor expansion for the exponential function,

$$\exp[-jkr] = 1 - jkr - \frac{k^2r^2}{2} + j\frac{k^3r^3}{6} + \cdots . \quad (4)$$

Later, we will see that, for our purpose, it is enough to keep the first four members of the series. We substitute the above expression into Eq. (3) and rewrite the expression for the magnetic field. Multiplying by the free-space permeability $\mu = \mu_0$, the magnetic flux density reduces to

$$\overline{B}_\theta|_{\theta=(\pi/2)} \approx jk^3\frac{\mu_0 a^2 I_0}{6}\left[1 + j\frac{3}{4kr} - j\frac{3}{2(kr)^3}\right]. \quad (5)$$

Let us assume that I_0 is a real value. Accordingly, Eq. (5) indicates that the expansion of magnetic flux density has a real part whose leading term is proportional to $1/r^3$, decreasing as the distance grows and singular at the origin. However, importantly, Eq. (5) also shows that there is an imaginary term of the complex amplitude which is nonsingular and uniform over the loop area (within the dipole-moment model, it does not depend on distance). As a result, this term gives rise to a finite and imaginary component in the total flux, that is readily found by multiplication by the area of the loop. Therefore, we have

$$\overline{\phi}_{\rm imag} = j\,{\rm Im}[\overline{B}_\theta|_{\theta=\pi/2}]A = jk^3\frac{\mu_0(\pi a^2)^2 I_0}{6\pi}, \quad (6)$$

where Im[ ] represents the imaginary part and $A = \pi a^2$ is the loop area.

In the frequency domain, the magnetic flux multiplied by "$-j\omega$" gives the electromotive force. Due to the imaginary unit "j," one can say that the real (imaginary) part of the electromotive force is determined by the imaginary (real) part of the magnetic flux. On the other hand, it is known that the active power radiated into space is related to the real part of the electromotive force, and the reactive power describing the stored energy near the dipole moment is associated with the imaginary part of the electromotive force. Consequently, based on this discussion, we can conclude that the imaginary term in Eq. (6) is the only contributing term in the total flux for obtaining the radiated power. Furthermore, since this term is finite, the radiated power is finite at any moment of time, although the source can have an arbitrarily small size and the fields are singular. This issue is well known in the context of self-field-based radiated power calculation using Green's tensor approach [7] [note that, in contrast, the real part of the flux should be linked to the reactive power because it corresponds to the imaginary part of the electromotive force, and it suffers from singularity which is explicitly seen from the real part of the magnetic flux density in Eq. (5)].

After finding the nonsingular component of the field and, subsequently, the finite component of the flux, we do not continue with the frequency analysis and move to the time domain. Pondering about the above equation, we discern that the flux is associated with the third derivative of the electric current. Why? Because we have the wave number in power three (or the angular frequency in power three since $k = \omega/c$, where c is the speed of light). Remember that, for an arbitrary function $f(t)$ having the Fourier transform $F(\omega)$, we write

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\exp(j\omega t)d\omega. \quad (7)$$

This relation clearly indicates that the Fourier transform corresponding to the third derivative of $f(t)$ is identical with $-j\omega^3 F(\omega)$,

$$\frac{d^3 f(t)}{dt^3} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[-j\omega^3 F(\omega)]\exp(j\omega t)d\omega. \quad (8)$$

Thus, according to this explanation and by taking the inverse Fourier from Eq. (6), the instantaneous flux associated with the radiated power is given by

$$\phi_{\rm imag}(t) = -\frac{\mu_0(\pi a^2)^2}{6\pi c^3}\frac{d^3 i(t)}{dt^3}. \quad (9)$$

Now, Faraday's law helps us to derive the electromotive force $v(t)$ induced due to the temporal variation of the flux,

$$v(t) = -\frac{d\phi_{\rm imag}(t)}{dt}. \quad (10)$$

It is worth noting that the electromotive force can be calculated also as the circulation of the nonsingular component of the electric field around the loop. Naturally, the final result is the same as explicitly shown in Appendix A of the paper. Finally, the instantaneous radiated power can be written as

$$P_{\mathrm{m}}(t)=v(t)i(t)=-\frac{d\phi_{\mathrm{imag}}(t)}{dt}i(t). \tag{11}$$

Since the radiated power is proportional to the first derivative of the flux, the radiated power is, consequently, proportional to the fourth derivative of the electric current and to the electric current itself. In terms of the magnetic moment, the same conclusion is true since the magnetic moment is related to the electric current as $m(t)=(\pi a^2)i(t)$. After performing some algebraic manipulations, we can write

$$P_{\mathrm{m}}(t)=\frac{\mu_0}{6\pi c^3}m(t)\frac{d^4m(t)}{dt^4}. \tag{12}$$

This is a key result, and we now discuss it. In the context of time-harmonic fields, if the magnetic moment is described by $m(t)=m_0\cos(\omega t)$, the radiated power is found as $P_{\mathrm{m}}(t)=(\mu_0/6\pi c^3)\omega^4m_0^2\cos^2\omega t$. If we write the instantaneous power in terms of the electric current and calculate the time-averaged radiated power (over one cycle), we see that

$$P_{\mathrm{m}}^{\mathrm{av}}=\frac{\mu_0}{12\pi c^3}\omega^4(\pi a^2)^2I_0^2. \tag{13}$$

Based on the terminology of antenna engineering, the averaged radiated power is expressed through the concept of the radiation resistance. Recalling that $\mu_0 c\approx 120\pi$, this resistance is found as

$$R_{\mathrm{m}}=\frac{2P_{\mathrm{m}}^{\mathrm{av}}}{I_0^2}=20\pi^2\left(\frac{C}{\lambda}\right)^4, \tag{14}$$

in which $C=2\pi a$ is the circumference of the loop. This expression is very well known for antenna engineers (see, e.g., Ref. [24]).

Let us point out the salient feature of Eq. (12). This equation is, indeed, more complex. We know that

$$m\frac{d^4m}{dt^4}=\frac{d}{dt}\left[m\frac{d^3m}{dt^3}\right]-\frac{dm}{dt}\frac{d^3m}{dt^3}, \tag{15}$$

where

$$\frac{dm}{dt}\frac{d^3m}{dt^3}=\frac{d}{dt}\left[\frac{dm}{dt}\frac{d^2m}{dt^2}\right]-\left(\frac{d^2m}{dt^2}\right)^2. \tag{16}$$

As a consequence, Eq. (15) reduces to

$$m\frac{d^4m}{dt^4}=\left(\frac{d^2m}{dt^2}\right)^2+\frac{d}{dt}\left[m\frac{d^3m}{dt^3}-\frac{dm}{dt}\frac{d^2m}{dt^2}\right]. \tag{17}$$

As far as we know, the last term (in the square brackets) is absent in all textbooks and papers (see, e.g., Refs. [4,26,27]) where the far-field approach is conventionally used to calculate the radiated power. Therefore, this term is a corrective term which results from contemplating the fields at the location of the dipole at the origin, and it means that, in fact, there is an additional instantaneous exchange of power between the dipole and the radiated fields. In the time domain, particularly for the case when the magnetic dipole moment is not a time-harmonic function, this additional term is important, and it can have a significant impact. For example, it is actually crucial for correct satisfaction of the instantaneous power balance. In Sec. IV, this critical point is explicitly shown for the electric dipole for which neglecting the corresponding additional term gives dramatically different and wrong results.

Let us go deeper and investigate the magnetic dipole radiation from one single electron rotating with acceleration. Itoh, in Ref. [28], wrote the corresponding electric field generated by such an electron (via the expansion method, the fourth-order term) as

$$\mathbf{E}=\frac{\mu_0}{12\pi c^3}\mathbf{R}\times\frac{d^4\mathbf{m}}{dt^4}, \tag{18}$$

where $\mathbf{R}$ is the position vector of the observation point. The magnetic moment is expressed in terms of the velocity as

$$\mathbf{m}=\frac{e}{2}\mathbf{R}\times\mathbf{v}. \tag{19}$$

The force due to the electric field acting on the electron is $\mathbf{F}=e\mathbf{E}$. Therefore, the force finally reduces to

$$\mathbf{F}=-e\left[\frac{\mu_0}{6\pi c^3}\left(\frac{R}{2}\right)^2\frac{d^4e\mathbf{v}}{dt^4}\right]. \tag{20}$$

Velocity corresponds to the electric current. Hence, from this point of view, the above expression is, indeed, quite similar to the electromotive force that was found for the magnetic dipole,

$$v(t)=-\frac{d\phi_{\mathrm{imag}}(t)}{dt}=\frac{\mu_0}{6\pi c^3}(\pi a^2)^2\frac{d^4i(t)}{dt^4}. \tag{21}$$

This similarity becomes more evident if we think about the radiated power from a single electron, which is given by

$$P_{\mathrm{electron}}(t)=-\mathbf{F}\cdot\mathbf{v}. \tag{22}$$

By comparing this equation with $P_{\mathrm{m}}(t)=v(t)i(t)$ written for the magnetic dipole, and remembering that velocity and electric current are related to each other, one concludes that the electromotive force $v(t)$ and the radiation reaction force $\mathbf{F}$ are equivalent.

III. ELECTRIC DIPOLE

Let us consider a radiating Hertzian dipole which has a dipole moment (in the frequency domain) along the z axis: $\overline{\mathbf{p}}=(I_0l/j\omega)\mathbf{a}_z$ in which l is the length of the dipole. The electric field corresponding to the elevation angle equal to $\pi/2$ is also parallel to the z axis and can be expressed as (e.g., Ref. [24])

$$\overline{\mathbf{E}}=-j\eta\frac{kI_0l}{4\pi r}\left[1+\frac{1}{jkr}-\frac{1}{(kr)^2}\right]\exp(-jkr)\mathbf{a}_z, \tag{23}$$

where η is the free-space intrinsic impedance. Similar to what we did for the magnetic dipole, we look at the field at the vicinity of the Hertzian dipole and expand the exponential function up to the fourth term, see Eq. (4). Hence, the electric field is simplified and expressed as

$$\overline{\mathbf{E}}\approx(jk)^2\frac{\eta I_0l}{6\pi}\left[1+j\frac{3}{4kr}-j\frac{3}{2(kr)^3}\right]\mathbf{a}_z. \tag{24}$$

Akin to the electric-current loop, the field has real and imaginary parts. However, here, the *real part* of the field is nonsingular and in the vicinity of the dipole does not depend on the distance. This component gives the real part of the electromotive force which is finite and determines the radiated power (electromotive force is simply the electric field multiplied by the length of the dipole). On the other hand, the imaginary part of the field suffering from the singularity results in an imaginary term in the electromotive force that is associated with the reactive power.

Let us continue and see what the instantaneous radiated power is for the electric dipole. Based on Eq. (24), the nonsingular component of the field is given by

$$\overline{\mathbf{E}}_{\rm ns}=(jk)^2\frac{\eta I_0 l}{6\pi}\mathbf{a}_z. \tag{25}$$

What does the second power of the wave number mean? It shows that the electric field is proportional to the second derivative of the electric current in time. Note that the wave number is the angular frequency divided by the speed of light. Ergo, the inverse Fourier transform gives

$$\mathbf{E}_{\rm ns}(t)=\frac{\mu_0}{6\pi c}l\frac{d^2 i(t)}{dt^2}\mathbf{a}_z. \tag{26}$$

Here, we have used the equality $\eta=\mu_0 c$. Now, since we have found the electric field, we can readily derive the induced electromotive force through

$$v_{\rm ED}(t)=-\int\mathbf{E}\cdot d\mathbf{l}=-\frac{\mu_0}{6\pi c}l^2\frac{d^2 i(t)}{dt^2}. \tag{27}$$

There are two explicit differences between the electromotive force for the electric dipole and for the magnetic dipole. First, $v(t)$ was inversely proportional to the third power of the speed of light whereas $v_{\rm ED}(t)$ is inversely proportional to the first power of c. Furthermore, $v(t)$ is related to the fourth derivative of the electric current, but $v_{\rm ED}(t)$ is related to the second derivative of the current. Comparing Eqs. (21) and (27) is intriguing and informative. Area is substituted by length, $1/c^3$ by $1/c$, and the fourth derivative by the second derivative. Knowing the electromotive force, we readily derive the instantaneous radiated power using

$$P_{\rm e}(t)=v_{\rm ED}(t)i(t). \tag{28}$$

Our purpose is to write the radiated power based on the electric dipole moment. We use the fact that the electric current multiplied by the length of the dipole is the time derivative of the electric dipole moment: $i(t)l=dp(t)/dt$. Accordingly, we eventually express the radiated power as

$$P_{\rm e}(t)=-\frac{\mu_0}{6\pi c}\frac{dp(t)}{dt}\frac{d^3p(t)}{dt^3}. \tag{29}$$

For a time-harmonic moment $p(t)=p_0\sin(\omega t)$, the radiated power is identical with $P_{\rm e}(t)=[\mu_0/6\pi c]\omega^4p_0^2\cos^2\omega t$. Consequently, the averaged radiated power is half of the amplitude, and we can write the radiation resistance of the radiator (in the frequency domain) as

$$R_{\rm e}=\frac{2P_{\rm e}^{\rm av}}{I_0^2}=\frac{\mu_0}{6\pi c}(\omega l)^2=80\pi^2\left(\frac{l}{\lambda}\right)^2, \tag{30}$$

which is the known expression in all antenna books, e.g., Ref. [24] (recalling that $\eta\approx 120\pi$).

The radiation reaction force acting on the accelerated electron which has a velocity $\mathbf{v}$ is [29]

$$\mathbf{F}_{\rm e}=-e\left(-\frac{\mu_0}{6\pi c}\frac{d^2e\mathbf{v}}{dt^2}\right). \tag{31}$$

Again, similar to the radiating magnetic moment, the electromotive force found above shows a strong link to the force exerted on one electron. The radiated power from an accelerated electron is given by $P_{\rm electron}=-\mathbf{F}_{\rm e}\cdot\mathbf{v}$, which is similar to Eq. (28) used for the Hertzian dipole. The total power radiated by the dipole is the product of the electromotive force and the electric current (that is related to the charge velocity).

IV. NUMERICAL RESULTS

To validate the theoretical results, we did full-wave simulations employing the frequency domain solver in the radio-frequency module in COMSOL MULTIPHYSICS software. As shown in Fig. 1(a), we excite an electrically small loop, made of a perfect conductor, with two symmetrically placed sources. The reason for that is to cancel the extra higher modes, mainly including the electric dipole moment (see Appendix B for more information). Therefore, only the magnetic dipole moment is predominantly excited. We employ the instantaneous power balance,

$$\Sigma P(t)=0, \tag{32}$$

in which $\Sigma P(t)$ includes the supplied power, the reactive power (the electric and magnetic energy per unit time), and the radiated power. One can see Appendix C for the definition of the reactive and supplied power terms. Figure 1(b) shows the reactive and supplied power. Based on the above equation, we must have

$$-[P_{\rm supplied}(t)+P_{\rm reactive}(t)]=P_{\rm radiative}(t). \tag{33}$$

The left side is extracted from the simulator and compared with the right side calculated theoretically according to Eq. (12). Note that, in our calculations, the magnetic dipole moment is extracted from the simulations using the current flowing through the ports, and the electric dipole moment is achieved using either the surface current or the charge density (see Appendix D for more information). Figure 1(c) demonstrates that the simulated and theoretical results are in agreement. We have also simulated different loops with different radii. Until the radius is electrically small enough, we have found that the error between simulations and theory is negligible (see Appendix B for more information). However, increasing the radius causes that higher-order modes become significant and they must be taken into account in the power balance, which is not straightforward.

For the electric dipole moment, we have simulated an electrically small wire, shown in Fig. 1(d), that is excited by one source located at the center. The supplied and the reactive powers are shown in Fig. 1(e), and the comparison between the summation of those powers with the radiated power is indicated by Fig. 1(f). As is seen, if we use the theoretical expression given by Eq. (29), the theory and the simulated results are in agreement.

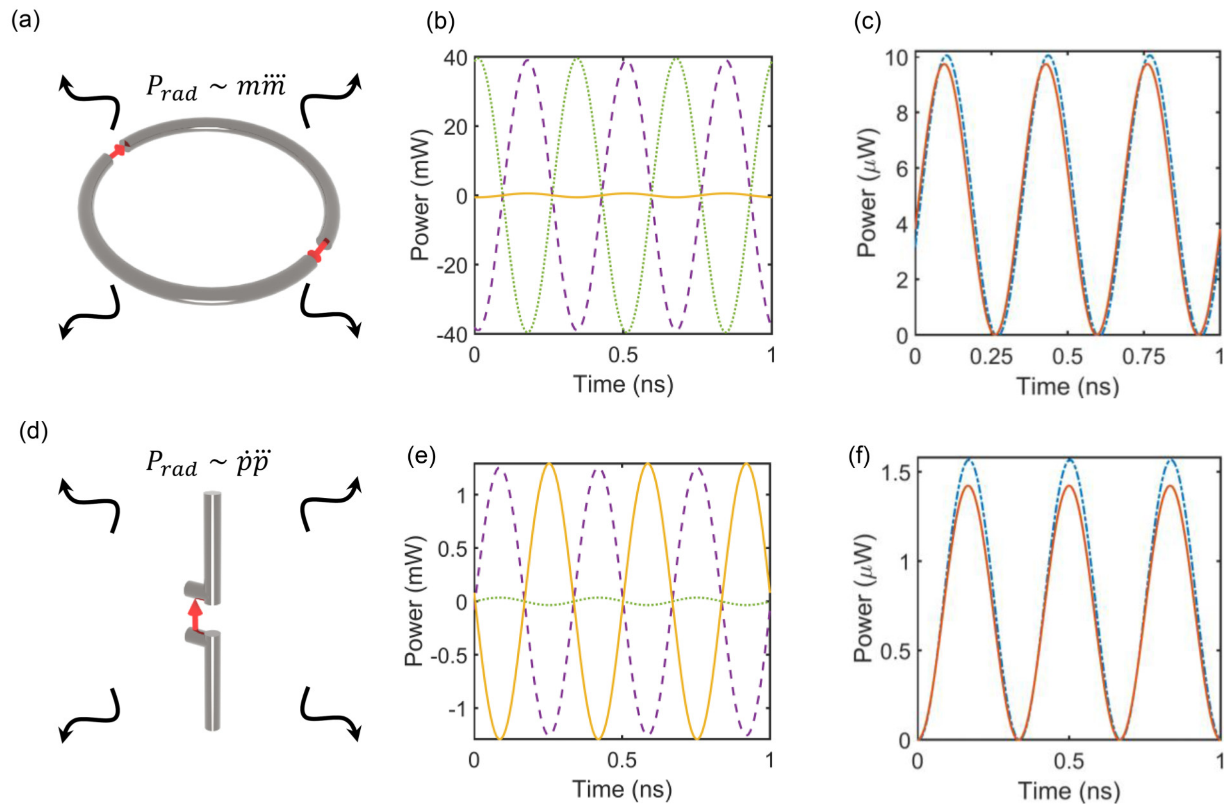

FIG. 1. (a)–(c) The instantaneous powers for the radiating magnetic dipole. (d)–(f) The instantaneous powers for the radiating electric dipole. The orange solid curve and the green dotted curve represent the electric and magnetic energies per unit time, respectively. Also, the purple dashed curve indicates the supplied energy per unit time. Finally, the red solid curve corresponds to the instantaneous radiated power, and the blue dot-dashed curve determines the sum of the electric, magnetic, and supplied energies per unit time taken with a negative sign. The radius of the loop is 3 mm, and the length of the cylindrical wire is 15 mm. Both the loop and the wire are made of perfect conductors and are much smaller than the operating wavelength that is 200 mm. The wire is fed in the center, and the loop is fed by two differential sources such that the electric dipole mode vanishes. The sources are determined by the red arrows.

Thus, we conclude that the correct form of the power radiated from the electric dipole is presented by Eq. (29). Here, it is worth mentioning another different point of view: One may say that Eq. (29) can be modified based on the expression,

$$\frac{dp}{dt}\frac{d^3p}{dt^3} = \frac{d}{dt}\left[\frac{dp}{dt}\frac{d^2p}{dt^2}\right] - \left(\frac{d^2p}{dt^2}\right)^2, \tag{34}$$

and interpret the first term on the right-hand side as not relevant to the radiated power. As a result, by considering only the second term $(d^2p/dt^2)^2$, he/she obtains the conventional expression: $P_e(t) = (\mu_0/6\pi c)(d^2p/dt^2)^2$ found in many textbooks and papers (see, e.g., Ref. [4]). However, our simulated results explicitly confirm the fact that completely neglecting the additional term $d/dt[(dp/dt)(d^2p/dt^2)]$ in Eq. (34) and using the conventional expression for the radiated power give rise to the violation of the instantaneous power balance Eq. (32). In other words, $\Sigma P(t)$ is zero only if we take into account the additional term.

Let us now demonstrate the generality of the power balance approach. In order to do so, instead of a time-harmonic excitation, a Gaussian pulse will be used. By placing a discrete port in the gap of a short wire antenna and applying a pulse excitation signal, we can again write the power balance equation. Note that, in this case, simulations are performed by the transient solver in the CST Studio Suite and the reactive power is calculated by using the effective lumped elements (for more information, see Appendix C). Figure 2 explicitly

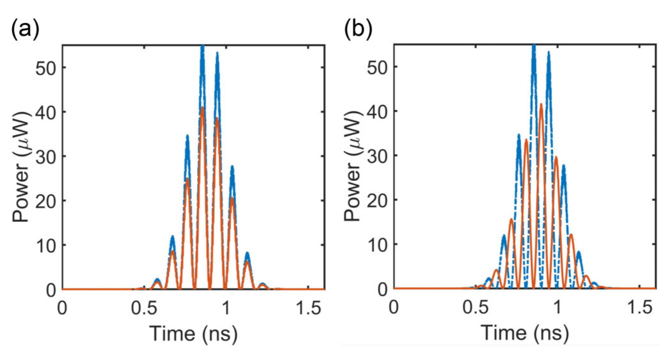

FIG. 2. Instantaneous power balance for electric dipole fed by a Gaussian pulse. Here, the radiated power is calculated by: (a) using Eq. (29) and (b) using the conventional formula which does not include the additional (corrective) term. The red solid curve corresponds to the instantaneous radiated power, and the blue dot-dashed curve shows the sum of electric, magnetic, and supplied energies per unit time taken with the negative sign.

indicates that the simulated results have good agreement with the theory. Importantly, this figure also shows the instantaneous power balance when the radiated power is obtained with and "without" the additional term in Eq. (34). Now, the importance of this additional term becomes obvious since, in the latter case, the power balance is not fulfilled.

V. TIME-VARYING ELECTRIC AND MAGNETIC DIPOLES

Although the derivations in previous sections are valid for instantaneous radiation, we basically focused on the fields which resulted from time-harmonic oscillations. However, we can arrive at general conclusions by carefully looking at the general expressions given for the time-varying electric and magnetic fields of arbitrary dynamic electric and magnetic dipoles. In Refs. [4,30,31], the time-varying electric field was derived and written for the electric dipole moment taking into account the retardation effect. According to those references, we have

$$\mathbf{E}(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\bigg[\frac{\ddot{\mathbf{p}}-\mathbf{a}_r(\mathbf{a}_r\cdot\ddot{\mathbf{p}})}{r} + c^2\frac{[\mathbf{p}+(r/c)\dot{\mathbf{p}}]-3\mathbf{a}_r\{\mathbf{a}_r\cdot[\mathbf{p}+(r/c)\dot{\mathbf{p}}]\}}{r^3}\bigg] - \frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r}), \tag{35}$$

where $\delta^3(\mathbf{r})$ denotes the three-dimensional Dirac δ function and ϵ_0 represents the free-space permittivity. In Eq. (35), the electric field at time t is due to the dipole moment at $t_\mathrm{r} = t - r/c$ (retardation effect). In other words,

$$\mathbf{p}(t_\mathrm{r}) = \mathbf{p}\Big(t-\frac{r}{c}\Big). \tag{36}$$

Applying the expansion technique, the dipole moment in the above is readily simplified to

$$\mathbf{p}\Big(t-\frac{r}{c}\Big) = \mathbf{p}(t) - \frac{r}{c}\dot{\mathbf{p}}(t) + \frac{1}{2}\Big(\frac{r}{c}\Big)^2\ddot{\mathbf{p}}(t) - \frac{1}{6}\Big(\frac{r}{c}\Big)^3\dddot{\mathbf{p}}(t) + \cdots . \tag{37}$$

Let us consider the limit as r approaches zero ($r \to 0$). Taking this limit in Eq. (35), we can neglect the $\mathbf{a}_r$ component of the electric field and only consider the component which is parallel to the electric dipole moment,

$$\mathbf{E}_\mathrm{p}(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\bigg[\frac{\ddot{\mathbf{p}}}{r} + c^2\frac{[\mathbf{p}+(r/c)\dot{\mathbf{p}}]}{r^3}\bigg] - \frac{1}{3\epsilon_0}\mathbf{p}\delta^3(\mathbf{r}), \tag{38}$$

and if we replace Eq. (37), after some algebraic manipulations, we deduce the nonsingular component of the electric field at the dipole location as

$$\mathbf{E}_\mathrm{ns}(\mathbf{r},t) = \frac{\mu_0}{6\pi c}\frac{d^3\mathbf{p}(t)}{dt^3}. \tag{39}$$

This expression is exactly identical with the expression given by Eq. (26) (recalling that the electric current multiplied by the electric dipole length is equal to the first time derivative of the electric dipole moment). Thus, the temporal electromotive force and the radiated power are, subsequently, the same as what we derived before.

Regarding magnetic dipole moments, the magnetic flux density is expressed as [4,30,31]

$$\mathbf{B}(\mathbf{r},t) = -\frac{\mu_0}{4\pi}\bigg[\frac{\ddot{\mathbf{m}}-\mathbf{a}_r(\mathbf{a}_r\cdot\ddot{\mathbf{m}})}{c^2 r} + \frac{[\mathbf{m}+(r/c)\dot{\mathbf{m}}]-3\mathbf{a}_r\{\mathbf{a}_r\cdot[\mathbf{m}+(r/c)\dot{\mathbf{m}}]\}}{r^3}\bigg] + \frac{2\mu_0}{3}\mathbf{m}\delta^3(\mathbf{r}), \tag{40}$$

that is in duality with the electric dipole moment as expected. Repeating the same procedure that we performed for electric dipoles, we obtain the following nonsingular component of the magnetic flux density that is parallel to the magnetic moment:

$$\mathbf{B}_\mathrm{ns}(\mathbf{r},t) = \frac{\mu_0}{6\pi c^3}\frac{d^3\mathbf{m}(t)}{dt^3}. \tag{41}$$

Comparing the above equation and Eq. (5), we see that the corresponding nonsingular components are quite similar [in Eq. (5), we wrote the magnetic flux density based on the electric current, and the expression is before the inverse Fourier transform].

Consequently, from the above two equations giving the nonsingular components of the electric and magnetic fields corresponding to the electric and magnetic moments, respectively, we understand that they are proportional to the third derivative of the dipole moment. However, since, for the magnetic dipole, the electromotive force is associated with the time derivative of the magnetic flux, the radiated power is proportional to the fourth derivative of the dipole moment contrary to the electric dipole whose radiated power is proportional to third time derivative of the dipole moment.

VI. CONCLUSIONS

Using the concept of electromotive force and contemplating the electromagnetic fields at the vicinity of the dipole moments, we obtained the instantaneous radiated power. We observed that the power is proportional to the magnetic moment and its fourth time derivative. However, regarding the electric moment, the power is proportional to the first and the third derivatives of the moment. These theoretical expressions were confirmed by several simulations of a loop and a cylindrical wire excited by time-harmonic or pulse sources. Also, we compared the electromotive force with the Lorentz-friction force and showed the similarity of these two concepts. Furthermore, based on the expansion technique, we analytically showed that our theoretical conclusions are not only valid for time-harmonic oscillation, but also valid for any nonstatic moments having arbitrary temporal variation. This paper may have a strong potential to influence engineers interested in antenna engineering and scattering from subwavelength particles both in the microwave and in the optical regimes. A possible future direction would be to consider an arbitrary volume current distribution and derive the instantaneous radiation including higher modes as well.

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APPENDIX A: ELECTROMOTIVE FORCE CALCULATION BASED ON CIRCULATION OF THE ELECTRIC FIELD

The electric field generated by an electric-current loop is expressed as

$$\overline{\mathbf{E}} = \eta \frac{(ka)^2 I_0}{4r}\left[1+\frac{1}{jkr}\right]\sin\theta \exp[-jkr]\mathbf{a}_\phi. \tag{A1}$$

By using the Taylor series for the exponential function and considering $\theta=\pi/2$, we find the real part of the electric field at the origin as

$$\overline{\mathbf{E}} \approx -\eta\frac{\pi a^2 I_0}{12\pi}k^4 r\mathbf{a}_\phi, \tag{A2}$$

in which $r=a$ around the loop. Remembering that $k=\omega/c$, in the time domain, this component of the electric field can be written as

$$\mathbf{E}(t) = -\eta\frac{\pi a^2}{12\pi c^4}\frac{d^4 i(t)}{dt^4} r\mathbf{a}_\phi\bigg|_{r=a}. \tag{A3}$$

The electromotive force, contributing in the radiated power, is found as

$$v(t) = -\oint \mathbf{E}\cdot d\mathbf{l} = \eta\frac{(\pi a^2)^2}{6\pi c^4}\frac{d^4 i(t)}{dt^4}. \tag{A4}$$

Here, we can substitute η by $\mu_0 c$ and simplify the electromotive force as

$$v(t) = \frac{\mu_0}{6\pi c^3}(\pi a^2)^2\frac{d^4 i(t)}{dt^4}, \tag{A5}$$

which is the same as derived in the main text by employing Faraday's law.

APPENDIX B: CURRENT DISTRIBUTION IN A LOOP ANTENNA

In this paper, the magnetic dipole is approximately realized as a small loop made of a thin perfectly conducting wire. Current distribution in a transmitting loop antenna fed by a single port is given by the formula [32],

$$I(\phi) = \frac{-jV_0}{\eta\pi}\left(\frac{1}{a_0} + 2\sum_1^\infty \frac{\cos n\phi}{a_n}\right), \tag{B1}$$

in which ϕ varies from 0 to 2π indicating the position on the loop, and V_0 is the amplitude of the voltage source placed at position $\phi=0$. Coefficients a_n can be calculated by using

$$a_n = \frac{kb}{2}(K_{n+1}+K_{n-1}) - \frac{n^2}{kb}K_n, \tag{B2}$$

where k is the wave number and b is the larger radius of the loop. Here, the coefficients K_n are given by the integrals,

$$K_n = \frac{1}{4\pi^2}\int_{-\pi}^{\pi} d\Psi \int_{-\pi}^{\pi} e^{jn\theta}e^{-jkbR(\theta)}\frac{d\theta}{R(\theta)}, \tag{B3}$$

in which $R(\theta)=[4\sin^2(\theta/2)+A^2/b^2]^{1/2}$ and $A=2r_0\sin(\Psi/2)$, where r_0 is the radius of the wire. Calculation of the coefficients a_n is quite complex, however, in the case of small loops all high-order terms can be neglected. In this case, the current distribution in Eq. (B1) reduces to

$$I(\phi) = I_0 + I_1(\phi), \tag{B4}$$

where

$$I_0 = \frac{-jV_0}{\eta\pi a_0}, \quad I_1(\phi) = \frac{-2jV_0}{\eta\pi a_1}\cos(\phi). \tag{B5}$$

Angular dependency can be canceled by employing two ports as shown in Fig. 1(a). Based on this scenario, only uniform current is excited which is responsible for the magnetic dipole moment. Figure 3 shows how discrepancy in the instantaneous power balance grows with the radius of the loop. When the radius of the loop approaches 13 mm, the total length of the loop becomes 82 mm which is almost half of the wavelength. In this case, the loop cannot be considered electrically small, and angle-dependent terms in Eq. (B1) are more complex and are not canceled by using symmetric excitation by the two ports.

APPENDIX C: REACTIVE AND SUPPLIED POWERS

Reactive power is the sum of electric and magnetic powers which can be calculated using the effective lumped elements, such as capacitance $C_{\rm eff}$ and inductance $L_{\rm eff}$. Assuming that all the electric energy is stored in the effective capacitance and all the magnetic energy is stored in the effective inductance enables us to express the corresponding powers via the current flowing in the antennas,

$$\begin{aligned} P^{\rm el}(t) &= \frac{1}{C_{\rm eff}} i(t)\int i(t)dt, \\ P^{\rm m}(t) &= L_{\rm eff}\frac{di(t)}{dt}i(t). \end{aligned} \tag{C1}$$

Considering, first, the time-harmonic case, the effective capacitance can be calculated as

$$C_{\rm eff} = \frac{|I_0|^2}{\omega^2}\frac{1}{4W^{\rm el}}. \tag{C2}$$

Analogously, the expression for the effective inductance reads

$$L_{\rm eff} = \frac{4W^{\rm m}}{|I_0|^2}. \tag{C3}$$

Alternatively, the effective lumped parameters can be calculated using the input impedance of the antenna in a range of frequencies. In the case of an electrically small antenna, the input impedance can be modeled as a series connection of the effective inductance and capacitance. Therefore, the input reactance of the antenna reads

$$X_{\rm in} = \frac{1}{j\omega C_{\rm eff}} + j\omega L_{\rm eff}. \tag{C4}$$

Extracting the value of reactance at two different frequencies gives a set of two equations. These equations can be solved assuming constant and nondispersive effective parameters. This is a valid assumption provided that a relatively narrow frequency range is used.

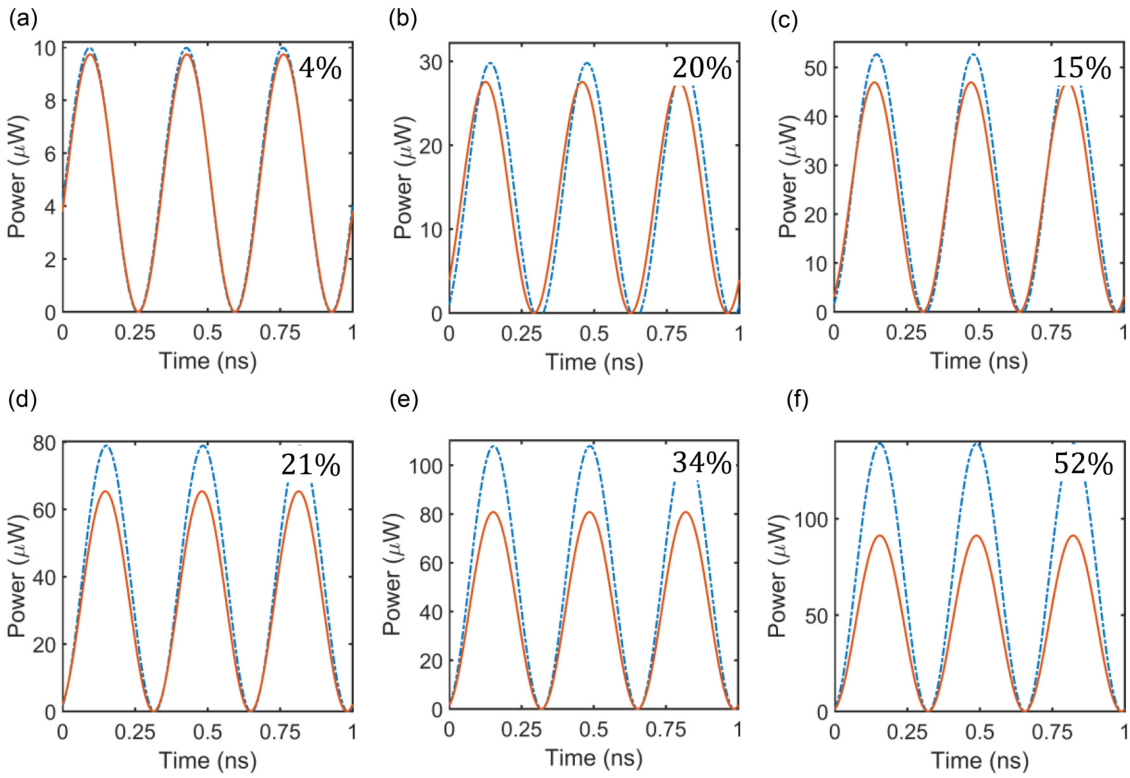

FIG. 3. Instantaneous power for magnetic dipoles of different radii: (a) 3 mm, (b) 5 mm, (c) 7 mm, (d) 9 mm, (e) 11 mm, and (f) 13 mm. The red solid curves in all the graphs correspond to the instantaneous radiated power, and the blue dot-dashed curves determine the sum of electric, magnetic, and supplied energies per unit time taken with the negative sign. The relative error, indicating the discrepancy, is shown in percentages on the graphs.

The supplied power to the antenna is simply the power supplied by the port, which is calculated as the current through the port $i(t)$ multiplied by the voltage on it $v_{\rm p}(t)$,

$$P_{\rm supplied}(t) = v_{\rm p}(t)i(t), \tag{C5}$$

where the voltage and current are extracted from the simulators. In the case of the magnetic dipole, two ports are used, therefore, the supplied power is the sum of powers supplied by each port.

APPENDIX D: ELECTRIC AND MAGNETIC DIPOLE MOMENTS

Radiated power in the time domain is calculated using Eq. (12) in the case of the magnetic dipole and Eq. (29) in the case of the electric dipole. The general definition of the electric dipole moment reads

$$\mathbf{p} = \int_V \mathbf{r}'\rho(\mathbf{r}', t)d\mathbf{r}', \tag{D1}$$

where $\rho(r, t)$ is the charge density. In the case of the electric dipole, in the frequency domain, it is very convenient to calculate it using this notation. However, in the transient analysis of the electric dipole, it is more convenient to use the definition of the dipole moment in terms of the surface current $\mathbf{J}(\mathbf{r}, t)$,

$$\mathbf{p} = \int dt \int \mathbf{J}(\mathbf{r}', t)d\mathbf{r}'. \tag{D2}$$

The value of the magnetic dipole moment is calculated through the current $i(t)$ flowing in the loop as

$$m(t) = \pi a^2 i(t), \tag{D3}$$

where a is the loop radius.

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