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SPARSE LOW-REDUNDANCY LINEAR ARRAY WITH UNIFORM SUM CO-ARRAY

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ABSTRACT

Sparse arrays can resolve vastly more scatterers than the number of sensors in tasks such as coherent source localization. This entails significant cost reductions compared to conventional arrays with uniformly spaced elements. In this paper, we introduce a parametric sparse linear array configuration called the Kløve array (KA). The KA has a contiguous sum and difference co-array, making it suitable for both active and passive sensing. We show that KAs of any size can have both low redundancy, and few closely spaced elements. This may improve robustness in the face of mutual coupling.[1]

Index Terms— Sparse arrays, sum co-array, active sensing, coherent imaging, mutual coupling

1. INTRODUCTION

Sensor arrays are a key technology routinely used in applications such as, radar, medical imaging, sonar, and many more [1]. However, the number of sensors N in a conventional *uniform linear arrays* (ULAs) grows proportionally to the desired electrical aperture. In contrast, sparse arrays only require $\mathcal{O}(\sqrt{N})$ sensors, which may significantly reduce costs of large arrays, as fewer costly RF-IF front ends are needed.

Sparse arrays can achieve comparable performance to filled arrays of equivalent aperture by utilizing a virtual array structure called the *co-array* [2]. This structure typically consists of the pairwise differences or sums of sensor positions. The co-array determines the *degrees of freedom* (DOF) available for many array processing applications, such as direction-of-arrival estimation [3] and coherent imaging [2].

Sparse array design commonly focuses on finding array configurations with a *contiguous* difference or sum co-array, and a minimal number of physical elements. A contiguous co-array maximizes the number of DOF for a given aperture, and facilitates the use of efficient array processing algorithms [3]. Arrays are often also designed to have few closely spaced sensors. This may improve performance in the presence of mutual coupling [4–7], since the coupling magnitude is usually inversely proportional to the inter-element distance [8].

The array configuration maximizing the size of the contiguous co-array, subject to a given number of elements, is called the *Minimum-redundancy array* (MRA) [9, 10]. MRAs are challenging to compute, as the search space of the associated optimization problem grows exponentially with the array size. Consequently, practically realizable large sparse arrays often have sub-optimal redundancy. One approach to designing such arrays is to extend known MRAs to larger apertures, as is done by *recursive arrays* [11–13], and *reduced redundancy arrays* [10, 11]. Another option is to find parametric designs based on the properties of the desired co-array. Examples include *nested* [3, 14] and *Wichmann* arrays [15, 16]. Parametric configurations often scale well for large apertures, since they can be easily optimized, e.g., for redundancy.

This paper introduces the *Kløve array* (KA), which is a parametric array configuration based on a class of restricted additive 2-bases originally studied in number theory [17]. The KA has a low redundancy and its element positions are given in closed form. This enables constructing large sparse arrays using very few elements. Furthermore, the KA is symmetric, with a contiguous sum and difference co-array.

We study in detail a subclass of KAs that have a low redundancy and a small constant number of closely spaced elements. For very large apertures, this *constant unit spacing KA* (KA_S) only has around 2.15% more elements than the *minimum-redundancy KA* (KA_R). However, unlike the KA_R and many other sparse array configurations, the number of closely spaced elements in the KA_S is independent of the number of array elements. This suggests that the KA_S may be more robust in the face of mutual coupling.

Notation: $\otimes$ and $\odot$ denote the Kronecker, and Khatri-Rao (column-wise Kronecker) products. Set $\{a : b : c\}$, denotes the integer line from a to c sampled in steps of $b \in \mathbb{N}_+$. When $b = 1$, we use shorthand $\{a : c\}$.

2. PRELIMINARIES

In this section, we introduce the coherent signal model giving rise to the sum co-array. We then briefly review two relevant array figures of merit, and two well-known sparse array configurations before our main results in sections 3 and 4.

2.1. Signal model and sum co-array

We consider estimating the angular directions $\{\varphi_k \in [-\pi/2, \pi/2]\}_{k=1}^K$ of K *coherent* far field scatterers with reflectivities

[1] This work was supported by the Academy of Finland project "Massive and Sparse Antenna Array Processing for mmWave Communications".

$\{\gamma_k \in \mathbb{C}\}_{k=1}^K$ using a linear array of N transceiving elements. Each transmitter illuminates the scattering scene using narrowband radiation in a sequential or simultaneous (orthogonal waveform MIMO) manner within the coherence time of the scene [18, 19]. The vectorized received signal vector after matched filtering then becomes [20]

$$\mathbf{y} = (\mathbf{A} \odot \mathbf{A})\boldsymbol{\gamma} + \mathbf{n}, \tag{1}$$

where $\mathbf{A} \in \mathbb{C}^{N\times K}$ is the array steering matrix, $\boldsymbol{\gamma} = [\gamma_1, \dots, \gamma_K]^{\mathrm{T}} \in \mathbb{C}^K$ the scattering coefficient vector, and $\mathbf{n} \in \mathbb{C}^{N^2}$ a noise vector.

We may interpret (1) as a virtual array with the manifold $\mathbf{A}_{\text{kr}} = \mathbf{A} \odot \mathbf{A}$ receiving signal $\boldsymbol{\gamma}$. Assuming omnidirectional elements free of mutual coupling, the (nm, k)th element of matrix $\mathbf{A}_{\text{kr}}$ becomes $[\mathbf{A}_{\text{kr}}]_{(n-1)N+m,k} = A_{n,k}A_{m,k} = e^{j2\pi \sin\varphi_k (d_n+d_m)/\lambda}$. Here, the physical element positions are given by set $\mathcal{D} = \{d_n\}_{n=1}^N \subseteq \mathbb{R}$, whereas the virtual elements of $\mathbf{A}_{\text{kr}}$ are supported on the sum co-array $\mathcal{D}_\Sigma = \mathcal{D} + \mathcal{D} = \{d_n + d_m \mid d_n, d_m \in \mathcal{D}\}$. Set $\mathcal{D}_\Sigma$ may have up to $N(N+1)/2$ unique elements. This implies that $\mathcal{O}(N^2)$ fully coherent scatterers can be resolved, for instance, using co-array MUSIC [3, 21] on the sum co-array. Note that $\mathcal{O}(N^4)$ *incoherent* scatterers could be resolved by leveraging second-order statistics and the *difference of the sum co-array* [20, 22].

We assume that any two elements are separated by a multiple of the unit inter-element spacing (typically half a carrier wavelength). For simplicity, we restrict $\mathcal{D} \supseteq \{0\}$ to the set of non-negative integers $\mathcal{D} \subseteq \mathbb{N}$ in the remainder of the paper.

2.2. Array figures of merit

We focus on two central figures of merit for sparse arrays, namely *redundancy* and *d-spacing multiplicity*. Redundancy, R quantifies the degree of repetition in the co-array. It is defined as the maximum number of unique pairwise sums that can be generated by $|\mathcal{D}| = N$ elements, divided by the number of unique elements in the sum co-array (which is contiguous).

Definition 1 (Redundancy [10])**.** *The redundancy of array $\mathcal{D}$ with a contiguous sum co-array is* $R = \frac{|\mathcal{D}|(|\mathcal{D}|+1)/2}{2\max\mathcal{D}+1} \geq 1$.

The smaller R is, the fewer elements the array has relative to its aperture $\max\mathcal{D}$. Since R is a function of the array size, it is often convenient to compute the asymptotical redundancy $R_\infty = \lim_{|\mathcal{D}|\to\infty} |\mathcal{D}|^2/(4\max\mathcal{D})$, which is a constant.

The d-spacing multiplicity, $S(d)$, enumerates the number of inter-element spacings of a given length d in the array. For linear arrays, this simplifies to the *weight* or *multiplicity function* of the difference co-array [3, 6].

Definition 2 (d-spacing multiplicity)**.** *The d-spacing multiplicity of array $\mathcal{D}$ is* $S(d) = \mathbb{1}(d_n - d_m = d \mid d_n, d_m \in \mathcal{D})$.

Mutual coupling may be mitigated by decreasing S for small d [4, 5, 7], as elements that are closer to each other typically interact more strongly [8]. The *number of unit spacings* $S(1)$ is therefore often used as a proxy for sensitivity to mutual coupling. This simplifies array design, but has its limitations, since it neglects important factors impacting coupling, such as the element gain patterns and the mounting platform.

2.3. Two sparse array configurations

We briefly review two well-established sparse array configurations with a contiguous sum co-array. The first is the *Minimum-redundancy array* (MRA), which maximizes the array aperture for a given number of elements N. This is equivalent to minimizing the redundancy of the array.

Definition 3 (Minimum-redundancy array (MRA))**.** *The element positions of the MRA are found from the solution to*

$$\underset{\mathcal{D}}{\text{maximize}} \max\mathcal{D} \text{ s.t. } |\mathcal{D}| = N; \mathcal{D}+\mathcal{D} = \{0 : 2\max\mathcal{D}\}. \tag{P1}$$

Solutions to (P1) are only known for $N \leq 48$ [23, 24], due to the challenging combinatorial nature of the problem. For larger arrays, sub-optimal solutions need to be considered. The *Concatenated nested array* (CNA) [14] is such an array configuration that can be generated for any N. The CNA consists of two dense ULAs separated by a sparse ULA.

Definition 4 (Concatenated nested array (CNA) [14])**.** *The element positions of the CNA with $N_1, N_2 \in \mathbb{N}$ are given by*

$$\mathcal{D}_{\text{CNA}} = \mathcal{D}_1 \cup (\mathcal{D}_2 + N_1) \cup (\mathcal{D}_1 + N_2(N_1+1)),$$

where $\mathcal{D}_1 = \{0 : N_1 - 1\}$, $\mathcal{D}_2 = \{0 : N_1+1 : (N_2(N_1+1)-1)\}$.

The CNA actually coincides with a *restricted additive 2-basis* studied by H. Rohrbach in the 1930's [25, Satz 2]. Unaware of this, we proposed the CNA much later in [14] as an extension to the *Nested array* (NA) [3]. Unlike the NA, which only has a contiguous difference co-array, the symmetry of the CNA ensures that also its sum co-array is contiguous.

3. KLøVE ARRAY

In this section, we present the Kløve array (KA), which is based on a class of restricted additive 2-bases proposed by T. Kløve in the context of additive combinatorics [17]. The KA has a nested structure, consisting of two CNAs that are connected by a sparse mid-section composed of N_3 widely separated and sub-sampled ULAs (Fig. 1). The spacing of these ULAs ensures that the sum co-array of the resulting KA is contiguous and as large as possible.

Definition 5 (Kløve array (KA) [17])**.** *The set of element positions of the KA with parameters $N_1, N_2, N_3 \in \mathbb{N}$ is*

$$\begin{aligned}\mathcal{D}_{\text{KA}} = \mathcal{D}_{\text{CNA}} &\cup (\mathcal{D}_3 + 2\max\mathcal{D}_{\text{CNA}} + 1)\\ &\cup (\mathcal{D}_{\text{CNA}} + 3\max\mathcal{D}_{\text{CNA}} + \max\mathcal{D}_3 + 2)\end{aligned}$$

where $\mathcal{D}_3 = \{0 : N_1 : N_1^2\} + \bigcup_{i=1}^{N_3}\{(i-1)(N_1^2 + \max\mathcal{D}_{\text{CNA}} + 1)\}$, *and $\mathcal{D}_{\text{CNA}}$ is given by Definition 4.*

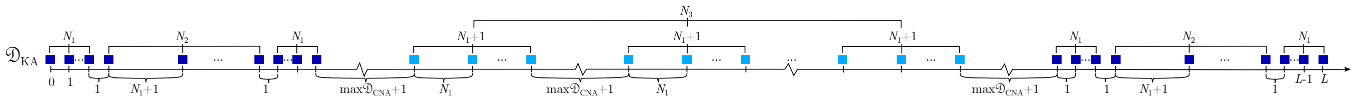

Fig. 1. Kløve array (KA). The KA consist of two identical CNAs (with parameters N_1, N_2) and N_3 undersampled ULAs.

The KA reduces to the CNA if $N_1 = 0$ or $N_2 = 0$. When $N_2 \geq 1$, the aperture L, and the number of elements N of the KA evaluate to [17, Lemma 3]:

$$L = (N_1 + 1)(N_3(N_1 + N_2) + 3N_2 + 3) - 5 \qquad (2)$$
$$N = 2(2N_1 + N_2) + N_3(N_1 + 1). \qquad (3)$$

Furthermore, using Fig. 1, we may establish that the number of unit spacings $S(1)$ of the KA is

$$S(1) = \begin{cases} N_3 + 1, & \text{when } N_1 = 0 \text{ and } N_2 = 1 \\ 2(N_2 - 1), & \text{when } N_1 = 0 \text{ and } N_2 \geq 2 \\ N_3 + 4, & \text{when } N_1 = 1 \\ 4N_1, & \text{when } N_1 \geq 2. \end{cases} \qquad (4)$$

The KA also has a contiguous sum and difference co-array.

Proposition 1 (Co-array of KA)**.** *The KA has a contiguous sum and difference co-array.*

Proof. The sum co-array is contiguous, since $|\mathcal{D}_{\mathrm{KA}} + \mathcal{D}_{\mathrm{KA}}| - 1 = 2L$ by [17, Lemma 3]. Because the KA is symmetric, also the difference co-array is contiguous [6, Lemma 1]. □

3.1. Minimum-redundancy Kløve array

The KA can achieve an asymptotic redundancy as low as $R_\infty = 23/12 \approx 1.92$ for appropriate choices of its parameters [17, Theorem]. This is in fact the lowest redundancy of any currently known parametric sparse array with a contiguous sum co-array. A drawback of this *minimum-redundancy* KA (KA_R) is that the number of unit spacings $S(1)$ grows linearly with the number of elements N (see Section 4). However, at the expense of a slightly higher redundancy, we may find a KA with an $S(1)$ that is small and independent of N.

3.2. Constant unit spacing Kløve array

We now propose a *constrained* minimum-redundancy KA, which maximizes the array aperture, subject to achieving a constant number of unit spacings. This *constant unit spacing* KA (KA_S) has the lowest $S(1)$ among all sufficiently large KAs ($N_1 \geq 2$) that achieve aperture $L = \mathcal{O}(N^2)$.

In deriving the KA_S, we first verify from (2) and (4) that $N_1 \geq 2$ is a necessary condition for the KA to satisfy $L = \mathcal{O}(N^2)$, when $S(1)$ is fixed. It follows from (4) that $S(1) = 4N_1$. Since $N_1 \geq 2$, the lowest value $S(1)$ can attain is then $S(1) = 8$. In order to minimize the redundancy of this array, we maximize the aperture in (2), subject to $N_1 = 2$ and a fixed N in (3). This yields the optimization problem

$$\underset{N_2 \in \mathbb{N}_+, N_3 \in \mathbb{N}}{\text{maximize}} \; 9N_2 + 3N_3(N_2 + 2) \text{ s.t. } 2N_2 + 3N_3 + 8 = N. \text{(P2)}$$

Although (P2) is a non-convex integer program, we obtain the solution in a closed form, as shown by the following theorem.

Theorem 1 (Minimum-redundancy parameters of KA_S)**.** *The optimal parameters of the KA_S solving* (P2) *are*

$$N_1 = 2, N_2 = (N + \alpha - 16)/4, \text{ and } N_3 = (N - \alpha)/6,$$

where $N = 12m + 10 + k \neq 11, m \in \mathbb{N}$*, and*

$$\alpha = \begin{cases} 17, & \textit{when } k = 1 \\ k + 10, & \textit{when } k \in \{0, 2, 4, 6, 8\} \\ k + 4, & \textit{when } k \in \{3, 5, 7, 9, 11\} \\ 8, & \textit{when } k = 10. \end{cases} \qquad (5)$$

Proof. See Appendix A. □

Corollary 1 (Properties of optimal KA_S)**.** *The aperture L, and number of elements N of the KA_S solving* (P2) *are*

$$L = (N^2 + 10N - 87)/8 - \beta$$
$$N = 2\sqrt{2}\sqrt{L + 14 + \beta} - 5$$

where $\beta = (\alpha - 13)^2/8$*, and α is given by* (5)*. Additionally, the number of unit spacings of the KA_S is $S(1) = 8$.*

Proof. This follows from (2), (3), (4), and Theorem 1. □

Notice that $\alpha = 13$ maximizes L (minimizes N) in Corollary 1. Also, note that $S(d \geq 2)$ can grow large as N increases.

4. NUMERICAL RESULTS

Next, we compare the redundancy and number of unit spacing of the KA, MRA and CNA. We also study the sensitivity of these arrays to mutual coupling effects numerically.

4.1. Comparison of array figures of merit

Fig. 2 shows the redundancy and number of unit spacings of the KA_S, KA_R, CNA, and MRA for large N. Asymptotically, the KA_S only has $\sqrt{24/23} \approx 2.15\%$ more elements than the KA_R. The difference is even smaller for finite N. In fact, the two KAs coincide when $33 \leq N \leq 52$. When $N \leq 32$, the redundancy of the KA_S is high due to the fact that $N_1 = 2$. Although this constraint could be relaxed, it is of lesser interest to do so, since MRAs with $S(1) \leq 10$ unit spacings are known for $N \leq 48$. Note that the KA_S has only $S(1) = 8$ for any N, whereas $S(1) \propto N$ for both the KA_R and CNA.

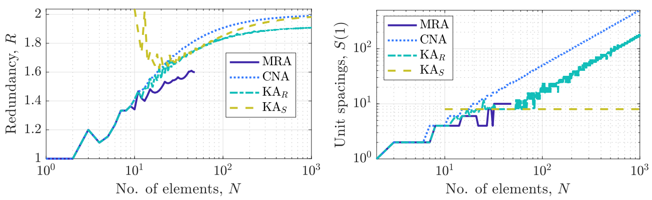

Fig. 2. The KA_S is slightly more redundant than the KA_R (left), but it has a constant number of unit spacings for any N (right). MRAs are only known for $N \le 48$ elements [23, 24].

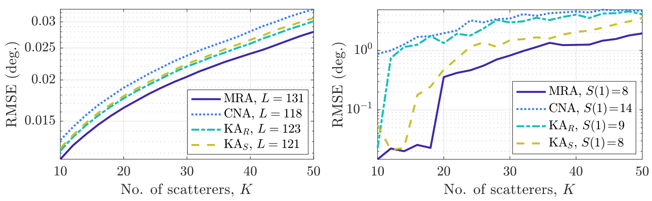

Fig. 3. In absence of mutual coupling (left), RMSE is mainly determined by aperture L. In presence of mutual coupling (right), a low number of unit spacings $S(1)$ is important.

4.2. Scatterer localization in presence of mutual coupling

Consider K equipower scatterers, whose angular directions φ_k uniformly sample the interval $[-\pi/3, \pi/3]$. We model mutual coupling by replacing the ideal steering matrix $\mathbf{A}$ in (1) by $\mathbf{CA}$, where the diagonal elements of coupling matrix $\mathbf{C} \in \mathbb{C}^{N\times N}$ are unity, and the non-diagonal elements are $C_{n,m} = 0.3 l_{n,m}^{-1} e^{-j(\phi l_{n,m}+\vartheta)}$. Here $l_{n,m} = |d_n - d_m| \in \mathbb{N}$ is the Euclidean distance between the nth and mth array elements, and ϕ, ϑ are the coupling phase *increment* and *bias*.

Fig. 3 shows the root-mean-squared-error (RMSE) of the angle estimates given by (sum) co-array MUSIC [21], when the number of physical elements is $N = 28$. Results are averaged over 10^4 i.i.d. realizations of the uniformly distributed coupling phases $\phi, \vartheta \sim \mathcal{U}(0, 2\pi)$, Gaussian noise $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \sigma^2\mathbf{I})$, and spherical scattering coefficients $\gamma_k = z_k/|z_k|$, with $z_k \sim \mathcal{CN}(0, 1)$. The SNR is $10\log(K/\sigma^2) = 10$ dB. In the absence of mutual coupling, the RMSE is primarily determined by the array aperture. However, in its presence, the RMSE is also influenced by the number of unit spacings, as the KA_S achieves the lowest RMSE among the parametric arrays. Note that each array is able to resolve more scatterers than sensors, unlike the ULA.

5. CONCLUSION

This paper proposed a linear sparse array configuration called the Kløve array (KA). The underlying additive basis was introduced in [17], but has not been considered in the context of array processing before. The KA has several desirable properties. For example, it is the sparsest known parametric array with a contiguous sum co-array. We showed that the KA can achieve low redundancy, in addition to having few unit spacings. We demonstrated that this can reduce sensitivity to mutual coupling. In future work, we wish to explore generalizations of the KA, e.g., with constant $S(d \ge 2)$, and provide a more comprehensive comparison of active array geometries.

A. PROOF OF THEOREM 1

We prove Theorem 1 by quantizing the solutions of a relaxation of (P2). We start by solving (3) for N_2 with $N_1 = 2$, yielding $N_2 = (N - 3N_3)/2 - 4$. Substitution into (2) with relaxation $N_3 \in \mathbb{R}$ leads to the concave optimization problem

$$\underset{N_3 \in \mathbb{R}}{\text{maximize}} \; -3N_3^2 + (N-13)N_3, \tag{P3}$$

which has the optimal solution $N_3 = (N-13)/6$.

We now show that the solution to (P2) is of the form $N_3 = \lceil (N-13)/6 \rfloor + k, k \in \mathbb{Z}$. For notational convenience, denote N_2 as $g(N_3) = (N - 3N_3)/2 - 4$, and the objective function in (P3) as $f(N_3, g(N_3)) = -3N_3^2 + (N-13)N_3$. Since $f(x, g(x))$ is concave for $x \in \mathbb{R}$, the maximizer of $f(x, g(x))$ for $x \in \mathbb{N}$ is $x = \lceil z \rfloor$. This is a global optimum of (P2), if $g(\lceil z \rfloor) \in \mathbb{N}$. By concavity of f, the smallest $|k|$ satisfying $g(\lceil z \rfloor + k) \in \mathbb{N}$ then yields the global optimum of (P2).

We proceed by evaluating the solution of (P2) for different N. Note from (3) that $N \ge 10$ is a necessary condition for $N_3 \ge 0$, since $N_1 = 2$ and $N_2 \ge 1$. Letting $N = 12m + 10 + k, m \in \mathbb{N}$, the closest integer valued solution to (P3) is

$$N_3 = \left\lceil \frac{N-13}{6} \right\rfloor \pm l = \begin{cases} 2m \pm l & \text{when } k \in \{0:5\} \\ 2m+1 \pm l & \text{when } k \in \{6:11\}, \end{cases}$$

where $l \in \mathbb{N}$, and $|l|$ should be as small as possible. We confirm from (3) that $N_2 \in \mathbb{N}_+$ either when i) $l = 0$ and $k \in \{0, 2, 4, 7, 9, 11\}$, or ii) $l = 1$ and $k \in \{1, 3, 5, 6, 8, 10\}$. Table 1 lists N_3, N_2, and $\alpha = N - 6N_3$ for different k. The loss in aperture incurred by quantizing the solution to (P3) is proportional to $|\alpha - 13|$ (cf. Corollary 1). Consequently, when k leads to a feasible solution for multiple values of N_3, we select the solution yielding the smallest $|\alpha - 13|$. In Table 1, this is reflected by the crossed out the values of k, which correspond to solutions with the smaller (or equal) L.

We may easily verify that $N = 10$ and $N \ge 12$ yield feasible N_2 and N_3 in Table 1. However, $N = 11$ yields the infeasible solution $N_3 = -1 \notin \mathbb{N}$, since $k = 1, m = 0$. Since $N_2 \ge 1$, we must have $N_3 \le 1/3$ by (3). This condition is only satisfied by $N_3 = 0$, which yields $N_2 = 3/2 \notin \mathbb{N}_+$, i.e., another infeasible solution. We therefore conclude that a solution to (P2) exists (and is given by Table 1) for any $N \ge 10, N \ne 11$.

Table 1. Solution to (P2) for $N = 12m+10+k, m \in \mathbb{N}, N \ne 11$.

N_3	N_2	α	$\lvert\alpha-13\rvert$	k
$2m$	$3m+1+k/2$	$k+10$	$\lvert k-3\rvert$	$0, 2, 4, 6, 8, \cancel{10}$
$2m+1$	$3m+1+(k-3)/2$	$k+4$	$\lvert k-9\rvert$	$\cancel{1}, 3, 5, 7, 9, 11$
$2m-1$	$3m+1+(k+3)/2$	$k+16$	$\lvert k+3\rvert$	$1, \cancel{3}, \cancel{5}$
$2m+2$	$3m-2+k/2$	$k-2$	$\lvert k-15\rvert$	$\cancel{6}, \cancel{8}, 10$

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