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On Two-Dimensional Polynomial Predictors

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***Abstract*—Many signals in nature and engineering systems can be locally modeled as relatively low degree polynomials, thus one-dimensional polynomial predictive filters are useful especially in time-critical systems. The goal of this paper is to introduce the two-dimensional polynomial predictive FIR filters and present few of their properties. First we discuss previous main results in one-dimensional polynomial predictive filters. Then we show how to find the coefficients and the system functions of the minimum area polynomial predictor, and we present the recursive form for the system function of a minimum area polynomial predictor. Finally, we approach the general form of 2D polynomial predictors.**

I. INTRODUCTION

A linear (discrete) predictive filter is an IIR (Infinite Impulse Response) or FIR (Finite Impulse response) linear filter that is able to calculate (predict) a future value of a signal that belongs to a particular class of signals and that the prediction is exact in the absence of noise. For example, available are $\ldots, x(n-2), x(n-1), x(n)$ and we wish to express $x(n+p)$, as

$$x(n+p) = \sum_{k=0}^{N} h_k x(n-k) \equiv y(n) \tag{1}$$

where

$$H(z) = \sum_{k=0}^{N} h_k z^{-k} \tag{2}$$

is the system function of the predictive filter and N is either finite or infinite.

From the theory of linear difference equations and the theory of formal power series [1], [2], it follows that in the case of finite N, a signal $x(n)$ satisfying (1) is of the form

$$x(n) = \sum_{k=0}^{K} \sum_{l=0}^{L_k} c_{kl} n^l z_k^n \tag{3}$$

where c_{kl}, z_k are complex numbers and K, L_k are positive integers. Thus the class of finitely (K, L_k are finite) linearly predictable signals consists of polynomially modulated exponential signals.

Let us consider a one-dimensional (1D) linear polynomial one-step ahead prediction filter of order L, i.e. a filter that performs polynomial prediction of

$$\pi(n) = \pi_0 + \pi_1 n + \ldots \pi_L n^L, \tag{4}$$

a polynomial of degree L.

Because many signals in nature and engineering systems, e.g. in control applications, can be locally modeled as relatively low degree polynomials, one-dimensional polynomial predictive filters are useful especially in time-critical systems. Their properties and design methods have been studied for decades [3], [4].

It has been shown [5] that a polynomial one-step ahead predicting filter can be decomposed in a natural way to a parallel structure

$$H(z) = \Theta(z) + \Phi(z)G(z) \tag{5}$$

where

- $\Theta(z)$ is the shortest one-step ahead predicting FIR filter of order L;
- $\Phi(z)$ is order $L+1$ difference operator and $G(z)$ is any filter.

For any degree L the minimum length predictor can be found e.g. noticing that the difference operator of order L reduces any polynomial signal of degree $\leq L$ to a constant [6] or by solving the linear equations that the requirement of one-step ahead prediction of polynomial signals $1, n, \ldots, n^L$ imposes on the coefficients of the predictive FIR filter [7], [8].

As far as we know, except [9], there was almost no study for the two-dimensional (2D) case. One reason may be that polynomial prediction is essentially a polynomial interpolation/extrapolation problem, but for the 2D case the existence of a unique solution depends not only on the number of distinct points relative to the degree of polynomials, but also on the configuration of the points on the plane [10].

The goal of this paper is to introduce the two-dimensional polynomial FIR predictive filters and present few of their properties. The paper is organized as follows. Section II discusses previous main results in one-dimensional polynomial predictive filters. In Section III we show how to find the coefficients in general case. For prediction vector $(p, q) = (1, 1)$ we compute the system function of a 2D minimum area polynomial predictor (Section III-A), then we present the recursive form for the system function of a minimum area polynomial predictor (Section IV). Finally we approach the general form of 2D polynomial predictors (Section V).

II. ONE-DIMENSIONAL POLYNOMIAL PREDICTIVE FILTERS

In this Section we discuss two main results in one-dimensional polynomial predictive filters that will be useful later: the minimum length polynomial predictor and the Heinonen-Neuvo predictor.

A. Minimum length polynomial predictor

The impulse response of the shortest FIR filter satisfying (1) with $p = 1$ has the z-transform

$$\Theta(z) = 1 + (1 - z^{-1}) + (1 - z^{-1})^2 + \ldots + (1 - z^{-1})^L. \quad (6)$$

If $\Delta(z) = (1 - z^{-1})$, then

$$\Theta(z) = \sum_{k=0}^{L} \Delta^l(z). \quad (7)$$

Remark 1: This has a nice formal interpretation. Consider the following

$$\begin{gathered}\Theta(z) = \{1 + (1 - z^{-1}) + (1 - z^{-1})^2 + \ldots\} \\ -(1 - z^{-1})^{L+1}\{1 + (1 - z^{-1}) + (1 - z^{-1})^2 + \ldots\}\end{gathered} \quad (8)$$

The first part (8) equals $(1 - (1 - z^{-1}))^{-1} = z$.

Because of the factor $(1 - z^{-1})^{L+1}$, the response of the second part (8) to any polynomial of degree at most L is zero. So from the point of view of a polynomial signal, $\Theta(z)$ looks like a one step advance operator.

In fact, it can be shown that the system function of any polynomial predictive filter "of order L" can be written in the form

$$\begin{aligned}\Theta(z) = &1 + (1 - z^{-1}) + (1 - z^{-1})^2 + \ldots + (1 - z^{-1})^L \\ &+(1 - z^{-1})^{L+1}G(z),\end{aligned} \quad (9)$$

where $G(z)$ is the transfer function of an FIR filter. The proof is based on the space of polynomials being a principal ideal domain, whence Euclidean algorithm can be applied [5].

B. Heinonen-Neuvo Predictor

Consider a linear polynomial predictor of length $N + 1$ predicting the value $\pi(n + 1)$ of any polynomial signal $\pi(n)$ of degree $\leq L$. That is, for all n:

$$\pi(n + 1) = \sum_{k=0}^{N} h_k \pi(n - k), \quad (10)$$

where $\pi(n)$ is any polynomial of degree $\leq L$.

Because $n^0, n^1, \ldots, n^L$ form a basis of the space of polynomials of degree $\leq L$, (10) holds if and only if for all n

$$(n + 1)^l = \sum_{k=0}^{N} h_k(n - k)^l, \ l = 0, 1, \ldots, L. \quad (11)$$

To find necessary conditions for h_k (that actually are also sufficient), substitute $n = -1$ to (11) to get [7]:

$$\begin{gathered}h_0 + h_1 + \ldots + h_N = 1 \\ h_0(-1) + h_1(-2) + \ldots + h_N(-N - 1) = 0 \\ \vdots \\ h_0(-1)^L + h_1(-2)^L + \ldots + h_N(-N - 1)^L = 0\end{gathered} \quad (12)$$

If $L > N$, the system consisting of the last $N + 1$ equations can have only trivial solution, because the column rank of the system is $N + 1$. Thus we have solutions if $N \geq L$.

The solution for $N = L$ is called minimum length predictor.

For $N \geq L$ the solution minimizing noise gain, i.e. minimizing

$$\sum_{k=0}^{N} h_k^2 \quad (13)$$

is called Heinonen-Neuvo predictor and minimizes output mean square error for input corrupted by white noise.

Remark 2: The system functions of the four first minimum length predictors are

$$\begin{aligned}\Theta_0 &= 1 \\ \Theta_1 &= 2 - z^{-1} \\ \Theta_3 &= 3 - 3z^{-1} + z^{-2} \\ \Theta_3 &= 4 - 6z^{-1} + 6z^{-2} - z^{-3}\end{aligned} \quad (14)$$

The corresponding noise gains are 1, 5, 19 and 89. We see that to have any noise attenuation, the predictor needs to be much longer than the minimum length predictor.

There are several ways [6], [11] to extend the minimum length predictors to useful FIR or IIR predictor that has optimized frequency response, tailored to the requirements of the application.

III. Two-dimensional polynomial predictive filters

Consider a two-dimensional causal linear FIR filter expressed in $\mathbb{Z}^2$ domain as

$$y(m, n) = \sum_{k=0}^{M} \sum_{l=0}^{N} h_{kl} x(m - k, n - l), \ (m, n) \in \mathbb{Z}^2 \quad (15)$$

If $y(m, n) = x(m + p, n + q)$ and (15) holds for any polynomial $\pi(m, n)$ of total degree $\leq D$, that is, for a polynomial

$$\pi(m, n) = \sum \pi_{rs} m^r n^s, \quad (16)$$

where the summation is over $r \geq 0, s \geq 0$ and $\pi_{rs} = 0$ for $r + s > D$, then h_{kl} is the impulse response of a two-dimensional polynomial predictive filter with prediction vector (p, q).

In the following we shall consider the case $(p, q) = (1, 1)$ that is the simplest, involving simultaneous prediction to both directions $(0, 1)$ and $(1, 0)$.

A. The coefficients and system function of the minimum area polynomial predictive filters for prediction vector $(p, q) = (1, 1)$

Let the filter window be as in (15), that is

$$W = \{0, 1, \ldots, M\} \times \{0, 1, \ldots, N\}.$$

As in the one-dimensional case we get the following necessary conditions for the filter coefficients by substituting

$$\pi(m, n) = m^r n^s, \ 0 \leq m \leq M, 0 \leq s \leq N, r + s \leq D \quad (17)$$

to

$$\pi(m+1,n+1)=\sum_{k=0}^{M}\sum_{l=0}^{N}h_{kl}\pi(m-k,n-l). \qquad (18)$$

We obtain the identities:

$$(m+1)^r(n+1)^s=\sum_{\substack{0\le k\le M\\0\le l\le N\\k+l\le D}}h_{kl}(m-k)^r(n-l)^s, \qquad (19)$$

for all $0\le r\le M$, $0\le s\le N$, $r+s\le D$.

When $M=N=D$, two-dimensional polynomial predictive filter is called the minimum area polynomial predictive filter, because there are $(D+1)(D+2)/2$ coefficients h_{kl}, i.e. the minimum number of coefficients.

By substituting $m=-1$, $n=-1$ in (19), we get the same number of linear equations for h_{kl}.

When $(r,s)=(0,0)$ we get

$$\sum_{\substack{0\le k\le M\\0\le l\le N\\k+l\le D}}h_{kl}=1 \qquad (20)$$

and when $(r,s)\ne(0,0)$ we get

$$\sum_{\substack{0\le k\le M\\0\le l\le N\\k+l\le D}}h_{kl}(-1)^{r+s}(k+1)^r(l+1)^s=0 \qquad (21)$$

The system of equations (20) and (21) provides the coefficients h_{kl}.

From the solutions of (20) and (21) we can immediately compute the values of the coefficients of the system function of the minimum area polynomial predictive filters for small values of $M=N=D$. The values of the coefficients h_{kl} are shown for polynomials of total degrees $D=1,2,3,4$ in Tables I, II and III. Note that h_{kl} are set 0 for $k+l>D$.

k \ l	0	1
0	3	-1
1	-1	0

k \ l	0	1	2
0	6	-4	1
1	-4	1	0
2	1	0	0

TABLE I
THE COEFFICIENTS h_{kl} FOR $D=1$ (LEFT) AND $D=2$ (RIGHT).

k \ l	0	1	2	3
0	10	-10	5	-1
1	-10	5	-1	0
2	5	-1	0	0
3	-1	0	0	0

TABLE II
THE COEFFICIENTS h_{kl} FOR TOTAL DEGREE $D=3$.

k \ l	0	1	2	3	4
0	15	-20	15	-6	1
1	-20	15	-6	1	0
2	15	-6	1	0	0
3	-6	1	0	0	0
4	1	0	0	0	0

TABLE III
THE COEFFICIENTS h_{kl} FOR TOTAL DEGREE $D=4$.

The system function of minimum area polynomial predictor for prediction vector $(p,q)=(1,1)$ follows:

$$\Theta_{(D)}(z_1,z_2)=\sum_{k,l}h_{kl}z_1^{-k}z_2^{-l}. \qquad (22)$$

They are for a polynomial of total degree 1

$$\Theta_{(1)}(z_1,z_2)=3-z_1^{-1}-z_2^{-1} \qquad (23)$$

and degree 2

$$\Theta_{(2)}(z_1,z_2)=6-4z_1^{-1}-4z_2^{-1}+z_1^{-2}+z_1^{-1}z_2^{-1}+z_2^{-2} \qquad (24)$$

and so on. The magnitude of the minimum area polynomial predictive filters are shown for polynomials of total degrees $D=1$ and 4 in Figure 1.

If we write

$$K_r(z_1,z_2)=\sum_{i=0}^{r}z_1^{-r+i}z_2^{-i} \qquad (25)$$

we can express minimum area predictor for polynomials of at most fixed total degree compactly:

$$\begin{aligned}\Theta_{(1)}(z_1,z_2)&=3K_0-K_1;\\ \Theta_{(2)}(z_1,z_2)&=6K_0-4K_1+K_2;\\ \Theta_{(3)}(z_1,z_2)&=10K_0-10K_1+5K_2-K_3;\\ \Theta_{(4)}(z_1,z_2)&=15K_0-20K_1+15K_2-6K_3+K_4.\end{aligned} \qquad (26)$$

IV. RECURSIVE FORM FOR THE SYSTEM FUNCTION OF A MINIMUM AREA POLYNOMIAL PREDICTOR FOR PREDICTION VECTOR $(p,q)=(1,1)$

The expressions (25) and (26) give a simple form of minimum area two-dimensional polynomial predictors with prediction vector $(p,q)=(1,1)$ for polynomial signals of total degree $\le D$. However, from these expressions we do not see immediately how the minimum length predictor for total degree $D+1$ evolves from the corresponding one for degree D.

Observing that $\Theta_{(0)}(z_1,z_2)=1$ is the required predictor for polynomials of total degree 0. Moreover, we have

$$\begin{aligned}\Theta_{(1)}(z_1,z_2)&=3-z_1^{-1}-z_2^{-1}=\\ &=1+[(1-z_1^{-1})+(1-z_2^{-1})]\end{aligned} \qquad (27)$$

and

$$\begin{aligned}\Theta_{(2)}(z_1,z_2)&=6-4z_1^{-1}-4z_2^{-2}\\ &+z_1^{-1}+z_1^{-1}z_2^{-2}+z_2^{-2}\\ &=1+[(1-z_1^{-1})+(1-z_2^{-1})]\\ +[(1-z_1^{-1})^2&+(1-z_1^{-1})(1-z_2^{-1})+(1-z_2^{-1})^2)]\end{aligned} \qquad (28)$$

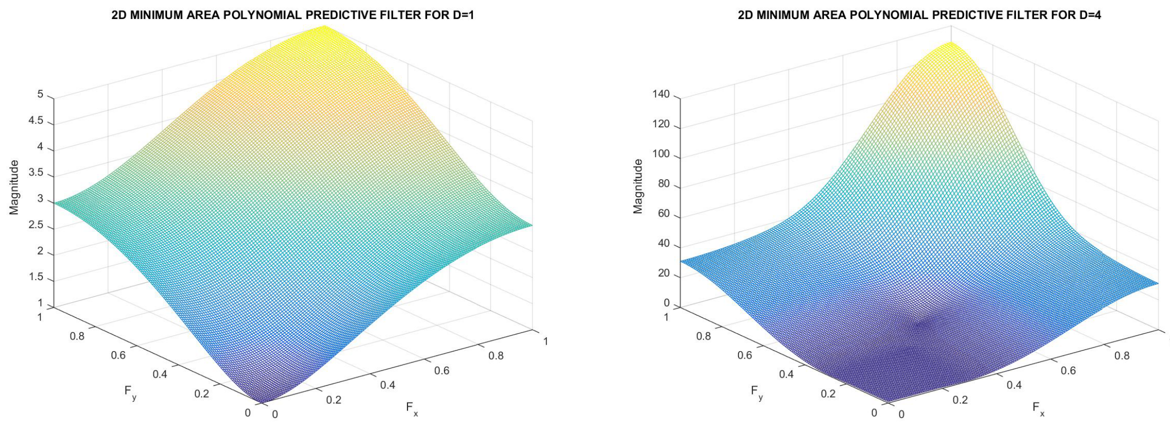

Fig. 1. Magnitude of minimum area polynomial predictive filter for $D = 1$ and 4.

and

$$\begin{aligned} \Theta_{(3)}(z_1, z_2) &= 10 - 10z_1^{-1} - 10z_2^{-1} + 5z_1^{-2} \\ 5z_1^{-1}z_2^{-1} + 5z_2^{-2} &+ z_1^{-3} + z_1^{-2}z_2^{-1} + z_1^{-1}z_2^{-2} + z_2^{-3} \\ &= 1 + [(1 - z_1^{-1}) + (1 - z_2^{-1})] \\ +[(1 - z_1^{-1})^2 &+ (1 - z_1^{-1})(1 - z_2^{-1}) + (1 - z_2^{-1})^2] \\ &+[(1 - z_1^{-1})^3 + (1 - z_1^{-1})^2(1 - z_2^{-1}) \\ &+(1 - z_1^{-1})^1(1 - z_2^{-1})^2 + (1 - z_2^{-1})^3] \end{aligned} \tag{29}$$

Thus we observe that the system function of a minimum area predictor with prediction vector $(p, q) = (1, 1)$ for polynomial signals of total degree $\leq D$ is

$$\Theta_{(D)}(z_1, z_2) = \sum_{k=0}^{D} \sum_{l=0}^{k} (1 - z_1^{-1})^{k-l}(1 - z_2^{-1})^l \tag{30}$$

For one-dimensional predictors it is relatively straightforward to give a combinatorial proof of the form (6) of the minimum length predictor.

For two-dimensional polynomial signals this would be more complicated. However, the approach using formal power series as in Remark 1 works. We can proceed as follows.

Write $\Delta_1(z_1) = (1 - z_1^{-1})$, $\Delta_2(z_2) = (1 - z_2^{-1})$.

Then

$$\Theta_{(D)}(z_1, z_2) = \sum_{k=0}^{D} \sum_{l=0}^{k} \Delta_1^{k-l}(z_1)\Delta_2^l(z_2) \tag{31}$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \Delta_1^{k-l}(z_1)\Delta_2^l(z_2) - \sum_{k=D+1}^{\infty} \sum_{l=0}^{k} \Delta_1^{k-l}(z_1)\Delta_2^l(z_2)$$

Because the second sum on the last line contains all mixed differences of order at least $D + 1$, its response to any polynomial of total degree $\leq D$ is zero. This means that the response of (31) to any polynomial signal of total degree at most D is

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_{l=0}^{k} \Delta_1^{k-l}(z_1)\Delta_2^l(z_2) \\ &= [1 + \Delta_1(z_1) + \Delta_1^2(z_1) + \ldots][1 + \Delta_2(z_2) + \Delta_2^2(z_2) + \ldots] \\ &= \frac{1}{1 - \Delta_1(z_1)} \cdot \frac{1}{1 - \Delta_2(z_2)} = z_1 z_2 \end{aligned} \tag{32}$$

showing that it realizes a polynomial predictor with prediction vector $(1, 1)$.

V. A General Form of 2D Polynomial Predictors

We can find a general form for 2D polynomial FIR predictors similarly to the 1D case given in (9). Following the reasoning for 1D case, we write the minimum area predictor for 2D polynomials of total degree $\leq L$ as

$$\begin{aligned} \Theta_{(L)}(z_1, z_2) \quad &= \quad \Phi^{(0)}(z_1, z_2) + \Phi^{(1)}(z_1, z_2) + \Phi^{(2)}(z_1, z_2) + \ldots \\ &+ \quad \Phi^{(L)}(z_1, z_2) \end{aligned}$$

where $\Phi^{(0)}(z_1, z_2) = 1$ and

$$\Phi^{(k)} = \sum_{l=0}^{k} (1 - z_1^{-1})^{k-l}(1 - z_2^{-1})^l. \tag{33}$$

Because $\Phi^{(L+1)}$ annihilates all polynomials of total degree $\leq L$, we notice that

$$\begin{aligned} H(z_1, z_2) &= [\Phi^{(0)}(z_1, z_2) + \Phi^{(1)}(z_1, z_2) + \Phi^{(2)}(z_1, z_2) + \ldots \\ &+\Phi^{(L)}(z_1, z_2)] + \Phi^{(L+1)}(z_1, z_2)G(z_1, z_2) \\ &= \Theta_{(L)}(z_1, z_2) + \Phi^{(L+1)}(z_1, z_2)G(z_1, z_2) \end{aligned} \tag{34}$$

is a 2D polynomial predictor for polynomials of total degree $\leq L$. However, unlike in 1D case, this representation is not unique.

We show now that by modifying $G(z_1, z_2)$, $H(z_1, z_2)$ changes too. Thus by moving the zeros of second filter

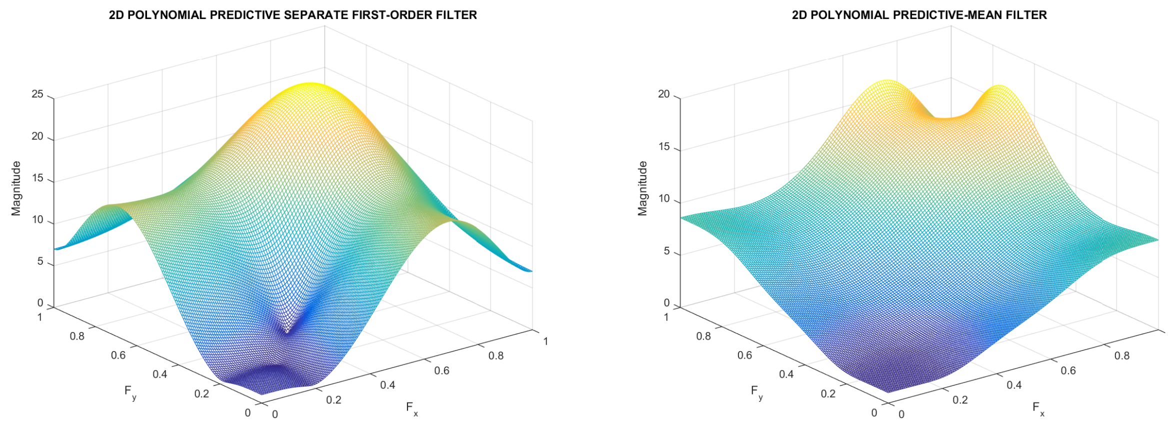

Fig. 2. Magnitude of $H_1(z_1, z_2)$ (left) and $H_2(z_1, z_2)$ (right).

$G(z_1, z_2)$, the magnitude of $H(z_1, z_2)$ changes clearly enough to indicate the existence of design possibility. As an example, we consider two cases for $L = 2$:

1) $G_1(z_1, z_2)$ is composed of two different separate first order filters that have zeros at -1:

$$G_1(z_1, z_2) = (1 + z_1^{-1})(1 + z_2^{-1}); \quad (35)$$

2) $G_2(z_1, z_2)$ is a mean filter:

$$G_2(z_1, z_2) = \frac{1}{10}(1 + z_1^{-1} + z_2^{-1} + z_1^{-2} + \\ + z_1^{-1}z_2^{-1} + z_2^{-2} + z_1^{-3} + z_1^{-2}z_2^{-1} + z_1^{-1}z_2^{-2} + z_2^{-3}). \quad (36)$$

The coefficients of $H_1(z_1, z_2)$ and $H_2(z_1, z_2)$ are shown in Tables IV and V. Their magnitude frequency plots are presented in Figure 2.

k \ l	0	1	2	3	4
0	10	-6	-1	3	1
1	-6	-1	1	2	-1
2	-1	1	2	-1	0
3	3	2	-1	0	0
4	-1	-1	0	0	0

TABLE IV
THE COEFFICIENTS h_{kl} OF $H_1(z_1, z_2)$.

k \ l	0	1	2	3	4	5	6
0	6.4	-4.2	1.2	0.1	-0.3	0.3	-0.1
1	-4.2	0.6	-0.1	-0.6	0.6	-0.2	0
2	1.2	-0.1	-0.2	0.9	-0.3	0	0
3	0.1	-0.6	0.9	-0.4	0	0	0
4	-0.3	0.6	-0.3	0	0	0	0
5	0.3	-0.2	0	0	0	0	0
6	-0.1	0	0	0	0	0	0

TABLE V
THE COEFFICIENTS h_{kl} OF $H_2(z_1, z_2)$.

VI. CONCLUSION

In this paper we have introduced the two-dimensional polynomial predictive FIR filters and we have presented few of their properties. Future work may focus on properties, extensions, and applications of multidimensional polynomial predictive filters utilizing both Heinonen-Neuvo approach and the decomposition (34). A real world example where 2D predictive filters can be applied is wind data. Sodar and Lidar based instruments are routinely used at airports and wind farms to measure the wind direction and strength as function of time at different heights. The desired signal can be the smoothed output signal or a measure of turbulence derived from the difference of the original and the smoothed signals.

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