



This is an electronic reprint of the original article. This reprint may differ from the original in pagination and typographic detail.

Lilja, Ville-Pekka; Polojärvi, Arttu; Tuhkuri, Jukka; Paavilainen, Jani

# Finite-discrete element modelling of sea ice sheet fracture

Published in: International Journal of Solids and Structures

DOI: 10.1016/j.ijsolstr.2020.11.028

Published: 15/05/2021

Document Version Peer-reviewed accepted author manuscript, also known as Final accepted manuscript or Post-print

Published under the following license: CC BY-NC-ND

Please cite the original version:

Lilja, V.-P., Polojärvi, A., Tuhkuri, J., & Paavilainen, J. (2021). Finite-discrete element modelling of sea ice sheet fracture. *International Journal of Solids and Structures*, *217-218*, 228-258. https://doi.org/10.1016/j.ijsolstr.2020.11.028

This material is protected by copyright and other intellectual property rights, and duplication or sale of all or part of any of the repository collections is not permitted, except that material may be duplicated by you for your research use or educational purposes in electronic or print form. You must obtain permission for any other use. Electronic or print copies may not be offered, whether for sale or otherwise to anyone who is not an authorised user.

# Journal Pre-proofs

Finite-discrete element modelling of sea ice sheet fracture

Ville-Pekka Lilja, Arttu Polojärvi, Jukka Tuhkuri, Jani Paavilainen

PII:	S0020-7683(20)30457-1
DOI:	https://doi.org/10.1016/j.ijsolstr.2020.11.028
Reference:	SAS 10925
To appear in:	International Journal of Solids and Structures
Received Date:	23 June 2020
Accepted Date:	21 November 2020



Please cite this article as: Lilja, V-P., Polojärvi, A., Tuhkuri, J., Paavilainen, J., Finite-discrete element modelling of sea ice sheet fracture, *International Journal of Solids and Structures* (2020), doi: https://doi.org/10.1016/j.ijsolstr.2020.11.028

This is a PDF file of an article that has undergone enhancements after acceptance, such as the addition of a cover page and metadata, and formatting for readability, but it is not yet the definitive version of record. This version will undergo additional copyediting, typesetting and review before it is published in its final form, but we are providing this version to give early visibility of the article. Please note that, during the production process, errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.

© 2020 Published by Elsevier Ltd.

# Finite-discrete element modelling of sea ice sheet fracture

Ville-Pekka Lilja<sup>a,\*</sup>, Arttu Polojärvi<sup>a</sup>, Jukka Tuhkuri<sup>a</sup>, Jani Paavilainen<sup>b</sup>

<sup>a</sup> Aalto University, School of Engineering, Department of Mechanical Engineering, P.O. Box 14300, FI-00076 Aalto, Finland <sup>b</sup>Rand Simulation Ltd., Vantaa, Finland

# Abstract

A rate-independent, de-cohesive damage model for the fracture modelling of large, cellular, plate-like, quasi-brittle structures is proposed. A hybrid, threedimensional finite-discrete element method to investigate sea ice sheet fracture is then introduced, followed by three applications. The uniaxial tensile fracture of an ice sheet of varying physical sizes is examined first. The effects of both the size of an ice sheet and the loading rate applied on the effective tensile strength are investigated. The vertical penetration fracture of an ice sheet loaded by a rigid, flat-ended, cylindrical indenter is examined next. The breakthrough loads and strengths of an ice sheet of varying physical sizes are computed, applicable scaling rules as regards to the vertical breakthrough strength searched for. To conclude, the breaking of an ice sheet containing a circular hole by a surfacing, rigid, truncated cone is studied (an axisymmetric contact problem). The loads on the cone are computed and then compared with loads that can be obtained analytically for a case in which a structure is stationary, a sheet moves, and the contact is unilateral. While computing the tensile and the breakthrough strengths, a set of self-similar sheet samples with an in-plane size range of 1:16 is examined. The samples are square; have a side length of either L = 10, 20,40, 80, or 160 m; and a thickness of either h = 0.5, 1.0, or 1.5 m. With the sheets containing holes, only the largest samples (L = 160 m) are investigated. The results indicate that i) both the tensile and the breakthrough strengths are strong functions of both L and h; ii) the tensile strength is a strong function of the applied loading rate; *iii*) the failure mode as regards to the vertical penetration fracture changes drastically as a function of L; iv) the model is able to demonstrate both radial and circumferential cracking; and that v) the proposed (in-direct) approach to compute ice loads on a conical offshore structure provides realistic results.

 $Keywords:\;$  Dynamic fracture, Plates, Numerical algorithms, Size effect, Ice and snow

<sup>\*</sup>Corresponding author

Email address: ville-pekka.lilja@aalto.fi (Ville-Pekka Lilja)

# 1. Introduction

#### 1.1. Background and context

The ice cover in the Arctic waters is decreasing. New shipping routes will thus become operational and the interest in industrial activities shall increase. A prime example of the latter is the growing interest to construct new wind farms, the foundations of which must be dimensioned to withstand loads caused by the, still existing, sea ice. Data on these loads can be obtained both through experimental campaigns and by using analytical or numerical methods.

To be able to compute ice loads requires a capability to model sea ice fracture. Modelling sea ice fracture (and failure), reliably, has been an ongoing quest for decades. A whole series of IUTAM symposia have been devoted to this and other closely related subjects (Tryde, 1980; Jones et al., 1991; Dempsey and Shen, 2001)<sup>1</sup>. A vast number of inelastic constitutive models with various levels of sophistication – both phenomenological and of an *ab initio* -type – have been proposed in the past (Schulson and Duval, 2009).



Figure 1: A snapshot from a simulation in which an originally intact finite-discrete element ice sheet sample is penetrated by a rigid, flat-ended, cylindrical indenter from below. The indenter surfaces at the centre. Several radial cracks nucleate at the contact point (on the upper surface) and start to propagate towards the free edges. A star-shaped pattern of cracks subsequently appears.

This paper develops a hybrid, three-dimensional finite-discrete element (FE-DE) method to investigate sea ice sheet fracture and to compute ice loads on

<sup>&</sup>lt;sup>1</sup>The last IUTAM symposium on the "Physics and Mechanics of Sea ice" was held in Espoo, Finland in June 2019 with the proceedings yet to be published.

an inclined offshore structure. With the method followed here, an ice sheet is modelled with rigid discrete elements. The mass centroids of the discrete elements are then connected via an in-plane beam lattice of de-cohesive, viscously damped, co-rotational Timoshenko beam finite elements, Figures 1 and 2. Fracture is thus modelled with de-cohesive beam finite elements. The paper is a direct continuation to (Lilja et al., 2019a), where the method was introduced, but which did not present the technique to describe fracture. This paper addresses fracture.

After a description of the technique to model fracture, three applications follow. The effective, uniaxial tensile strength of an ice sheet of varying physical sizes will be computed first, the breakthrough loads and strengths next. The effect of the loading rate applied on the effective tensile strength will be examined, applicable scaling rules for both the tensile and the breakthrough strengths searched for. To conclude, a new "in-direct" approach to study ice-structure interaction is introduced: a truncated, rigid, ascending cone<sup>2</sup> is breaking a sea ice sheet containing a circular hole. The breaking loads are computed and then compared with known analytical results of a "direct" case.

Traditionally the scheme to study ice-structure interaction has been such that a moving ice sheet collides unilaterally (one-sidedly) with a stationary offshore structure (ship-ice interaction is here excluded). Such an approach is referred to herein as a "direct" approach. The present study reverses the roles and somewhat extends the scheme. In the new scheme, a sea ice sheet containing a circular hole remains stationary, whereas a structure is moving vertically upwards, surfaces through the hole, and then breaks the sheet. Such a scheme is referred to herein as an "in-direct" approach. The method is probably best illustrated by an example: the attached animations, Animation 1 and Animation 2, depict the scheme but via a generic case; a sheet with a side length of L = 50 m and a thickness of h = 0.1 m is interacting with a cone having an upper diameter of five meters and a cone angle of 45 degrees. It can be seen that with an in-direct approach, a cone is surrounded by ice and the contact zone extends around the whole circumference (an axisymmetric contact problem). It will thus be of an interest to explore whether the solution of a seemingly unrelated problem with an axisymmetric contact yields results that are comparable to those of the direct case with a unilateral contact.

The three applications presented above are considered in that order for a specific purpose. A lattice-based model contains an internal length scale that affects the mechanical response on the other scales. It is roughly represented here by the mean size of the discrete elements (that is, the internal heterogeneities). The constitutive properties of a lattice on a microscopic scale (i.e. the properties of individual beams, both elastic and inelastic) are generally different from the "effective" constitutive properties on the meso or macroscopic scale. The first computed effective tensile strengths of sheets of different sizes will thus prove to be useful in interpreting the breakthrough strengths, which

 $<sup>^2\</sup>mathrm{A}$  conical frustum.

are then required in interpreting the cone loads.

As to why a conical offshore structure is in this paper examined is motivated by the following: many offshore structures employ a conical shape near the waterline because it helps in preventing ice from being crushed. Ice crushing is known to lead to high local and global loads which must then be transferred to, e.g., the sea bottom. A conical shape is advantageous as it allows an ice sheet to fail in bending instead. An ice sheet that fails in bending results in significantly lower structural loads (Brown and Määttänen, 2009).

# 1.2. Scope and objectives

The paper is divided in four main "parts." The first main part, section 3, presents the development of a new rate-independent, de-cohesive damage model and then describes its algorithmic implementation. In the second part, subsection 6.1, the effective, uniaxial tensile strength of an FE-DE ice sheet sample of varying physical sizes is studied. The dependence of the strength on both the size of an ice sheet and the loading rate applied is investigated. In the third part, subsection 6.2, the vertical penetration fracture of an ice sheet loaded by a rigid, flat-ended, cylindrical indenter is examined. The associated breakthrough loads and strengths are computed, appropriate scaling rules as regards to the vertical breakthrough strength searched for. In the fourth part, subsection 6.3, the breaking of an FE-DE ice sheet sample containing a circular hole by a rigid, truncated, ascending cone is investigated. The ice loads on the cone associated with the breaking of the surrounding ice cover are computed and then compared with loads that can be obtained analytically. These are for the direct case.

The main contents of the paper can be summarized as follows:

- 1) a new rate-independent, de-cohesive damage model for the fracture modelling of large, cellular, plate-like, quasi-brittle structures is proposed;
- the effective, uniaxial tensile strength of an FE-DE ice sheet sample of varying physical sizes is computed;
- the emerging size and rate dependencies as regards to the effective, uniaxial tensile strength are investigated;
- 4) the breakthrough loads and strengths of an FE-DE ice sheet sample of varying physical sizes are computed;
- 5) applicable scaling rules as regards to the vertical breakthrough strength are searched for;
- 6) the change in the observed failure mode as regards to the vertical penetration fracture is examined; and
- 7) a new in-direct technique to compute ice loads on an inclined offshore structure, as well as to investigate the ability of a numerical method to describe both radial and circumferential cracking, is proposed.

The whole study aims, in essence, to investigate the applicability of the proposed hybrid FE-DE scheme in modelling sea ice sheet fracture.

#### 1.3. A brief literature survey and the state-of-the-art

For a brief historical survey of applying an in-plane beam lattice to emulate plate- or sheet-like behaviour, see (Lilja et al., 2019a). For a fracture-based overview, see (Bažant and Planas, 1997, Ch. 14.4).

The de-cohesive damage model proposed has its roots in the mid 1970's. The kind of an approach was coined, apparently, by (Dougill, 1976). Dougill developed his theory for a "progressively fracturing solid." Other exemplary works of a similar type are, for example, those of Carol et al. (1997); Gálvez et al. (2002) and Schreyer et al. (2006). In (Carol et al., 1994), such developments were called as "elastic-degrading" models. What these models have in common is the fact that the basic machinery of small strain hardening plasticity is readily applicable (Carol et al., 1994, sec. 3). Softening can be modelled as well; an adequate element length, however, is required in order to avoid snapback instability, see (Bažant and Oh, 1983, Eq. 21, p. 164). Note that the "crack band model" of Bažant and Oh (1983) and the "fictitious crack model" of Hillerborg et al. (1976), of which the latter will be here partly adopted, are conceptually equivalent. The main difference is that Bažant and Oh (1983) assume the existence of a finite width "crack band," whereas Hillerborg et al. (1976) postulate the fracture to localize onto a plane.

A concise historical survey on the topic in the second main part (i.e. the uniaxial tensile strength of a cellular, plate-like structure) is given in (Bažant et al., 1990), see also (Jirásek and Bažant, 1995). A number of papers, both analytical and numerical, have been published about the problem in the third part (i.e. the vertical penetration fracture/failure of a plate) (Bažant and Li, 1994; Bažant and Kim, 1998a,b; Bažant, 2002; Beltaos, 2002; Dempsey et al., 1995; Dempsey and Vasileva, 2006b,a; Dempsey et al., 2006; Kerr, 1976, 1996; Li and Bažant, 1994; Lu et al., 2015, 2016; McGilvary et al., 1990; Pushkin et al., 1991; Slepyan, 1990; Sodhi, 1989, 1995, 1998; Vasileva and Dempsey, 2006). A hybrid finite-discrete element method has not been, to the authors' best knowledge, applied before. No papers appear to exist studying a problem similar to that in the fourth part (i.e. the breaking of a floating plate containing a circular hole by a surfacing cone). Nevel (1992) gives a clear account of the subject but with a direct approach. It is believed that the approach here proposed is new. No historical account is given as regards to the experimental work done on the problem in the third part. In that matter, see the review papers by Kerr (1976, 1996). Often cited treatises are, however, those of Black (1958); Gold et al. (1958); Frankenstein (1963, 1966); Meyerhof (1960), and Lichtenberger et al. (1974).

# 1.4. Outline of the paper

The rest of the paper is organized in six sections. In section 2, a brief description of the FE-DE model constructed is given. The de-cohesive damage model is explained in detail in section 3. In section 4, the simulation setups are described. A short account of the existing analytical results as regards to the vertical bearing capacity of an infinite ice sheet and the ice loads on an inclined offshore structure are given in section 5. In section 6, the results computed with the FE-DE approach are presented and discussed. The uniaxial tensile fracture is studied first. The vertical penetration fracture of a sea ice sheet loaded by a rigid, flat-ended, cylindrical indenter is examined next. The breaking of a sea ice sheet containing a circular hole by a rigid, truncated, ascending cone is investigated last. In section 7, the paper is then drawn to a close.

#### 2. A finite-discrete element model of an ice sheet

This section provides an overview of the model features. A complete description is given in (Lilja et al., 2019a), except of the de-cohesive damage model that is introduced in this paper.

# 2.1. Construction of the numerical model

As has been already stated, a sheet is modelled with rigid discrete elements. The mass centroids of the discrete elements connect then via an in-plane beam lattice of Timoshenko beam finite elements, Figures 2 and 3. The beam formulation adopted follows mainly (Crisfield, 1997, Ch. 17.1-2): a local, elemental triad of vectors,  $\mathbf{T}_{p,q}$ , associated with a beam finite element p, q, tracks the average, incremental motions of the discrete elements p and q, and deforms. The strains in a beam element p, q are given by i) the stretch of the axial vector component and ii) the changes between the mutual orientations of the discrete elements p and twist in a beam element p, q are due to the differing nodal orientations of the discrete elements p and q. Geometrical non-linearity is taken into account, is due to the finite displacements and rotations of the discrete elements, yet the deformations in a beam element are assumed to be small.

The material of the beam finite elements follows Hooke's law. To dissipate energy, a viscous damping model is used. A rheological equivalent of the material model implemented would thus be that of Kelvin-Voigt (a viscoelastic solid).<sup>3</sup> Due to the Timoshenko beam finite elements following a co-rotational kinematical description (a co-rotational description extracts the rigid body displacement components from the total displacements), the application of a linearised stress-strain relationship – within the co-rotational, elemental frame – is justified. Once the de-cohesive cracking initiates, the constitutive behaviour, however, changes. A linearly softening, rate-independent traction-separation law is subsequently applied.

$$\vec{F}_{\mathbf{T}_{p,q}}^{\mathrm{mat,c}} = c \cdot \mathrm{diag} \begin{bmatrix} A & A & J_{x^{\prime\prime}x^{\prime\prime}} & I_{y^{\prime\prime}y^{\prime\prime}} & I_{z^{\prime\prime}z^{\prime\prime}} \end{bmatrix}_{\mathbf{T}_{p,q}} \left\{ \begin{matrix} \dot{\vec{\varepsilon}} \\ \dot{\vec{\chi}} \end{matrix} \right\}_{\mathbf{T}_{p,q}}.$$

 $<sup>^{3}</sup>$ There is an unfortunate typo in (Lilja et al., 2019a). The viscous force vector should there read as (the diagonal matrix had been inadvertently omitted):



Figure 2: Discrete elements p and q connected with a Timoshenko beam finite element p, q (adapted from (Lilja et al., 2019a, Fig. 2, p. 2)). The local, nodal triads of a discrete element, **de**, and of a beam finite element, **be**, are depicted in a red and a black colour, respectively. A global triad of base vectors (with an associated co-ordinate system) in a fixed, inertial frame of reference is denoted with a symbol **g**. The symbols x'', y'', and z'' denote the local co-ordinates of a beam finite element. For a detailed explanation of the nomenclature used, see (Lilja et al., 2019a).

#### 2.1.1. Mesh creation via a centroidal Voronoi tessellation procedure

A centroidal Voronoi tessellation procedure with a random generating point set and Lloyd's algorithm, see (Du et al., 1999), is used to produce an unstructured mesh, Figure 3. Lloyd's algorithm includes an iteration loop during which the seeds acting as the generators of a Voronoi diagram are incrementally displaced to merge with the centroids of the contiguous Voronoi cells. The iteration continues until some termination criterion is met. A criterion based on the minimisation of an associated energy functional with a cut-off tolerance of  $\epsilon = 10^{-5}$  or 500 iteration cycles is here adopted (Talischi et al., 2012, sec. 3). The meshes that had holes were, in this paper, created by the meshing tool "Polymesher" presented in (Talischi et al., 2012).

As an "end product," Lloyd's algorithm produces an in-plane tessellation that consists of only highly regular polygons (convex hexagons, pentagons, and quadrilaterals). The polygons are next extruded in the out-of-plane direction to become prismatic polyhedrons. These polyhedrons denote the discrete elements.

With the aid of a Delaunay triangulation scheme, a triangular lattice mesh of Timoshenko beam finite elements is finally created. On a large enough relative size scale  $L_{\rm rel}$  ( $L_{\rm rel} = L/l$ ), a CVT-tessellated, unstructured Timoshenko beam lattice behaves as an (in-plane) isotropic medium. This is because the nuclei of the Voronoi cells define the beam end nodes and the beam cross-sectional areas are equal to the areas of the interfaces separating the adjacent discrete elements (Bolander and Saito, 1998). For a further discussion on mesh dependence and other mesh related issues, see (Lilja et al., 2019a).



Figure 3: A CVT-tessellated FE-DE sheet sample with a side length of L = 160 m, a thickness of h = 1.5 m, and a discrete element size of l = 3h (adapted from (Lilja et al., 2019b, Fig. 3, p. 3)). The red lines denote the longitudinal axes of the Timoshenko beam finite elements. The cross-sectional area of each Timoshenko beam finite element gets determined by the area of the joint surface separating the adjacent discrete elements (in cyan). Figure 26 shows two examples of the meshes with holes.

#### 2.1.2. Equations of motion

The equations to be solved, for each discrete element (discretely in time), are the three translational and the three rotational equations of motion (i.e. the Newton-Euler equations). The semi-discretised equations of motion of an ice sheet modelled with a hybrid FE-DE method can be written as (for full details with the coupling terms explicitly presented, see (Lilja et al., 2019a)):

$$\vec{F}_{\mathbf{g},\mathbf{de}}^{\mathrm{int}}(\vec{x}(t), \dot{\vec{x}}(t), t) + \vec{F}_{\mathbf{g},\mathbf{de}}^{\mathrm{ext}}(\vec{x}(t), \dot{\vec{x}}(t), t) = \mathbf{M}_{\mathbf{g},\mathbf{de}}\ddot{\vec{x}}(t),$$
(1)

where  $\vec{F}_{\mathbf{g},\mathbf{de}}^{\mathrm{int}}$  is an internal force vector containing forces and moments (both viscous and non-viscous) due to the deforming and fracturing beam finite elements;  $\vec{F}_{\mathbf{g},\mathbf{de}}^{\mathrm{ext}}$  an external force vector containing forces and moments due to the contacts, buoyancy, drag, and gravity;  $\vec{x}(t)$  a vector containing the translational and the angular positions of the discrete elements (t denotes time);  $\dot{\vec{x}}(t)$  and  $\ddot{\vec{x}}(t)$  the first and the second time derivatives of  $\vec{x}(t)$ , respectively; and  $\mathbf{M}_{\mathbf{g},\mathbf{de}}$ 

a diagonal matrix containing the translational masses and the mass moments of inertia of the discrete elements. The subindexes "g" and "de" refer here to the bases the translational and the rotational equations of motion have been written with respect to, respectively, see Figure 2.

As was stated above,  $\vec{F}_{g,de}^{ext}$  is an external force vector containing forces and moments due to the contacts, buoyancy, drag, and gravity. The contact scheme here adopted is identical to the one presented in (Feng et al., 2005) and is thus not repeated. A linear-type contact energy function,  $W(V) = k_c V$ , is applied in which the overlap volume V is being penalized by a contact stiffness parameter  $k_c$ , Table 1. The contact damping forces as well as the buoyant-, drag-, and the frictional forces are computed as is presented in (Polojärvi et al., 2012, pp. 22-24).

The translational equations of motion are integrated (in time) with a  $2^{nd}$  order accurate central difference time integration scheme. The rotational equations of motion with three-dimensional, finite, non-commutative rotations are integrated with a (modified)  $4^{th}$  order accurate Runge-Kutta scheme (Munjiza et al., 2003).

The length of a critical time step,  $\Delta t_{\rm cr}$  (stability limit), can be estimated via  $\Delta t_{\rm cr} = L_{\rm e}/c_{\rm s} = L_{\rm e}/\sqrt{E_{\rm b}/\rho_{\rm i}}$ . Here,  $L_{\rm e}$  denotes the length of a Timoshenko beam finite element;  $c_{\rm s}$  a (material-dependent) dilatational/bar wave speed (with  $\nu_{\rm b} = 0$  and no damping) (Hughes, 1983, pp. 102-104);  $E_{\rm b}$  and  $\nu_{\rm b}$  the Young's modulus and Poisson's ratio given to the Timoshenko beam finite elements, respectively; and  $\rho_{\rm i}$  the density of ice. For the actual numerical values, see Table 1 below. Because of the added viscous damping and the contact computations, the length of the time step employed,  $\Delta t$ , has to be, in practice, manually adjusted.

While computing the internal, nodal force vector of an undamaged Timoshenko beam finite element, a resultant-based beam formulation can be employed (Lilja et al., 2019a). A total, internal stress resultant vector read there as (with a slightly different notation though):

$$\vec{F}_{\mathbf{T}_{p,q}}^{\text{tot}} = \left\{ \vec{P} \\ \vec{M} \right\}_{\mathbf{T}_{p,q}}^{\text{tot}} = \vec{F}_{\mathbf{T}_{p,q}}^{\text{mat}} + \vec{F}_{\mathbf{T}_{p,q}}^{\text{mat,c}} = \left\{ \vec{P} \\ \vec{M} \right\}_{\mathbf{T}_{p,q}}^{\text{mat}} + \left\{ \vec{P} \\ \vec{M} \right\}_{\mathbf{T}_{p,q}}^{\text{mat,c}},$$
(2)

where  $\vec{F}_{\mathbf{T}_{p,q}}^{\text{mat}}$  is a non-viscous and  $\vec{F}_{\mathbf{T}_{p,q}}^{\text{mat,c}}$  a viscous, internal stress resultant vector. Eq. (2) gave then, with the principle of virtual work, as the internal, nodal force vector:

$$\vec{F}_{\mathbf{g},p,q}^{\text{int}} = \left\{ \begin{matrix} \vec{P} \\ \vec{M} \end{matrix} \right\}_{\mathbf{g}}^{\text{int}} = \mathbf{B}_{p,q,\mathbf{g}} \left\{ \begin{matrix} \vec{f} \\ \vec{m} \end{matrix} \right\}_{\mathbf{g}},\tag{3}$$

where  $\vec{f}_{\mathbf{g}} = \mathbf{T}_{p,q,\mathbf{g}} \vec{P}_{\mathbf{T}_{p,q}}^{\text{tot}}$ ,  $\vec{m}_{\mathbf{g}} = \mathbf{T}_{p,q,\mathbf{g}} \vec{M}_{\mathbf{T}_{p,q}}^{\text{tot}}$ , and  $\mathbf{B}_{p,q,\mathbf{g}}$  a strain-displacement matrix connecting a vector of varied, generalized strains (strains, curvatures, and twist) with a vector of varied, global, nodal degrees of freedom. For further details, explanations, and references, see (Lilja et al., 2019a). Eqs. (2) and (3) can be utilised for a beam undergoing damage as well, but – now – the internal stress resultants have to be integrated numerically. This is examined next.

# 3. Description of fracture

To model damage, the Timoshenko beam finite elements are able to fracture. A de-cohesive crack (a fictitious crack of Hillerborg et al. (1976)) is allowed to propagate along an interface separating the adjacent discrete elements, Figure 4. The moment the cracking initiates, a rate-independent, linearly softening traction-separation law is activated in order to allow the discrete elements to separate. The activation is done independently at each integration point along the cross-section of a beam finite element. Because of the change of the constitutive behaviour once the damage begins, the model can be categorized as "extrinsic" (Seagraves and Radovitzky, 2010). No new cohesive finite elements are introduced, just the constitutive behaviour is changed.



Figure 4: A schematic illustration of an advancing de-cohesive (fictitious) crack (adapted from (Lilja et al., 2017, Fig. 1)). Note that the deformations are highly exaggerated. The white crosses denote fully damaged integration points. A traction-free (true) crack tip advances along the cross-section as the integration points get fully damaged. The other features have been presented and explained in Figure 2.

What will be given in the next subsections follows mostly (Paavilainen et al., 2009), but takes into account the three-dimensionality of the present model. In their model, a crack was able to propagate only vertically due to the modelling space being two-dimensional. Since the space is here three-dimensional, a crack is able to propagate both horizontally and vertically. A mixed mode fracture gets treated with by first reducing, via a change of variables, the original three-component state of stress at an integration point to a one-dimensional "effective" state of stress and then representing that state of stress with respect to a loading surface (a failure surface) which degrades (evolves) as a function of an effective crack-opening displacement measure. The adjacent discrete elements separate once a beam has fully eroded.

A note before proceeding further is probably in place: a description of fracture with cohesive finite elements – when the crack path is not known in advance – fails to exhibit energetic convergence. A convergent result is unobtainable with any reasonable number of elements used (Seagraves, 2013). In other words, the amount of dissipated fracture energy grows indefinitely as a function of the number of elements employed. Seagraves (2013) utilises a DG-based (Discontinuous Galerkin) scheme; modelling fracture with extrinsic cohesive finite elements yields results equivalent to those by a DG-based method.

If a mesh is structured, crack growth with cohesive elements is, in addition, prone to preferred directions. This is because the artificial toughness induced by the mesh varies with direction (Rimoli et al., 2012). If a crack cannot evolve freely, is bound to the interelement (here, the discrete element) boundaries, a method is, by its very nature, mesh dependent. Crack growth can be rendered approximately direction independent, i.e. isotropic (but still mesh dependent), if a CVT-tessellated, unstructured mesh is used (Leon et al., 2014; Spring et al., 2014). Such meshes exhibit isotropic crack growth although the mesh induced toughness is still a bit high. For an overview of the use of cohesive finite elements to describe dynamic fracture, see (Seagraves and Radovitzky, 2010).

#### 3.1. Internal stress resultant vectors

This subsection states the equations of the internal stress resultant vectors that are needed in computing the internal, nodal force vector  $\vec{F}_{\mathbf{g},p,q}^{\text{int}}$ , Eq. (3). In place of a resultant-based beam formulation that is applicable for an undamaged beam,(Lilja et al., 2019a), a component-based formulation is now required.

# 3.1.1. Viscous and non-viscous traction vectors

In this paper, a Timoshenko beam finite element p, q has three both viscous and non-viscous stress components. A non-viscous traction vector of an integration point i, j reads as (with base vectors omitted here and also subsequently):

$$\vec{\sigma}_{\mathbf{T}_{p,q}}^{i,j,\text{mat}} = \begin{cases} \sigma_{x''x''}^{i,j} \\ \tau_{x''y''}^{i,j} \\ \tau_{x''z''}^{i,j} \end{cases}_{\mathbf{T}_{p,q}}^{\text{mat}} = \begin{cases} E_{\mathrm{b}}\epsilon_{x''x''}^{i,j} \\ G_{\mathrm{b}}\gamma_{x''y''}^{i,j} \\ G_{\mathrm{b}}\gamma_{x''z''}^{i,j} \end{cases}_{\mathbf{T}_{p,q}}^{\text{mat}} \\ = \begin{cases} E_{\mathrm{b}}\left(\epsilon_{x''x''}^{b} + \frac{i,j}{y''}\chi_{x''y''}^{b} + \frac{i,j}{y''}\chi_{x''z''}^{b} \right) \\ G_{\mathrm{b}}\left(\gamma_{x''y''}^{b} - \frac{i,j}{z''}\chi_{x''x''}^{b}\right) \\ G_{\mathrm{b}}\left(\gamma_{x''z''}^{b} + \frac{i,j}{y''}\chi_{x''x''}^{b}\right) \\ G_{\mathrm{b}}\left(\gamma_{x''z''}^{b} + \frac{i,j}{y''}\chi_{x''x''}^{b}\right) \end{cases}_{\mathbf{T}_{p,q}}^{\text{mat}},$$

$$(4)$$

whereas a viscous traction vector (a vector of strain rates multiplied by a damping constant c) is given by:

$$\vec{\sigma}_{\mathbf{T}_{p,q}}^{i,j,\mathrm{mat},c} = \begin{cases} \sigma_{x''x''}^{i,j,c} \\ \tau_{x''z''}^{i,j,c} \\ \tau_{x''z''}^{i,j,c} \\ \tau_{x''z''}^{i,j,c} \end{cases} \mathbf{T}_{p,q} = c \begin{cases} \dot{\epsilon}_{x''x''}^{i,j} \\ \dot{\gamma}_{x''z''}^{i,j} \\ \dot{\gamma}_{x''z''}^{i,j} \\ \dot{\gamma}_{x''z''}^{i,j} \\ \dot{\gamma}_{x''z''}^{i,j} \end{cases} \mathbf{T}_{p,q} \\ \mathbf{T}_{p,q} \end{cases} \mathbf{T}_{p,q}$$

$$= c \begin{cases} \dot{\epsilon}_{x''x''}^{b} + i j y'' \dot{\chi}_{x''y''}^{b} + i j z'' \dot{\chi}_{x''z''}^{b} \\ \dot{\gamma}_{x''y''}^{b} - i j z'' \dot{\chi}_{x''x''}^{b} \\ \dot{\gamma}_{x''z''}^{b} + i j y'' \dot{\chi}_{x''x''}^{b} \\ \dot{\gamma}_{x''z''}^{b} + i j y'' \dot{\chi}_{x''x''}^{b} \\ \dot{\gamma}_{x''z''}^{b} + i j y'' \dot{\chi}_{x''x''}^{b} \end{cases} \mathbf{T}_{p,q}$$

$$(5)$$

In above, the symbols  $\epsilon_{x''x''}^{b}$ ,  $\gamma_{x''y''}^{b}$ , and  $\gamma_{x''z''}^{b}$  denote the components of a local strain vector  $\vec{\varepsilon}_{\mathbf{T}_{p,q}}$ ; the symbols  $\chi_{x''x''}^{b}$ ,  $\chi_{x''y''}^{b}$ , and  $\chi_{x''z''}^{b}$  the components of a local curvature vector  $\vec{\chi}_{\mathbf{T}_{p,q}}$ ; and the lowercase x'', y'', and z'' the local coordinates of a Timoshenko beam finite element, see Figure 2. The superscript "b" is used to stress that the respective quantities are computed on a beam level. For a succinct description of the kinematics of a Timoshenko beam finite element, as well as of the sign convention adopted, see (Luo, 2008).

A remark: the uppercase co-ordinates  ${}^{i,j}y''$  and  ${}^{i,j}z''$  measure the distances between the points of interest, i.e. the integration points i, j and the neutral axis. When a cohesive crack advances from one integration point to another, the position of the neutral axis should get updated accordingly. The implemented routine here is such that the neutral axis is made to always pass through the centroid of the undamaged part of a beam's cross-section. Notice also that in what follows, the description of fracture is kinematically consistent because the cracking strains follow directly from the local strain and the curvature vectors. The components of the local curvature vector are computed by the nodal orientations of the discrete elements, see (Lilja et al., 2019a), and then summed with the local strains, see Eq. (4).

# 3.1.2. Viscous and non-viscous internal stress resultant vectors

With the aid of the non-viscous stress components, Eq. (4), a non-viscous, internal stress resultant vector is formed and reads as (the rightmost vector is written in a format directly amenable to a numerical integration with the symbols  $\alpha^{i,j}$  denoting integration weights):

$$\vec{F}_{\mathbf{T}_{p,q}}^{\mathrm{mat}} = \begin{cases} N_{x''x''} \\ Q_{x''y''} \\ Q_{x''z''} \\ M_{x''x''} \\ M_{x''y''} \\ M_{x''z''} \\ M_{x''z'''} \\ M_{x''z''''} \\ M_{x''z''''} \\ M_{x''z''''} \\ M_{x''z''''''''''''''''$$

In Eq. (6), the symbol  $f(\mu)^{i,j}$  denotes a switch function yielding an output of  $f(\mu)^{i,j} = \mu$  if an existing true crack is currently closed, whereas it is equal to one otherwise. The function operates independently at each integration point. The symbols  $\kappa_{y''}$ ,  $\kappa_{z''}$ , and  $\kappa_{x''}$  denote, then, shear correction factors with respect to the local xy- and xz-planes and a warping correction factor, respectively. Note that for a solid, square cross-section, the shear and the warping correction

factors (applied here for non-square cross-sections as well) are practically equal, i.e.  $\kappa_{x''} \approx \kappa_{y''} = \kappa_{z''} = \kappa = 10(1 + \nu)/(12 + 11\nu)$ , (Freund and Karakoç, 2015; Cowper, 1966). Only this one specific value is used for all  $\kappa_{x''}$ ,  $\kappa_{y''}$ , and  $\kappa_{z''}$ .

In a likewise manner, the viscous stress components, Eq. (5), yield a viscous, internal stress resultant vector, which reads as:

$$\vec{F}_{\mathbf{T}_{p,q}}^{\mathrm{mat},c} = \begin{cases} N_{x''x''}^{\mathrm{c}} \\ Q_{x''y''}^{\mathrm{c}} \\ Q_{x''y''}^{\mathrm{c}} \\ Q_{x''x''}^{\mathrm{c}} \\ M_{x''x''}^{\mathrm{c}} \\ M_{x''y''}^{\mathrm{c}} \\ M_{x''y''}^{\mathrm{c}} \\ M_{x''z''}^{\mathrm{c}} \\ M_{x''z''}^{\mathrm{c}} \\ M_{x''z''}^{\mathrm{c}} \\ M_{x''z''}^{\mathrm{c}} \\ M_{x''z''}^{\mathrm{c}} \\ M_{x''z''}^{\mathrm{c}} \\ f(\mu)^{i,j} \left( \tau_{x''z''}^{i,j,c} i,j y'' - \tau_{x''y''}^{i,j,c} i,j z'' \right) \alpha^{i,j} \\ \sum_{i,j} f(\mu)^{i,j} \sigma_{x''x''}^{i,j,c} i,j z'' \alpha^{i,j} \\ \sum_{i,j} f(\mu)^{i,j} \sigma_{x''x''}^{i,j,c} i,j y'' \alpha^{i,j} \\ \sum_{i,j} f(\mu)^{i,j} \sigma_{x''x''}^{i,j,c} i,j y'' \alpha^{i,j} \\ \sum_{i,j} f(\mu)^{i,j} \sigma_{x''x''}^{i,j,c} i,j y'' \alpha^{i,j} \\ \\ \end{array} \right\}_{\mathbf{T}_{p,q}}^{\mathrm{mat},c}$$
(7)

The computation of the viscous, internal stress resultant vector  $\vec{F}_{\mathbf{T}_{p,q}}^{\mathrm{mat},c}$  proceeds, in practice, as follows: the viscous (generalized) strain components (strain rates)  $\dot{\epsilon}_{x''x''}^{i,j}, \dot{\gamma}_{x''y''}^{i,j}$ , and  $\dot{\gamma}_{x''z''}^{i,j}$ , Eq. (5), are computed at each integration point as simple time difference quotients through the relations of  $\dot{\epsilon}_{x''x''}^{i,j} = (\epsilon_{x''x''}^{i,j,t+\Delta t} - \epsilon_{x''y''}^{i,j,t})/\Delta t$ ,  $\dot{\gamma}_{x''y''}^{i,j} = (\gamma_{x''x''}^{i,j,t+\Delta t} - \gamma_{x''y''}^{i,j,t})/\Delta t$ , and  $\dot{\gamma}_{x''z''}^{i,j} = (\gamma_{x''z''}^{i,j,t+\Delta t} - \epsilon_{x''x''}^{i,j,t})/\Delta t$ . The strains with a superscript  $(\cdot)^t$  indicate here strains that are known from the previous time step, whereas those with a superscript  $(\cdot)^{t+\Delta t}$  denote strains at the end of the current time step. The latter will be returned by the decohesive damage algorithm. The strain rates so computed will next yield the components of the viscous traction vector,  $\vec{\sigma}_{\mathbf{T}_{p,q}}^{i,j,\mathrm{mat},c}$ , which, when substituted in Eq. (7), produce the viscous, internal stress resultant vector  $\vec{F}_{\mathbf{T}_{p,q}}^{\mathrm{mat},c}$ .

Simpson's two-dimensional, composite numerical integration rule is used in integrating the internal stress resultant vectors. The integrations are performed over the original cross-sectional areas of the beam finite elements. In order to have a consistent integration scheme, the number of integration points is set to depend on the thickness of a sheet sample considered. A grid of  $7 \times 7$  evenly distributed integration points (along the cross-sectional co-ordinate directions y'' and z'') is constructed for the sheets with a thickness of h = 0.5 m,<sup>4</sup> a grid of  $15 \times 15$  points for the sheets with a thickness of h = 1.0 m, and a grid of  $21 \times 21$  points for the sheets with a thickness of h = 1.5 m. Note that Simpson's

 $<sup>^4\</sup>mathrm{A}$  grid of 9  $\times$  9 points was tested and observed to produce results with a negligible difference.

composite integration rule requires an odd number of integration points (or even number of integration intervals) to be used. It should, however, be noted that because the cross-sections of the Timoshenko beam finite elements are not necessarily square, and vary from one beam to another, the discretisation densities may not be the same in the local y'' and z'' co-ordinate directions. The discretisation of the cohesive process zone may thus unavoidably vary depending on the direction a crack is advancing and also from one element to another.

The numerically integrated viscous and non-viscous internal stress resultant vectors  $\vec{F}_{\mathbf{T}_{p,q}}^{\mathrm{mat},c}$  and  $\vec{F}_{\mathbf{T}_{p,q}}^{\mathrm{mat}}$ , respectively, produce then the total, internal stress resultant vector,  $\vec{F}_{\mathbf{T}_{p,q}}^{\mathrm{tot}}$ , which, when substituted in Eq. (3), yields the internal, nodal force vector  $\vec{F}_{\mathbf{g},p,q}^{\mathrm{int}}$ . The ingredients of the de-cohesive damage model implemented are described next.

# 3.2. De-cohesive damage model

This subsection gives the details of the (effective) de-cohesive damage model. Described are, first, the conversion from the three-component state of stress to a one-dimensional "effective" state of stress, next the methodology to deal with an irreversible damage, and finally the procedure to return back to the three-component state in order to compute the internal stress resultant vectors  $\vec{F}_{\mathbf{T}_{p,q}}^{\text{mat}}$  and  $\vec{F}_{\mathbf{T}_{p,q}}^{\text{mat,c}}$ . The relations that follow are first given in a somewhat general format. Some simplifications deemed to be appropriate, and the implications thereof, are discussed last in sub-subsection 3.2.4. Note that all the relations below hold on an integration point level. This is depicted with a symbol "*i*, *j*".

#### 3.2.1. Effective constitutive model

Box 1 gives the basic constituents of the constitutive model. The made assumptions are that *i*) the inelastic deformations are small; *ii*) the total, effective strain measure,  $\epsilon_{\text{eff}}^{i,j}$ , can be – additively – decomposed into an elastic,  $\epsilon_{\text{e,eff}}^{i,j}$ , and an inelastic,  $\epsilon_{\text{c.s.,eff}}^{i,j}$  (c.s. ~ "cracking strain"), part, Eq. (8a); and that *iii*) an elastic stress-strain relationship holds, Eq. (8b). Eqs. (8c)-(8g) state relations that are needed in order to deal with an irreversible damage (a secant unloading/reloading relationship is given as well) and are treated later in sub-subsections 3.2.2 and 3.2.3.

Figure 5 gives a schematic illustration of the effective stress-strain relationship an integration point i, j follows. The figure has been divided in two parts. The right hand side gives the full, effective stress-strain relationship, whereas the left hand side depicts only the softening part. The scheme follows, in essence, that of Hillerborg et al. (1976). While in position "O," a state of zero effective stress prevails. On the ascending branch "O-A," Hooke's law is active and a point is virginal. The stress of an integration point evolves there linearly – either increases or decreases – but the point experiences no damage. Damage takes place only after point "A" and gets triggered the moment a critical, effective stress limit,  $\sigma_{cr,eff}^{i,j}$ , is reached. This critical stress limit is an effective quantity because both axial and shear stresses can initiate damage and because the limit evolves (decreases) as damage progresses. A reduced, critical stress limit is then denoted by a symbol  $\tilde{\sigma}_{\rm cr,eff}^{i,j}$ . On the descending branch "A-C," a point undergoes irreversible damage (softens linearly) such that at "C," a point is considered to be fully damaged. If, while on the descending branch "A-C," a point unloads, a secant unloading/reloading branch "O-B" is followed instead.

**Box 1.** An effective (one-dimensional), rate-independent, elastic-de-cohesive constitutive model with a linear softening law.

1) Additive decomposition of elastic and inelastic (cracking) strains:  $\epsilon_{\text{eff}}^{i,j} = \epsilon_{\text{e,eff}}^{i,j} + \epsilon_{\text{c.s.,eff}}^{i,j} = \frac{\sigma_{\text{eff}}^{i,j}}{r^{i,j}} + \frac{\delta_{\text{eff}}^{i,j}}{r}.$ (8a)

 $\left( \begin{array}{c} \delta^{i,j} \end{array} \right)$ 

$$\sigma_{\rm eff}^{i,j} = E_{\rm eff}^{i,j} \left( \epsilon_{\rm eff}^{i,j} - \frac{\delta_{\rm eff}^{\circ}}{L_{0,p,q}} \right).$$
(8b)

3) Flow rules, (linear) softening law, and secant unloading/reloading relation:

$$\dot{\epsilon}_{\rm c.s.,eff}^{i,j} = \frac{\Delta \delta_{\rm eff}^{i,j}}{L_{0,p,q}}, \ \dot{\delta}_{\rm eff}^{i,j} \simeq \Delta \delta_{\rm eff}^{i,j} \ ({\rm cracking strain and separation}),$$
(8c)

$$\sigma_{\rm eff}^{i,j} = \begin{cases} \sigma_{\rm cr,eff}^{i,j} \left( 1 - \frac{\delta_{\rm eff}^{i,j}}{\delta_{\rm cr,eff}^{i,j}} \right) \text{ (linear softening),} \\ \sigma_{\rm cr,eff}^{i,j} \left( 1 - \frac{\delta_{\rm max,eff}^{i,j}}{\delta_{\rm cr,eff}^{i,j}} \right) \frac{\delta_{\rm eff}^{i,j}}{\delta_{\rm max,eff}^{i,j}} \text{ (unloading/reloading).} \end{cases}$$
(8d)

4) Loading function:

$$F\left(\sigma_{\text{eff}}, \delta_{\text{eff}}\right)^{i,j} = \sigma_{\text{eff}}^{i,j} - \sigma_{\text{cr,eff}}^{i,j}(\delta_{\text{eff}}^{i,j}) \le 0.$$
(8e)

5) Karush-Kuhn-Tucker complementarity conditions:

$$\dot{\delta}_{\text{eff}}^{i,j} \ge 0, \ F \left(\sigma_{\text{eff}}, \delta_{\text{eff}}\right)^{i,j} \le 0, \text{ and } \dot{\delta}_{\text{eff}}^{i,j} F \left(\sigma_{\text{eff}}, \delta_{\text{eff}}\right)^{i,j} = 0.$$
  
**6)** Consistency (persistency) condition:
  
(8f)

$$\dot{\delta}_{\text{eff}}^{i,j} \dot{F} \left(\sigma_{\text{eff}}, \delta_{\text{eff}}\right)^{i,j} = 0 \text{ if } F \left(\sigma_{\text{eff}}, \delta_{\text{eff}}\right)^{i,j} = 0.$$
(8g

After an integration point has damaged fully, a true crack advances and frictional contact takes place. Frictional contact continues as long as there are other integration points along the same cross-section that have not yet damaged fully. After a beam has fully degraded, the discrete elements joined by the beam separate and a DEM contact scheme is activated. In practice, a beam is considered to be fully degraded as soon as a critical number of fully damaged integration points is reached. A criterion of a form  $a^2 - b = a$  is applied, where a is equal to the square root of the total number of integration points along a cross-section and b denotes the number of fully damaged integration points. For

example, for the sheets with a thickness of h = 0.5 m, a quadrature of  $7 \times 7$  points was constructed. When the total number of fully damaged integration points reaches 42 ( $7^2 - 42 = 7$ ), a beam is considered to be fully degraded. Such a simplification is necessary in order to avoid a situation where, for instance, a beam in pure bending does not degrade fully unless a large shear or tensile stress acts that causes the last integration points along the outermost section of a beam to fully damage. Pure bending alone cannot damage the last integration points because the position of the neutral axis is updated when a crack is propagating.

Before proceeding further, it is noted that the model, in fact, closely resembles a one-dimensional, rate-independent plasticity model that includes a linear, isotropic hardening law (a strain-driven problem in an effective stress space). See, for example, (Simo and Hughes, 1998, Box 1.2, p. 11). A few differences, however, exist. In the present model, the material does not harden but softens; no purely compressive failure is modelled; no permanent strains remain if an integration point unloads completely (a fictitious crack closes); and upon full damage, a true crack forms with frictional contact taking place afterwards.

Box 2 states the relationships the constitutive parameters shown in Figure 5 and given in Box 1 follow. The parameter  $E_{\text{eff}}^{i,j,\star}$  is an effective secant modulus (in terms of effective stresses and strains),  $D_{\text{eff}}^{i,j}$  a damage index,  $E_{\text{T,eff}}^{i,j}$  a tangent modulus,  $K_{\text{S,eff}}^{i,j}$  a softening modulus,  $K_{\text{eff}}^{i,j}$  an effective secant modulus (in terms of effective stresses and separations), and  $G_{\rm eff}^{i,j}$  an effective, specific fracture energy parameter. The symbols  $\delta_{\rm eff}^{i,j}$ ,  $\delta_{\rm max,eff}^{i,j}$ ,  $\delta_{\rm cr,eff}^{i,j}$ ,  $\sigma_{\rm cr}$ ,  $\tau_{\rm cr}$ , and  $L_{0,p,q}$  denote, in addition, an effective crack-opening displacement measure (COD); an effective, maximum crack-opening displacement measure; an effective, critical crack-opening displacement measure; a critical axial stress parameter; a critical shear stress parameter; and the length of a Timoshenko beam finite element p, q in the initial, strain-free configuration; respectively. Note that  $\delta_{\max, eff}^{i,j}$  is, in essence, an internal state variable keeping record of the maximum damage an integration point i, j has suffered thus far. It is, in this respect, perfectly analogous to, for example, the effective plastic strain in small strain hardening plasticity. Notice also that the effective, specific fracture energy parameter  $G_{\text{eff}}^{i,j}$ corresponds to the total amount of energy dissipated during de-cohesive softening and is equal to the area highlighted in red in Figure 5. It is, as well, an effective quantity because of its association with a mixed mode fracture (viz., a fracture does not necessarily occur in mode I only).

Box 3 gives the relationships the effective stress measure, the effective (elastic) strain measure, and the effective Young's modulus depicted in Box 1 and shown in Figure 5 follow. The expression of the effective stress measure follows from a manipulation of the loading function given in Eq. (11a) below. By adopting an expression of the form given in Eq. (9a), a particularly simple one-dimensional representation for the loading function results, Eq. (11b). The expression of the effective, elastic strain measure, Eq. (9d), follows then simply by assuming a form similar to that of the effective stress measure. The relationships  $\sigma_{x''x''}^{i,j} = E_b \epsilon_{x''x''}^{i,j}, \tau_{x''y''}^{i,j} = G_b \gamma_{x''y''}^{i,j}$ , and  $\tau_{x''z''}^{i,j} = G_b \gamma_{x''z''}^{i,j}$  have been utilised in the manipulations, see Eq. (4). The expression of the effective Young's modulus,  $E_{\rm eff}^{i,j}$ , follows finally by taking a quotient of the effective stress and strain measures, Eq. (9e). It has been here given as a function of the local strains and the critical stresses. Note that  $E_{\rm eff}^{i,j}$  relates the effective stress,  $\sigma_{\rm eff}^{i,j}$ , to the effective, elastic strain,  $\epsilon_{\rm e,eff}^{i,j}$ , on the ascending branch "O-A," see Figure 5.



Box 2. The relationships the constitutive parameters shown in the figure follow.

$$\begin{split} E_{\rm eff}^{i,j,\star} &= E_{\rm eff}^{i,j} D_{\rm eff}^{i,j}, \ D_{\rm eff}^{i,j} = \left(1 - \frac{\delta_{\rm max,eff}^{i,j}}{\delta_{\rm cr,eff}^{i,j}}\right), \ E_{\rm T,eff}^{i,j} &= \frac{E_{\rm eff}^{i,j} K_{\rm S,eff}^{i,j}}{E_{\rm eff}^{i,j} - K_{\rm S,eff}^{i,j}}, \\ K_{\rm S,eff}^{i,j} &= \frac{(\sigma_{\rm cr,eff}^{i,j})^2 L_{0,p,q}}{2G_{\rm eff}^{i,j}}, \ K_{\rm eff}^{i,j} &= \frac{\tilde{\sigma}_{\rm cr,eff}^{i,j}}{\delta_{\rm max,eff}^{i,j}}, \ G_{\rm eff}^{i,j} &= \int_0^{\delta_{\rm cr,eff}^{i,j}} \sigma_{\rm eff}^{i,j} d\delta_{\rm eff}^{i,j} &= \frac{\sigma_{\rm cr,eff}^{i,j} \delta_{\rm cr,eff}^{i,j}}{2}. \end{split}$$

Figure 5: A schematic illustration of an effective, rate-independent, elastic-de-cohesive constitutive model with a linear softening law. On the left: an effective, linearly softening traction-separation law in terms of  $\sigma_{\rm eff}^{i,j}$  and  $\delta_{\rm eff}^{i,j}$ ; on the right: an effective stress-strain law in terms of  $\sigma_{\rm eff}^{i,j}$  and  $\epsilon_{\rm eff}^{i,j}$ . Box 2 gives the relationships the different constitutive parameters shown in the figure follow. The expressions of the effective stress measure,  $\sigma_{\rm eff}^{i,j}$ ; effective, elastic strain measure,  $\epsilon_{\rm e,eff}^{i,j}$ ; and the effective Young's modulus,  $E_{\rm eff}^{i,j}$ , on the ascending branch "O-A," have been given in Box 3. The expressions of the effective stress measures on the descending, "A-C," and on the unloading/reloading, "O-B," branches have been given in Box 1.

Box 3. Effective stress, strain, and Young's modulus.

Effective stress in terms of local stresses:  $\sigma_{\text{eff}}^{i,j} = \sigma_{x''x''}^{i,j} + \frac{\sigma_{\text{cr}}\left(\left(\tau_{x''y''}^{i,j}\right)^2 + \left(\tau_{x''z''}^{i,j}\right)^2\right)}{\tau_{\text{cr}}^2}.$ (9a) Effective stress in terms of local strains:

$$\sigma_{\rm eff}^{i,j,\dagger} = E_{\rm b} \epsilon_{x''x''}^{i,j} + \frac{G_{\rm b}^2 \sigma_{\rm cr} \left( (\gamma_{x''y''}^{i,j})^2 + (\gamma_{x''z''}^{i,j})^2 \right)}{\tau_{\rm cr}^2}.$$
(9b)

Effective, elastic strain in terms of local strains:

$$\epsilon_{\rm e,eff}^{i,j} = \epsilon_{x''x''}^{i,j} + \frac{\epsilon_{x''x'',\rm cr}^{i,j} \left( (\gamma_{x''y''}^{i,j})^2 + (\gamma_{x''z''}^{i,j})^2 \right)}{(\gamma_{x''y'',\rm cr}^{i,j})^2 + (\gamma_{x''z'',\rm cr}^{i,j})^2}.$$
(9c)

Effective, elastic strain in terms of local strains and critical stresses:

$$\epsilon_{\rm e,eff}^{i,j,\dagger} = \epsilon_{x''x''}^{i,j} + \frac{G_{\rm b}^2 \sigma_{\rm cr} \left( (\gamma_{x''y''}^{i,j})^2 + (\gamma_{x''z''}^{i,j})^2 \right)}{2E_{\rm b} \tau_{\rm cr}^2}.$$
(9d)

Effective Young's modulus in terms of local strains/critical stresses:

$$E_{\rm eff}^{i,j} = \frac{\sigma_{\rm eff}^{i,j,\dagger}}{\epsilon_{\rm e,eff}^{i,j,\dagger}} = \frac{2E_{\rm b} \left(\tau_{\rm cr}^2 E_{\rm b} \epsilon_{x''x''}^{i,j} + G_{\rm b}^2 \sigma_{\rm cr} \left((\gamma_{x''y''}^{i,j})^2 + (\gamma_{x''z''}^{i,j})^2\right)\right)}{2E_{\rm b} \tau_{\rm cr}^2 \epsilon_{x''x''}^{i,j} + G_{\rm b}^2 \sigma_{\rm cr} \left((\gamma_{x''y''}^{i,j})^2 + (\gamma_{x''z''}^{i,j})^2\right)\right)}.$$
 (9e)

# 3.2.2. Loading function

A de-cohesive loading function introduced by Schreyer et al. (2006) is here adopted. Schreyer's loading function is advantageous as it enables both the axial and the shear stresses to be taken into account. Box 4 gives the equations of the loading function, the effective loading function, as well as an interpretation of the effective loading function in terms of a flow stress. A schematic representation of an evolving loading surface (a failure surface) is then given in Figure 6.

The modelling of the ice sheet failure is, in the present model, limited to the modelling of failure either in tension, shear, their combination, or the combination of compression and shear. No purely compressive failure (i.e. ice crushing with  $\sigma_{x''x''} < 0$  while  $\tau_{x''y''} = \tau_{x''z''} = 0$  simultaneously holds) is modelled. Some compressive inelasticity is heuristically modelled through the contact damping, but it becomes effective only after a beam has fully eroded (i.e. during the DEM-phase). Notice that the fictitious crack model of Hillerborg et al. (1976) is not meant for modelling purely compressive failure. Ice crushing is an important topic, see (Schulson and Duval, 2009), but ought not to play a significant role in the present context.

The constitutive status of an integration point can be monitored by following

the value the loading function attains. The following three values are possible:

 $F^{i,j} = \begin{cases} < 0 \,, \, \text{if a virgin or in a secant unloading/reloading/"stationary" state,} \\ 0 \quad, \, \text{if in a damage loading, secant unloading, or a neutral state, (10)} \\ > 0 \,, \, \text{if the state of stress is inadmissible.} \end{cases}$ 

In above, and also below,  $F^{i,j} \sim F(\sigma_{\text{eff}}, \delta_{\max,\text{eff}})^{i,j}$ . About how is it in practice determined, in which state a point is, is explained in the next sub-subsection.

Box 4. Schreyer's loading function.

Loading function:  

$$F(\vec{\sigma}_{\mathbf{T}_{p,q}}^{\text{mat}}, \delta_{\max,\text{eff}})^{i,j} = \frac{\sigma_{x''x''}^{i,j}}{\sigma_{\text{cr}}} + \frac{(\tau_{x''y''}^{i,j})^2 + (\tau_{x''z''}^{i,j})^2}{\tau_{\text{cr}}^2} + \frac{\delta_{\max,\text{eff}}^{i,j}}{\delta_{\text{cr,eff}}^{i,j}} - 1.$$
(11a)

Effective loading function:

$$F\left(\sigma_{\text{eff}}, \delta_{\text{max,eff}}\right)^{i,j} = \sigma_{\text{eff}}^{i,j} - \sigma_{\text{cr,eff}}^{i,j} \left(1 - \frac{\delta_{\text{max,eff}}^{i,j}}{\delta_{\text{cr,eff}}^{i,j}}\right) \le 0.$$
(11b)

Interpretation in terms of a flow stress:

$$F\left(\sigma_{\text{eff}}, \delta_{\text{max,eff}}\right)^{i,j} = \sigma_{\text{eff}}^{i,j} - \sigma_{\text{cr,eff}}^{i,j}(\delta_{\text{max,eff}}^{i,j}) \le 0.$$
(11c)

In addition to the loading function, relations describing restrictions that must be satisfied, whether the material is suffering damage or not, are required. Such restrictions are provided by the Karush-Kuhn-Tucker (KKT) complementarity (optimality) conditions, Eq. (8f), and the persistency condition, Eq. (8g). The first two KKT conditions assert that dissipation (i.e. damage growth) should occur only when the loading function  $F(\sigma_{\text{eff}}, \delta_{\max,\text{eff}})^{i,j} = 0$ . The last condition declares that the current stress point must lie on or below the current loading surface. The consistency (persistency) condition demands finally that the relaxed stress point must be consistent with, and therefore lie, on the reduced, final loading surface. The condition yields, analogously to plasticity, a crack opening displacement increment (~ a consistency parameter/plastic multiplier), Eq. (15a). Most of the different constitutive states an integration point can be in have been illustrated in Eq. (12) (no rates of  $F^{i,j}$  have been here considered). Those that have not, have been described in the next paragraph.

$$\begin{cases} F^{i,j} > 0 & \Leftrightarrow & \text{inadmissible state of stress,} \\ F^{i,j} = 0 \text{ and } \begin{cases} \dot{\delta}_{\text{eff}}^{i,j} < 0 \Leftrightarrow & \text{secant unloading,} \\ \dot{\delta}_{\text{eff}}^{i,j} = 0 \Leftrightarrow & \text{neutral loading,} \\ \dot{\delta}_{\text{eff}}^{i,j} > 0 \Leftrightarrow & \text{damage loading,} \\ \dot{\delta}_{\text{eff}}^{i,j} < 0 \Leftrightarrow & \text{secant unloading,} \\ \dot{\delta}_{\text{eff}}^{i,j} < 0 \Leftrightarrow & \text{secant unloading,} \\ \dot{\delta}_{\text{eff}}^{i,j} = 0 \text{ and } \begin{cases} \delta_{\text{max,eff}}^{i,j} = 0 & \Leftrightarrow & \text{virginal state,} \\ 0 < \delta_{\text{max,eff}}^{i,j} < \delta_{\text{cr,eff}}^{i,j} \Leftrightarrow & \text{"stationary" loading,} \\ \dot{\delta}_{\text{eff}}^{i,j} > 0 \Leftrightarrow & \text{secant reloading.} \end{cases} \end{cases}$$

Note that existing damage is required for an integration point to be in a secant unloading, secant reloading, or a "stationary" loading state if  $F^{i,j} < 0$ . A further detail to be noticed (not stated in Eq. (12)) is that if  $F^{i,j} < 0$  and  $\sigma_{x''x''}^{i,j} < 0$ , the assumption of a virginal material is made regardless the previous state (either a virgin or partly damaged). If in a partly damaged state, the material is thus able to heal. For a point to be in a frictional loading state (frictional contact in Figure 5, also not stated in Eq. (12)) requires that a true crack exists, is currently closed, and is under a compressive state of stress. The frictional contact forces are then computed through Eqs. (6) and (7) by setting  $f(\mu)^{i,j} = \mu$ . For a virginal material – as well as for a fully unloaded, partly damaged material –  $f(\mu)^{i,j} = 1$ .

# 3.2.3. Evolving damage – stress state computation

In order to determine whether a stress state is within an elastic domain.  $F(\sigma_{\text{eff}}, \delta_{\text{max,eff}})^{i,j} < 0$ , on the loading surface,  $F(\sigma_{\text{eff}}, \delta_{\text{max,eff}})^{i,j} = 0$ , or inadmissible,  $F(\sigma_{\text{eff}}, \delta_{\text{max,eff}})^{i,j} > 0$ , a trial, elastic stress predictor is computed via an "elastic predictor step," Box 5. If it holds that  $F(\sigma_{\text{eff}}, \delta_{\text{max,eff}})_{t+\Delta t}^{i,j,\text{trial}} < 0$ , a point is either a virgin or in a secant unloading, secant reloading, or a stationary loading state. In the simplest case of a virginal material, the stress state is accepted as it is, no updates or transformations are necessary. If on the secant unloading/reloading branch, an existing effective, fictitious crack either closes or opens. The effective crack opening displacement,  $\delta_{\text{eff}}^{i,j}$ , can in this case be computed directly by Eq. (8a) because the total, effective strain,  $\epsilon_{\text{eff}}^{i,j}$ , is known and requires no corrections. The effective stress follows then from Eq. (8d) and is next transformed back to the local stress and strain components following a transformation similar to that explained below, see Box 7. Note that on the unloading/reloading branch, an effective crack opening displacement increment does not lead to an increased damage. The maximum, effective crack opening displacement,  $\delta_{\max,\text{eff}}^{i,j}$ , increases only on the descending branch "A-C," Figure 5.

If, on the contrary, it holds that  $F(\sigma_{\text{eff}}, \delta_{\max,\text{eff}})_{t+\Delta t}^{i,j,\text{trial}} > 0$ , damage progresses. The state of stress is such that a relaxation (return) onto a new, reduced loading surface (to be simultaneously determined) is required. This is accomplished via a "damage corrector step," Box 6. Note that in the two(three)-dimensional, local stress space, Figure 6, the return direction will not be perpendicular to the loading surface. The flow rule is non-associative, the existence of a spherical damage potential function  $Q^{i,j}, Q^{i,j} \neq F^{i,j}$ , is thus postulated. In an effective stress space, the return direction will, on the contrary, be perpendicular to the loading "surface," see Figure 7. With the aid of the damage corrector step, a new i) (true) crack opening displacement increment; ii) relaxed, effective stress; and iii) a (true) maximum, effective crack opening displacement can then be computed. A new, reduced loading surface,  $F(\sigma_{\text{eff}}, \delta_{\max,\text{eff}})_{t+\Delta t}^{i,j} = 0$ , is simultaneously determined.

A remark: the denominator in Eq. (15a), Box 6, yields an upper limit for the element length. The condition must be satisfied in order to avoid snapback



Figure 6: A schematic representation of evolving de-cohesive loading and damage potential surfaces in a two-dimensional, local stress space. A three-dimensional representation follows from a simple rotation around the " $\tau_{x''y''}^{i,j}/\tau_{\rm cr} = 0$ " axis. Note that no purely compressive failure (i.e. ice crushing with  $\sigma_{x''x''} < 0$  while  $\tau_{x''y''} = \tau_{x''z''} = 0$  simultaneously holds) is modelled. The stress components with a superscript " $\star$ " denote stresses that have been relaxed to the new, reduced loading surface  $F\left(\sigma_{\rm eff}, \delta_{\max,{\rm eff}}\right)_{t+n\Delta t}^{i,j} = 0$ . Note also that the flow rule is non-associative in a two(three)-dimensional, local stress space, but can be interpreted to be associative in an effective (one-dimensional) stress space, see Figure 7.

instability. Written out full, the condition reads as:

$$L_{0,p,q} < \frac{2G_{\text{eff}}^{i,j} E_{\text{eff}}^{i,j}}{(\sigma_{\text{cr.eff}}^{i,j})^2} = 2I_{\text{ch.}}$$
(13)

The limiting case, i.e.  $L_{0,p,q} = 2G_{\text{eff}}^{i,j}E_{\text{eff}}^{i,j}/(\sigma_{\text{cr,eff}}^{i,j})^2$ , corresponds to a sudden vertical stress drop because the tangential modulus  $E_{\text{T,eff}}^{i,j} \mapsto -\infty$  (since  $K_{\text{S,eff}}^{i,j} \mapsto -E_{\text{eff}}^{i,j}$ ), (Bažant and Oh, 1983). Note that the parameter  $I_{\text{ch}}$  is, in fact, Irwin's (or Hillerborg's) characteristic length, see subsection 6.1 below.

Box 5. Elastic predictor step.

1) trial crack-opening displacement (t denotes previous time step):  $\delta_{\max,\text{eff},t+\Delta t}^{i,j,\text{trial}} = \delta_{\max,\text{eff},t}^{i,j},$ 

2) trial stress:

Compute:

$$\sigma_{\text{eff},t+\Delta t}^{i,j,\text{trial}} = E_{\text{eff}}^{i,j} \left( \epsilon_{\text{eff},t+\Delta t}^{i,j} - \frac{\delta_{\max,\text{eff},t+\Delta t}^{i,j,\text{trial}}}{L_{0,p,q}} \right), \text{ and}$$
(14b)

(14a)

3) test the loading function:

$$F\left(\sigma_{\rm eff}, \delta_{\rm max, eff}\right)_{t+\Delta t}^{i,j,\rm trial} = \sigma_{\rm eff, t+\Delta t}^{i,j,\rm trial} - \sigma_{\rm cr, eff}^{i,j} \left(1 - \frac{\delta_{\rm max, eff, t+\Delta t}^{i,j,\rm trial}}{\delta_{\rm cr, eff}^{i,j}}\right).$$
(14c)

Box 6. Damage corrector step.

Find the new: 1) crack opening displacement increment (consistency parameter):  $\Delta \delta_{\max,\text{eff}}^{i,j} = \frac{F\left(\sigma_{\text{eff}}, \delta_{\max,\text{eff}}\right)_{t+\Delta t}^{i,j,\text{trial}} L_{0,p,q}}{E_{\text{eff}}^{i,j} - K_{\text{S,eff}}^{i,j}} > 0, \qquad (15a)$ 

2) relaxed, effective stress:

$$\sigma_{\text{eff},t+\Delta t}^{i,j} = \left[1 - \frac{\Delta \delta_{\max,\text{eff}}^{i,j} E_{\text{eff}}^{i,j}}{L_{0,p,q} |\sigma_{\text{eff},t+\Delta t}^{i,j,\text{trial}}|}\right] \sigma_{\text{eff},t+\Delta t}^{i,j,\text{trial}},$$
(15b)

3) maximum, effective crack opening displacement:

$$\delta_{\max,\text{eff},t+\Delta t}^{i,j} = \delta_{\max,\text{eff},t}^{i,j} + \Delta \delta_{\max,\text{eff}}^{i,j}, \text{ and}$$
(15c)

4) the reduced loading surface 
$$F(\sigma_{\text{eff}}, \delta_{\max, \text{eff}})_{t+\Delta t}^{i,j} = 0.$$
 (15d)

After a "damage corrector step" has been taken, the value of the relaxed, effective stress (a scalar) is known, Eq. (15b). The values of the relaxed strain components are, however, not known. In order to be able to compute the internal stress resultant vectors, Eqs. (6) and (7), these must be determined. Their values can be found via a requirement that the return mapping satisfies the

conditions given in Eqs. (16) below (the function sgn (·) denotes there a signum function, i.e. sgn (·) = (·) /| (·) |). The conditions state, in essence, that the two traction vectors (trial and relaxed) must be coaxial and that their components shall have equal signs. Note that these requirements are enforced in the three-dimensional, local stress space, Figure 6. In terms of analytic geometry, the problem reduces to finding the intersection point of the new, reduced loading surface,  $F(\sigma_{\text{eff}}, \delta_{\max,\text{eff}})_{t+\Delta t}^{i,j} = 0$ , and the trial traction vector. The end result is a set of three quadratic equations whose solutions are given in Box 7. Note that these solutions are for the stress and strain components in the octants I, IV, V, and VIII, Figure 6. Similar relations hold in the other four octants as well. The strains in Box 7 are subsequently substituted in Eqs. (6) and (7) for a numerical integration of the internal stress resultant vectors  $\vec{F}_{\text{Tp},q}^{\text{mat}}$  and  $\vec{F}_{\text{Tp},q}^{\text{mat},c}$ . These, and Eq. (2), produce then the internal, nodal force vector  $\vec{F}_{\text{g},p,q,t+\Delta t}^{\text{int}}$  is substituted in the equations of motion, Eq. (1), to evolve the system in time. This accomplishes the formulation.



Figure 7: A schematic illustration of an evolving de-cohesive loading "surface" in an effective stress space. Presented are the trial,  $F^{\text{trial}} = F\left(\sigma_{\text{eff}}, \delta_{\max,\text{eff}}\right)_{t+\Delta t}^{i,j,\text{trial}}$ , and the reduced,  $F^{\text{red.}} = F\left(\sigma_{\text{eff}}, \delta_{\max,\text{eff}}\right)_{t+\Delta t}^{i,j} = 0$ , loading surfaces. A return mapping is "radial" in an effective stress space (the relaxed stress point is on a plane tangent to the reduced loading surface, the normal of which is given by the gradient  $\left|\frac{\partial F^{\text{red.}}}{\partial \sigma_{\text{eff},t+\Delta t}}\right|$ ).

Algorithm 1 below gives a simplified pseudo-code of the constitutive algorithm. On a beam level, the loop runs as long as the number of fully damaged integration points is below the criterion given in sub-subsection 3.2.1 above. After a beam has fully eroded (the criterion has been satisfied), the hybrid FE-DE scheme turns into a pure DEM scheme – adjacent discrete elements separate and start to interact through contacts. On a mesh level, the loop runs as long as there are beams that have not yet degraded fully.

$$\begin{cases} \frac{\tau_{x''y''}^{i,j,\star}}{\sigma_{x''x''}^{i,j,\star}} = \frac{\tau_{x''y''}^{i,j,\text{trial}}}{\sigma_{x''x''}^{i,j,\text{trial}}}, \frac{\tau_{x''z''}^{i,j,\star}}{\sigma_{x''x''}^{i,j,\star}} = \frac{\tau_{x''z''}^{i,j,\text{trial}}}{\sigma_{x''x''}^{i,j,\text{trial}}}, \frac{\tau_{x''z''}^{i,j,\star}}{\tau_{x''y''}^{i,j,\star}} = \frac{\tau_{x''z''}^{i,j,\text{trial}}}{\tau_{x''y''}^{i,j,\text{trial}}}, \\ \operatorname{sgn}(\sigma_{x''x''}^{i,j,\star}) = \operatorname{sgn}(\sigma_{x''x''}^{i,j,\text{trial}}), \operatorname{sgn}(\tau_{x''y''}^{i,j,\star}) = \operatorname{sgn}(\tau_{x''y''}^{i,j,\text{trial}}), \\ \operatorname{sgn}(\tau_{x''z''}^{i,j,\star}) = \operatorname{sgn}(\tau_{x''z''}^{i,j,\text{trial}}). \end{cases}$$
(16)

# Box 7. Relaxed, local stress and strain components.

Compute new:

- 1) relaxed, local stress components:  $\sigma_{x''x''}^{i,j,\star} = \sigma_{x''x''}^{i,j,\circ} \tau_{cr}A, \ \tau_{x''y''}^{i,j,\star} = \tau_{x''y''}^{i,j,\circ} \tau_{cr}A, \text{ and } \tau_{x''z''}^{i,j,\star} = \tau_{x''z''}^{i,j,\circ} \tau_{cr}A,$ (17a)
- 2) relaxed, local strain components:  $\epsilon_{x''x''}^{i,j,\star} = \sigma_{x''x''}^{i,j,\star}/E_{\rm b}, \ \gamma_{x''y''}^{i,j,\star} = \tau_{x''y''}^{i,j,\star}/G_{\rm b}, \text{ and } \gamma_{x''z''}^{i,j,\star} = \tau_{x''z''}^{i,j,\star}/G_{\rm b},$ (17b) where

$$A = \frac{-\sigma_{x''x''}^{i,j,\circ}\tau_{\rm cr} + \sqrt{(\sigma_{x''x''}^{i,j,\circ}\tau_{\rm cr})^2 + 4\sigma_{\rm cr}\sigma_{\rm eff}^{i,j,\star}\left((\tau_{x''y''}^{i,j,\circ})^2 + (\tau_{x''z''}^{i,j,\circ})^2\right)}{2\sigma_{\rm cr}\left((\tau_{x''y''}^{i,j,\circ})^2 + (\tau_{x''z''}^{i,j,\circ})^2\right)},$$
 (17c)  
and  $(\cdot)^{i,j,\circ} = (\cdot)^{i,j,{\rm trial}}.$ 

# 3.2.4. Some simplifications and observations

It is, in practice, computationally heavy – and perhaps even unnecessary – to update the values of the effective Young's moduli  $E_{\text{eff}}^{i,j}$  at each integration point and at each time step taken separately. A look in Eq. (9e) reveals that in a state of uniaxial tension (i.e. when  $\gamma_{x''y''}^{i,j} = \gamma_{x''z''}^{i,j} = 0$ )  $E_{\text{eff}}^{i,j} = E_{\text{b}}$ . The effective Young's moduli were thus not computed and updated as was just discussed, but were, bluntly, set to equal the Young's modulus  $E_{\text{b}}$  given to the Timoshenko beam finite elements. The scheme should work well in tension – which ought to be the dominant fracture mode – but to result in an artificially brittle response in shear. This is because  $E_{\text{b}}/G_{\text{b}} \approx 2.6$  and so smaller strains lead to a damage. The ratio  $E_{\text{eff}}/G_{\text{b}}$  would be, in fact, twice as large (if not scaled to yield a unity). It is to be noticed, however, that because the stiffness of an ice sheet modelled with a lattice-based approach is a relative quantity, (Lilja et al., 2019a), it is a

rather complicated task (and possibly impossible if a lattice is unstructured) to predict the "true" onset of damage. The effective stiffnesses (the values of which depend on the relative sheet size parameter L/l) affect the computed strains, which are, then, driving the damage. In other words, damage will generally not initiate on strain levels one would expect. The problem belongs, therefore, to the realm of multiscale problems.

Another question is that of the secant modulus  $E_{\text{eff}}^{i,j,\star}$ . Notice that the elastic stiffness applied in computing the stresses does not evolve as the damage progresses, Eqs. (8a) and (8b). A damaged, effective secant modulus  $E_{\text{eff}}^{i,j,\star}$  is employed (effectively) only on the secant unloading/reloading branch "O-B," Figure 5. Once back on the effective stress-strain curve (point "B" in Figure 5), the virginal stiffness, i.e.  $E_{\text{b}}$ , is fully recovered. In terms of different "types" of uniaxial softening models, see (Bažant and Planas, 1997, sec. 8.4, p. 228), the model thus falls into the category of "mixed" models. For such models, it is typical that both the strength limit and the elastic moduli suffer damage. In the current model, a decreased stiffness is applied (effectively), however, only on the secant unloading/reloading branch and not otherwise. Note that in plasticity, from where most of the formulation has been adopted from, the elastic stiffness does not change.

It is interesting to note what would happen if the elastic stiffness would be decreased. In order for the softening process to remain stable (i.e. no snapback instability), Eq. (13) must be satisfied. A look into the equation reveals that the upper limit for the element length is, in fact, a monotonically decreasing function of  $E_{\text{eff}}^{i,j}$ . In case the stiffness would be decreased, the element length should thus get (adaptively) decreased as well. In the limit, the element length  $L_{0,p,q} \mapsto 0$  because  $E_{\text{eff}}^{i,j} \mapsto 0$ . Such a requirement is obviously not acceptable. It would thus be necessary to either stop updating the modulus well before the stability limit is reached or choose not to update it at all. This last option is, in a sense, what was done also here.

A final remark: if the effective Young's modulus  $E_{\text{eff}}^{i,j}$  of an integration point would be computed by Eq. (9e), it would necessarily be a function of the distance from the neutral axis because the local strains  $\epsilon_{x''x''}^{i,j}$ ,  $\gamma_{x''y''}^{i,j}$ , and  $\gamma_{x''z''}^{i,j}$  are linear functions of the distance parameters  ${}^{i,j}y''$  and  ${}^{i,j}z''$ , see Eqs. (4). The flexural rigidity of an ice sheet could thus be made to vary across the thickness, an aspect that was discussed in (Kerr and Palmer, 1972). Note, however, that the Young's modulus of an ice sheet varies usually linearly across the thickness because of the temperature difference between the bottom and the top surfaces.

Ir	<b>put:</b> $\delta_{\text{cr.eff}}^{i,j}, \delta_{\max,\text{eff},t}^{i,j}, \vec{\varepsilon}_{\mathbf{T}_{p,q},t+\Delta t}$ , and the parameters in Box 1-Box 7.				
0	<b>Putput:</b> assembled internal, nodal force vector $\vec{F}_{\sigma t+\Delta t}^{\text{int}}$ .				
1 W	hile $t < t_{end}$ do				
2	for each Timoshenko beam finite element $p, q$ do				
3	if not fully degraded then				
4	for each integration point $i, j$ do				
5	Local stresses and strains, Eq. (4).				
6	if $\delta_{\max,\text{eff},t}^{i,j} < \delta_{\text{cr,eff}}^{i,j}$ then				
7	Elastic predictor step, Box 5.				
8	if $F(\sigma_{\text{eff}}, \delta_{\max, \text{eff}})_{t+\Delta t}^{i,j, \text{trial}} < 0$ then				
9	if $\delta_{\max \text{ eff } t+\Delta t}^{i,j,\text{trial}} = 0$ then				
10	Virginal state. Set $f(\mu)^{i,j} = 1$ .				
11	else if $\delta_{max,\sigma}^{i,j,\text{trial}} = 0$ and $\sigma_{-l'-l'}^{i,j} > 0$ then				
12	Secant unloading/reloading state, Box 1.				
13	else				
14	Virginal state. Set $f(\mu)^{i,j} = 1$ .				
15	end				
16	else				
17	Damage loading state.				
18	Damage corrector step, Box 6.				
19	Relaxed stresses and strains, Box 7.				
20	end				
21	else				
22	Frictional state. Set $f(\mu)^{i,j} = \mu$ .				
23	end				
24	end				
25	else				
26	Internal, nodal force vector $F_{\mathbf{g},p,q,t+\Delta t}^{\text{int}} = \vec{0}$ , continue.				
27	end				
28	if new fully damaged integration points then				
29	A crack advances.				
30	Update the position of the neutral axis.				
31	end				
32	Stress resultant vectors $F_{\mathbf{T}_{p,q},t+\Delta t}^{\text{max}}$ and $F_{\mathbf{T}_{p,q},t+\Delta t}^{\text{max},\circ}$ , Eqs. (6)-(7).				
33	Internal, nodal force vector $\vec{F}_{\mathbf{g},p,q,t+\Delta t}^{\text{int}}$ , Eq. (3).				
34	end				
35	Assembled internal, nodal force vector $\vec{F}_{\mathbf{g},\mathbf{de},t+\Delta t}^{\text{int}}$ .				
36	Evolve the system in time, Eq. $(1)$ .				
37 ei	end				

Algorithm 1: A simplified pseudocode of the constitutive algorithm.

#### 4. The sheet samples examined and the simulation setups

This section gives the simulation setups of the load cases analysed. The main features are presented and the boundary conditions applied discussed. The section begins, however, with a description of the sheet samples investigated, after which the simulation setups follow.

#### 4.1. The sheet samples examined

Figure 8 presents the sheet samples examined. The samples are square; have a side length of either L = 10, 20, 40, 80, or 160 m; a thickness of either h = 0.5, 1.0, or 1.5 m; a discrete element size of either l = 2h or 3h (defined as an average diameter of the circumscribed circles of the polygonal discrete elements in each mesh); and float on a buoyant foundation. Due to CVT, the beam finite elements are of approximately the same "sizes" as the discrete elements (i.e.  $L_e \approx l$ ). In the case of a fracture, each broken fragment should thus have, at the minimum, a size comparable with actual block sizes measured from ridge sails (Kankaanpää, 1988; Høyland, 2007; Kulyakhtin, 2014). Ten randomized CVT meshes were created for each case ( $\sim$  a sheet "type"), excluding the case with L = 160 m, h = 0.5 m, and l = 2h in Load cases A1 and A2 (to be defined below) for which only six meshes were produced.



Figure 8: The FE-DE sheet samples examined. The sheets have an in-plane size range of 1:16 and are shown here in scale (reproduced from (Lilja et al., 2019a, Fig. 4, p. 6)). From left to right: L = 10, 20, 40, 80, and 160 m. The three thin horizontal lines atop each sample denote the thicknesses studied. From top to bottom: h = 1.5, 1.0, and 0.5 m. The largest sample on the right shows, as an example, sections of the two meshes that have the most and the least amount of discrete elements for that particular sheet "type." Mesh "A" has 29561 discrete elements (L = 160 m, h = 0.5 m, and l = 2h), whereas mesh "B" has 1460 discrete elements (L = 160 m, h = 1.5 m, and l = 3h). In Load cases A1, A2, and B1 all the sheets shown are examined. In Load cases B2, C1, and C2 only the largest sheets (L = 160 m) are studied. In Load cases C1 and C2, the sheets have, in addition, circular holes, see Figure 26.

#### 4.2. Uniaxial tensile fracture

The effective, uniaxial tensile strength of an FE-DE sheet sample was computed by conducting a tensile test, Figure 9. All the sheets in Figure 8 were examined. The nodes on the right and left boundaries of a specimen were pulled into the positive and negative global x co-ordinate directions, respectively, until a sheet was completely fractured. Each sheet was loaded in displacement control. In order to prevent early fracture near the boundaries (and to start the simulations with an approximately constant rate of initial longitudinal strain). a linearly changing initial velocity field,  $\vec{v}_x(x, y, 0) = 2|v_x|x\vec{i}/L$ , was established in each specimen. A similar procedure was followed in (Miller et al., 1999). In order to examine the effect of the applied loading rate, the computations were repeated at two different displacement rates. The rates had the magnitudes of  $|v_x| = 0.1$  (Load case A1) and 0.01 m/s (Load case A2). As L was varied, the strain rates varied then as  $\dot{\epsilon} = 2|v_x|/L$ . Post-fracture contacts were not computed, nor were the buoyant-, drag, or the gravitational forces considered. The Timoshenko beam finite elements were, however, allowed to fracture. For an explanation as to why post-fracture contacts were not considered, see sub-subsection 6.1.2.



Figure 9: A schematic illustration of a simulation setup in Load case A (adapted from (Lilja et al., 2019a, Fig. 5, p. 8)).

The strength of an FE-DE sheet sample was next evaluated through  $\sigma_{\rm cr,eff} = F_x^{\rm max}/A_0$ , where  $F_x^{\rm max}$  denotes the scalar component of a computed resultant reaction force vector in the global x co-ordinate direction (the recorded maximum value) and  $A_0$  the initial cross-sectional area of the sheet sample  $(A_0 = Lh)$ . The force component  $F_x^{\rm max}$  was computed for each sheet sample by summing up the global x co-ordinate direction components of the internal, nodal force vectors of the beams having nodes either on the right or left boundary. The choice, in practice, proved to be immaterial, see Figure 10. Figure 10 depicts, as an example, typical resultant force time histories recorded. The force  $F_x$  (recorded every 0.001 s) evolved, in each case, approximately linearly up until the onset



Figure 10: Example resultant force time histories recorded in Load case A, Figure 9. There are 20 results in both figures (10 sheets with results from both ends; L = 160 m, h = 1.5 m, and l = 3h). Note that the force components  $F_x$  have been plotted here with equal signs and that the recorded values do not start from zero because of the applied initial conditions, see Figure 9.

of damage. Afterwards, it dropped sharply. Before the onset of damage, the responses thus appeared quasi-static.

# 4.3. Vertical penetration fracture

Two load cases were studied, denoted as B1 and B2. In Load case B1, the samples were let to float freely, whereas in Load case B2, the outer boundaries of an FE-DE sheet sample were constrained with pinned boundary conditions (i.e. it was set that the displacements  $u_x = u_y = u_z = 0$  along the outer boundaries). In Load case B1, all the sheets shown in Figure 8 were studied. In Load case B2, only the largest sheets (L = 160 m) were examined. The purpose of Load case B2 was to examine whether the largest sheets are large enough to mimic an "infinite" ice sheet. If so, the boundary conditions should have no significant effects. Notice that a pinned boundary condition prevents in-plane Poisson's effect in contrast to, for example, a simply supported boundary condition.

In each simulation, a sheet sample examined was in an initial equilibrium; the gravitational-, buoyant-, drag-, and the contact forces were taken into account; and the Timoshenko beam finite elements were allowed to fracture.



Figure 11: A schematic illustration of a simulation setup in Load case B. A rigid, flat-ended, cylindrical indenter penetrates an FE-DE ice sheet sample from below.

At the beginning of each simulation, a rigid, flat-ended, cylindrical indenter was positioned directly underneath the geometrical midpoint of a specimen. No initial gap between the bottom surface of the FE-DE sheet sample and the upper, horizontal face of the indenter was set, Figure 11. Once a simulation got started, the indenter was accelerated (linearly) up to a constant speed of  $|v_z| = 0.025$  m/s. The speed was ramped up within the first second of a simulation. As a result, the indenter displaced vertically upwards and penetrated through the sheet. Each simulation was then continued until a smooth, horizontal post-fracture loading plateau was reached (either a zero or a significantly lower recorded resultant contact force, see below). Depending somewhat on the size of a specimen, the simulation times varied between 30 to 55 seconds.

The indenter dimensions were chosen as follows: if the diameter D of a cylindrical indenter is greater than 2h, a bending failure is likely to result (Sodhi, 1995). Because the maximum discrete element size l equals here 3h, a diameter of D = 3h was chosen. Note that with D = 3h, the ratio L/D is about 2.2 for the samples with a side length of L = 10 m and a thickness of h = 1.5 m. Such a low ratio yields results that are probably highly approximate (as regards to the fracture) – a finite rigid body displacement component results. Note also that shear may dominate with the sheets with l = 3h because for those sheets D/l = 1. The ratio should – ideally – give  $D/l \gg 1$ , which was thus not achievable.

The vertical break through load,  $F_{\rm cr},$  of an FE-DE sheet sample was then computed as a maximum resultant force of the vertical contact forces experienced by the indenter during a simulation. The force was recorded as a moving average for each successive 0.025 seconds. It was observed, in practice, that the force evolved approximately linearly up until the onset of damage, and the response thus to exhibit a quasi-static behaviour, see Figure 12. After the onset of damage, the load dropped sharply to either zero, see for example Figure 12e, or to a value approximately equal to the buoyancy-reduced weight of those discrete elements that had not fallen off from atop the rising indenter, Figure 12f. Note that no hydrodynamic effects were (or could have been with the current model) considered, except some through drag.<sup>5</sup> The penetration speed was maintained, in addition, as low as possible in trying to exclude all dynamic effects related to cracking. After fracture, the discrete elements - of course - accelerated. The reported breakthrough strengths were, in subsection 6.2, computed then as  $\sigma_{\rm F} = F_{\rm cr}/h^2$ , see (Bažant, 2002). Note that  $\sigma_{\rm F}$  is not a true stress, but a nominal stress with a correct dimension.

# 4.4. Ice-structure interaction

Two load cases were studied, denoted as C1 and C2. In both load cases, only the largest sheets with a side length of L = 160 m were examined, Figure 8. Note that Figure 8 does not display any holes. In that matter, see Figure 13. The holes had "effective" diameters of five meters. The word "effective" is here used in order to emphasize that the holes were not perfectly circular but angular and that the diameters actually changed during the simulations. For the samples with a side length of L = 160 m, a thickness of h = 1.5 m, and a discrete element

<sup>&</sup>lt;sup>5</sup>For an analytic treatment of hydrodynamic effects as regards to the forced sub-surface uplift of a floating ice sheet (with no failure though), see (Dempsey and Zhao, 1993).



Figure 12: Example resultant contact force time histories recorded in Load case B1, Figure 11. Note that both  $t_{\rm end}$  and  $F_{\rm cr}$  are generally different in each case.

size of l = 3h, the holes were, in fact, very angular. Subsection 5.2 below gives an explanation as to why the diameters changed during the simulations.



The boundary conditions applied were similar to those in Load case B. In Load case C1, the sheets were let to float freely, whereas in Load case C2, the outer boundaries of an FE-DE sheet sample were pinned.

Figure 13: A schematic illustration of a simulation setup in Load case C. A surfacing, rigid, truncated cone is breaking an FE-DE ice sheet sample containing a circular hole. Note that although a dotted line is drawn to indicate axisymmetry, the FE-DE sheet samples were not axisymmetric but square. Only the cone is axisymmetric.

In the beginning of each simulation, a rigid, truncated cone was positioned directly underneath the geometrical midpoint of a specimen. An initial vertical gap of 20 mm between the bottom surface of the sheet sample and the upper, horizontal face of the cone was set. Once a simulation got started, the cone was displaced with a constant speed,  $|v_z| = 0.1 \text{ m/s}$ , directly upwards and through the hole. In contrast to Load case B, no initial accelerations were applied. The cone had an upper diameter of five meters and a cone angle of 45 degrees. With the chosen parametrisation, horizontal velocity components with the magnitudes of  $|v_x| = |v_y| = 0.05 \text{ m/s}$  thus result. These velocities are "effective" velocities with which an ice sheet is being pushed in the plane because of the contacts with the cone. Each simulation was then continued until a clear drop in the recorded load was observed, see Figure 14. In each simulation (in both load cases considered), a sheet sample examined was in an initial equilibrium; the gravitational-, buoyant-, drag-, and the contact forces were taken into account; and the Timoshenko beam finite elements were allowed to fracture.

The maximum, vertical ice load on the cone,  $F_{V,FE-DE}$ , was then computed as a maximum resultant force of the vertical contact forces experienced by the cone during a simulation. The force was recorded as a moving average for each successive 0.025 seconds, as in Load case B. In practice, the recorded vertical force was observed to evolve approximately linearly up until the onset of damage and then to suddenly drop to a significantly lower value, see Figure 14. The computed maximum force was then divided with the effective circumference of the cone,  $\pi D_{\rm w}$ , in order to get a load per unit circumference. The diameter  $D_{\rm w}$  corresponded to the time instant the maximum load occurred at, see Figure 25 and the equations in Box 8.

It is good to note that the frictional contact forces act here mostly in directions opposite to those with a direct approach. As regards to the inertial forces, the discrete elements tend here to accelerate, whereas with a direct approach, they decelerate in the horizontal and accelerate in the vertical directions. Further notice that the problems depicted in Figures 11 and 13 are, in fact, illposed (Dempsey and Vasileva, 2006b). In their paper, a sensitivity to the data was found. This possible sensitivity was not here investigated.

To conclude the current section, Table 1 gives the values of the main simulation parameters in each of the load cases studied.

	Parameter	Symbol	Unit	Value or range
General	Gravitational acceleration	g	${ m m/s}^2$	9.81
	Drag coefficient <sup><math>a</math></sup>	$c_{\rm d}$		1.0
	Damping coefficient <sup><math>b</math></sup>	c		critical
	Coefficient of friction <sup><math>c</math></sup>	$\mu_{\mathrm{i,s}}$		0.05
	Coefficient of friction <sup><math>d</math></sup>	$\mu_{\mathrm{i,i}}$		0.05
	Contact stiffness	$k_{ m c}$	GPa	4
	Contact damping	$d_{\rm c}$		0.95
	Cone angle	$\alpha$		$45^{\circ}$
	Cone top width	D	m	5
	Time step	$\Delta t$	s	$15.0 \times 10^{-5}$
Ice sheet	Side length	L	m	10, 20, 40, 80, 160
	Thickness	h	m	0.5, 1.0, 1.5
	Young's modulus <sup><math>e</math></sup>	$E_{\rm b}$	GPa	4
	Poisson's ratio	$ u_{ m b}$		0.3
	Density	$ ho_{\rm i}$	$\mathrm{kg/m}^3$	920
	Critical axial stress	$\sigma_{ m cr}$	kPa	125
	Critical shear stress	$ au_{ m cr}$	kPa	125
	Specific fracture energy	$G_{\rm eff}$	$\mathrm{J/m}^2$	15
Water	Density	$ ho_{ m w}$	$\mathrm{kg/m}^3$	1010

Table 1: Main simulation parameters in Load cases A, B, and C.

<sup>a</sup>Both in translation and rotation.  ${}^{b}c = 2\sqrt{m_{\rm eff}E_{\rm b}}$ , where the "effective" mass,  $m_{\rm eff}$ , is taken to be the average of the translational masses of the two discrete elements a beam finite element connects. <sup>c</sup>Coefficient of friction between ice and structure. <sup>d</sup>Coefficient of friction between two pieces of ice. <sup>e</sup>"Apparent" or "effective" elastic modulus (Timco and Weeks, 2010, sec. 13).



Figure 14: Example resultant contact force time histories recorded in Load case C, Figure 13. Note that both  $t_{end}$  and  $F_{V,FE-DE}$  are generally different in each case.

#### 5. Analytical reference results

This section gives analytical relations for i) the vertical breakthrough load of an infinite ice sheet loaded by a uniform load distributed over a circular area and ii) the breaking load a level ice sheet imposes on an inclined offshore structure. The first result was derived by Wyman (1950), the second by Croasdale and Cammaert (1994). This section repeats their results because the numerical FE-DE results are compared with them in subsections 6.2 and 6.3.

#### 5.1. Vertical breakthrough load of an infinite ice sheet

In (Wyman, 1950), an analytical relation was derived giving the vertical breakthrough load of an infinite ice sheet loaded by a uniform load distributed over a circular area. The solution is based on the theory of elasticity and thus on an assumption that the maximum load is reached as soon as the stresses are equal to the proportionality (elastic) limit. The solution reads as:

$$F_{\rm cr,Wyman} = \frac{\pi \sigma_{\rm cr} b h^2}{3 \left(1 + \nu\right) \, {\rm kei}'\left(b\right)},\tag{18}$$

where  $\sigma_{\rm cr}$  denotes a critical stress (the strength),  $b = D/2l_{\rm ch}$ ,  $l_{\rm ch}$  a characteristic length (defined below),  $\nu$  the Poisson's ratio of the plate material, kei' (b) =  $(0.6159 - \ln (b)) (b/2) + \pi b^3/64 + \ldots$ , and kei a Kelvin function. The truncated expansion for the derivative (as denoted by the prime) as shown is used. Note that Eq. (18) describes, in fact, the "bearing capacity" of an infinite ice sheet because the loading direction was opposite to that used in this paper.

Notice that in (Wyman, 1950) a sheet was loaded in a load control, whereas in the FE-DE simulations in a displacement control. The contact pressures experienced by the FE-DE sheet samples thus differ from those assumed in (Wyman, 1950). High contact pressure zones exist near the indenter edges, the resultant forces are, however, assumed to be comparable. Notice finally that many alternatives to Eq. 18 exist, see (Kerr, 1976) and (Kerr, 1996).

### 5.2. Ice loads on an inclined offshore structure

Following (Croasdale and Cammaert, 1994), the horizontal load a level ice sheet imposes on an inclined offshore structure is given by:

$$F_{\rm H,Croasdale} = \frac{H_B}{1 - \frac{H_B}{\sigma_{\rm cr} l_{\rm ch} h}},\tag{19}$$

where  $H_B$  denotes a breaking load component,  $l_{\rm ch} = (K/k)^{1/4}$  a characteristic length (decay length of flexural waves),  $K = Eh^3/12(1 - \nu^2)$  the cylindrical flexural rigidity of a Kirchhoff-Love plate, k the foundation modulus (specific weight of water  $-k = \rho_{\rm w}$ g), and E the Young's modulus of the plate material. The breaking load component  $H_B$  reads as:

$$H_B = 0.68\xi \sigma_{\rm cr} \left(\frac{\rho_{\rm w} g h^5}{E}\right)^{1/4} \left(\pi D_{\rm w} + \frac{\pi^2 l_{\rm ch}}{4}\right),\tag{20}$$

where  $\xi = \frac{\tan(\alpha) + \mu_{i,s}}{1 - \mu_{i,s} \tan(\alpha)}$ . The vertical breaking load component (per unit circumference) follows then from Eq. (20) through:

$$F_{\rm V,Croasdale} = \frac{F_{\rm H,Croasdale}}{\pi D_{\rm w} \xi}.$$
(21)

Note that only the breaking load component has been here considered. Rubbling and other subsequent effects have been ignored. The exclusion is justified because of the termination of the simulations right after the occurrence of the first flexural failure. Notice also that in place of  $\sigma_{\rm cr}$ ,  $l_{\rm ch}$ , and E, "effective" values (specific to the lattices considered) will be used below because their values are all known, see sub-subsection 6.3.1. While computing the vertical breakthrough load above, Eq. (18), such a procedure was not possible because the values of the characteristic lengths of the smallest sheets (L = 10 m) were not available. The symbols  $\sigma_{\rm cr}$  and E thus refer there to the critical axial stress parameter and the Young's modulus given to the Timoshenko beam finite elements, respectively, whereas the symbol  $l_{\rm ch}$  denotes the characteristic length of a Kirchhoff-Love plate computed using the relations given above.

A remark: the symbol  $D_{\rm w}$  in Eqs. (20) and (21) denotes the waterline diameter (or the "effective width") of an inclined offshore structure. It is here multiplied with  $\pi$  to give the circumference. Normally, a contact between an ice sheet and a structure is unilateral: an ice sheet advances and collides with a structure only one-sidedly. Here, the contact zone extends over the entire circumference and so the "effective width" becomes equal to  $\pi D_{\rm w}$ . Note further that the width  $D_{\rm w}$  is not a constant but, in fact, an unknown variable. While a simulation is in progress, the "effective width" changes due to the conical shape of the surfacing structure and the fact that the peak load may not occur right after the occurrence of the first contact. A schematic illustration of the interaction between an initially stationary ice sheet and an ascending, upward breaking cone is given in Figure 25. Equations to compute the "effective width"  $D_{\rm w}$  are given there in Box 8.

Notice finally that other results regarding the breaking load component exist as well, see, for example, (Ralston, 1977) and (Nevel, 1992). Ralston (1977) gives an upper bound plastic limit analysis of an elastic-perfectly plastic plate interacting with a cone on an elastic-perfectly plastic foundation while Nevel (1992) treats the failure of a point-loaded, elastic wedge on a Winkler-type foundation. The analysis in (Ralston, 1977) was suggested, in fact, by J.R. Rice.

# 6. Results

In this section, the computed uniaxial tensile strengths, breakthrough loads and strengths, as well as the ice-structure interaction loads by the FE-DE approach are reported and discussed. Subsection 6.1 discusses the tensile strengths, subsection 6.2 the breakthrough loads and strengths while subsection 6.3 gives an account on the computed ice-structure interaction loads.

# 6.1. Effective tensile strengths

### 6.1.1. Rate and size effects

Figure 15 shows the computed strengths,  $\sigma_{\rm cr,eff}$ , as well as their standard deviations. Each strength has been normalised with respect to the critical axial stress parameter,  $\sigma_{\rm cr}$ , Table 1. Figure 15a depicts the strengths at the higher displacement rate ( $|v_x| = 0.1 \text{ m/s}$ , Load case A1), whereas Figure 15b gives the strengths at the lower displacement rate ( $|v_x| = 0.01 \text{ m/s}$ , Load case A2). The results have been given in an order of ascending sheet size L. For each L, the results have been arranged in an order of ascending sheet thickness h. For each h, the strengths are averages over 20 simulation results (10 randomized CVT meshes with both l), except the case with L = 160 m, h = 0.5 m, and l = 2h for which only 16 meshes were produced (10 with l = 3h and six with l = 2h).

It is observed that the effective tensile strength grows as a function of sheet thickness h and decreases as a function of sheet side length L, Figure 15a. The sheets that are smaller or thicker have, in general, a higher effective tensile strength than those that are thinner or larger. At the largest sheet size, the strength, however, appears to saturate. Somewhat similar conclusions can be drawn from the results in Figure 15b as well, except that - now - i) for the two largest sheet sizes (L = 80 and 160 m), the strength appears to decrease as a function of h and that ii) the sheets with L = 40 m produce practically equal results. For the two smallest sheet sizes, the differences between the results, whether a sheet thickness of either h = 0.5, 1.0, or 1.5 m is considered, are in both load cases rather large. Except of the smallest sheets (L = 10 m), the standard deviations are close to null.

Figure 16 presents the results in a slightly different, now in a fully nondimensional, format. The logarithms of the ratios  $\sigma_{\rm cr,eff}/\sigma_{\rm cr}$  have been plotted as functions of the logarithms of  $L/I_{\rm ch}$ . The parameter  $I_{\rm ch}$  denotes here Irwin's (or Hillerborg's) characteristic length. Irwin's characteristic length approximates the "length" of a fracture process zone ahead of a crack tip and reads as  $I_{\rm ch} = E_{\rm eff}G_{\rm eff}/\sigma_{\rm cr}^2$ . Note that the Young's modulus  $E_{\rm eff}$  is not here equal to the Young's modulus  $E_{\rm b}$  given to the Timoshenko beam finite elements, but denotes the effective Young's modulus of a particular sheet "type" (i.e. a sheet with specific L, h, and l). These were computed in (Lilja et al., 2019a). Note also that the effective, specific fracture energy parameter  $G_{\rm eff}$  is an effective quantity only in a sense that it represents the energy dissipated under a mixed mode fracture. It does not represent a fracture energy specific to a particular sheet type. Such values are not known. Strengths so computed have then been given separately for each sheet thickness h considered.

It is found that the non-dimensional effective strength decreases as the nondimensional sheet size increases, whereas the thickness h has an opposite effect. These observations apply at the higher displacement rate,  $|v_x| = 0.1$  m/s (Load case A1), Figure 16a. At the lower displacement rate (Load case A2), Figure 16b, the non-dimensional strength appears to be a similar decreasing function of the non-dimensional sheet size, but as regards to the effect of the thickness h, the situation is more complex. It appears that an inflection point is located somewhere near the point  $L/I_{\rm ch} = 1$ , and that for  $L/I_{\rm ch} > 1$ , the thicker sheets give lower strengths. This last "branch" yields results that are intuitively "correct." thicker sheets exhibit a lower strength.



Figure 15: Averaged, normalised, effective tensile strengths of the FE-DE sheet samples examined. Each result (except those with L = 160 m and h = 0.5 m for which the results are averages over 16 simulation results – six meshes were created with a discrete element size of l = 2h) is an average over 20 simulation results (10 with both l), each with a different, randomized CVT mesh.



(b)  $|v_x| = 0.01 \text{ m/s}$  (Load case A2).

Figure 16: Non-dimensional effective tensile strengths (log-log values on a linear chart).

Figure 16 gives also linear regression lines (with slopes m and coefficients of determination, i.e. goodness of fits,  $R^2$ ) fitted to the data.<sup>6</sup> The fits correlate reasonably well, given the parabolic appearance of the data curves. This is especially so in Figure 16a. A "truly" linear response is found only for the red data points in Figure 16b. It is observed that |m| is a monotonically increasing function of h and that the trend continues – uninterruptedly – from Figure 16b to Figure 16a. Rate effects dominate. The slope of the blue regression line in Figure 16a gives eventually a scaling law approximately equal to the LEFMtype scaling law of  $(\cdot)^{-1/2}$ . In other words, for a thick enough sheet and with a large enough displacement rate applied, the response appears to emulate a LEFM-type response. Notice, however, that the strain rates were not kept here as constants and that at higher rates the slopes may have got even steeper. It also seems, as was noted above, that the data curves tend to curve up as Lis getting smaller because the strain rates are getting there higher. For more accurate results, a larger simulation set with several different displacement rates applied – so that the strain rates could have been made equal for sheets having different side lengths L – would have been required. In any case, the effective tensile strength appears to be a strong function of both L and h.

A usual assumption for quasi-brittle materials is that a larger sample is less strong than a smaller sample. If it is thus taken as a "rule" that thicker sheets must produce lower strengths, the strain rate has to be as low as around  $0.25 \times 10^{-3}$ /s, see the results for the sheets with the side lengths of L = 80 and 160 m in Figure 15b. For those sheets, the responses appear to be most closely quasi-static and the thicker sheets exhibit lower strengths. For the sheets with a side length of L = 160 m, a regression gives an exponent of about -0.05 ( $\sim h^{-0.05}$ ) – far from LEFM.

Figure 17 portrays, as an example, a completely fractured ice sheet with a side length of L = 160 m, a thickness of h = 0.5 m, and a discrete element size of l = 2h. The mesh has, in total, 29561 discrete elements. Broad areas are softening and are highlighted in purple. Several fully grown cracks appear and are highlighted in grey. Their orientation appears to be, for the most part, approximately perpendicular to the loading direction. A usual assumption is that cracks grow in a direction perpendicular to the maximum principal stress trajectory. Areas remaining still undamaged are, then, highlighted in blue. It appears that the cracks tend to branch, bridge, and have, in general, a rather tortuous pattern. The cracks, in short, bifurcate. Figure 18 shows, as a further example, a sequence of snapshots illustrating how damage evolves as a function of time. Damage initiates quickly and fully developed cracks emerge, which then start to propagate towards the free edges.

 $<sup>^{6}</sup>$ Recall that the slope of a linear function (a straight line) on a linear chart with log-log values gives the exponent of a power-law function (a curve following a power-law) on the same chart with lin-lin values.

### 6.1.2. Discussion

A lucid *a priori* assumption would have been that all rate-related effects are of a minor importance: the de-cohesive damage model implemented was rateindependent and the displacement rates applied rather low. Such an assumption, however, would have proved to be false – a significant rate effect emerged. The ratio between the effective tensile strengths of the smallest thickest sheets, for the two displacement rates considered, is, for example, approximately four. It then reduces to an average of around 1.3 for the largest thickest sheets, Figure 15. For the thinner sheets, and so with more elements, the effect somewhat attenuates, but is still present. A possible explanation for the observed increase in the effective strength at the higher loading rates is that of diffuse cracking. Instead of a single dominant crack, several smaller cracks (microcracks) appear, which then produces an increase in the effective strength. This is not only because of an increase in the dissipated fracture energy, but due to inertia as well. A plausible assumption would have been that inertia plays no role because of the initial (linearly changing) velocity field established in each specimen, see subsection 4.2. It may, however, be that as a result of elastic restoring forces, viscous damping forces, and microcracking, the "microinertia" on the micro-scale (i.e. the inertia of individual discrete elements) becomes significant. The microcrack nucleation, growth, interaction, and coalescence inevitably either accelerates or decelerates individual discrete elements, not just those with beams undergoing damage, but the nearby elements as well. Inertia, in conjunction with an intrinsic time scale, leads then to an apparent strength increase.

It has been shown that a cohesive finite element with a rate-independent traction-separation law exhibits not only a characteristic length scale but a characteristic time scale,  $t_{\rm ch}$ , as well (Camacho and Ortiz, 1996). The expression of the characteristic (relaxation) time  $t_{\rm ch}$  is a linear function of both the density (here, the inertia of a discrete element) and the longitudinal wave speed,  $c_{\rm s}$ , in an element, see (Camacho and Ortiz, 1996, Eq. 69, p. 2916). The effective stress amplitude to cause fracture, on the other hand, is an exponentially decaying function of  $\tau/t_{\rm ch}$ , where  $\tau$  denotes pulse duration. An increase in the loading rate (so that the pulse duration decreases) results then in an increase in the apparent strength. The observed effective strength increase is thus interpreted to result from both the increased fracture energy dissipation and the microinertia.

Diffuse cracking per se may be best explained by following lines similar to those set forth by Mott (1947). Mott analysed detonation-driven wave propagation and fracture in a ring-shaped specimen made out of a ductile metal (a one-dimensional problem). At a sufficiently low strain rate, a stress release wave from a single crack is able to unload the rest of the specimen and so no further damage ensues. Conversely, at a sufficiently high strain rate, the wave is too slow, which then leads to further damage. Whether Mott's theory holds also for a lattice, especially because the strain rates were here rather low (i.e.  $c_s/L > \dot{\epsilon}$  should have held), is questionable, but the principle is plausible. Another explanation could be that of bifurcation instability. Schardin (1959) found experimentally that at higher loading rates, the cracks in a brittle solid tend to

bifurcate and the limit speed to be far less than  $c_{\rm s}$ . Such an observation appears to support the current results. Instead of a single dominant crack, rather distributed damage occurs while  $c_{\rm s}/L > \dot{\epsilon}$  simultaneously holds, see Figure 17.

Note that the omission of computation of the post-fracture contacts (as well as not considering drag forces) gave probably lower estimates for the effective tensile strengths by not taking into account effects due to friction (say interlocking). Not considering contacts was due to computational reasons. A look in Figure 17 reveals that softening, fracturing, and thus contacts occur not only locally but globally. Running all the simulations on a Fujitsu Celsius W530 Power workstation (with an Intel Xeon E3-1245V3 processor) took about eight months.



Figure 17: A fractured ice sheet with a side length of L = 160 m, a thickness of h = 0.5 m, and a discrete element size of l = 2h (adapted from (Lilja et al., 2017, Fig. 6)). The completely fractured beams are highlighted in grey, damaged (softening) beams in purple, and the beams that are still virginal in blue. There are, in total, 29561 discrete elements in the mesh. The sheet was loaded in uniaxial tension in the horizontal direction by applying a forced displacement depicted in Figure 9 ( $|v_x| = 0.1$  m/s, Load case A1).



Figure 18: Evolving damage as a function of time (L = 160 m, h = 0.5 m, and l = 2h),  $t_{\rm end} = 0.3$  s, Load case A1. For an explanation of the colouring scheme, see Figure 17.

#### 6.2. Breakthrough loads and strengths

This subsection presents and discusses the computed breakthrough loads and strengths. In Load case B1, an FE-DE sheet sample penetrated by a rigid, flat-ended, cylindrical indenter was examined. All the sheets shown in Figure 8 were studied. In Load case B2, only the largest sheets (L = 160 m), and with pinned boundary conditions applied, were considered.

#### 6.2.1. Breakthrough loads

Figure 19 presents the loads,  $F_{\rm cr}$ , as well as their standard deviations, for each side length L and thickness h considered. For each L, the results have been arranged in an order of ascending sheet thickness h, like before. The results shown are averaged breakthrough loads computed with the aid of 20 randomized CVT meshes (10 for both l). Each load has finally been normalised with respect to the vertical breakthrough load of an infinite ice sheet, Eq. (18), with a corresponding thickness h.

Before proceeding, recall that the Wyman's formula, Eq. (18), includes a critical axial stress parameter in the nominator and a characteristic length parameter (including E as well) both in the nominator and the denominator. The values of the effective characteristic lengths of the smallest sheets are not known. Those that are, are given in (Lilja et al., 2019b). The breakthrough loads in Figure 19 were, therefore, normalised with respect to reference loads computed with the aid of  $\sigma_{\rm cr}$ ,  $E_{\rm b}$ , and  $l_{\rm ch}$  of which the first two are the critical axial stress parameter and the Young's modulus given to the Timoshenko beam finite elements, respectively, whereas  $l_{\rm ch}$  denotes the characteristic length of a Kirchhoff-Love plate computed using the relations given after Eq. (19). In other words, no effective values (tensile strengths, Young's moduli, or characteristic lengths) were here employed. The concave appearance, to be shortly discussed, may have "straightened" had all the effective values been available. Note, however, that this comment applies only to the visual appearance of the results in Figure 19. The fracture characteristics *per se* remain, of course, unchanged.

It transpires that the breakthrough load is a "concave up" function of L for each sheet thickness h considered and that the location of the relative minimum depends on h. For the thinnest sheets, a minimum appears to be located at about L = 20 m, whereas for the thicker sheets, at around L = 40 m. Note that for the smallest sheets (L = 10 m), the response is strongly affected by the dimensions of the indenter. The thickest sheets tend to rise as rigid bodies and then suddenly fail when the buoyant (supporting) forces have nearly vanished. See, for example, Figure 12e. The rising trend there (as regards to the thickness h) is explained by the increased mass due to the thicker sheets. For the larger sheets, the trend appears, on the contrary, to be reversed – thinner sheets exhibit higher (relative) loads. These two observations (the reversed effect of h and the concavity as  $L = L_{\min} \mapsto L_{\max}$ ) can be interpreted to be due to two reasons: i) a change in the mode of failure and ii) inertia. The sheets with a side length of  $L \leq 20$  m exhibit mostly radial cracking (the thinnest sheets show also some circumferential cracking, see Figure 20), whereas for  $L \geq 40$  m, the failure is



Figure 19: Averaged, normalised breakthrough loads of the FE-DE sheet samples examined. Each result is an average over 20 simulation results (10 with both l), each with a different, randomized CVT mesh. The three rightmost columns – "160 (pinned)" – are for Load case B2, the rest for Load case B1.

nearly always accompanied by circumferential cracking as well. Circumferential cracking necessarily means higher loads, which explains the rising trend. The other contributor to the higher observed loads (as L increases) is the fact that the larger sheets have a higher inertia. While a simulation is in progress, an indenter is trying to get through the sheet, an effort which is made much more difficult if the broken pieces of ice tend not to give way. This is an aspect that is discussed more below, see sub-subsection 6.2.3. An interesting additional observation is that the load appears to saturate for the thinnest largest sheets. The pinned boundary conditions have no effects on their effective breakthrough strengths. A simple explanation could be that a sheet with a side length of L = 160 m and a thickness of h = 0.5 m resembles an infinite ice sheet. For the thicker sheets, the boundary conditions still have a noticeable effect.

# 6.2.2. Breakthrough strengths

Figure 21 presents the averaged breakthrough strengths,  $\sigma_{\rm F}$ , as well as their standard deviations, for each side length L and thickness h considered. The strengths have been computed as  $\sigma_{\rm F} = F_{\rm cr}/h^2$ , see subsection 4.3. Each result



Figure 20: A sequence of snapshots from a simulation in Load case B1. An originally intact FE-DE ice sheet sample (L = 20 m, h = 0.5 m, and l = 2h) is penetrated by a rigid, flat-ended, cylindrical indenter from below. The indenter surfaces at the centre.

is an average over 20 simulation results (10 for each l), as before. Unlike above, the results have been here, however, normalised with respect to the effective tensile strength,  $\sigma_{\rm cr,eff}$ , of a sheet sample having the corresponding L and h. These were computed in subsection 6.1 above and correspond to Load case A1 ( $|v_x| = 0.1 \text{ m/s}$ ), Figure 15a. The chosen normalisation, of course, affects the results. Had the strengths been normalised with respect to the data produced by Load case A2, the results would have looked as in Figure 19. An appreciable change would have occurred in only the smallest sheets (the absolute values, of course, change in each case). Similar conclusions hold if the data is normalized with respect to the critical axial stress parameter  $\sigma_{\rm cr}$ . A line  $\sigma_{\rm F}/\sigma_{\rm cr,eff} = 1$  is plotted, in addition, to indicate that the breakthrough strengths are not equal to the effective tensile strengths, and because for most of the samples  $\sigma_{\rm cr,eff}/\sigma_{\rm cr} \gtrsim 1$ , not necessarily equal to the critical axial stress either.



Figure 21: Averaged, normalised breakthrough strengths of the FE-DE sheet samples examined. Each result has been computed as an average over 20 simulation results (10 with both *l*), each with a different, randomized CVT mesh. The three rightmost columns – "160 (pinned)" – are for the Load case B2, the others for the Load case B1.

There appears to be a clear trend of thicker sheets exhibiting a lower (relative) breakthrough strength. A strong dependence on the side length L is found as well. The larger and thinner the sheet, the higher the (relative) breakthrough strength. An interesting observation is that the strength appears to saturate if L = 160 m and h = 0.5 m. The boundary conditions have no effect. For the thicker sheets, the strength, on the contrary, keeps on increasing from Load case B1 to B2. Recall that a similar observation was done already for the breakthrough loads above. The sheets with a side length of L = 160 m and a thickness of h = 0.5 m thus appear to resemble an infinite ice sheet. An alternative interpretation could be that the thinnest sheets are thin enough for the breakthrough strengths to settle on the small size asymptotic tail of Bažant's generalized size effect law on which the failure is governed by a strength- or a yield-type criterion. The data points in Figure 23 for the largest sheets (L = 160m), in fact, resemble a typical size effect plot, cf. (Bažant, 2002, Figure 3g, p. 16). Note that the experimental data of Frankenstein (1963, 1966), and Lichtenberger et al. (1974) clearly suggest that a size effect exists also in nature, see Bažant and Kim (1998b) and Bažant (2002).

Before turning to Figure 23 more closely, the data in Figure 21 is interpreted again but from a slightly different perspective. Figure 22 presents the normalised breakthrough strengths but now on a linear chart and with regression lines included. The results of Load case B2 have been here omitted. It transpires that the data settles on nearly straight lines but with slopes m close to zero and approximately equal. Weak or no scaling is thereby implied. Notice that plotting the data in the same manner as in Figure 16a would have been redundant (would not have shown a meaningful size effect) because the data points in Figures 15a and 21 do not show much symmetry. The data in Figure 15a increases as a function of h and decreases as a function of L, whereas in Figure 21, the situation is vice versa. Had the breakthrough strengths been normalised with respect to the critical axial stress parameter  $\sigma_{\rm cr}$  (or, in that matter, to the data produced by Load Case A2), an appreciable change would have occurred in only the smallest sheets, as was stated already above.

It is stated in (Bažant and Kim, 1998b) and (Bažant, 2002) that the breakthrough strength of a floating, point-loaded, infinite ice sheet scales as  $h^{-1/2}$  if the cracks are partially through and as  $h^{-3/8}$  if they are fully through. The latter rule was first found, in fact, by Slepyan (1990).<sup>7</sup> These results are, more specifically, for a notched Kirchhoff-Love plate resting on a Winkler-type foundation. Following (Bažant and Guo, 2002), a statically indeterminate in-plane frame with inelastic softening hinges and which rests on a Winkler-type foundation exhibits a similar strong, monotonic size effect of a type  $h^{-1/2}$ . A sea ice sheet modelled with a hybrid finite-discrete element method, with which the fracture is modelled with de-cohesive Timoshenko beam finite elements, ought to exhibit a similar response as the model of Bažant and Guo (2002). It thus seems justified to assume that the relationship between the breakthrough strength,  $\sigma_{\rm F}$ , and the thickness, h, of a finite FE-DE ice sheet sample follows a power-law as well,

 $<sup>^{7}</sup>$ The result by Slepyan, i.e. the exponent of -3/8, is apparently because of not considering crack closure. In the Slepyan's model, the crack faces can interpenetrate in contrast to the present simulations.



Figure 22: Averaged, normalised breakthrough strengths of the FE-DE sheet samples examined. The results have been here plotted as functions of L. Each result has been computed as an average over 20 simulation results (10 with both l), each with a different, randomized CVT mesh.

i.e.  $\sigma_{\rm F}(h) = bh^m$ , and that the strength scales (asymptotically) as  $h^{-1/2}$ . This is examined next. Before proceeding, it is good to note that the breakthrough strength scales also as  $l_{\rm ch}^{-3/8}$ , but this is for the case of an infinite, notched Kirchhoff-Love plate resting on a Winkler-type foundation (Bažant, 2002). It has not been studied, to the authors' best knowledge, whether the breakthrough strength scales as a function of L for a finite, unnotched, free-edged ice sheet.

Figure 23 shows, in a fully non-dimensional format, the logarithms of the averaged, normalised breakthrough strengths,  $\sigma_{\rm F}/\sigma_{\rm cr,eff}$ , as functions of the logarithms of  $h/I_{\rm ch}$ . Note that both  $\sigma_{\rm cr,eff}$  and  $E_{\rm eff}$  (recall that  $I_{\rm ch} = E_{\rm eff}G_{\rm eff}/\sigma_{\rm cr,eff}^2$ ) refer here to the case with the higher displacement rate ( $|v_x| = 0.1 \text{ m/s}$ , Load case A1), Figure 15a and (Lilja et al., 2019a), respectively.

It is clearly visible that the data settles on nearly straight lines. It appears thus evident that the breakthrough strength of a finite, free-edged ice sheet does scale according to a power-law. It is found, in addition, that the slope mchanges from one L to the other. As the side length L increases, the slope m (and thereby the scaling rule with an exponent m) visibly asymptotes to the known analytical result of Bažant and Guo (2002), i.e.  $\sigma_{\rm F} \propto h^{-1/2}$ . Notice, however, that the result holds true only for the two thickest sheet sizes considered and was obtained by reading the slope of the line joining the two points. The observed saturation for the thinnest sheets causes the fit of the regression over the full data set to deteriorate. For more accurate results, a larger simulation set with both thinner and thicker, as well as larger, ice sheets would have been required. The model, nevertheless, appears to have an ability to demonstrate size effects. Such a result is typical for models that fail only after stable crack growth associated with the development of a fracture process zone (i.e. quasibrittle fracture via softening damage). Notice that the results of Load case B2 were not here plotted because of the found saturation, see Figures 19 and 21. The end result would have been a line with a positive slope, which would have resulted in an inverted scaling rule.

A remark: if the data would have been normalized with respect to the data produced by Load case A2 (and the corresponding  $E_{\text{eff}}$ ), the results would have looked otherwise similar to those in Figure 23, except that the scaling rule of the smallest sheets would have inverted (a line with a positive slope, m = 0.09, results). The other slopes, from L = 20, 40, 80, and 160, read as m = -0.18, -0.31, -0.36, and -0.08, respectively. The slope of the line connecting the data points of the two thickest sheet sizes (for L = 160 m) equals -0.44 and is thus approximately equal to that computed with the data from Load case A1.



Figure 23: Logarithms of the averaged, normalised breakthrough strengths of the FE-DE sheet samples examined. The results have been plotted as functions of the logarithms of normalised h. Each result has been computed as an average over 20 simulation results (10 with both l), each with a different, randomized CVT mesh.

Note finally that it was not possible to include a theoretical size effect curve in Figure 23 because of the fact that not only is the breakthrough strength  $\sigma_{\rm F}$ dependent on size (on both L and h) but the critical axial stress  $\sigma_{\rm cr,eff}$  and the Young's modulus  $E_{\rm eff}$  as well. There is a size effect not only with respect to the fracture (both in the in-plane and the out-of-plane directions) but to the elastic properties as well. In other words, different  $\sigma_{\rm F}$ ,  $\sigma_{\rm cr,eff}$ , and  $E_{\rm eff}$  exist for each L and h studied. In order to have been able to plot such a curve would have required that only the breakthrough strength shows a size dependence and only with respect to h so that a unique critical axial stress parameter would have existed on the vertical axis and a unique "characteristic length" parameter on the horizontal axis. Further notice that the data of the largest samples could not be used for fitting purposes because the data had not fully converged yet. A set of larger sheets would have been required.

#### 6.2.3. Discussion

There has been much discussion in the pertinent literature as to whether the radial cracks are, at maximum load, fully through or not (Dempsey et al., 1995; Bažant and Kim, 1998a,b; Bažant, 2002). The last three results are for an infinite ice sheet, whereas the first one is for a finite, clamped plate (not on an elastic foundation). Here, in Load case B1 (for a finite, freely-floating ice sheet), the cracks appeared to be fully through. The observation is based solely on visual inspection of animated crack growth, but appeared evident for the sheets with the side lengths of L = 10, 20, and 40 m. For the larger sheets, and especially for the sheets with the pinned boundary conditions in Load case B2, the situation was much more complex. A closer look was obscured by the fact that a beam was judged to be fully degraded as soon as a critical number of fully damaged integration points was reached, see sub-subsection 3.2.1. The bottommost row of integration points (i.e. those closest to the indenter) got thus probably destroyed when reached by the crack front.

Another interesting question is that of crack closure. Some crack closure may have occurred in Load case B1, but only instantly, and is because of the boundary conditions. The outer boundaries of a sheet sample were free in Load case B1. To be more precise, no "Dirichlet" -type boundary conditions were applied. Because of the free boundary conditions, the radial cracks tend to propagate all the way to the free boundaries. This is especially so in thick sheets, see Figure 26a. Due to the momentum imparted by the penetrating indenter to the sheet, the broken pieces tend then to drift apart. There is nothing holding them back, which is necessary for the dome (or the arching) effect – and thus the crack closure – to take place. The dome effect ought to dominate when the boundaries are not free and with samples that have a large enough inertia. For an infinite ice sheet (and the sheets in Load case B2) this, indeed, is the case. The attached animations, Animation 3 and Animation 4, illustrate these phenomena quite clearly. In the first animation, a sheet with a side length of L = 40 m and a thickness of h = 1.0 m is penetrated by an indenter from below (Load case B1). In the latter animation, a surfacing cone is breaking a sheet with a side length of L = 160 m and a thickness of h = 1.5 m (Load case C1). Similar conclusions can be drawn from both simulations: the broken pieces start to drift apart and crack closure takes place only instantly. It can be argued, of course, that at least partly the observed movement is because of force transmit through crack closure.

A final remark: one may assume – based on intuition – that the vertical load a semi-infinite ice sheet with a free edge imposes on an inclined offshore structure is about one half of the breakthrough load of an infinite ice sheet if the loading "widths" are equal. For a narrow structure this, indeed, has been shown to be the case (Gold et al., 1958; Black, 1958; Meyerhof, 1960). The breakthrough loads of the largest sheets with a thickness of h = 0.5 m should thus be applicable in approximating the vertical load imposed on an inclined offshore structure (of a width similar to that of the indenter) by a semi-infinite ice sheet with a free edge and of the same thickness.

# 6.3. Ice-structure interaction loads

This subsection presents and discusses the computed cone ice loads. The typical cracking characteristics observed are described as well. In Load case C1, the breaking of a freely-floating FE-DE ice sheet sample containing a circular hole by a rigid, truncated, ascending cone was examined. Only the largest sheets (with a side length of L = 160 m), Figure 8, were studied. In Load case C2, the same sheets, but with pinned boundary conditions applied, were investigated.

#### 6.3.1. Breaking loads

Figure 24 presents the loads (with their standard deviations) for both of the load cases. As before, the results are averages over 20 simulation results (10 with both l). The left-hand side is for Load case C1, whereas the right-hand side is for Load case C2.

In both load cases, the computed breaking loads are (if taken as an average over all h) approximately equal to those by Eq. (21). In Load case C1, a slight increasing trend is found, but the standard deviations overlap. At a rough estimate, the presented loads occurred at time instants the circumferential cracks started to form. The observation is, however, based on visual inspection of animated crack growth and was not systematically studied.

Based on the foregoing, it thus appears that an ice-structure interaction scheme in which a stationary ice sheet containing a circular hole is interacting with a surfacing cone yields loads that are approximately equal to the case in which an ice sheet is moving, a structure is held stationary, and the contacts occur only unilaterally. Recall that the loading widths were, however, assumed to be equal. The found result ought to hold, at least, at moderately low penetration speeds. At higher speeds, adverse effects due to inertia may occur (Pushkin et al., 1991).

A remark: while computing the vertical breaking load component  $F_{V,Croasdale}$ , it was assumed that:

i) the flexural strength of an ice sheet,  $\sigma_{\rm cr}$ , corresponds to the breakthrough strength (~ modulus of rupture ~ flexural strength),  $\sigma_{\rm F}$ , computed for each sheet in sub-subsection 6.2.2 (Load case B1);



Figure 24: Averaged, normalised, vertical cone loads. Each result has been computed as an average over 20 simulation results (10 with both l), each with a different, randomized CVT mesh. The bars on the left denote Load case C1, whereas those on the right Load case C2.

- *ii*) the Young's modulus of an ice sheet, E, corresponds to the effective Young's modulus,  $E_{\text{eff}}$ , computed for each sheet in (Lilja et al., 2019a) (*Load case I*,  $|v_x| = 0.1 \text{ m/s}$ ); and that
- *iii*) the characteristic length of an ice sheet,  $l_{\rm ch}$ , corresponds to the effective characteristic length,  $l_{\rm ch,eff}$ , computed for each sheet in (Lilja et al., 2019b).

These facts are here emphasized because a direct usage of the data given in Table 1 (as applied on the "microscopic" scale of a lattice, i.e. with the beams) would have yielded loads with magnitudes of about  $F_{\rm V,FE-DE}/F_{\rm V,Croasdale} \approx 4...5$ . This would have held for each h in both load cases. A situation of not knowing the values of the "real" constitutive parameters one (implicitly) applies while computing the loads may thus occur and is likely to lead to erratic interpretations. If, on the other hand, the model has been tested and the (effective) constitutive properties are known, the computed ice-structure interaction loads appear consistent. What data sets one uses with the items above affects, of course, the results. It was here estimated that because the vertical cone speed was 0.1 m/s and the effective in-plane speeds 0.05 m/s (see sub-subsection 4.4), the made selections ought to provide the best approximation.

An interesting feature of the results is that the boundary conditions appear to have no effect. This is interpreted to indicate that the sheets studied are large enough (for the current load cases) to resemble an infinite ice sheet. The outer boundaries are far enough from the process zones as regards to the computation of the breaking loads. Note that an opposite observation was done in subsection 6.2 above. There the boundary conditions had an effect. The breakthrough loads and strengths were significantly higher in Load case B2 for the two thickest sheet sizes considered.



**Box 8.** Computation of the effective waterline diameter  $D_{\rm w}$ .

$D_{\rm w} = D - 2D_x = D - 2l_\alpha \cos\left(\alpha\right)$	
$= D - \frac{2h_2\cos\left(\alpha\right)}{\sin\left(\alpha\right)} = D - \frac{2h_2}{\tan\left(\alpha\right)}$	
$= D - \frac{2\left(\rho_{\rm i}h/\rho_{\rm w} + \delta_0 - v_z\Delta t\right)}{\tan\left(\alpha\right)}$	

Figure 25: A schematic illustration of an ice-structure interaction process in Load case C – computation of the effective waterline diameter  $D_{\rm w}$ . A surfacing, rigid, truncated cone is breaking an ice sheet containing a circular hole. Note that although a dotted line is drawn to indicate axisymmetry, the FE-DE sheet samples were not axisymmetric but square. Only the cone is axisymmetric.

#### 6.3.2. Cracking characteristics

Figure 26 displays, as an example, two fractured ice sheets studied in Load case C1. Several radial cracks propagate outwards from the contact zones approximately perpendicular to the cone surfaces. Circumferential cracks joining the radial cracks emerge and complete the failure mechanisms. Notice (in both examples) that the first one or two layers of the discrete elements immediately adjacent to the holes provide paths for the cracks to advance nearly radially outwards. This is a byproduct of the meshing scheme. An alternative technique to create a hole would have been, for example, to first create a regular mesh (i.e. without a hole) and then to remove from it all those discrete elements whose centres of mass lie within a certain radial distance, say D/2, from the centre. Such an approach would have, however, created holes with extremely irregular boundaries and was thus not considered.

An interesting observation is that in a thick sheet, the cracks tend to propagate all the way to the free boundaries, Figure 26a, whereas in a thin sheet, Figure 26b, they tend to arrest. This looks like a clear size effect (the size and number of elements may have an effect though). In a thick sheet, cracks tend to grow longer due to the greater amount of stored potential energy being released from the water-ice sheet system (i.e. the crack driving force is larger). Otherwise, the responses appear actually quite similar. The effective "sizes" of the broken off wedges, for example, are in both cases approximately equal – about  $3l \dots 5l$  – and the failure patterns near the cones seem to be similar. Cracking characteristics should be similar for geometrically similar structures.

# 6.3.3. Discussion

It is clear that with the proposed in-direct approach no downdrift (wake) region forms. The applicability of the approach may thus be limited to the first occurrence of a flexural failure. Even so, the approach should provide a reasonable approximation of the breaking loads as well as a means to investigate the failure as regards to the ability of a numerical method to produce both radial and circumferential cracking. It is not limited to the hybrid FE-DE method applied in this paper, but is generally applicable to examine the ability of a numerical method to describe the typical cracking characteristics observed in nature.

Since it was found that a scheme in which a stationary ice sheet containing a circular hole is interacting with a surfacing cone produces loads approximately equal to the conventional case of a moving ice sheet interacting with a one-sided, stationary structure, it may be possible to devise an experimental setting (for measuring ice loads imposed on a model structure by a model ice sheet – for instance) that is more easily controllable: a sheet may remain stationary while a model structure (a cone) is driven through a hole in it.

One special feature of the proposed approach deserves a final mention: it should provide a device to compute bifurcation-type buckling loads. It is evident that a sheet is under an approximately axisymmetric, compressive, membrane state of stress (with other than free boundary conditions) through the contacts with the cone, see Figure 25.



Figure 26: FE-DE ice sheet samples interacting with rigid, truncated cones (not shown) in Load case C1. The cones are surfacing from below at the centres of the samples. Radial cracks propagate from the contact zones towards the free edges approximately perpendicular to the cone surfaces. Circumferential cracks joining the radial cracks emerge and complete the failure mechanisms.

#### 7. Summary and conclusions

In this paper, a rate-independent, de-cohesive damage model for the fracture modelling of large, cellular, plate-like, quasi-brittle structures was proposed. A three-dimensional, hybrid finite-discrete element (FE-DE) method was then introduced in order to study sea ice sheet fracture. This was followed by three applications. The uniaxial tensile fracture of an ice sheet was examined first. The effects of both the size and the loading rate applied on the effective, uniaxial tensile strength of an ice sheet were studied. The vertical penetration fracture of an ice sheet loaded by a rigid, flat-ended, cylindrical indenter was investigated next. The associated breakthrough loads and strengths were computed. It was studied whether the loads or strengths were functions of either the thickness h or the macroscopic in-plane size (side length L) of a square-shaped sheet. The breaking of an ice sheet containing a circular hole by a rigid, truncated, ascending cone was investigated last. The loads on the cone were computed and then compared with known analytical results of a "direct" case. It was explored whether a scheme in which a stationary ice sheet containing a circular hole interacting with a surfacing cone yields loads that are comparable to the case in which an ice sheet is moving, a structure is held stationary, and the contacts occur only unilaterally (i.e. one-sidedly). The approach is believed to be new and may be categorized as "in-direct" due to the reversed nature of the interaction scenario. The approach should provide a means not only to compute cone ice loads but also to investigate the failure as regards to the ability of a numerical method to describe both radial and circumferential cracking. It should, in addition, provide a device to compute bifurcation-type buckling loads.

While computing the tensile and the breakthrough strengths, a set of selfsimilar sheet samples with an in-plane size range of 1:16 was examined. The samples were square; had a side length of either L = 10, 20, 40, 80, or 160 m; and a thickness of either h = 0.5, 1.0, or 1.5 m. With the sheets containing holes, only the largest samples (L = 160 m) were investigated.

The results indicated that i) both the tensile and the breakthrough strengths are strong functions of both L and h; i) the tensile strength is a strong function of the applied loading rate; iii) the failure mode as regards to the vertical penetration fracture changes drastically as a function of L; iv) the model is able to demonstrate both radial and circumferential cracking; and that v) the proposed in-direct approach to compute ice loads on a conical offshore structure provides realistic (i.e. comparable to the loads computed with a direct approach) results.

#### Acknowledgements

The first three authors gratefully acknowledge the financial support by the Academy of Finland through the project *Discrete Numerical Simulation and Statistical Analysis of the Failure Process of a Non-homogeneous Ice Sheet Against an Offshore Structure (DICE)*. The second author gratefully acknowledges

the financial support by Business Finland through the project Arktisten Rakenteiden Jääkuormat: Mallikokeista Teollisuuden Mittakaavaan (ARAJÄÄ). Professor John Dempsey from Clarkson University, U.S.A. (Finland Distinguished Professor (FiDiPro) at Aalto University from November 2015 through August 2018, as part of the project Scaling of Ice Strength: Measurements and Modelling (ICESCALE), and funded by Business Finland) suggested the authors to study the breaking of an ice sheet containing a circular hole by an ascending cone and is thereby acknowledged with thanks. The financial support from the industrial partners Aker Arctic Technology Ltd., Arctech Helsinki Shipyard Ltd., Arctia Ltd., ABB Marine Ltd., Technip Offshore Finland Ltd., Suomen Hyötytuuli Ltd., Ponvia Ltd., and the Finnish Transport Agency, through the projects ICESCALE and ARAJÄÄ, is acknowledged with gratitude.

# References

- Bažant, Z.P., 2002. Scaling of Sea Ice Fracture Part I: Vertical Penetration. Journal of Applied Mechanics 69, 11–18.
- Bažant, Z.P., Guo, Z., 2002. Size Effect on Strength of Floating Sea Ice under Vertical Line Load. Journal of Engineering Mechanics 128, 254–263.
- Bažant, Z.P., Kim, J.J.H., 1998a. Size Effect in Penetration of Sea Ice Plate with Part-Through Cracks. I: Theory. Journal of Engineering Mechanics 124, 1310–1315.
- Bažant, Z.P., Kim, J.J.H., 1998b. Size Effect in Penetration of Sea Ice Plate with Part-Through Cracks. II: Results. Journal of Engineering Mechanics 124, 1316–1324.
- Bažant, Z.P., Li, Y.N., 1994. Penetration Fracture of Sea Ice Plate: Simplified Analysis and Size Effect. Journal of Engineering Mechanics 120, 1304–1321.
- Bažant, Z.P., Oh, B.H., 1983. Crack Band Theory for Fracture of Concrete. Materials and Structures 16, 155–177.
- Bažant, Z.P., Planas, J., 1997. Fracture and Size Effect in Concrete and Other Quasibrittle Materials. CRC Press.
- Bažant, Z.P., Tabbara, M.R., Kazemi, M.T., Pijaudier-Cabot, G., 1990. Random Particle Model for Fracture of Aggregate or Fiber Composites. Journal of Engineering Mechanics 116, 1686–1705.
- Beltaos, S., 2002. Collapse of Floating Ice Covers Under Vertical Loads: Test Data vs. Theory. Cold Regions Science and Technology 34, 191–207.
- Black, L.D., 1958. Relative Strengths of Plates on Elastic Foundation. Transactions of the Engineering Institute of Canada 2, 129–131.
- Bolander, J.E., Saito, S., 1998. Fracture Analyses Using Spring Networks with Random Geometry. Engineering Fracture Mechanics 61, 569–591.

- Brown, T.G., Määttänen, M., 2009. Comparison of Kemi-I and Confederation Bridge Cone Ice Load Measurement Results. Cold Regions Science and Technology 55, 3–13.
- Camacho, G.T., Ortiz, M., 1996. Computational Modelling of Impact Damage in Brittle Materials. International Journal of Solids and Structures 33, 2899– 2938.
- Carol, I., Prat, P.C., López, C.M., 1997. Normal/Shear Cracking Model: Application to Discrete Crack Analysis. Journal of Engineering Mechanics 123, 765–773.
- Carol, I., Rizzi, E., Willam, K., 1994. A Unified Theory of Elastic Degradation and Damage Based on a Loading Surface. International Journal of Solids and Structures 31, 2835–2865.
- Cowper, G.R., 1966. The Shear Coefficient in Timoshenko's Beam Theory. Journal of Applied Mechanics 33, 335–340.
- Crisfield, M.A., 1997. Non-Linear Finite Element Analysis of Solids and Structures, Volume 2: Advanced Topics. John Wiley & Sons.
- Croasdale, K.R., Cammaert, A.B., 1994. An Improved Method for the Calculation of Ice Loads on Sloping Structures in First-Year Ice. Hydrotechnical Construction 28, 174–179.
- Dempsey, J.P., Shen, H.H. (Eds.), 2001. Scaling Laws in Ice Mechanics and Ice Dynamics: Proceedings of the IUTAM Symposium Held in Fairbanks, Alaska, U.S.A., 13-16 June 2000, Springer Science & Business Media.
- Dempsey, J.P., Slepyan, L.I., Shekhtman, I.I., 1995. Radial Cracking with Closure. International Journal of Fracture 73, 233–261.
- Dempsey, J.P., Zhao, Z.G., 1993. Elastohydrodynamic Response of an Ice Sheet to Forced Sub-Surface Uplift. Journal of the Mechanics and Physics of Solids 41, 487–506.
- Dempsey, K.M., Grossman, N., Vasileva, I.V., 2006. Force Inversion in Floating Plate Dynamics. Inverse Problems 22, 381–397.
- Dempsey, K.M., Vasileva, I.V., 2006a. Dynamic Uplift Scenarios for Floating Ice. Journal of Cold Regions Engineering 20, 39–51.
- Dempsey, K.M., Vasileva, I.V., 2006b. Ill-Posedness in Floating Plate Dynamics, in: Proceedings of the 18<sup>th</sup> International Conference on Port and Ocean Engineering Under Arctic Conditions Held in Potsdam, New York, U.S.A., 26-30 June 2005, pp. 1117–1130.
- Dougill, J.W., 1976. On Stable Progressively Fracturing Solids. Journal of Applied Mathematics and Physics (ZAMP) 27, 423–437.

- Du, Q., Faber, V., Gunzburger, M., 1999. Centroidal Voronoi Tessellations: Applications and Algorithms. SIAM review 41, 637–676.
- Feng, Y.T., Han, K., Owen, D.R.J., 2005. An Energy-Based Polyhedron-to-Polyhedron Contact Model, in: Proceedings of the Third M.I.T. Conference on Computational Fluid and Solid Mechanics Held in Cambridge, U.S.A., 14-17 June 2005, pp. 210–214.
- Frankenstein, E.G., 1963. Load Test Data for Lake Ice Sheet, Rep. No. 89. Technical Report. U.S. Army Corps of Engineers, Cold Regions Research & Engineering Laboratory, Hanover, New Hampshire.
- Frankenstein, E.G., 1966. Strength of Ice Sheets, in: Proceedings of the Conference on Ice Pressures Against Structures Held at Laval University, Quebec, 10-11 November 1966, National Research Council of Canada. pp. 79–87.
- Freund, J., Karakoç, A., 2015. Shear and Torsion Correction Factors of Timoshenko Beam Model for Generic Cross Sections. Research on Engineering Structures & Materials 2, 19–27.
- Gálvez, J.C., Červenka, J., Cendón, D.A., Saouma, V., 2002. A Discrete Crack Approach to Normal/Shear Cracking of Concrete. Cement and Concrete Research 32, 1567–1585.
- Gold, L.W., Black, L.D., Trofimenkoff, F.N., Matz, D., 1958. Deflections of Plates on Elastic Foundation. Transactions of the Engineering Institute of Canada 2, 123–130.
- Hillerborg, A., Modéer, M., Petersson, P.E., 1976. Analysis of Crack Formation and Crack Growth in Concrete by Means of Fracture Mechanics and Finite Elements. Cement and Concrete Research 6, 773–781.
- Høyland, K.V., 2007. Morphology and Small-Scale Strength of Ridges in the North-Western Barents Sea. Cold Regions Science and Technology 48, 169– 187.
- Hughes, T.J.R., 1983. Analysis of Transient Algorithms with Particular Reference to Stability Behavior, in: Belytschko, T., Hughes, T.J.R. (Eds.), Computational Methods for Transient Analysis.. North-Holland. chapter 2, pp. 67–155.
- Jirásek, M., Bažant, Z.P., 1995. Macroscopic Fracture Characteristics of Random Particle Systems. International Journal of Fracture 69, 201–228.
- Jones, S.J., McKenna, R.F., Tillotson, J., Jordaan, I.J. (Eds.), 1991. Ice-Structure Interaction: Proceedings of the IUTAM/IAHR Symposium Held in St. John's, Newfoundland, Canada, 14-17 August 1989, Springer-Verlag.
- Kankaanpää, P., 1988. Morphology of Sea Ice Pressure Ridges in the Baltic Sea. Geophysica 24, 15–44.

- Kerr, A.D., 1976. The Bearing Capacity of Floating Ice Plates Subjected to Static or Quasi-Static Loads. Journal of Glaciology 17, 229–268.
- Kerr, A.D., 1996. Bearing Capacity of Floating Ice Covers Subjected to Static, Moving, and Oscillatory Loads. Applied Mechanics Reviews 49, 463–476.
- Kerr, A.D., Palmer, W.T., 1972. The Deformations and Stresses in Floating Ice Plates. Acta Mechanica 15, 57–72.
- Kulyakhtin, S., 2014. Distribution of Ice Block Sizes in Sails of Pressure Ice Ridges, in: Proceedings of the  $22^{nd}$  IAHR International Symposium on Ice Held in Singapore, 11-15 August 2014.
- Leon, S.E., Spring, D.W., Paulino, G.H., 2014. Reduction in Mesh Bias for Dynamic Fracture Using Adaptive Splitting of Polygonal Finite Elements. International Journal for Numerical Methods in Engineering 100, 555–576.
- Li, Y.N., Bažant, Z.P., 1994. Penetration Fracture of Ice Plate: 2D Analysis and Size Effect. Journal of Engineering Mechanics 120, 1481–1498.
- Lichtenberger, G.J., Jones, J.W., Stegall, R.D., Zadow, D.W., 1974. Static Ice Loading Tests: Resolute Bay – Winter 1973/74, APOA Project No. 64, Report No. 745B-74-14. Technical Report. Sunoco Science and Technology.
- Lilja, V.P., Polojärvi, A., Tuhkuri, J., Paavilainen, J., 2017. Effective Tensile Strength of an Ice Sheet Using a Three-Dimensional FEM-DEM Approach, in: Proceedings of the 24<sup>th</sup> International Conference on Port and Ocean Engineering under Arctic Conditions Held in Busan, South Korea, 11-16 June 2017.
- Lilja, V.P., Polojärvi, A., Tuhkuri, J., Paavilainen, J., 2019a. Effective Material Properties of a Finite Element-Discrete Element Model of an Ice Sheet. Computers and Structures 224.
- Lilja, V.P., Polojärvi, A., Tuhkuri, J., Paavilainen, J., 2019b. A Free, Square, Point-Loaded Ice Sheet: A Finite Element-Discrete Element Approach. Marine Structures 68.
- Lu, W., Lubbad, R., Løset, S., 2015. Out-of-Plane Failure of an Ice Floe: Radial-Crack-Initiation-Controlled Fracture. Cold Regions Science and Technology 119, 183–203.
- Lu, W., Lubbad, R., Løset, S., Kashafutdinov, M., 2016. Fracture of an Ice Floe: Local Out-of-Plane Flexural Failures Versus Global In-Plane Splitting Failure. Cold Regions Science and Technology 123, 1–13.
- Luo, Y., 2008. An Efficient 3D Timoshenko Beam Element with Consistent Shape Functions. Advances in Theoretical and Applied Mechanics 1, 95–106.

- McGilvary, W.R., Sodhi, D.S., Lever, J.H., 1990. Dynamic analysis of a floating ice sheet undergoing vertical indentation, in: Proceedings of the Ninth International Conference on Offshore Mechanics and Arctic Engineering Held in Houston, Texas, U.S.A., 18-23 February 1990, pp. 195–203.
- Meyerhof, G.G., 1960. Bearing Capacity of Floating Ice Sheets. Journal of the Engineering Mechanics Division 86, 113–146.
- Miller, O., Freund, L.B., Needleman, A., 1999. Modeling and Simulation of Dynamic Fragmentation in Brittle Materials. International Journal of Fracture 96, 101–125.
- Mott, N.F., 1947. Fragmentation of Shell Cases. Proceedings of the Royal Society of London, Series A: Mathematical, Physical and Engineering Sciences 189, 300–308.
- Munjiza, A.A., Latham, J.P., John, N.W.M., 2003. 3D Dynamics of Discrete Element Systems Comprising Irregular Discrete Elements – Integration Solution for Finite Rotations in 3D. International Journal for Numerical Methods in Engineering 56, 35–55.
- Nevel, D.E., 1992. Ice forces on cones from floes, in: Proceedings of the 11<sup>th</sup> IAHR Symposium on Ice Held in Banff, Alberta, Canada, 15-19 June 1992, pp. 1391–1404.
- Paavilainen, J., Tuhkuri, J., Polojärvi, A., 2009. 2D Combined Finite-Discrete Element Method to Model Multi-Fracture of Beam Structures. Engineering Computations 26, 578–598.
- Polojärvi, A., Tuhkuri, J., Korkalo, O., 2012. Comparison and Analysis of Experimental and Virtual Laboratory Scale Punch Through Tests. Cold Regions Science and Technology 81, 11–25.
- Pushkin, A.V., Slepyan, L.I., Zlatin, A.N., 1991. Dynamical Problems of Ice Cover Fracture. International Journal of Offshore and Polar Engineering 1, 212–219.
- Ralston, T.D., 1977. Ice Force Design Considerations for Conical Offshore Structures, in: Proceedings of the Fourth International Conference on Port and Ocean Engineering under Arctic Conditions Held in St. John's, Newfoundland, Canada, 26-30 September 1977, pp. 741–752.
- Rimoli, J.J., Rojas, J.J., Khemani, F.N., 2012. On the Mesh Dependency of Cohesive Zone Models for Crack Propagation Analysis, in: Proceedings of the 53<sup>rd</sup> AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference Held in Honolulu, Hawaii, U.S.A., 23-26 April 2012.
- Schardin, H., 1959. Velocity Effects in Fracture, in: Averbach, B.L. (Ed.), Fracture. MIT Press, pp. 297–330.

- Schreyer, H.L., Sulsky, D.L., Munday, L.B., Coon, M.D., Kwok, R., 2006. Elastic-Decohesive Constitutive Model for Sea Ice. Journal of Geophysical Research 111.
- Schulson, E.M., Duval, P., 2009. Creep and Fracture of Ice. Cambridge University Press.
- Seagraves, A., Radovitzky, R., 2010. Advances in Cohesive Zone Modeling of Dynamic Fracture, in: Shukla, A., Ravichandran, G., Rajapakse, Y.D.S. (Eds.), Dynamic Failure of Materials and Structures. Springer Science, pp. 349–405.
- Seagraves, A.N., 2013. Next Generation Computational Tools for Extreme-Scale Simulation of Dynamic Fracture and Fragmentation in Three Dimensions. Ph.D. thesis. Massachusetts Institute of Technology, Cambridge, U.S.A.
- Simo, J.C., Hughes, T.J.R., 1998. Computational Inelasticity. Springer-Verlag.
- Slepyan, L.I., 1990. Modeling of Fracture of Sheet Ice. Rossijskaya Akademiya Nauk Izvestiya, Mekhanika Tverdogo Tela (Transactions of the Academy of Sciences of the U.S.S.R., Mechanics of Solids) 25, 151–157.
- Sodhi, D.S., 1989. Interaction Forces During Vertical Penetration of Floating Ice Sheets With Cylindrical Indentors, in: Proceedings of the Eighth International Conference of Offshore Mechanics and Arctic Engineering Held in The Hague, The Netherlands, 19-23 March 1989, pp. 377–382.
- Sodhi, D.S., 1995. Breakthrough Loads of Floating Ice Sheets. Journal of Cold Regions Engineering 9, 4–22.
- Sodhi, D.S., 1998. Vertical Penetration of Floating Ice Sheets. International Journal of Solids and Structures 35, 4275–4294.
- Spring, D.W., Leon, S.E., Paulino, G.H., 2014. Unstructured Polygonal Meshes with Adaptive Refinement for the Numerical Simulation of Dynamic Cohesive Fracture. International Journal of Fracture 189, 33–57.
- Talischi, C., Paulino, G.H., Pereira, A., Menezes, I.F.M., 2012. Polymesher: A General-Purpose Mesh Generator for Polygonal Elements Written in Matlab. Structural and Multidisciplinary Optimization 45, 309–328.
- Timco, G.W., Weeks, W.F., 2010. A Review of the Engineering Properties of Sea Ice. Cold Regions Science and Technology 60, 107–129.
- Tryde, P. (Ed.), 1980. Physics and Mechanics of Ice: Proceedings of the IU-TAM Symposium Held in Copenhagen, Denmark, 6-10 August 1979, Springer-Verlag.

- Vasileva, I.V., Dempsey, K.M., 2006. Numerical Force Inversion in Floating Plate Dynamics, in: Proceedings of the 18<sup>th</sup> International Conference on Port and Ocean Engineering Under Arctic Conditions Held in Potsdam, New York, U.S.A., 26-30 June 2005, pp. 1103–1116.
- Wyman, M., 1950. Deflections of an Infinite Plate. Canadian Journal of Research 28, Sec. A., 293–302.