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Consensus of General Linear Multi-Agent Systems with Heterogeneous Input and Communication Delays

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***Abstract*—This paper studies the consensus tracking control for multi-agent systems (MASs) of general linear dynamics considering heterogeneous constant known input and communication delays under a directed communication graph containing a spanning tree. First, for open-loop stable MASs, a distributed predictive observer is proposed to estimate the consensus tracking error and to construct the control input that does not involve any integral term (which is time-efficient in calculation). Then, using the generalized Nyquist criterion, we derive the conditions for asymptotic convergence of the closed-loop system and show that is delay-independent. Subsequently, another observer is designed that allows the MASs to be open-loop unstable. Next, we use the generalized Nyquist criterion to compute the observer's gain matrix. Towards this end, we choose a specific structure with which the problem boils down to computing a single parameter, herein called the predictive observer parameter. Two algorithms are proposed for choosing this parameter: one for general linear systems and one for monotone systems. To the best of the authors' knowledge, this is the first work for which asymptotic convergence of consensus is proven for general linear MASs with arbitrary heterogeneous delays. Finally, the validity of our results is demonstrated via a vehicle platooning example.**

***Index Terms*—Consensus, heterogeneous delays, delay-independent stability, generalized Nyquist criterion.**

I. INTRODUCTION

CONSENSUS problems appear in many disciplines, such as biology, engineering, computer science and physics. Typical applications are vehicle formations, distributed optimization, flocking of animals or robots, sensor networks, distributed estimation/filtering, and social networks. Multi-agent consensus tracking control, which aims at controlling followers' states to track the leader's state, has been investigated for decades (see, e.g., [1], [2] and references therein).

A fundamental challenge in designing feedback controllers to be implemented over communication networks is to cope with time-delays [3]. There are two types of time-delays in multi-agent consensus: *input delay* and *communication delay*. Input delays are related to connecting and processing times for packets arriving at an agent and they also occur when actuators and controllers are connected by networks. Communication delays occur when agents receive delayed information from neighboring agents via the underlying communication network. Both types of delays play a vital role in the stability of the whole system. For instance, for unstable systems with fast dynamics, like flying robots, even small input delays from the controller to actuator could still destabilize the whole system.

Many consensus controllers have been proposed to tackle homogeneous communication delays, e.g., in [1], [4]. One key advantage of addressing the problem of having homogeneous communication delays is the easiness to put the MAS dynamics into a compact mathematical form related to the Laplacian matrix of the communication graph. For heterogeneous communication delays, however, the above advantage disappears and linear matrix inequality (LMI) conditions are often proposed for consensus in MASs, e.g., in [5]. However, these LMI conditions are not scalable to arbitrarily large networks as the size of the LMI increases with the number of delays and the number of agents. Alternatively, the heterogeneous delays can be transformed into the Laplace domain and approaches in the frequency domain (e.g., generalized Nyquist criterion [6]) can be utilized to design controllers for specific dynamics of MASs, i.e., single-input-single-output (SISO) [7]–[10], first order [11], and double integrator [12]. It should be noted here that all the above agent dynamics are quite simple, which can deliver the advantage of forming the multiple transfer functions in the compact diagonal matrix format (off-diagonal elements being zero) and thus calculating each heterogeneous delay related to the corresponding single agent dynamics independently. However, when dealing with MASs of general linear dynamics, the aforementioned results are no longer feasible since the compact diagonal matrix cannot be constructed any more, which is the main challenge and also one motivation of this work. To the authors' best knowledge, this is the first time asymptotic convergence of consensus related to arbitrary heterogeneous communication delays can be achieved for the general linear MASs.

In addition to communication delays, heterogeneous input delays are challenging to deal with as Kronecker format dynamics for MASs cannot be constructed like in the case of homogeneous ones; see, e.g., in [4], [13]. To solve this challenging problem, Xu *et al.* in [14] use the output regulation approach in the time domain to transform the consensus problem into studying a single system with its own input delay. However, this method cannot apply to homogeneous MASs with heterogeneous input delays and the output regulation equation has constraints on leader's dynamics. Following the research line in [7], [10], [15], a frequency analysis approach

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is utilized in [16] with single/double integrator dynamics. However, the controller gain is delay-dependent (inverse proportional to the upper bound of delays) meaning delays cannot be large (gain would be very small). To author's best knowledge, to deal with arbitrary heterogeneous input delays without strong constraints on the leader's dynamics for general linear MASs is still an open challenge, which is the other motivation of this work.

In this paper, first, for open-loop stable MASs, a distributed predictive observer is proposed that does not involve any integral term (which is time-efficient in calculation). By ignoring the effects of delays, we extract the necessary conditions for stability of MASs. Then, using frequency domain arguments (specifically, the generalized Nyquist stability criterion) we derive the conditions with which consensus tracking is achieved, showing at the same time that these conditions are independent of the size of the delay, unlike [10], [12], [14], [16]. Subsequently, another observer is designed that allows the MASs to be open-loop unstable, thus broadening the range of possible applications. Last, we use the stability conditions to compute the observer's gain matrix. Towards this end, we choose a specific structure with which the problem boils down to computing a single parameter, herein called the *predictive observer parameter*. For a general linear system, a numerical blind search algorithm is proposed. For monotone[1] systems, a simpler and more efficient algorithm is designed, which guarantees the existence of the parameter.

The main contributions are as follows. (i) This is the first result that shows consensus with arbitrary heterogeneous input or communication delays for agents with general linear dynamics. (ii) The predictive observers for open-loop stable/unstable dynamics are designed. (iii) We propose a specific structure for the predictive observer parameter. Then, a blind search algorithm for the corresponding parameter is proposed. We also find out that the existence of the parameter is guaranteed for monotone systems.

Notation: Throughout this letter, $\mathbb{R}^{m\times n}$ and $\mathbb{R}^n$ are respectively the $m\times n$ real matrix space and n-dimensional Euclidean vector space. The set of real (positive integer) numbers is denoted by $\mathbb{R}$ ($\mathbb{N}$). I denotes the identity matrix (of appropriate dimensions) and $\otimes$ is the Kronecker product. For the square matrix A, $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ and $\mathrm{Re}(\lambda(A))$ represent the minimal, maximal and the real part of eigenvalues of A, respectively. $\det(A)$ and $\rho(A)$ are respectively the determinant and spectral radius of A. $\mathrm{diag}\{a_1,\ldots,a_n\}$ represents a diagonal matrix with diagonal elements being $a_1,\ldots,a_n$. Matrices are assumed to have compatible dimensions if not explicitly stated. $|x|$ denote its absolute value. For any integer $a\le b$, denote $\mathbf{I}_a^b=\{a,a+1,\ldots,b\}$.

II. Problem setup

A. Graph theory

In a weighted graph $\mathcal{G}=(\mathcal{N},\mathcal{E},\mathcal{A})$, $\mathcal{N}=\{1,2,\ldots,N\}$ and $\mathcal{E}\subseteq\mathcal{N}\times\mathcal{N}$ are the nodes and edges, respectively. $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ is the weighted adjacency matrix, where $a_{ij}=1,(i,j)\in\mathcal{E}$ and $a_{ij}=0$ otherwise. An edge $(j,i)\in\mathcal{E}$ means agent j can get information from agent i but not necessarily conversely. A directed path from node i to j is a sequence of nodes $i=l_1,l_2,\ldots,l_t=j$ such that link $(l_{m+1},l_m)\in\mathcal{E}$ for all $m=1,2,\ldots,t-1$. The Laplacian matrix $\mathcal{L}=[l_{ij}]\in\mathbb{R}^{N\times N}$ is defined as $l_{ij}=-a_{ij},i\ne j$ and $l_{ii}=\sum_{j\ne i}a_{ij}$. All nodes that can transmit information to node i directly are said to be in-neighbors of node i and belong to the set $\mathcal{N}_i^-=\{i\in\mathcal{N}\,|\,(i,j)\in\mathcal{E}\}$. The nodes that receive information from node i belong to the set of out-neighbors of node i, denoted by $\mathcal{N}_i^+=\{j\in\mathcal{N}\,|\,(j,i)\in\mathcal{E}\}$. A directed graph contains a directed spanning tree if there is a node from which a directed path exists to each other node.

B. System model

Consider a group of N followers as

$$\begin{aligned}\dot{x}_i(t)=&Ax_i(t)+Bu_i(t-\tau_{u_i}),\\ y_i(t)=&Cx_i(t),i\in\mathbf{I}_1^N,\end{aligned}\tag{1}$$

where $x_i(t)=[x_{i1}(t),\ldots,x_{in}(t)]^T\in\mathbb{R}^n$, $u_i(t)\in\mathbb{R}^p$ and $y_i(t)\in\mathbb{R}^q$ are respectively the state, input and measured output of the i-th follower. τ_{u_i} is a known constant heterogeneous input delay. The dynamics of leader indexed by 0 is

$$\dot{x}_0(t)=Ax_0(t)\,,\,y_0(t)=Cx_0(t),\tag{2}$$

where $x_0(t)\in\mathbb{R}^n$ and $y_0(t)\in\mathbb{R}^q$. It is reasonable to have the leader without neighbors or input, i.e., $u_0(t)=0$.

Assumption 1. *All the eigenvalues of A are in the open left-half plane.*

Assumption 2. *Graph $\mathcal{G}$ contains a directed spanning tree in which the leader acts as the root node.*

Assumption 1 is needed for the deployment of generalized Nyquist criterion and also appears in [4], [7]. From Assumption 2, the Laplacian matrix of $\mathcal{G}$ can be partitioned as $\mathcal{L}=\begin{bmatrix}0 & 0_{1\times N}\\ \mathcal{L}_2 & \mathcal{L}_1\end{bmatrix}$, where $\mathcal{L}_2\in\mathbb{R}^{N\times 1}$ and $\mathcal{L}_1\in\mathbb{R}^{N\times N}$. Under Assumption 2, $\mathcal{L}_1$ is a nonsingular M-matrix. Therefore, all the eigenvalues of $\mathcal{L}_1$ have positive real parts.

As followers can receive the output information from the in-neighbors, the measurement at a node i can be synthesized as a single signal as

$$\bar{y}_i(t)=\sum_{j\in\mathcal{N}_i^-}a_{ij}[y_i(t-\tau_{c_{ij}})-y_j(t-\tau_{c_{ij}})],i\in\mathbf{I}_1^N,\tag{3}$$

where a_{ij} is the ij-th entry of the adjacent matrix $\mathcal{A}$ of graph $\mathcal{G}$ and $\tau_{c_{ij}}$ is the known constant communication delays from agent j to agent i.

Remark 1. *In applications where only relative information can be measured (e.g., $y_i(t)$ is not directly measured but only $y_i(t)-y_j(t)$ is measured) and the communication delay is involved in the measurement, taking the same communication delay for node i and node j is inevitable, i.e., $y_i(t-\tau_{c_{ij}})-y_j(t-\tau_{c_{ij}})$ in* (3)*; see, for example in platooning. In applications where absolute output rather than relative output is available, by using timestamps at the transmitted packets of node j, the receiving node i is able to measure*

[1]The system $\dot{x}=f(x)$ is called *monotone* if $a\le b\Rightarrow x(t,a)\le x(t,b),\forall t\ge 0$.

the delay $\tau_{c_{ij}}$. Then, the output signal is delayed before being applied (see, e.g., [17]).

We denote the largest communication delay to agent i from all its neighbors by τ_{c_i}, i.e., $\tau_{c_i} \triangleq \max_{j\in\mathcal{N}_i^-} \tau_{c_{ij}}$. Also, τ_i, denotes the total delay of node i, i.e.,

$$\tau_i \triangleq \tau_{u_i} + \tau_{c_i}, i \in \mathbf{I}_1^N. \tag{4}$$

Then, the initial conditions for followers (1) is $x_i(t) = \phi_i(t)$ when $t \in [-\tau_i, 0]$. Similarly, define $x_0(t) = \phi_0(t), t \in [-\tau_0, 0]$ where $\tau_0 = \max_{j\in\mathcal{N}_0^+}(\tau_{u_j} + \tau_{c_{j0}})$.

Based on (1) and (2), denote the state consensus tracking error for follower i as $\tilde{x}_i(t) = x_i(t) - x_0(t)$ and we have

$$\dot{\tilde{x}}_i(t) = A\tilde{x}_i(t) + Bu_i(t - \tau_{u_i}), i \in \mathbf{I}_1^N. \tag{5}$$

Problem 1. *Considering arbitrarily large known constant heterogeneous input and communication delays, for any given initial states $x_i(0)\cup x_0(0)$, design a distributed controller such that the consensus tracking error $\tilde{x}_i(t), i \in \boldsymbol{I}_1^N$ in (5) converges to zero asymptotically.*

III. Main results

A. Distributed predictive observer

The idea is to design a predictive observer as $\hat{x}_i(t) \in \mathbb{R}^n$ with $\hat{x}_i(t) = 0, \forall t \in [-\tau_i, 0]$ to estimate $\tilde{x}_i(t)$ in (5). To achieve prediction, the classic Artstein's model reduction technique [18] involves the integral term whose calculation is very time-consuming. Therefore, designing an observer without any integral term is preferred. Based on (3), $\hat{x}_i(t)$ is proposed as follows:

$$\begin{aligned}\dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + Bu_i(t) + L\big\{\bar{y}_i(t)\\ &+ \sum_{j=1,j\neq i}^{N} a_{ij}C\big[\hat{x}_j(t-\tau_{u_j}-\tau_{c_{ij}}) - \hat{x}_i(t-\tau_{u_i}-\tau_{c_{ij}})\big]\\ &- a_{i0}C\hat{x}_i(t-\tau_{u_i}-\tau_{c_{i0}})\big\},\end{aligned} \tag{6}$$

where $L \in \mathbb{R}^{n\times q}$ will be designed later.

Remark 2. *$\hat{x}_j(t-\tau_{u_j}-\tau_{c_{ij}})$ means the neighbor's observer $\hat{x}_j(t)$ needs to be self delayed by its own input delay τ_{u_j} first, and then, send this delayed information $\hat{x}_j(t-\tau_{u_j})$ to agent i via communication topology edge (i, j) which has communication delay $\tau_{c_{ij}}$. $\hat{x}_i(t-\tau_{u_i}-\tau_{c_{ij}})$ represents the observer $\hat{x}_i(t)$ of agent i should be self delayed by $\tau_{u_i}+\tau_{c_{ij}}$.*

There exists $\tau_{c_{ij}}$ to agent i from its neighbor j in communication link (i, j). Hence, based on the definition of τ_{c_i}, we can even retard the communicated value as follows:

$$\begin{aligned}\dot{\hat{x}}_i(t) &= A\hat{x}_i(t) + Bu_i(t) + L\big\{\sum_{j=1,j\neq i}^{N} a_{ij}\big[y_i(t-\tau_{c_i})\\ &- y_j(t-\tau_{c_i}) - C\hat{x}_i(t-\tau_{u_i}-\tau_{c_i}) + C\hat{x}_j(t-\tau_{u_j}-\tau_{c_i})\big]\\ &+ a_{i0}\big[y_i(t-\tau_{c_i}) - y_0(t-\tau_{c_i}) - C\hat{x}_i(t-\tau_{u_i}-\tau_{c_i})\big]\big\}.\end{aligned} \tag{7}$$

Remark 3. *Compared to error dynamics* (5)*, there exists no input delay in predictive observer* (7)*. As there is no integral term inside* (7)*, the predictive observer calculation is not time-consuming. Thanks to the adoption of stored historical values, this predictive observer $\hat{x}_i(t)$ can predict $\tilde{x}_i(t)$ with τ_{u_i} delay in advance, i.e., $\hat{x}_i(t) \to \tilde{x}_i(t+\tau_{u_i})$.*

Inspired by [19], denote the observer estimating error $\xi_i(t) \in \mathbb{R}^n$ as

$$\xi_i(t) = \tilde{x}_i(t) - \hat{x}_i(t-\tau_{u_i}), i \in \mathbf{I}_1^N. \tag{8}$$

Then, (7) changes to $\dot{\hat{x}}_i(t) = A\hat{x}_i(t) + Bu_i(t) + LC\sum_{j=1}^N l_{ij}\xi_j(t-\tau_{c_i})$, where $\xi_j(t) = \tilde{x}_j(t) - \hat{x}_j(t-\tau_{u_j})$ is used in the middle of calculation. Combining the above $\hat{x}_i(t)$ dynamics with Eqs. (4) and (5), the dynamics $\xi_i(t)$ is

$$\dot{\xi}_i(t) = A\xi_i(t) - LC\sum_{j=1}^{N} l_{ij}\xi_j(t-\tau_i). \tag{9}$$

Now, the control input is chosen to be of the form as

$$u_i(t) = -K\hat{x}_i(t), i \in \mathbf{I}_1^N, \tag{10}$$

where matrix $K \in \mathbb{R}^{p\times n}$ will be designed later.

Based on $\xi_i(t)$ in (8), integrating input (10) into consensus tracking error dynamics (5) gives

$$\dot{\tilde{x}}_i(t) = (A - BK)\tilde{x}_i(t) + BK\xi_i(t). \tag{11}$$

B. Necessary conditions for stability

When there is no delays, error dynamics (9) changes to

$$\dot{\xi}_i(t) = A\xi_i(t) - LC\sum_{j=1}^{N} l_{ij}\xi_j(t). \tag{12}$$

Denote $\xi(t) = [\xi_1^T(t), \ldots, \xi_N^T(t)]^T$. The Kronecker format of (12) is $\dot{\xi}(t) = (I_N \otimes A - \mathcal{L}_1 \otimes (LC))\xi(t)$. So in order to get $\lim_{t\to\infty}\xi(t) = 0$, the observer parameter L should satisfy that all eigenvalues of $I_N \otimes A - \mathcal{L}_1 \otimes (LC)$ have negative real parts, i.e.,

$$\mathrm{Re}(\lambda(I_N \otimes A - \mathcal{L}_1 \otimes (LC))) < 0. \tag{13}$$

Hence, $\lim_{t\to\infty}\tilde{x}_i(t) = 0$ in (11) if parameter K satisfies

$$\mathrm{Re}(\lambda(A - BK)) < 0. \tag{14}$$

Remark 4. *Eqs.* (13) *and* (14) *are necessary conditions for the asymptotic convergence of consensus tracking error.*

The following sections will focus on proving the stability of observer error dynamics (9) and parameter L design.

C. Observer parameter law for stability

Due to the heterogeneous nature of delays considered in this paper, $\dot{\xi}_i(t)$ in (9) cannot have the Kronecker form as in [1], [4] in the time domain. We, therefore, switch to the use of a frequency domain method to deal with τ_i. After taking the Laplace transform of both sides of (9), we have

$$s\xi_i(s) - \xi_i(0) = A\xi_i(s) - LC\sum_{j=1}^{N} l_{ij}e^{-s\tau_i}\xi_j(s), \tag{15}$$

where $\xi_i(s)$ denotes the Laplace transform of $\xi_i(t)$ for all $i \in \mathbf{I}_1^N$. From Assumption 1, $(sI - A)$ is invertible. Therefore, (15) can be written as

$$\xi_i(s) + e^{-s\tau_i}(sI - A)^{-1}LC\sum_{j=1}^{N} l_{ij}\xi_j(s) = (sI - A)^{-1}\xi_i(0).$$

In (4), $\tau_i = \tau_{u_i} + \tau_{c_i}$ means each agent can have its own input and communication delays. Here, the largest delay (i.e., $\bar{\tau} = \max_{i\in \mathbf{I}_1^N} \tau_i$) that every agent can tolerate is of interest in this work. Thus, the above equation can change to

$$(I_{Nn} + \mathcal{L}_1 \otimes \underbrace{(e^{-s\bar{\tau}}(sI - A)^{-1}LC)}_{Q(s)})\xi(s) = (I_N \otimes (sI - A)^{-1})\xi(0). \quad (16)$$

The above closed-loop system is stable if $\det(I_{Nn} + \mathcal{L}_1 \otimes Q(s))$ has no zero in the closed right-half plane. From (16) one has $\xi(s) = (I_{Nn} + \mathcal{L}_1 \otimes Q(s))^{-1}(I_N \otimes (sI - A)^{-1})\xi(0)$. To achieve $\xi(s) \to 0$, due to $\xi(0) \neq 0$, the right hand side of (16) is the reason for Assumption 1 ($\mathrm{Re}(\lambda(A)) < 0$) which will be relaxed in the next section.

The computation of the roots of (16) is extremely difficult because matrix $Q(s)$ depends on s. Instead of computing these roots, we use the generalized Nyquist stability criterion to determine the stability of $\xi(t)$ and then, asymptotic convergence of $\tilde{x}(t)$.

Theorem 1. *Under Assumptions 1-2, Problem 1 is solved by controller* (10) *and observer* (7) *if* $A - BK$ *is Hurwitz,* L *satisfies Re*$(\lambda(I_N \otimes A - \mathcal{L}_1 \otimes (LC))) < 0$ *and*

$$\rho(\underbrace{(j\omega I - A)^{-1}LC}_{\hat{G}(j\omega)}) < \frac{1}{\lambda_{\max}(\mathcal{L}_1)}, \omega \in \mathbb{R}. \quad (17)$$

Proof. From the generalized Nyquist criterion, based on Assumption 1, it is sufficient to verify that the eigenvalues of $\mathcal{L}_1 \otimes Q(j\omega)$,[2] $\omega \in \mathbb{R}$, do not encircle the point -1 to guarantee the stability of the closed-loop system (16), which means (16) is stable if

$$\rho(\mathcal{L}_1 \otimes Q(j\omega)) = \max |\lambda(\mathcal{L}_1 \otimes Q(j\omega))| < 1, \omega \in \mathbb{R}. \quad (18)$$

Note that $\mathcal{L}_1$ is a nonsingular M-matrix based on Assumption 2 with $\mathrm{Re}(\lambda(\mathcal{L}_1)) > 0$. Recall the eigenvalue property that if the eigenvalues of $S \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ are $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$, respectively, then the eigenvalues of $S \otimes R$ are $\lambda_i\mu_k, i \in \mathbf{I}_1^n, k \in \mathbf{I}_1^m$. As a consequence, we have

$$\begin{aligned}\rho(\mathcal{L}_1 \otimes Q(j\omega)) =& \lambda_{\max}(\mathcal{L}_1) \max |\lambda(Q(j\omega))| \\ =& \lambda_{\max}(\mathcal{L}_1)\rho(Q(j\omega)), w \in \mathbb{R}.\end{aligned} \quad (19)$$

Due to $|e^{-j\omega\bar{\tau}}|_{\omega\in\mathbb{R}} = 1, \forall \bar{\tau} > 0$, based on (18), the closed-loop system $\xi_i(t)$ (9) is stable for arbitrary delay $\bar{\tau}$ under condition (17).

From the result in Section III-B, one can see that zero is the equilibrium of $\xi_i(t)$ dynamics if condition (13) is satisfied. As a result, $\lim_{t\to\infty} \xi_i(t) = 0$ can be achieved under conditions (13) and (17). In addition to that, from (11), one can prove $\lim_{t\to\infty} \tilde{x}_i(t) = 0$ with condition (14). ■

Remark 5. *Controller parameters in Theorem 1 are delay-independent under directed communication graphs. However, parameters in [10], [12], [14], [16] are delay-dependent, e.g., they are inverse proportional to the upper bound of delays in [10], [16] meaning delays cannot be arbitrarily large. Besides, works in [7], [8], [11] require undirected graphs.*

[2]We denote the imaginary unit $j^2 = -1$ for distinguishing j from the index j of the agents.

D. Relaxation of Assumption 1

Our results so far required that the state matrix A is already stable (Assumption 1), which is restrictive. However, for relaxing this assumption, we need the following assumptions:

Assumption 3. *The output matrix* C *is full rank.*

Assumption 4. (A, B) *is controllable.*

Based on Assumption 3 (i.e., C is full rank), Assumption 1 can be relaxed to Assumption 4 now. The motivation behind this is that several real-world scenarios may involve system dynamics A that are open-loop unstable (e.g., fight aircrafts).

The methodology is to replace matrix A in (16) by $(A - BK)$ which is Hurwitz. By adding a state predictor $x_i(t+\tau_{u_i})$ in the following, the observer (7) is modified as follows:

$$\begin{aligned}\dot{\hat{x}}_i(t) =& A\hat{x}_i(t) + Bu_i(t) + L\{\sum_{j=1,j\neq i}^{N} a_{ij}[y_i(t - \tau_{c_i}) \\ &- y_j(t - \tau_{c_i}) - C\hat{x}_i(t - \tau_{u_i} - \tau_{c_i}) + C\hat{x}_j(t - \tau_{u_j} - \tau_{c_i})] \\ &+ a_{i0}[y_i(t - \tau_{c_i}) - y_0(t - \tau_{c_i}) - C\hat{x}_i(t - \tau_{u_i} - \tau_{c_i})]\} \\ &+ BK[x_i(t + \tau_{u_i}) - e^{A\tau_{u_i}}x_0(t) - \hat{x}_i(t)],\end{aligned} \quad (20)$$

where $x_i(t + \tau_{u_i})$ can be computed by

$$x_i(t + \tau_{u_i}) = e^{A\tau_{u_i}}x_i(t) + \int_{t-\tau_{u_i}}^{t} e^{A(t-s)}Bu_i(s)ds.$$

Since the input delay τ_{u_i} is known to agent i, the designed observer (20) is available with historical values of input $u_i(t), t \in [t - \tau_{u_i}, t]$. Assumption 3 can guarantee the feasibility of observer (20) in which both state and output measurements are used.

By designing the same input $u_i(t)$ in (10), based on $e^{A\tau_{u_i}}x_0(t - \tau_{u_i}) = x_0(t)$ from (2), Eqs. (4), (5) and (20), one can calculate that $\xi_i(t)$ in (9) will change to

$$\dot{\xi}_i(t) = (A - BK)\xi_i(t) - LC\sum_{j=1}^{N} l_{ij}\xi_j(t - \tau_i). \quad (21)$$

Consequentely, necessary conditions for the asymptotic convergence of consensus tracking error $\tilde{x}(t)$ will be (14) and

$$\mathrm{Re}(\lambda(I_N \otimes (A - BK) - \mathcal{L}_1 \otimes (LC))) < 0. \quad (22)$$

Similarly, (16) turns to

$$\begin{aligned}&\big(I_{Nn} + \mathcal{L}_1 \otimes (e^{-s\bar{\tau}}(sI - (A - BK))^{-1}LC)\big)\xi(s) \\ &\qquad = (I_N \otimes (sI - (A - BK))^{-1})\xi(0).\end{aligned} \quad (23)$$

Theorem 2. *Under Assumptions 2-4, Problem 1 is solved by controller* (10) *and observer* (20) *if* $A - BK$ *is Hurwitz,* L *satisfies* $\mathrm{Re}(\lambda(I_N \otimes (A - BK) - \mathcal{L}_1 \otimes (LC))) < 0$ *and*

$$\rho((j\omega I - (A - BK))^{-1}LC) < \frac{1}{\lambda_{\max}(\mathcal{L}_1)}, \omega \in \mathbb{R}. \quad (24)$$

Proof. In Theorem 1, $\mathrm{Re}(\lambda(A)) < 0$, which is guaranteed by Assumption 1, is needed. In Theorem 2, similarly, $A - BK$ needs to be Hurwitz, i.e., $\mathrm{Re}(\lambda(A - BK)) < 0$, which is guaranteed by Assumption 4. Other calculations are the same as in the proof of Theorem 1 and, thus, omitted here. ■

E. Algorithms to design observer parameter L

Theorems 1 and 2 guarantee that the consensus tracking can be achieved if conditions (17) and (24) are satisfied. However, it is not easy to design L to satisfy conditions (17) or (24) since the value space for ω is $[-\infty, +\infty]$. From the necessary condition (13) for stability of $\xi(t)$, since A is negative definite from Assumption 1 and $\mathcal{L}_1$ is positive definite from Assumption 2, the intuition is that if we design the term LC being non-negative definite, then condition (13) has high possibility to be satisfied. So here, we propose one solution of L as

$$L = \alpha C^T, \alpha > 0. \tag{25}$$

Thus, the objective is simplified to design α.

1) Design L by blind search: We take condition (17) for designing L first. By intuition, when $\omega \to \infty$, $\rho((j\omega I - A)^{-1})$ will go to zero. Let ω_s be a relatively small sample frequency and ϵ a sufficiently small non-negative scalar. We propose Algorithm 1 to find α by blind search.

Algorithm 1 Choice of parameter α using blind search

Input: A, C, $\Delta\omega$ and ϵ

initialization Set $i = 0$ and $k = 0$
$\omega_s = \Delta\omega$, $\rho_0 = \rho((-A)^{-1}C^TC)$ and $\rho_1^+ = \rho((j\omega_s I - A)^{-1}C^TC)$
while not $|\rho_{i+1} - \rho_i| \leqslant \epsilon$ **do**
  $\rho_{i+2}^+ = \rho((j(i+2)\omega_s I - A)^{-1}C^TC)$
  $i = i + 1$
end while
$\omega_s = -\Delta\omega$ and $\rho_1^- = \rho((j\omega_s I - A)^{-1}C^TC)$
while not $|\rho_{k+1} - \rho_k| \leqslant \epsilon$ **do**
  $\rho_{k+2}^- = \rho((j(k+2)\omega_s I - A)^{-1}C^TC)$
  $k = k + 1$
end while
$i^* = i + 1$ and $k^* = k + 1$
$\rho^* = \max(\rho_{k^*}^-, ..., \rho_1^-, \rho_0, \rho_1^+, ..., \rho_{i^*}^+)$
return $\alpha < 1/(\rho^* \lambda_{\max}(\mathcal{L}_1))$

Algorithm 1 uses a numerical blind search method to obtain the maximum value ρ^* of $\rho((j\omega I - A)^{-1}C^TC)$ over the frequency domain from $\omega = 0$ until its convergence at $i^*\Delta\omega$ and $-k^*\Delta\omega$. Note that when $\omega_s > 0$, ω goes from 0 to $+\infty$; when $\omega_s < 0$, ω goes from 0 to $-\infty$. In such way, $\omega \in \mathbb{R}$ is searched. Since $L = \alpha C^T$, the choice $\alpha < 1/(\rho^* \lambda_{\max}(\mathcal{L}_1))$ should be made to satisfy the condition (13). To design L for condition (24), we replace matrix A as $A - BK$ in Algorithm 1.

2) Design L for a monotone system: In (17), $\hat{G}(j\omega) := (j\omega I - A)^{-1}LC$ is the complex matrix valued function of $G := (sI - A)^{-1}LC$. One can see that G is the Laplacian transform function of

$$\dot{z}(t) = Az(t) + LCu_z(t), \quad \hat{y}(t) = z(t), \tag{26}$$

where $z(t) \in \mathbb{R}^n, u_z(t) \in \mathbb{R}^n$ and $\hat{y}(t) \in \mathbb{R}^n$ are the corresponding state, input and output. Now, before we cite some lemmas related to dynamics (26), the following notations are presented. $L_p(a, b), p \in \mathbb{N}$ represents the space of functions $\phi : (a, b) \to \mathbb{R}^n$ with the norm $\|\phi\|_{L_p} = [\int_a^b |\phi(\theta)|^p \mathrm{d}\theta]^{1/p}$. Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a proper cone[3] and $\mathcal{H}_\infty$ be the Hardy space. $L_2^{\mathcal{K}}[0, \infty)$ is a cone in Lebesgue 2-space $L_2[0, \infty)$ as

$$L_2^{\mathcal{K}}[0, \infty) := \{v \in L_2[0, \infty) : \ v(t) \in \mathcal{K}\}.$$

Define a space of maps preserving $L_2^{\mathcal{K}}[0, \infty)$ by

$$\mathcal{H}_\infty^{\mathcal{K}} := \{G \in \mathcal{H}_\infty : \ Gu \in L_2^{\mathcal{K}}[0, \infty), \ \forall \ u \in L_2^{\mathcal{K}}[0, \infty)\}.$$

More explanations about above notations can refer to [20]. Note any system operator $G \in \mathcal{H}_\infty^{\mathcal{K}}$ is considered as a cone-preserving operator, i.e., it maps any input signal in the cone space $L_2^{\mathcal{K}}[0, \infty)$ to obtain the system state in the same space.

Lemma 1 ([20]). *Let $\mathcal{K} \subseteq \mathbb{R}^n$ be a proper cone. If $G \in \mathcal{H}_\infty^{\mathcal{K}}$, then $\rho(\hat{G}(j\omega)) \leq \rho(\hat{G}(0)), \forall \omega \in \mathbb{R}$.*

Lemma 2 ([20]). *The class of monotone dynamic systems has cone-preserving property, i.e., $G \in \mathcal{H}_\infty^{\mathcal{K}}$.*

Using the above lemmas, the following theorem provides the choice of α for specific class of systems.

Theorem 3. *Under Assumptions 1-2, if A is a Metzler[4] matrix and L is set as (25), dynamics (26) constitutes a monotone system. To satisfy law (17), α is designed as*

$$\alpha < \frac{1}{\lambda_{\max}(\mathcal{L}_1) \max |\lambda(A^{-1}C^TC)|}. \tag{27}$$

Proof. Since A is a Metzler matrix and $LC = \alpha C^TC$ is a square entry-wise nonnegative matrix, the system is a positive system (see the mathematical definition in Introduction Section of [20]), which is a special monotone system. According to Lemmas 1 and 2, we can furthermore have

$$\rho(\hat{G}(j\omega)) = \rho(\hat{G}(0)) = \max |\lambda(A^{-1}LC)|, \forall \omega \in \mathbb{R}, \tag{28}$$

from condition (17), which hence yields the choice (27). ■

Proposition 1. *Under Assumptions 2-4, if $A - BK$ is Hurwitz and is a Metzler matrix and L is set as (25), then, to satisfy condition (24), α is designed as*

$$\alpha < \frac{1}{\lambda_{\max}(\mathcal{L}_1) \max |\lambda((A - BK)^{-1}C^TC)|}. \tag{29}$$

Theorem 3 and Proposition 1 can guarantee the existence of L. To sum up, the following algorithm is proposed to design parameters K and L to solve Problem 1.

Algorithm 2 Parameters design procedure

1. Design K to satisfy $\mathrm{Re}(\lambda(A - BK)) < 0$ (14).
2. Design $L = \alpha C^T$ with α satisfying the choice obtained by Algorithm 1 or for monotone systems, by (27) for Theorem 1 and by (29) for Theorem 2.
3. Verify L to satisfy (13) for Theorem 1 or to satisfy (22) for Theorem 2.

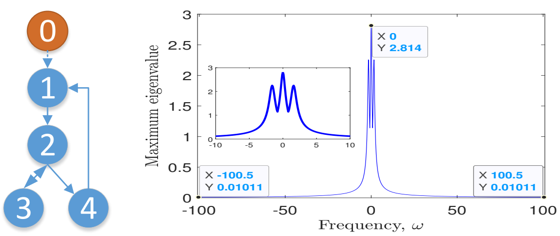
IV. NUMERICAL EXAMPLE

The vehicle platooning scenario is considered here. From [21], the linearized vehicle platooning dynamics is $A = \begin{bmatrix} 0_{2\times 1} & I_2 \\ 0 & [0 \ \ -1/\tau_{pt}] \end{bmatrix}, B = \begin{bmatrix} 0_{2\times 1} \\ 1/\tau_{pt} \end{bmatrix}$, where $x_i(t) = [p_i(t), v_i(t), a_i(t)]^T \in \mathbb{R}^3$ are the position, velocity, and acceleration, respectively; $\tau_{pt} = 0.5s$ is the inertial time lag in

[3]A cone K is said to be proper if it is convex ($x, y \in K; \beta, \gamma \geq 0 \Rightarrow \beta x + \gamma y \in K$), pointed ($(K \cap (-K) = \{0\})$, closed and solid [20].

[4]A matrix $A \in \mathbb{R}^{n\times n}$ is called Metzler if every off-diagonal entry of A is non-negative.

the powertrain. A is a Metzler matrix with eigenvalues $\lambda(A) = 0, 0, -1/\tau_{pt}$, which means Assumption 1 is not satisfied. By setting $C = I_3$, Assumptions 3 and 4 are satisfied. Therefore, this example verifies the results for open-loop unstable systems. The string stability in platooning is not discussed here due to the page limit and research topic. So the time headway is set as zero and the average distance between vehicles is set as $D = 10m$. Correspondingly, the observer (20) for Theorem 2 is changed, i.e., replacing $y_i(t-\tau_{c_i})$ as $y_i(t-\tau_{c_i}) + C[iD, 0, 0]^T$, $y_j(t-\tau_{c_i})$ as $y_j(t-\tau_{c_i}) + C[jD, 0, 0]^T$ and $x_i(t+\tau_{u_i})$ as $x_i(t+\tau_{u_i}) + [iD, 0, 0]^T$. The line formation error is $p_i(t) - p_0(t) + iD$. The communication graph is shown in Fig. 1 (a). Design $K = [0.4843, 1.5205, -0.3790]$ such that $A - BK$ is Hurwitz. Parameters in Algorithm 1 for condition (24) are shown in Fig. 1 (b) with $\alpha < 1/(2.814 \times \rho(\mathcal{L}_1)) = 0.1425$. Set the input/communication delays as $[\tau_{u_1}, \tau_{u_2}, \tau_{u_3}, \tau_{u_4}] = [0.55s, 0.65s, 0.75s, 0.85s]$ and $[\tau_{c_1}, \tau_{c_2}, \tau_{c_3}, \tau_{c_4}] = [1.5s, 20s, 30s, 40s]$. Fig 2 shows performance of different values of α to the line formation of vehicle platooning, which verifies the results in this paper.



(a) (b) $\rho((j\omega I - A)^{-1}C^TC), \omega \in \mathbb{R}$.

Fig. 1. (a) Directed communication graph; (b) Algorithm 1: $\Delta\omega = 0.1, \epsilon = 0.00001$, $i^* = 1017, k^* = 1017$.

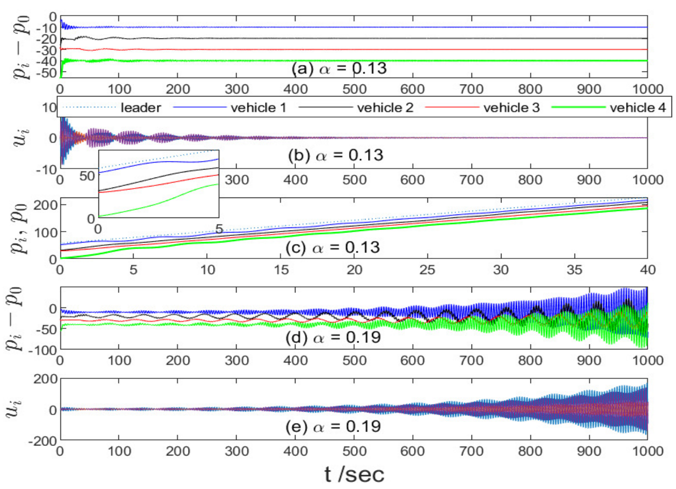


Fig. 2. (a) and (d) are the distance between vehicle $i, i \in \mathbf{I}_1^4$ and the leader vehicle with different values of α and (b) and (e) are their corresponding inputs. (c) is the position trajectories of five vehicles. One can see when $\alpha = 0.13 < 0.1425$, $p_i - p_0 = -iD$, i.e., the line formation of vehicle platooning is achieved and stable while $\alpha = 0.19 > 0.1425$ is not.

V. Conclusions

In this work, we proposed a distributed observer which can handle arbitrary heterogeneous (input and communication) delays during consensus tracking in multi-agent systems with general linear dynamics and under a directed graph representing the communication topology. The design of this observer was transformed to a single parameter design which is chosen such that the stability of the close-loop system is guaranteed. Two algorithms were proposed for choosing this parameter: one for general linear systems and one for monotone systems.

Ongoing research focuses on relaxing the constraint requiring each follower to possess the leader's state for designing its observer for open-loop unstable systems with/without uncertainties.

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