Hitruhin, Lauri; Lindberg, Sauli

Lamination Convex Hull of Stationary Incompressible Porous Media Equations

Published in:
SIAM Journal on Mathematical Analysis

DOI:
10.1137/20M134633X

Published: 01/01/2021

Document Version
Publisher's PDF, also known as Version of record

SIAM J. MATH. ANAL.
Vol. 53, No. 1, pp. 491–508

LAMINATION CONVEX HULL OF STATIONARY INCOMPRESSIBLE POROUS MEDIA EQUATIONS*

LAURI HITRUHIN† AND SAULI LINDBERG†

Abstract. We compute the lamination convex hull of the stationary incompressible porous media (IPM) equations. We also show in bounded domains that for subsolutions of stationary IPM taking values in the lamination convex hull, velocity vanishes identically and density depends only on height. We relate the results to the infinite time limit of non-stationary IPM.

Key words. incompressible porous media equations, relaxation, lamination convex hull

AMS subject classifications. 35Q35, 76S05, 76B03

DOI. 10.1137/20M134633X

1. Introduction.

We consider the flow of two immiscible incompressible fluids with equal viscosities and different densities in a porous medium. This can be modelled by the incompressible porous media (IPM) equations which consist of conservation of mass, incompressibility, and Darcy's law:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1.1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1.2}$$

$$\frac{\mu}{\kappa}\mathbf{v} = -\nabla p - \rho \mathbf{g}, \tag{1.3}$$

where $\rho(x,t) \in \mathbb{R}$ is the fluid density, $\mathbf{v}(x,t) \in \mathbb{R}^2$ is the fluid velocity, and $\mathbf{g} = (0, g)$ is gravity [7]. In the case of a smooth (simply connected) domain $\Omega \subset \mathbb{R}^2$, we assume the impermeability condition $\mathbf{v} \cdot \nu = 0$ on $\partial\Omega$. Without loss of generality, we set $\mu/\kappa = g = 1$.

Córdoba, Faraco, and Gancedo proved the nonuniqueness of spatially periodic weak solutions $\mathbf{v} = (v_1, v_2) \in L^\infty(0,T;L^2)$ and $\rho \in L^\infty(0,T;L^\infty)$ of IPM in [6]. The proof employs the method of convex integration which was first adapted to hydrodynamics by De Lellis and Székelyhidi in their ground-breaking paper [8]. The construction of [6], built on degenerate T4 configurations, provides a robust method of constructing bounded weak solutions in inviscid fluid dynamics without determining the exact Λ-convex hull; for an application to more general active scalar equations with an even multiplier, see [22]. The existence and nonuniqueness of spatially periodic $C^{1/9-\epsilon}$ solutions of IPM for smooth initial data was shown by Isett and Vicol in [15].

In [23], Székelyhidi computed the Λ-convex hull and showed it to be the exact relaxation of IPM equations. He also used it to construct infinitely many admissible weak solutions to the unstable Muskat problem in $\Omega = (-1,1)^2$ with a flat interface

*Received by the editors June 17, 2020; accepted for publication (in revised form) October 30, 2020; published electronically January 21, 2021.
https://doi.org/10.1137/20M134633X
Funding: The first author was supported by ICMAT Severo Ochoa project SEV-2015-0554 grant MTM2017-85934-C3-2-P and the ERC grant 307179-GFTIPFD and the ERC grant 834728 Quamap and by a grant from The Emil Aaltonen Foundation. The second author was supported by the ERC grant 307179-GFTIPFD and by the AtMath Collaboration at the University of Helsinki.
†Department of Mathematics and Systems Analysis, Aalto University, FI-00076 Aalto, Finland (lauri.hitruhin@aalto.fi, sauli.lindberg@aalto.fi).

as initial data. Székelyhidi also computed a differently normalized hull that leads to solutions with a bounded velocity. Admissible mixing solutions to the unstable Muskat problem with a nonflat H^5-regular interface were constructed by Castro, Córdoba, and Faraco in [2]. For further developments see [1, 4, 13, 19, 20]; here, also, the construction of admissible weak solutions relies on the exact hull and the construction of an admissible subsolution.

In stationary IPM, in contrast, if $\mathbf{v} \in L^2(\Omega, \mathbb{R}^2)$ with $\mathbf{v} \cdot \nu|_{\partial\Omega} = 0$ and $\rho \in L^\infty(\Omega)$ form a weak solution, then $\mathbf{v} \equiv 0$ and $\partial_1 \rho \equiv 0$; a proof of this simple fact by Elgindi appears in [10]. As a main result, Elgindi showed on $\mathbb{R}^2$ and $\mathbb{T}^2$ that whenever solutions of nonstationary IPM have initial data near certain stationary solutions, they must converge to the stationary solution in H^3 when $t \to \infty$. The global well-posedness of nonstationary IPM is open, but Elgindi showed it around said stationary solutions. In [3], Castro, Córdoba, and Lear proved structurally similar results for the *confined IPM* case $\Omega = \mathbb{T}^1 \times (-1, 1)$, overcoming new difficulties to do with the boundary.

Nevertheless, in [5], Constantin, La, and Vicol constructed solutions of stationary IPM that are smooth and vanish outside a strip that has finite width in the direction $x_1 = kx_2$, $k \in \mathbb{R}$. The result is one example of their construction which uses Grad–Shafranov-like equations to obtain smooth, localized solutions in hydrodynamics, motivated by Gavrilov's construction of smooth, compactly supported solutions of stationary Euler equations in [14]. The solutions of stationary IPM in [5] are functions of the variable $z = x_1 - kx_2$ and, as such, they are periodic (even constant) in the axial direction of the strip. If the direction of finite width of the strip is $(0, 1)$, i.e., in the case $\Omega = \mathbb{T}^1 \times (-1, 1)$, an easy adaptation of Elgindi's proof (see section 5) rules out such a construction. This dichotomy highlights the role of the direction of gravity in IPM and is discussed briefly in Remark 5.1.

The problem we address is the determination of the *relaxation* of stationary IPM. One of our aims is to shed light on the following question: How are the differences between stationary and nonstationary IPM as well as the somewhat surprising combination of the results of [5] and [10] reflected in the relaxation? We also wish to use information on the relaxation to better understand the infinite time limit of nonstationary IPM.

We set the stage by briefly describing convex integration in the Tartar framework; the relevant definitions are recalled in section 2. One first decouples a system of nonlinear constant-coefficient PDEs into a system of first-order linear PDEs $\mathscr{L}(z) = \mathbf{0}$ and the pointwise constraint that $z(x)$ takes values in a constitutive set K. In the case of stationary IPM, $z = (\rho, \mathbf{v}, \mathbf{m})$, the set of linear equations $\mathscr{L}(z) = \mathbf{0}$ is

$$\nabla \cdot \mathbf{m} = 0, \tag{1.4}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{1.5}$$

$$\nabla^\perp \cdot (\mathbf{v} + (0, \rho)) = 0, \tag{1.6}$$

and the constitutive set is

$$K = \{(\rho, \mathbf{v}, \mathbf{m}) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \colon |\rho| = 1, \ \mathbf{m} = \rho \mathbf{v}\}, \tag{1.7}$$

where the constraint $\rho \in \{-1, 1\}$ codifies the densities of the two immiscible fluids. Returning to the general Tartar framework, given initial/boundary data, one attempts to construct a strict *subsolution*, that is, z_0 satisfying $\mathscr{L}(z_0) = \mathbf{0}$ and taking values in a suitable subset $\mathscr{U}$ of the Λ-convex hull K^Λ; usually, $\mathscr{U} = \operatorname{int}(K^\Lambda)$. One then forms perturbations z_j of z_0 by adding localized plane waves where the admissible directions of oscillation are dictated by the *wave cone* Λ, and z_j take values in $\mathscr{U}$.

By a limiting argument, one intends to find infinitely many subsolutions with the prescribed initial/boundary conditions and with values in K, which would then yield nonuniqueness of the original system of PDEs for the given boundary/initial data (see [9]).

The *relaxation* of K can be given slightly different meanings, but it is defined here as the smallest set $\tilde{K} \supset K$ that is stable under weak convergence for solutions of (1.4)–(1.6), essentially following Tartar [24]. As such, it models macroscopic averages of solutions of stationary IPM. By a result of Tartar, $\tilde{K}$ contains the lamination convex hull $K^{lc,\Lambda}$ [24, Theorem 8]. We compute the lamination convex hull of stationary IPM in Theorem 1.1, and we believe that as in nonstationary IPM, the lamination and Λ-convex hulls and the relaxation coincide.

As emphasised in [9, 23], precise information on the hull is crucial in identifying the boundary/initial data for which one can run convex integration. As an example of this we mention that the hull of compressible Euler is notoriously difficult to compute and that to the authors' knowledge, due to insufficient information on the hull, lack of uniqueness has so far only been shown for a set of data where one is able to reduce to an incompressible system; see [12, 18]. However, in [18], Markfelder computed the Λ-convex hull of a suitably normalized constraint set K.

Furthermore, the physical relevance of subsolutions was already emphasised in [23] in the case of the Muskat problem. The unstable Muskat problem with a flat interface is ill-posed, but in a pioneering work [21], Otto had used mass transport techniques to construct macroscopically averaged relaxed solutions that arise as an entropy solution of a scalar conservation law. At a certain asymptotic limit [23, p. 505], Székelyhidi's subsolutions converge to Otto's relaxed solution. A subsolution can be viewed as a kind of coarse-grained average; this interpretation is explored in detail, e.g., in [4, 9, 23].

(Topological) smallness of the hull seems to reflect uniqueness of bounded solutions under trivial initial/boundary datas and (in the case of evolutionary models) existence of robust conserved quantities. As an example, in IPM and other active scalar equations with an even Fourier multiplier, K^Λ has a nonempty interior [17, 22] and there exist nontrivial bounded (even Hölder continuous) solutions with compact support in time [22, 15]. Surface quasi-geostrophic equations, in contrast, has an odd multiplier and a trivial hull $K^\Lambda = K$ (defining Λ as in [17, 22]), and the Hamiltonian is conserved by L^3 solutions, ruling out bounded solutions with compact support in time [15].

Quadratic Λ-affine functions are a simple and powerful tool in determining the size of K^Λ. To illustrate this, while two-dimensional (2D) and three-dimensional (3D) ideal magnetohydrodynamics (MHD) look superficially similar to Euler equations, both possess a nontrivial quadratic Λ-affine function which vanishes in K, making $\operatorname{int}(K^\Lambda)$ empty. As a direct reflection of this, bounded solutions conserve the mean-square magnetic potential in 2D and the magnetic helicity in 3D. This rules out solutions with a nontrivial, compactly supported magnetic field in 2D but, perhaps surprisingly, not in 3D [11]. By Tartar's Theorem (see [24, Theorem 11]), quadratic Λ-affine functions are weakly continuous, and as such, they also aid the understanding of various asymptotic regimes such as weak limits of (sub)solutions or the inviscid limit; see also [4, p. 58].

In nonstationary IPM, (1.4) is replaced by $\partial_t \rho + \nabla \cdot \mathbf{m} = 0$ and the Λ-convex hull consists of triples $(\rho, \mathbf{v}, \mathbf{m})$ such that $|\rho| \leq 1$ and $\left|\mathbf{m} - \rho\mathbf{v} + \left(0, (1-\rho^2)/2\right)\right| \leq (1-\rho^2)/2$ [23]. In particular, the Λ-convex hull has a nonempty interior.

In stationary IPM, however, $G(\rho, \mathbf{v}, \mathbf{m}) := \mathbf{m} \cdot \mathbf{v}^\perp$ vanishes in $K \cap \Lambda$, enforcing $\operatorname{int}(K^\Lambda) = \emptyset$. Other quadratic Λ-affine functions of stationary IPM include $|\mathbf{v}|^2 + \rho v_2$

and $\mathbf{m}\cdot(\mathbf{v}+(0,\rho))$—in fact, these three functions determine Λ (see Proposition 2.1). If $(\rho,\mathbf{v},\mathbf{m})\in K^{\Lambda}$ with $\mathbf{v}\neq\mathbf{0}$, then $\mathbf{m}\cdot\mathbf{v}^{\perp}=0$ yields $\mathbf{m}=k\mathbf{v}$ for some $k\in\mathbb{R}$. The main challenge in the computation of $K^{lc,\Lambda}$ is the determination of the exact range of the constant of proportionality k in $\mathbf{m}=k\mathbf{v}$.

THEOREM 1.1. *$K^{lc,\Lambda}=\cup_{j=1}^{4}X_j$, where*

$$X_1:=\left\{\left(\rho,\mathbf{0},\frac{1-\rho^2}{2}[\mathbf{e}-(0,1)]\right):\ |\rho|\le 1,\ |\mathbf{e}|\le 1\right\},$$

$$X_2:=\left\{(\rho,\mathbf{v},k\mathbf{v})\colon\ |\rho|\le 1,\ \mathbf{v}\neq\mathbf{0},\ 1\le k\le \rho-\frac{(1-\rho^2)v_2}{|\mathbf{v}|^2}\right\},$$

$$X_3:=\left\{(\rho,\mathbf{v},k\mathbf{v})\colon\ |\rho|<1,\ \mathbf{v}\neq\mathbf{0},\ -1<k=\rho-\frac{(1-\rho^2)v_2}{|\mathbf{v}|^2}<1\right\},$$

$$X_4:=\left\{(\rho,\mathbf{v},k\mathbf{v})\colon\ |\rho|\le 1,\ \mathbf{v}\neq\mathbf{0},\ \rho-\frac{(1-\rho^2)v_2}{|\mathbf{v}|^2}\le k\le -1\right\}.$$

We interpret $K^{lc,\Lambda}$ geometrically. The projections of X_2 and X_4 into $\mathbb{R}\times\mathbb{R}^2$ are cones where $\rho\in[-1,1]$ and

$$\left|\mathbf{v}+\left(0,\frac{1+\rho}{2}\right)\right|\le\frac{1+\rho}{2}\quad\Longleftrightarrow\quad|\mathbf{v}|^2+(\rho+1)v_2\le 0,\quad (X_2)\tag{1.8}$$

$$\left|\mathbf{v}-\left(0,\frac{1-\rho}{2}\right)\right|\le\frac{1-\rho}{2}\quad\Longleftrightarrow\quad|\mathbf{v}|^2+(\rho-1)v_2\le 0.\quad (X_4)\tag{1.9}$$

The *power balance* $|\mathbf{v}|^2+\rho v_2$ can be interpreted as the balance between the density of energy per unit time consumed by friction and the density of work per unit time done by gravity [4]. Thus Theorem 1.1 shows that $\{(\rho,\mathbf{v},\mathbf{m})\in K^{lc,\Lambda}\colon \mathbf{v}\neq\mathbf{0}\}$ divides into two subsets: the *flexible region* where $|\mathbf{v}|^2+\rho v_2\le|v_2|$ (the cones) and the parameter k in $\mathbf{m}=k\mathbf{v}$ lies on a nondegenerate interval, and the *rigid region* where the power balance dominates the vertical speed $|v_2|$ and k is uniquely determined. This rigid region is just the projection of X_3 into $\mathbb{R}\times\mathbb{R}^2$. When $(\rho,\mathbf{0},\mathbf{m})\in K^{lc,\Lambda}$, and thus $z=(\rho,\mathbf{0},\mathbf{m})\in X_1$, the component $\mathbf{m}$ has the same range of values as in nonstationary IPM. Furthermore, the projection of X_1 into $\mathbb{R}\times\mathbb{R}^2$ is the line segment that is formed as an intersection of the cones resulting from X_2 and X_4.

The main technical difficulties of the proof involve the smallness of the set X_3. Note that for any suitable pair $(\rho,\mathbf{v})$ there exists exactly one $\mathbf{m}\in\mathbb{R}^2$ such that $(\rho,\mathbf{v},\mathbf{m})\in X_3$. This makes it very challenging to construct Λ-convex functions that would show for these $(\rho,\mathbf{v})$ that the lamination convex and Λ-convex hull coincide. We nevertheless manage to show coincidence for all other points; see (4.1). The difficulties are also present in Propositions 4.7–4.10, most notably when showing the lamination convexity of X_3; this is the technically most difficult part of the paper.

As another main result, we show in bounded domains that if a subsolution of stationary IPM takes values in $K^{lc,\Lambda}$, it has a vanishing velocity. Recall that $L^2_\sigma(\Omega,\mathbb{R}^2):=\{\mathbf{w}\in L^2(\Omega,\mathbb{R}^2)\colon \nabla\cdot\mathbf{w}=0,\ \mathbf{w}\cdot\nu|_{\partial\Omega}=0\}$.

THEOREM 1.2. *Suppose $\Omega\subset\mathbb{R}^2$ is bounded, strongly Lipschitz, and simply connected. Suppose $\rho\in L^\infty(\Omega)$ and $\mathbf{v},\mathbf{m}\in L^2_\sigma(\Omega,\mathbb{R}^2)$ satisfy* (1.4)–(1.6). *Suppose $(\rho,\mathbf{v},\mathbf{m})(x)\in K^{lc,\Lambda}$ for almost every $x\in\Omega$. Then $\mathbf{v}=\mathbf{0}$ and $\partial_1\rho=0$.*

The conclusion of Theorem 1.2 also holds in $\mathbb{T}^1 \times (-1, 1)$, extending the dichotomy on solutions vanishing outside a strip into subsolutions (see Remark 5.1).

Motivated by Elgindi's computations in [10] as well as Theorem 1.2, we also discuss the infinite time limit of nonstationary IPM. We show in Proposition 6.1 that if $\rho \in L^\infty(0,\infty; L^\infty)$ and $\mathbf{v}, \mathbf{m} \in L^\infty(0,\infty; L^2_\sigma)$ form a subsolution in a bounded domain Ω, then $\mathbf{v} \in L^2(0,\infty; L^2_\sigma)$; in particular, $\lim_{M\to\infty} \int_M^\infty \int_\Omega |\mathbf{v}(x,t)|^2 \,\mathrm{d}x\,\mathrm{d}t = 0$.

The structure of the paper is as follows. The relevant definitions are recalled in section 2, where we also compute the wave cone. The inclusion $K^{lc,\Lambda} \supset \cup_{j=1}^4 X_j$ is proved in section 3, whereas $K^{lc,\Lambda} \subset \cup_{j=1}^4 X_j$ is proved in section 4. The proof of Theorem 1.2 is presented in section 5, and the limit $t \to \infty$ of nonstationary IPM is studied in section 6.

2. Relevant notions.

We briefly recall some notions from the theory of differential inclusions; a thorough discussion of related topics can be found in [16].

The *wave cone* Λ consists of directions $z = (\rho, \mathbf{v}, \mathbf{m}) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$ such that for some $\xi \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, plane waves of the form $x \mapsto h(x\cdot\xi)z \colon \mathbb{R}^2 \to \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$ satisfy (1.4)–(1.6) for all $h \in C^\infty(\mathbb{R})$. Denoting $(\xi_1,\xi_2)^\perp = (-\xi_2, \xi_1)$, the wave cone conditions are thus

$$\mathbf{m}\cdot\xi = 0, \tag{2.1}$$

$$\mathbf{v}\cdot\xi = 0, \tag{2.2}$$

$$(\mathbf{v} + (0,\rho))\cdot\xi^\perp = 0. \tag{2.3}$$

An explicit form of Λ is given in Proposition 2.1 and Corollary 2.2.

Proposition 2.1. *The wave cone of stationary IPM is*

$$\Lambda = \{(\rho,\mathbf{v},\mathbf{m})\colon |\mathbf{v} + (0,\rho/2)| = |\rho|/2, \quad \mathbf{m}\cdot\mathbf{v}^\perp = 0, \quad \textit{and} \quad \mathbf{m}\cdot(\mathbf{v} + (0,\rho)) = 0\}.$$

Proof. First assume $(\rho,\mathbf{v},\mathbf{m}) \in \Lambda$. The conditions (2.2)–(2.3) imply that $\mathbf{v} = k\xi^\perp$ and $\mathbf{v} + (0,\rho) = \ell\xi$ for some $k, \ell \in \mathbb{R}$. Thus $|\mathbf{v}|^2 + \rho v_2 = k\ell\xi\cdot\xi^\perp = 0$, giving $|\mathbf{v} + (0,\rho/2)| = |\rho|/2$. If $\rho = 0$, then $\mathbf{v} = \mathbf{0}$, and so clearly $\mathbf{m}\cdot(\mathbf{v}+(0,\rho)) = \mathbf{m}\cdot\mathbf{v}^\perp = 0$. If $\rho \neq 0$, then (2.1)–(2.3) give $\mathbf{m}\cdot(\mathbf{v}+(0,\rho)) = \ell\mathbf{m}\cdot\xi = 0$ and $\mathbf{m}\cdot\mathbf{v}^\perp = -k\mathbf{m}\cdot\xi = 0$.

Conversely, if $|\mathbf{v}|^2 + \rho v_2 = \mathbf{m}\cdot(\mathbf{v}+(0,\rho)) = \mathbf{m}\cdot\mathbf{v}^\perp = 0$, then we get $(\rho,\mathbf{v},\mathbf{m}) \in \Lambda$ by choosing $\xi = \mathbf{v} + (0,\rho)$ if $\mathbf{v} \neq (0,-\rho) \neq \mathbf{0}$, $\xi = \mathbf{v}^\perp$ if $\mathbf{v} = (0,-\rho) \neq \mathbf{0}$, $\xi = \mathbf{m}^\perp$ if $(0,\rho) = \mathbf{0} = \mathbf{v}$ and $\mathbf{m} \neq \mathbf{0}$, and finally $\xi = (1,1)$ if $\mathbf{v} = (0,\rho) = \mathbf{m} = \mathbf{0}$. ∎

Corollary 2.2. *The wave cone Λ consists of vectors $z \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$ of the following three forms:*

$$z = \left(\rho, \frac{\rho}{2}(\mathbf{e} - (0,1)), \ell(\mathbf{e} - (0,1))\right), \quad \rho \neq 0,\ \mathbf{e} \in S^1 \setminus \{(0,1)\},\ \ell \in \mathbb{R},$$
$$z = (\rho, \mathbf{0}, (m_1, 0)), \quad \rho \neq 0,\ m_1 \in \mathbb{R},$$
$$z = (0, \mathbf{0}, \mathbf{m}), \quad \mathbf{m} \in \mathbb{R}^2.$$

Remark 2.3. The first condition $|\mathbf{v} + (0,\rho/2)| = |\rho|/2$ in Proposition 2.1 says, equivalently, that the power balance $|\mathbf{v}|^2 + \rho v_2$ vanishes in Λ. The second condition $\mathbf{m}\cdot\mathbf{v}^\perp = 0$, in turn, expresses the fact that for plane wave solutions of (1.4)–(1.6), the velocity and the the momentum are perfectly aligned. In nonstationary IPM, the parallelism breaks down, leading to a substantially larger hull. Unless $\mathbf{v} = \mathbf{0}$, the third condition $\mathbf{m}\cdot(\mathbf{v} + (0,\rho)) = 0$ reduces to the first two by the computation $\mathbf{m}\cdot(\mathbf{v}+(0,\rho)) = k\mathbf{v}\cdot(\mathbf{v}+(0,\rho)) = 0$.

Given any compact set $C \subset \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$, the *laminates* $C^{k,\Lambda}$, $k \in \mathbb{N}_0$, of C are defined as follows:

$$C^{0,\Lambda} := C,$$
$$C^{k+1,\Lambda} := \{(\lambda z_1 + (1-\lambda) z_2 \colon z_1, z_2 \in C^{k,\Lambda},\ z_1 - z_2 \in \Lambda,\ \lambda \in [0,1]\}.$$

The *lamination convex hull* of C is defined as

$$C^{lc,\Lambda} := \cup_{k=0}^{\infty} C^{k,\Lambda}.$$

Recall also that a function $G \colon \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is said to be Λ-*convex* if $t \mapsto G(z_0 + tz) \colon \mathbb{R} \to \mathbb{R}$ is convex for every $z_0 \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$ and $z \in \Lambda$. The Λ-*convex hull* C^Λ consists of points $z \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$ that cannot be separated from C by a Λ-convex function. More precisely, $z \notin C^\Lambda$ if and only if there exists a Λ-convex function G such that $G|_C \leq 0$ but $G(z) > 0$.

We always have $C^\Lambda \supset C^{lc,\Lambda}$. A classical example of strict inclusion $C^\Lambda \supsetneq C^{lc,\Lambda}$ is given by the Tartar square; see [16, pp. 86–87].

Remark 2.4. Denote the wave cone of nonstationary IPM by Λ_{ns}. The constitutive set K is the same in stationary and nonstationary IPM but $\Lambda \subset \Lambda_{\text{ns}}$, so that we immediately get $K^{lc,\Lambda} \subset K^{lc,\Lambda_{\text{ns}}}$ and $K^\Lambda \subset K^{\Lambda_{\text{ns}}}$.

If $\rho \in L^\infty(\Omega)$ and $\mathbf{v}, \mathbf{m} \in L^2_\sigma(\Omega, \mathbb{R}^2)$ satisfy (1.4)–(1.6) and $z(x) = (\rho, \mathbf{v}, \mathbf{m})(x) \in K^\Lambda$ a.e. $x \in \Omega$, then z is called a *subsolution of stationary IPM*.

3. Estimating the hull from below. We wish to first show that $K^{3,\Lambda}$ contains the set $\cup_{j=1}^4 X_j$ described in Theorem 1.1. We begin by computing the first laminate.

PROPOSITION 3.1. *We have*

$$K^{1,\Lambda} = \left\{ (\rho, \mathbf{0}, \mathbf{m}) \colon |\rho| \leq 1, \quad \mathbf{m} = \frac{1-\rho^2}{2}(\mathbf{e} - (0,1)), \quad |\mathbf{e}| = 1 \right\}$$
$$\bigcup \left\{ (\rho, \mathbf{v}, \mathbf{m}) \colon |\rho| \leq 1, \quad \mathbf{m} = \left[\rho - \frac{(1-\rho^2) v_2}{|\mathbf{v}|^2} \right] \mathbf{v} \right\}.$$

Proof. A general convex combination of two elements of K is either an element of K or of the form

$$\big(\rho, \mathbf{v}, \rho \mathbf{v} + (1-\rho^2)\mathbf{w}\big) = \frac{1+\rho}{2}\left(1, \mathbf{v} + (1-\rho)\mathbf{w}, \mathbf{v} + (1-\rho)\mathbf{w}\right) \tag{3.1}$$
$$+ \frac{1-\rho}{2}\left(-1, \mathbf{v} - (1+\rho)\mathbf{w}, -\mathbf{v} + (1+\rho)\mathbf{w}\right), \tag{3.2}$$

where $|\rho| < 1$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$.

The linear combination in (3.1)–(3.2) is Λ-convex if and only if $(2, 2\mathbf{w}, 2\mathbf{v} - 2\rho\mathbf{w}) \in \Lambda$. By Proposition 2.1, this occurs precisely when $|\mathbf{w} + (0, 1/2)| = 1/2$, $\mathbf{w} \cdot \mathbf{v}^\perp = 0$ and $\mathbf{v} \cdot (\mathbf{w} + (0,1)) = 0$.

If $\mathbf{v} = \mathbf{0}$, the wave cone conditions are equivalent to $\mathbf{w} = (\mathbf{e} - (0,1))/2$ with $|\mathbf{e}| = 1$, whereas in the case $\mathbf{v} \neq \mathbf{0}$ they are equivalent to $\mathbf{w} = -(v_2/|\mathbf{v}|^2)\mathbf{v}$, which completes the proof. ∎

By Corollary 2.2 and Proposition 3.1, $X_1 \subset K^{2,\Lambda}$ and $X_3 \subset K^{1,\Lambda}$. The next two propositions, combined with Corollary 2.2 and Proposition 3.1, show that $X_2 \cup X_4 \subset K^{3,\Lambda}$.

PROPOSITION 3.2. *Suppose* $|\rho| < 1$ *and* $\mathbf{v} \neq \mathbf{0}$ *with*

$$\rho - \frac{(1-\rho^2)v_2}{|\mathbf{v}|^2} \geq 1. \tag{3.3}$$

Then

$$(\rho, \mathbf{v}, \mathbf{v}) \in K^{2,\Lambda}.$$

Proof. Suppose (3.3) holds. As a consequence, $v_2 < 0$. Let us write

$$(\rho, \mathbf{v}, \mathbf{v}) = \lambda\left(1, \frac{\mathbf{v}}{\lambda}, \frac{\mathbf{v}}{\lambda}\right) + (1-\lambda)(\psi, \mathbf{0}, \mathbf{0}),$$

where $0 < \lambda < 1$ and

$$\psi = \frac{\rho - \lambda}{1 - \lambda}. \tag{3.4}$$

We need to choose λ in such a way that $-1 \leq \psi < \rho$ and $z_1 - z_2 = (1-\psi, \mathbf{v}/\lambda, \mathbf{v}/\lambda) \in \Lambda$.

By Proposition 2.1 and Remark 2.3, $z_1 - z_2 \in \Lambda$ is equivalent to

$$\frac{\mathbf{v}}{\lambda} \cdot \left[\frac{\mathbf{v}}{\lambda} + (0, 1-\psi)\right] = 0.$$

In conjunction with (3.4), this leads to the choices

$$\lambda = \frac{|\mathbf{v}|^2}{|\mathbf{v}|^2 - (1-\rho)v_2}, \qquad \psi = \frac{|\mathbf{v}|^2 + \rho v_2}{v_2}.$$

Note that (3.3) holds if and only if $|\mathbf{v}|^2 + (1+\rho)v_2 \leq 0$ if and only if $\psi \geq -1$. Since $v_2 < 0$, we also have $0 < \lambda < 1$. Furthermore, $\psi = (\rho - \lambda)/(1-\lambda) < \rho$ since $\rho < 1$. ◻

PROPOSITION 3.3. *Suppose* $|\rho| \leq 1$ *and* $\mathbf{v} \neq \mathbf{0}$ *with*

$$\rho - \frac{(1-\rho^2)v_2}{|\mathbf{v}|^2} \leq -1. \tag{3.5}$$

Then

$$(\rho, \mathbf{v}, -\mathbf{v}) \in K^{2,\Lambda}.$$

Proof. The proof is entirely analogous to that of Proposition 3.2; we write $(\rho, \mathbf{v}, -\mathbf{v}) = \lambda(-1, \mathbf{v}/\lambda, -\mathbf{v}/\lambda) + (1-\lambda)(\psi, \mathbf{0}, \mathbf{0})$ and set

$$\lambda = \frac{|\mathbf{v}|^2}{|\mathbf{v}|^2 + (1+\rho)v_2}, \qquad \psi = \frac{|\mathbf{v}|^2 + \rho v_2}{v_2}.$$

Now (3.5) is equivalent to $|\mathbf{v}|^2 - (1-\rho)v_2 \leq 0$, which in turn is equivalent to $\psi \leq 1$. In addition, (3.5) implies $v_2 > 0$, which in turn gives $0 < \lambda < 1$. ◻

4. Estimating the hull from above. We now intend to show that $K^{lc,\Lambda} \subset \cup_{j=1}^4 X_j$. The steps of the proof are as follows:

- When $\mathbf{v} = \mathbf{0}$, Corollary 4.2 shows that if $z = (\rho, \mathbf{0}, \mathbf{m}) \in K^\Lambda$, then $z \in X_1$.
- When $(\rho, \mathbf{v}, \mathbf{m}) \in K^\Lambda$ and $\mathbf{v} \neq \mathbf{0}$, Corollary 4.4 shows that $\mathbf{m} = k\mathbf{v}$ for some $k \in \mathbb{R}$.

• When $z = (\rho, \mathbf{v}, k\mathbf{v}) \in K^\Lambda$ and $|\rho - (1-\rho^2)v_2/|\mathbf{v}|^2| \geq 1$, Corollary 4.6 yields $z \in X_2 \cup X_4$.

• When $z = (\rho, \mathbf{v}, k\mathbf{v}) \in K^{lc,\Lambda}$ and $|\rho - (1-\rho^2)v_2/|\mathbf{v}|^2| < 1$, Propositions 4.7–4.10 imply that $z \in X_3$. This is the only result that we are not able to show for K^Λ but only for $K^{lc,\Lambda}$.

We begin by recalling a proposition from [23] which also applies to stationary IPM in view of Remark 2.4.

Proposition 4.1. *The function*

$$G_1(\rho, \mathbf{v}, \mathbf{m}) := \left|\mathbf{m} - \rho\mathbf{v} + \left(0, \frac{1-\rho^2}{2}\right)\right| - \frac{1-\rho^2}{2}$$

is Λ-convex and vanishes in K. Consequently,

$$K^\Lambda \subset \left\{(\rho, \mathbf{v}, \mathbf{m})\colon\ |\rho| \leq 1, \left|\mathbf{m} - \rho\mathbf{v} + \left(0, \frac{1-\rho^2}{2}\right)\right| \leq \frac{1-\rho^2}{2}\right\}.$$

Propositions 3.1 and 4.1 have the following consequence.

Corollary 4.2. *Let $|\rho| \leq 1$. Then*

$$(\rho, \mathbf{0}, \mathbf{m}) \in K^\Lambda \quad \Longleftrightarrow \quad \mathbf{m} = \frac{1-\rho^2}{2}(\mathbf{e} - (0,1)),\ |\mathbf{e}| \leq 1.$$

We then consider the case $\mathbf{v} \neq \mathbf{0}$. The following result follows immediately from Proposition 2.1.

Proposition 4.3. *The function*

$$G_2(\rho, \mathbf{v}, \mathbf{m}) := \mathbf{m} \cdot \mathbf{v}^\perp$$

is Λ-affine and vanishes in K.

Corollary 4.4. *If $(\rho, \mathbf{v}, \mathbf{m}) \in K^\Lambda$ with $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{m} = k\mathbf{v}$ for some $k \in \mathbb{R}$.*

In view of Proposition 4.2 and Corollary 4.4, the hull $K^{lc,\Lambda}$ is determined by finding the exact range of the parameter $k = k(\rho, \mathbf{v})$ in $\mathbf{m} = k\mathbf{v}$. Proposition 4.1 implies that k lies between ρ and $\rho - (1-\rho^2)v_2/|\mathbf{v}|^2$, giving the optimal range in the case of non-stationary IPM. However, in the case of *stationary* IPM, the range of $k(\rho, \mathbf{v})$ is smaller, as stated in Theorem 1.1.

We divide the set of points $(\rho, \mathbf{v}) \in \mathbb{R} \times (\mathbb{R}^2 \setminus \{\mathbf{0}\})$ into the cones described by (1.8)–(1.9) and the complement of their union. We first address the points of the two cones.

Proposition 4.5. *The functions defined by*

$$G_3(\rho, \mathbf{v}, \mathbf{m}) := -[\mathbf{v} - \mathbf{m}] \cdot [\mathbf{v} + (0, 1+\rho)] + \frac{|\mathbf{v} - \mathbf{m}|^2}{2},$$
$$G_4(\rho, \mathbf{v}, \mathbf{m}) := -[\mathbf{v} + \mathbf{m}] \cdot [\mathbf{v} - (0, 1-\rho)] + \frac{|\mathbf{v} + \mathbf{m}|^2}{2},$$

are Λ-convex and satisfy $G_3|_K = G_4|_K = 0$.

Proof. We prove the claims for G_3; the proofs for G_4 are analogous. Let us fix $z_0 = (\rho_0, \mathbf{v}_0, \mathbf{m}_0) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$, $z = (\rho, \mathbf{v}, \mathbf{m}) \in \Lambda$, and $t \in \mathbb{R}$. Then

$$\begin{aligned} G_3(z_0+tz) &= -[\mathbf{v}_0 - \mathbf{m}_0 + t(\mathbf{v}-\mathbf{m})]\cdot[\mathbf{v}_0 + (0,1+\rho_0) + t(\mathbf{v}+(0,\rho))] \\ &\quad + \frac{|\mathbf{v}_0-\mathbf{m}_0+t(\mathbf{v}-\mathbf{m})|^2}{2} \\ &= G_3(z_0) + C_{z_0,z}t + \frac{|\mathbf{v}-\mathbf{m}|^2}{2}t^2 \end{aligned}$$

since $(\mathbf{v}-\mathbf{m})\cdot(\mathbf{v}+(0,\rho)) = 0$ in view of Proposition 2.1 and Remark 2.3. Furthermore, $G_3(1,\mathbf{v},\mathbf{v}) = 0$ and $G_3(-1,\mathbf{v},-\mathbf{v}) = 0$ for all $\mathbf{v}\in\mathbb{R}^2$ so that $G_3|_K = 0$. ∎

COROLLARY 4.6. *Suppose* $|\rho| < 1$, $\mathbf{v}\neq\mathbf{0}$, *and* $(\rho,\mathbf{v},\mathbf{m})\in K^\Lambda$. *If* $|\rho-(1-\rho^2)v_2/|\mathbf{v}|^2\,|\geq 1$, *then* $z\in X_2\cup X_4$.

Proof. Assume $\rho-(1-\rho^2)v_2/|\mathbf{v}|^2\geq 1$; the proof of the case $\rho-(1-\rho^2)v_2/|\mathbf{v}|^2\leq -1$ is analogous. By Corollary 4.4, $\mathbf{m}=k\mathbf{v}$ for some $k\in\mathbb{R}$. Our aim is to show that $(\rho,\mathbf{v},k\mathbf{v})\in X_2$, i.e., $1\leq k\leq \rho-(1-\rho^2)v_2/|\mathbf{v}|^2$.

The inequality $k\leq\rho-(1-\rho^2)v_2/|\mathbf{v}|^2$ follows from Proposition 4.1. For the claim $k\geq 1$ note that $\rho-(1-\rho^2)v_2/|\mathbf{v}|^2\geq 1$ can be written as $(1-\rho)|\mathbf{v}|^2+(1-\rho^2)v_2\leq 0$. We compute

$$\begin{aligned} 0 &\geq (1-\rho)G_3(\rho,\mathbf{v},\mathbf{m}) \\ &= (k-1)\mathbf{v}\cdot\left((1-\rho)\mathbf{v}+(0,1-\rho^2)+\frac{(1-\rho)(k-1)\mathbf{v}}{2}\right) \\ &= (k-1)\left((1-\rho)|\mathbf{v}|^2+(1-\rho^2)v_2+\frac{(1-\rho)(k-1)|\mathbf{v}|^2}{2}\right), \end{aligned}$$

which implies the claim. ∎

Corollaries 4.2 and 4.6 show that

$$K^\Lambda \setminus \left\{(\rho,\mathbf{v},\mathbf{m})\colon \mathbf{v}\neq\mathbf{0},\ -1<\rho-\frac{(1-\rho^2)v_2}{|\mathbf{v}|^2}<1\right\} = X_1\cup X_2\cup X_4. \tag{4.1}$$

In other words, we have computed the exact range of the $\mathbf{m}$ component for the Λ-convex hull (and, therefore, for the lamination convex hull) in all cases except $\mathbf{v}\neq\mathbf{0}$, $\rho-(1-\rho^2)v_2/|\mathbf{v}|^2\in(-1,1)$. We finish the proof of Theorem 1.1 by showing that, for the lamination convex hull,

$$K^{lc,\Lambda}\cap\left\{(\rho,\mathbf{v},\mathbf{m})\colon \mathbf{v}\neq\mathbf{0},\ -1<\rho-\frac{(1-\rho^2)v_2}{|\mathbf{v}|^2}<1\right\} = X_3; \tag{4.2}$$

combining (4.1) and (4.2) yields $K^{lc,\Lambda}=\cup_{j=1}^4 X_j$.

The proof of (4.2) consists of two parts. First, Proposition 4.7 says that $X_3^{1,\Lambda}=X_3$. Then, if $z=(\rho,\mathbf{v},k\mathbf{v})\in(\cup_{j=1}^4X_j)^{1,\Lambda}$, where $\mathbf{v}\neq\mathbf{0}$ and $-1<\rho-(1-\rho^2)v_2/|\mathbf{v}|^2<1$, we write z as a Λ-convex combination of $z_1\in X_i$ and $z_2\in X_j$, where $i,j\in\{1,2,3,4\}$. We show in Propositions 4.8–4.10 that we cannot have $i\neq j$. Now, since each X_i is lamination convex, we get $i=j=3$, so that $z\in X_3$, as claimed.

PROPOSITION 4.7. $X_3^{1,\Lambda}=X_3$.

Proof. Suppose $z_1,z_2\in X_3$ satisfy $\mathbf{0}\neq z_1-z_2\in\Lambda$. We already mention that by Propositions 4.8–4.9 below, for every $(\rho,\mathbf{v},\mathbf{m})\in[z_1,z_2]$ we have $-1<\rho-(1-\rho^2)v_2/|\mathbf{v}|^2<1$.

Let $0 < \lambda < 1$ and $\lambda + \mu = 1$. We write

$$\begin{aligned} z &= \lambda z_1 + \mu z_2 \\ &= \lambda\left(\rho + \mu t, \mathbf{v} + \mu\mathbf{w}, k_1(\mathbf{v} + \mu\mathbf{w})\right) + \mu\left(\rho - \lambda t, \mathbf{v} - \lambda\mathbf{w}, k_2(\mathbf{v} - \lambda\mathbf{w})\right) \\ &= (\rho, \mathbf{v}, k\mathbf{v}) \end{aligned}$$

and wish to show that $k = \rho - (1 - \rho^2)v_2/|\mathbf{v}|^2$. We write

$$z_1 - z_2 = (t, \mathbf{w}, (k_1 - k_2)\mathbf{v} + (\mu k_1 + \lambda k_2)\mathbf{w}) \in \Lambda. \tag{4.3}$$

Corollary 2.2 and the assumption $z_1 - z_2 \neq \mathbf{0}$ imply that $t \neq 0$. Assume, without loss of generality, that $t > 0$.

We first note that if $\mathbf{w} = \mathbf{0}$, then Corollary 2.2 yields $(k_1 - k_2)v_2 = 0$. First, in the case $v_2 = 0$, then the assumption $z_1, z_2 \in X_3$ yields $k_1 = \rho + \mu t$ and $k_2 = \rho - \lambda t$, so that $k = \rho$ and $z \in X_3$.

We then treat the rest of the cases. Suppose, therefore, that either $\mathbf{w} \neq \mathbf{0}$ or $k_1 - k_2 = |\mathbf{w}| = 0$. In each case, by (4.3) and Corollary 2.2, we may write

$$z_1 - z_2 = \left(t, \frac{t}{2}(\mathbf{e} - (0,1)), \ell(\mathbf{e} - (0,1))\right)$$

for some $\mathbf{e} \in S^1$ and $\ell \in \mathbb{R}$.

We intend show that $k_1 < k_2$. (In particular, this rules out the case $k_1 - k_2 = |\mathbf{w}| = 0$.) This reduces to showing a claim that we next specify. Suppose

$$\zeta_1 = \left(\rho, \mathbf{v}, \left[\rho - \frac{(1-\rho^2)v_2}{|\mathbf{v}|^2}\right]\mathbf{v}\right) =: (\rho, \mathbf{v}, \ell_1\mathbf{v}) \in X_3;$$

that is,

$$|\mathbf{v}|^2 + \rho v_2 - |v_2| > 0. \tag{4.4}$$

Suppose $\epsilon > 0$ is small and $\zeta_2 = (\rho + \epsilon, \mathbf{v} + \epsilon(\mathbf{e} - (0,1))/2, \ell_2[\mathbf{v} + \epsilon(\mathbf{e} - (0,1))/2]) \in X_3$; that is,

$$\begin{aligned} \zeta_2 = \Big(&\rho + \epsilon, \mathbf{v} + \frac{\epsilon}{2}[\mathbf{e} - (0,1)], \\ &\left[\rho + \epsilon - \frac{[1 - (\rho+\epsilon)^2]\left(v_2 + \frac{\epsilon}{2}(e_2 - 1)\right)}{\left|\mathbf{v} + \frac{\epsilon}{2}[\mathbf{e} - (0,1)]\right|^2}\right]\left(\mathbf{v} + \frac{\epsilon}{2}[\mathbf{e} - (0,1)]\right)\Big), \end{aligned}$$

where $|\mathbf{e}| = 1$. We claim that $\ell_1 < \ell_2$.

We write $\ell_2 - \ell_1$ as a Taylor series:

$$\begin{aligned}
\ell_2 - \ell_1 &= \rho + \epsilon - \frac{(1-\rho^2)v_2 + \epsilon[(1-\rho^2)(e_2-1)/2 - 2\rho v_2] + O(\epsilon^2)}{|\mathbf{v}|^2 + \epsilon \mathbf{v}\cdot[\mathbf{e} - (0,1)] + O(\epsilon^2)} \\
&\quad - \rho + \frac{(1-\rho^2)v_2}{|\mathbf{v}|^2} \\
&= \epsilon\left[1 - \frac{(1-\rho^2)(e_2-1)/2 - 2\rho v_2}{|\mathbf{v}|^2} + \frac{(1-\rho^2)v_2\mathbf{v}\cdot[\mathbf{e}-(0,1)]}{|\mathbf{v}|^4}\right] + O(\epsilon^2) \\
&= \frac{\epsilon}{|\mathbf{v}|^2}\left[|\mathbf{v}|^2 + 2\rho v_2 + \frac{(1-\rho^2)v_1v_2e_1}{|\mathbf{v}|^2} + \frac{(1-\rho^2)(v_2^2 - v_1^2)(e_2-1)/2}{|\mathbf{v}|^2}\right] \\
&\quad + O(\epsilon^2) \\
&= \frac{\epsilon}{|\mathbf{v}|^2}\left[|\mathbf{v}|^2 + 2\rho v_2 + \frac{1-\rho^2}{2}\left(\frac{2v_1v_2}{|\mathbf{v}|^2}, \frac{v_2^2-v_1^2}{|\mathbf{v}|^2}\right)\cdot[\mathbf{e}-(0,1)]\right] + O(\epsilon^2).
\end{aligned}$$

Thus it suffices to show that

$$H(\tilde{\mathbf{e}}) := |\mathbf{v}|^2 + 2\rho v_2 + \frac{1-\rho^2}{2}\left(\frac{2v_1v_2}{|\mathbf{v}|^2}, \frac{v_2^2-v_1^2}{|\mathbf{v}|^2}\right)\cdot[\tilde{\mathbf{e}}-(0,1)] > 0 \quad \text{for all } \tilde{\mathbf{e}} \in S^1.$$

Note that H is minimized when $\tilde{\mathbf{e}}\cdot(2v_1v_2/|\mathbf{v}|^2, [v_2^2 - v_1^2]/|\mathbf{v}|^2)$ is minimized, that is, when $\tilde{\mathbf{e}} = -(2v_1v_2/|\mathbf{v}|^2, [v_2^2-v_1^2]/|\mathbf{v}|^2)$. The minimum value

$$\begin{aligned}
H\left(\frac{-2v_1v_2}{|\mathbf{v}|^2}, \frac{v_1^2-v_2^2}{|\mathbf{v}|^2}\right) &= |\mathbf{v}|^2 + 2\rho v_2 - \frac{1-\rho^2}{2} - \frac{1-\rho^2}{2}\frac{v_2^2-v_1^2}{|\mathbf{v}|^2} \\
&= |\mathbf{v}|^2 + 2\rho v_2 - (1-\rho^2)\frac{v_2^2}{|\mathbf{v}|^2} \\
&= \frac{[|\mathbf{v}|^2 + (\rho-1)v_2][|\mathbf{v}|^2 + (\rho+1)v_2]}{|\mathbf{v}|^2} \\
&> 0
\end{aligned}$$

by (4.4). Thus $\ell_1 < \ell_2$. We conclude that $k_1 < k_2$ in (4.3), and so $\mathbf{w} \neq \mathbf{0}$.

Recall that $\mathbf{w} = t(\mathbf{e} - (0,1))/2$ for some $\mathbf{e} \in S^1$; since $\mathbf{w} \neq \mathbf{0}$, we have $\mathbf{e} \neq (0,1)$. Since we already showed that $k_1 < k_2$, we conclude from (4.3) that $\mathbf{v}\cdot\mathbf{w}^{\perp} = 0$.

Proposition 4.9 below implies that $\mathbf{v} \neq \mathbf{0}$. Thus $\mathbf{w} = \ell\mathbf{v}$, where $|\mathbf{w} + (0, t/2)| = |t/2|$ gives $\ell = -tv_2/|\mathbf{v}|^2$, so that we can write

$$\begin{aligned}
z_1 &= \left(\rho + \mu t, \left(1 - \mu\frac{tv_2}{|\mathbf{v}|^2}\right)\mathbf{v}, k_1\left(1 - \mu\frac{tv_2}{|\mathbf{v}|^2}\right)\mathbf{v}\right), \\
z_2 &= \left(\rho - \lambda t, \left(1 + \lambda\frac{tv_2}{|\mathbf{v}|^2}\right)\mathbf{v}, k_2\left(1 + \lambda\frac{tv_2}{|\mathbf{v}|^2}\right)\mathbf{v}\right).
\end{aligned}$$

Since $z_1, z_2 \in X_3$, we have

$$k_1 = \rho + \mu t - \frac{[1-(\rho+\mu t)^2]v_2}{\left(1-\mu\frac{tv_2}{|\mathbf{v}|^2}\right)|\mathbf{v}|^2},$$
$$k_2 = \rho - \lambda t - \frac{[1-(\rho-\lambda t)^2]v_2}{\left(1+\lambda\frac{tv_2}{|\mathbf{v}|^2}\right)|\mathbf{v}|^2}$$

so that

$$\begin{aligned} k &= \lambda\left(1-\mu\frac{tv_2}{|\mathbf{v}|^2}\right)k_1 + \mu\left(1+\lambda\frac{tv_2}{|\mathbf{v}|^2}\right)k_2 \\ &= \rho - \lambda\mu\frac{t^2v_2}{|\mathbf{v}|^2} - \frac{v_2}{|\mathbf{v}|^2}[\lambda[1-(\rho+\mu t)^2] + \mu[1-(\rho-\lambda t)^2]] \\ &= \rho - \frac{(1-\rho^2)v_2}{|\mathbf{v}|^2}, \end{aligned}$$

as claimed. ∎

PROPOSITION 4.8. $[X_3 - (X_2 \cup X_4)] \cap \Lambda = \emptyset$.

Proof. Suppose

$$z_1 = \left(\rho, \mathbf{v}, \left[\rho - \frac{(1-\rho^2)v_2}{|\mathbf{v}|^2}\right]\mathbf{v}\right) =: (\rho, \mathbf{v}, k\mathbf{v}) \in X_3,$$
$$z_2 = (\psi, \mathbf{w}, \ell\mathbf{w}) \in X_2 \cup X_4,$$

so that $\mathbf{v}, \mathbf{w} \neq \mathbf{0}$. Seeking a contradiction, assume that

$$z_1 - z_2 = (\rho-\psi, \mathbf{v}-\mathbf{w}, k(\mathbf{v}-\mathbf{w}) + (k-\ell)\mathbf{w}) \in \Lambda.$$

By the definitions of X_2 and X_4, we get $k \neq \ell$, so that Proposition 2.1 gives $\mathbf{w}\cdot\mathbf{v}^\perp = 0$. Now $\mathbf{v} = (1+t)\mathbf{w}$ for some $t \in \mathbb{R} \setminus \{-1, 0\}$; if we had $t = 0$, then $z_1 - z_2 \in \Lambda$ would imply $\rho = \psi$, in contradiction with the definitions of X_2, X_3, and X_4.

Now, since $z_1 - z_2 \in \Lambda$, we have

$$0 = |\mathbf{v}-\mathbf{w}|^2 + (\rho-\psi)(v_2-w_2) = t^2|\mathbf{w}|^2 + t(\rho-\psi)w_2$$

so that $\mathbf{v} = (1+t)\mathbf{w} = [1 + (\psi-\rho)w_2/|\mathbf{w}|^2]\mathbf{w}$ and $\rho \neq \psi$. We therefore obtain

$$k = \rho - \frac{(1-\rho^2)v_2}{|\mathbf{v}|^2} = \rho - \frac{(1-\rho^2)w_2}{|\mathbf{w}|^2 + (\psi-\rho)w_2}. \tag{4.5}$$

We divide the rest of the proof into separate cases.

Suppose first $z_2 \in X_2$ (that is, $|\mathbf{w}|^2 + (1+\psi)w_2 \leq 0$) and $1+t > 0$ (i.e., $|\mathbf{w}|^2 + (\psi-\rho)w_2 > 0$). By (4.5), the assumption $k < 1$ can be written as $|\mathbf{w}|^2 + (1+\psi)w_2 > 0$, which gives a contradiction.

Suppose next $|\mathbf{w}|^2 + (1+\psi)w_2 \leq 0$ and $|\mathbf{w}|^2 + (\psi-\rho)w_2 < 0$. Thus $w_2 < 0$. Now $k > -1$ can be written as $|\mathbf{w}|^2 + (\psi-1)w_2 < 0$, yielding a contradiction.

Similarly, if $z_2 \in X_4$ (i.e., $|\mathbf{w}|^2 + (1+\psi)w_2 \leq 0$) and $1+t > 0$, then $k < 1$ is in contradiction with the assumption $z_2 \in X_4$. Finally, if $z_2 \in X_4$ and $1+t < 0$, then $k > -1$ contradicts $z_2 \in X_4$. ∎

PROPOSITION 4.9. *Suppose* $z_1 \in X_1$, $z_2 \in X_2 \cup X_3 \cup X_4$, *and,* $z_2 - z_1 \in \Lambda$. *Then the half-open interval* $(z_1, z_2] \subset X_2 \cup X_4$.

Proof. Suppose $z_1 = (\rho, \mathbf{0}, \mathbf{m}) \in X_1$ and $z_2 \in X_2 \cup X_3 \cup X_4$ satisfy $z_2 - z_1 \in \Lambda$. Let $z = (\rho + \epsilon, \mathbf{v}, \tilde{\mathbf{m}}) \in (z_1, z_2]$; thus

$$(z - z_1) = (\epsilon, \mathbf{v}, \tilde{\mathbf{m}} - \mathbf{m}) \in \Lambda.$$

Also note that $z_2 \in X_2 \cup X_3 \cup X_4$ implies that $\mathbf{v} \neq \mathbf{0}$.

If $\epsilon = 0$, we get $z - z_1 = (0, \mathbf{v}, \tilde{\mathbf{m}} - \mathbf{m}) \in \Lambda$, which contradicts Corollary 2.2. We then assume that $0 < \epsilon \leq 1 - \rho$. By Proposition 2.1, $|\mathbf{v}|^2 + \epsilon v_2 = 0$. Since $\mathbf{v} \neq \mathbf{0}$, we conclude that $v_2 < 0$. Thus

$$|\mathbf{v}|^2 + (\rho + \epsilon + 1) v_2 = (\rho + 1) v_2 \leq 0,$$

which, combined with Corollary 4.6, yields $z \in X_2$. Similarly, if $-1 - \rho \leq \epsilon < 0$, then $z_2 \in X_4$. ∎

We finish the proof of Theorem 1.1 by showing that a Λ-segment between $z_1 \in X_2$ and $z_2 \in X_4$ cannot contain $(\rho, \mathbf{v}, k\mathbf{v})$ with $\mathbf{v} \neq \mathbf{0}$ and $-1 < \rho - (1-\rho^2) v_2 / |\mathbf{v}|^2 < 1$.

PROPOSITION 4.10. $(X_2 \cup X_4)^{1,\Lambda} \subset X_1 \cup X_2 \cup X_4$.

Proof. Suppose

$$z_1 = (\rho, \mathbf{v}, k\mathbf{v}) \in X_2, \qquad z_2 = (\psi, \mathbf{w}, \ell\mathbf{w}) \in X_4,$$

and

$$z_1 - z_2 = (\rho - \psi, \mathbf{v} - \mathbf{w}, k(\mathbf{v} - \mathbf{w}) + (k - \ell)\mathbf{w}) \in \Lambda.$$

Thus

$$1 \leq k \leq \rho - \frac{(1-\rho^2) v_2}{|\mathbf{v}|^2}, \qquad \psi - \frac{(1-\psi^2) w_2}{|\mathbf{w}|^2} \leq \ell \leq -1,$$

giving $|\mathbf{v}|^2 + (\rho+1) v_2 \leq 0$ and $|\mathbf{w}|^2 + (\psi - 1) w_2 \leq 0$, which in turn yields $v_2 < 0 < w_2$. Now $\rho \neq \psi$, as otherwise $z_1 - z_2 \in \Lambda$ would give $\mathbf{v} - \mathbf{w} = \mathbf{0}$, contradicting $v_2 < 0 < w_2$.

Choose the unique $\tilde{\psi} = \lambda\psi + \mu\rho \in [\psi, \rho]$ (where $0 \leq \lambda \leq 1$ and $\lambda + \mu = 1$) such that $\tilde{z} = (\tilde{\psi}, \tilde{\mathbf{w}}, \tilde{\mathbf{m}}) := \lambda z_1 + \mu z_2$ satisfies $\tilde{\mathbf{w}} = \mathbf{0}$ or

$$\tilde{\psi} - \frac{(1 - \tilde{\psi}^2) \tilde{w}_2}{|\tilde{\mathbf{w}}|^2} = -1. \tag{4.6}$$

If $\tilde{\mathbf{w}} = \mathbf{0}$, then $\tilde{z} \in X_1$, and we are reduced to the situation of Proposition 4.9. Assume, therefore, $\tilde{\mathbf{w}} \neq \mathbf{0}$ and (4.6) hold. Consequently, $|\tilde{\mathbf{w}}|^2 + (\tilde{\psi} - 1)\tilde{w}_2 = 0$, giving $\tilde{w}_2 > 0$. Note that (4.6) and Corollary 4.6 give $\tilde{z} = (\tilde{\psi}, \tilde{\mathbf{w}}, -\tilde{\mathbf{w}})$.

Now, by assumption,

$$z_1 - \tilde{z} = (\rho - \tilde{\psi}, \mathbf{v} - \tilde{\mathbf{w}}, k(\mathbf{v} - \tilde{\mathbf{w}}) + (k+1)\tilde{\mathbf{w}}) \in \Lambda,$$

so that $\tilde{\mathbf{w}} \cdot \mathbf{v}^\perp = 0$ since $k \geq 1$. Let us write $\mathbf{v} = (1+t)\tilde{\mathbf{w}}$; now $z_1 - z_2 \in \Lambda$ gives $t = (\tilde{\psi} - \rho)\tilde{\mathbf{w}}_2 / |\tilde{\mathbf{w}}|^2$. On the other hand, $v_2 < 0 < \tilde{w}_2$ and $\mathbf{v} = (1+t)\tilde{\mathbf{w}}$ yield $1 + t < 0$, so that

$$0 > \frac{|\tilde{\mathbf{w}}|^2 + (\tilde{\psi} - \rho)\tilde{\psi}_2}{|\tilde{\mathbf{w}}|^2} = \frac{(1-\rho)\tilde{w}_2}{|\tilde{\mathbf{w}}|^2},$$

giving a contradiction with $\tilde{w}_2 > 0$. ∎

Proposition 4.10 completes the proof of Theorem 1.1 and gives the exact lamination convex hull of the stationary IPM equations.

Remark 4.11. The proof of Theorem 1.1 shows that

$$K^{\Lambda} \setminus \left\{ (\rho, \mathbf{v}, \mathbf{m}) \colon \mathbf{v} \neq \mathbf{0},\ -1 < \rho - \frac{(1-\rho^2)v_2}{|\mathbf{v}|^2} < 1 \right\} = X_1 \cup X_2 \cup X_4,$$

where the condition of the removed set describes the "rigid region," that is, the projection $\mathrm{pr}_{\mathbb{R}\times\mathbb{R}^2}(X_3)$.

If we could calculate the exact range of $\mathbf{m}$ also for the triples $(\rho, \mathbf{v}, \mathbf{m}) \in K^{\Lambda}$, where $(\rho, \mathbf{v}) \in \mathrm{pr}_{\mathbb{R}\times\mathbb{R}^3}(X_3)$, we could formulate Theorem 1.2 for the Λ-convex hull instead of the lamination convex hull.

5. Nonexistence of nontrivial subsolutions in bounded domains. As observed in [10] (although stated under different hypotheses), if $\mathbf{v} \in L^2_\sigma(\Omega, \mathbb{R}^2)$ and $\rho \in L^\infty(\Omega)$ form a solution of stationary IPM, then

$$\int_\Omega |\mathbf{v}|^2 = \int_\Omega \mathbf{v} \cdot [-\nabla p - (0,\rho)] = -\int_\Omega \rho v_2 = -\int_\Omega \rho \mathbf{v} \cdot \nabla y = 0. \tag{5.1}$$

We adapt the proof to subsolutions with values in $K^{lc,\Lambda}$ by using the exact form of $K^{lc,\Lambda}$ computed in Theorem 1.1.

Proof of Theorem 1.2. Since $\mathbf{m} \in L^2_\sigma(\Omega, \mathbb{R}^2) = [\nabla W^{1,2}(\Omega)]^\perp$ and $(\rho, \mathbf{v}, \mathbf{m})(x) \in K^{lc,\Lambda}$ a.e. $x \in \Omega$, we may write

$$0 = \int_\Omega \mathbf{m} \cdot \nabla y = \int_{\mathbf{v}=\mathbf{0}} \frac{1-\rho^2}{2}(e_2 - 1) + \sum_{j=2}^{4} \int_{(\rho,\mathbf{v}) \in X_j} k v_2. \tag{5.2}$$

If $(\rho, \mathbf{v}) \in X_2$, then $1 \leq k \leq \rho - (1-\rho^2)v_2/|\mathbf{v}|^2$ so that either $\rho = 1$ or $v_2 < 0$. In both cases, $kv_2 \leq v_2$. Thus

$$\int_{(\rho,\mathbf{v}) \in X_2} k v_2 \leq \int_{(\rho,\mathbf{v}) \in X_2} v_2. \tag{5.3}$$

Similarly, if $(\rho, \mathbf{v}) \in X_4$, then $\rho - (1-\rho^2)v_2/|\mathbf{v}|^2 \leq k \leq -1$ so that either $\rho = -1$ or $v_2 > 0$, giving $kv_2 \leq -v_2$ and

$$\int_{(\rho,\mathbf{v}) \in X_4} k v_2 \leq -\int_{(\rho,\mathbf{v}) \in X_4} v_2. \tag{5.4}$$

Furthermore, since $(\rho, \mathbf{v}, \mathbf{m})(x) \in K^{lc,\Lambda}$ a.e. $x \in \Omega$, we get

$$\int_{(\rho,\mathbf{v}) \in X_3} k v_2 = \int_{(\rho,\mathbf{v}) \in X_3} \rho v_2 - \int_{(\rho,\mathbf{v}) \in X_3} \frac{(1-\rho^2)v_2^2}{|\mathbf{v}|^2}.$$

Using (5.2)–(5.4),

$$\begin{aligned}
-\int_{(\rho,\mathbf{v}) \in X_3} \rho v_2 &= -\int_{(\rho,\mathbf{v}) \in X_3} k v_2 - \int_{(\rho,\mathbf{v}) \in X_3} \frac{(1-\rho^2)v_2^2}{|v|^2} \\
&= \int_{\mathbf{v}=\mathbf{0}} \frac{1-\rho^2}{2}(e_2 - 1) + \int_{(\rho,\mathbf{v}) \in X_2 \cup X_4} k v_2 - \int_{X_3} \frac{(1-\rho^2)v_2^2}{|v|^2} \\
&\leq \int_{(\rho,\mathbf{v}) \in X_2} v_2 - \int_{(\rho,\mathbf{v}) \in X_4} v_2 - \int_{(\rho,\mathbf{v}) \in X_3} \frac{(1-\rho^2)v_2^2}{|\mathbf{v}|^2},
\end{aligned}$$

and so, using the assumption that $\mathbf{v} \in L^2_\sigma(\Omega, \mathbb{R}^2)$,

$$\begin{aligned}
0 &\leq \int_\Omega |\mathbf{v}|^2 = \int_\Omega \mathbf{v} \cdot [-\nabla p - (0,\rho)] = -\int_\Omega \rho v_2 = -\sum_{j=2}^{4} \int_{(\rho,\mathbf{v})\in X_j} \rho v_2 \\
&\leq \int_{(\rho,\mathbf{v})\in X_2} v_2 - \int_{(\rho,\mathbf{v})\in X_4} v_2 - \int_{(\rho,\mathbf{v})\in X_3} \frac{(1-\rho^2)v_2^2}{|\mathbf{v}|^2} \\
&\quad - \int_{(\rho,\mathbf{v})\in X_2} \rho v_2 - \int_{(\rho,\mathbf{v})\in X_4} \rho v_2 \\
&= \int_{(\rho,\mathbf{v})\in X_2} (1-\rho) v_2 - \int_{(\rho,\mathbf{v})\in X_4} (1+\rho) v_2 - \int_{(\rho,\mathbf{v})\in X_3} \frac{(1-\rho^2)v_2^2}{|\mathbf{v}|^2} \\
&\leq 0,
\end{aligned}$$

where in the last inequality we have used $v_2 \leq 0$ in X_2 and $v_2 \geq 0$ in X_4. We thus conclude that $\mathbf{v} = \mathbf{0}$. Now (1.6) gives $\partial_x \rho = \nabla^\perp \cdot (0,\rho) = 0$. ◻

Remark 5.1. The proof of Theorem 1.2 also works essentially verbatim with impermeable walls in the vertical direction and periodic boundary conditions in the horizontal direction. Thus, the dichotomy on directions of strips that we mentioned in the introduction extends to subsolutions with values in $K^{lc,\Lambda}$.

Adapting (5.1) to a strip with finite width in the direction $(0,1)$, we briefly indicate the role that the direction $(0,1)$ plays. The second equality in (5.1) uses the boundary conditions that $\mathbf{v} \cdot \nu|_{\partial\Omega} = 0$ when $y = 0$ and $\mathbf{v}$ is periodic in x; this part works equally in the setting of [5]. However, the fourth equality in (5.1) uses the fact that $(x,y) \mapsto y$ is periodic in x. It is here that the adaptation to all other strips breaks down, and thus there is no geometric obstruction to the solutions of [5]. In the proof of Theorem 1.2, the fourth equality of (5.1) is necessarily replaced by a weaker condition, and the proof requires the precise computation of $K^{lc,\Lambda}$ in Theorem 1.1.

Remark 5.2. If the lamination and Λ-convex hull coincide, as we have predicted, it seems unlikely that one can use standard methods of convex integration even in unbounded or periodic domains, despite the solutions constructed in [5]. Indeed, one would need to find potentials satisfying the extremely rigid constraint $\mathbf{m} \equiv [\rho - (1-\rho^2)v_2/|\mathbf{v}|^2]\mathbf{v}$ in the rigid region where $(\rho, \mathbf{v}) \in \mathrm{pr}(X_3) \subset \mathbb{R} \times \mathbb{R}^2$ or avoid the (rather large) rigid region altogether. In 3D MHD, by contrast, the lamination convex hull is, loosely, speaking, 1-codimensional, which left enough room for running convex integration in [11].

6. Relation to the infinite time limit of nonstationary IPM.

As the last topic of this paper, we show that Theorem 1.2 reflects the behavior of subsolutions of nonstationary IPM at the limit $t \to \infty$. The proof is a straightforward application of [10, Corollary 1.2] which states that $\partial_t \int_\Omega \rho x_2 \, \mathrm{d}x = 2^{-1} \partial_t \int_\Omega |\rho - x_2|^2 \, \mathrm{d}x = -\int_\Omega |\mathbf{v}|^2 \, \mathrm{d}x$ for smooth solutions of nonstationary IPM.

PROPOSITION 6.1. *Suppose $\rho \in L^\infty(0,\infty; L^\infty)$ and $\mathbf{v}, \mathbf{m} \in L^\infty(0,\infty; L^2_\sigma)$ form a subsolution of nonstationary IPM in a smooth, bounded, simply connected domain $\Omega \subset \mathbb{R}^2$. Then $\mathbf{v} \in L^2(0,\infty; L^2_\sigma)$.*

Proposition 6.1 and its proof work equally well in the confined IPM case $\Omega = \mathbb{T}^1 \times (-1,1)$. Before presenting the proof, we recall the definition of a subsolution in this context. Under the integrability assumptions of Theorem 6.1, $z = (\rho, \mathbf{v}, \mathbf{m})$ is a subsolution of nonstationary IPM if

$$z(x) \in K^\Lambda = \left\{(\bar\rho, \bar{\mathbf{v}}, \bar{\mathbf{m}})\colon |\bar\rho| \le 1, \left|\bar{\mathbf{m}} - \bar\rho\bar{\mathbf{v}} + \left(0, \frac{1-\bar\rho^2}{2}\right)\right| \le \frac{1-\bar\rho^2}{2}\right\} \tag{6.1}$$

a.e. $x \in \Omega \times [0,\infty)$ and

$$\int_0^\infty \int_\Omega (\rho\,\partial_t\varphi + \mathbf{m}\cdot\nabla\varphi)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \rho_0\varphi(\cdot,0)\,\mathrm{d}x = 0 \quad \forall\varphi \in C_c^\infty(\bar\Omega\times[0,\infty)), \tag{6.2}$$

$$\int_0^\infty \int_\Omega \mathbf{v}\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = 0 \quad \forall\varphi \in C_c^\infty(\bar\Omega\times[0,\infty)), \tag{6.3}$$

$$\int_0^\infty \int_\Omega (\mathbf{v} + (0,\rho))\cdot\nabla^\perp\varphi\,\mathrm{d}x\,\mathrm{d}t = 0 \quad \forall\varphi \in C_c^\infty(\Omega\times[0,\infty)). \tag{6.4}$$

Note that (6.2)–(6.3) incorporate the condition $\mathbf{v}\cdot\nu|_{\partial\Omega} = \mathbf{m}\cdot\nu|_{\partial\Omega} = 0$.

Proof of Proposition 6.1. Let $\eta \in C_c^\infty(0,\infty)$, and set $\varphi(x,t) := \eta(t)x_2$ in (6.2), so that

$$\int_0^\infty \eta'(t)\int_\Omega \rho(x,t)x_2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\infty \eta\int_\Omega m_2(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

As a consequence, $\partial_t \int_\Omega \rho(x,\cdot)x_2\,\mathrm{d}x = \int_\Omega m_2(x,\cdot)\,\mathrm{d}x \in L^\infty(0,\infty)$ in the sense of distributions. Thus, after possibly modifying ρ on a set of measure zero, $F(t) := \int_\Omega \rho(x,t)x_2\,\mathrm{d}x$ is Lipschitz continuous and

$$F(t) = \int_\Omega \rho_0(x)x_2\,\mathrm{d}x + \int_0^t\int_\Omega m_2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \tag{6.5}$$

for all $t \in [0,\infty)$.

We use (6.1) to get $m_2 = \rho v_2 + (1-\rho^2)(e_2-1)/2$, where $\mathbf{e} = (e_1,e_2)$ takes values in $\bar B(0,1)$, so that

$$\int_0^t\int_\Omega m_2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau \le \int_0^t\int_\Omega \rho(x,\tau)v_2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \tag{6.6}$$

Now, approximating $\mathbf{v}$ in $L^2(0,t;L^2_\sigma)$ by mappings $\nabla^\perp\varphi_j$, $\varphi_j \in C_c^\infty(\Omega\times[0,t))$, the assumption (6.4) gives

$$\int_0^t\int_\Omega \rho(x,\tau)v_2(x,\tau)\,\mathrm{d}x\,\mathrm{d}\tau = -\int_0^t\int_\Omega |\mathbf{v}(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau. \tag{6.7}$$

Combining (6.5)–(6.7), we conclude that

$$\begin{aligned}\int_0^t\int_\Omega |\mathbf{v}(x,\tau)|^2\,\mathrm{d}x\,\mathrm{d}\tau - \int_\Omega \rho_0(x)x_2\,\mathrm{d}x \le -F(t) &\le \int_\Omega |\rho(x,t)x_2|\,\mathrm{d}x \\ &\le \|\rho\|_{L^\infty(0,\infty;L^\infty)}\int_\Omega |x_2|\,\mathrm{d}x\end{aligned}$$

for all $t \in [0,\infty)$. The claim follows. ∎

Acknowledgments. We thank Ángel Castro, Daniel Faraco, and Francisco Mengual for several useful comments. We also thank the anonymous referee for helpful comments which improved the exposition.

REFERENCES

[1] V. Arnaiz, A. Castro, and D. Faraco, *Semiclassical Estimates for Pseudodifferential Operators and the Muskat Problem in the Unstable Regime*, preprint, arXiv:2001.06361, 2020, https://arxiv.org/abs/2001.06361.

[2] A. Castro, D. Córdoba, and D. Faraco, *Mixing Solutions for the Muskat Problem*, preprint, arXiv:1605.04822, 2016, https://arxiv.org/abs/1605.04822.

[3] A. Castro, D. Córdoba, and D. Lear, *Global existence of quasi-stratified solutions for the confined IPM equation*, Arch. Ration. Mech. Anal., 232 (2019), pp. 437–471, https://doi.org/10.1007/s00205-018-1324-3.

[4] A. Castro, D. Faraco, and F. Mengual, *Degraded mixing solutions for the Muskat problem*, Calc. Var. Partial Differential Equations, 58 (2019), 58, 29 pp., https://doi.org/10.1007/s00526-019-1489-0.

[5] P. Constantin, J. La, and V. Vicol, *Remarks on a paper by Gavrilov: Grad-Shafranov equations, steady solutions of the three dimensional incompressible Euler equations with compactly supported velocities, and applications*, Geom. Funct. Anal., 29 (2019), pp. 1773–1793, https://doi.org/10.1007/s00039-019-00516-1.

[6] D. Córdoba, D. Faraco, and F. Gancedo, *Lack of uniqueness for weak solutions of the incompressible porous media equation*, Arch. Ration. Mech. Anal., 200 (2011), pp. 725–746, https://doi.org/10.1007/s00205-010-0365-z.

[7] D. Córdoba and F. Gancedo, *Contour dynamics of incompressible* 3*D fluids in a porous medium with different densities*, Comm. Math. Phys., 273 (2007), pp. 445–471, https://doi.org/10.1007/s00220-007-0246-y.

[8] C. De Lellis and L. Székelyhidi, Jr., *The Euler equations as a differential inclusion*, Ann. of Math., 170 (2009), pp. 1417–1436, https://doi.org/10.4007/annals.2009.170.1417.

[9] C. De Lellis and L. Székelyhidi, Jr., *The h-principle and the equations of fluid dynamics*, Bull. Amer. Math. Soc. (N.S.), 49 (2012), pp. 347–375, https://doi.org/10.1090/S0273-0979-2012-01376-9.

[10] T. M. Elgindi, *On the asymptotic stability of stationary solutions of the inviscid incompressible porous medium equation*, Arch. Ration. Mech. Anal., 225 (2017), pp. 573–599, https://doi.org/10.1007/s00205-017-1090-7.

[11] D. Faraco, S. Lindberg, and L. Székelyhidi Jr., *Bounded Solutions of Ideal MHD with Compact Support in Space-time*, preprint, arXiv:1909.08678, 2020.

[12] E. Feireisl, C. Klingenberg, and S. Markfelder, *On the Density of "wild" Initial Data for thE Compressible Euler System*, preprint, arXiv:1812.11802v2, 2019, https://arxiv.org/abs/1812.11802v2.

[13] C. Förster and L. Székelyhidi, Jr., *Piecewise constant subsolutions for the Muskat problem*, Comm. Math. Phys., 363 (2018), pp. 1051–1080, https://doi.org/10.1007/s00220-018-3245-2.

[14] A. V. Gavrilov, *A steady Euler flow with compact support*, Geom. Funct. Anal., 29 (2019), pp. 190–197, https://doi.org/10.1007/s00039-019-00476-6.

[15] P. Isett and V. Vicol, *Hölder continuous solutions of active scalar equations*, Ann. PDE, 1 (2015), 2, 77 pp., https://doi.org/10.1007/s40818-015-0002-0.

[16] B. Kirchheim, *Rigidity and Geometry of Microstructures*, Lecture Notes, Max-Planck-Inst. für Mathematik in den Naturwiss., 2003.

[17] G. Knott, *Oscillatory Solutions to Hyperbolic Conservation Laws and Active Scalar Equations*, Ph.D. thesis, Universität Leipzig, 2013, https://ul.qucosa.de/api/qucosa%3A12122/attachment/ATT-0/.

[18] S. Markfelder, *On the* Λ*-Convex Hull for Convex Integration Applied to the Isentropic Compressible Euler System*, preprint, arXiv:2001.04373, 2020, https://arxiv.org/abs/2001.04373.

[19] F. Mengual, *H-principle for the* 2*D incompressible porous media equation with viscosity jump*, preprint, arXiv:2004.03307, 2020, https://arxiv.org/abs/2004.03307.

[20] F. Noisette and L. Székelyhidi, Jr., *Mixing Solutions for the Muskat Problem with Variable Speed*, preprint, arXiv:2005.08814, 2020, https://arxiv.org/abs/2005.08814.

[21] F. Otto, *Evolution of microstructure in unstable porous media flow: A relaxational approach*, Comm. Pure Appl. Math., 52 (1999), pp. 873–915, https://doi.org/10.1002/(SICI)1097-0312(199907)52:7⟨873::AID-CPA5⟩3.3.CO;2-K.

[22] R. Shvydkoy, *Convex integration for a class of active scalar equations*, J. Amer. Math. Soc., 24 (2011), pp. 1159–1174, https://doi.org/10.1090/S0894-0347-2011-00705-4.

[23] L. Székelyhidi, Jr., *Relaxation of the incompressible porous media equation*, Ann. Sci. Éc. Norm. Supér., 45 (2012), pp. 491–509, https://doi.org/10.24033/asens.2171.

[24] L. Tartar, *Compensated Compactness and Applications to Partial Differential Equations*, in Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. IV, Res. Notes in Math.. 39, Pitman, Boston, Mass.-London, 1979, pp. 136–212.