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# CONVERGENCE OF THE WEIL-PETERSSON METRIC ON THE TEICHMÜLLER SPACE OF BORDERED RIEMANN SURFACES

DAVID RADNELL, ERIC SCHIPPERS, AND WOLFGANG STAUBACH

ABSTRACT. Consider a Riemann surface of genus  $g$  bordered by  $n$  curves homeomorphic to the unit circle, and assume that  $2g - 2 + n > 0$ . For such bordered Riemann surfaces, the authors have previously defined a Teichmüller space which is a Hilbert manifold and which is holomorphically included in the standard Teichmüller space.

We show that any tangent vector can be represented as the derivative of a holomorphic curve whose representative Beltrami differentials are simultaneously in  $L^2$  and  $L^\infty$ , and furthermore that the space of  $(-1, 1)$  differentials in  $L^2 \cap L^\infty$  decomposes as a direct sum of infinitesimally trivial differentials and  $L^2$  harmonic  $(-1, 1)$  differentials. Thus the tangent space of this Teichmüller space is given by  $L^2$  harmonic Beltrami differentials. We conclude that this Teichmüller space has a finite Weil-Petersson Hermitian metric. Finally, we show that the aforementioned Teichmüller space is locally modelled on a space of  $L^2$  harmonic Beltrami differentials.

## 1. INTRODUCTION

**1.1. Literature and motivation for the results.** In [19] the authors defined an (infinite-dimensional) quasiconformal Teichmüller space of Riemann surfaces of genus  $g$  bordered by  $n$  closed curves, modelled on certain  $L^2$  spaces, and proved that it has a complex Hilbert manifold structure. In this paper, we show that the tangent space is modelled on  $L^2$  harmonic Beltrami differentials, that this Teichmüller space is locally modelled on  $L^2$  harmonic Beltrami differentials and that it possesses a finite Hermitian Weil-Petersson metric.

We briefly give some background on Weil-Petersson class Teichmüller theory. Although the Weil-Petersson metric of course converges on finite-dimensional Teichmüller spaces, it has long been known to diverge on more general Teichmüller spaces. S. Nag and A. Verjovsky [14] showed that the metric does in fact converge in directions tangent to the subset of the universal Teichmüller space which correspond to analytic parametrizations of  $\mathbb{S}^1$ . G. Cui [2] found a completion of Nag and Verjovsky's subset of the universal Teichmüller space, modelled on  $L^2$  spaces of either Schwarzians of conformal maps of the disk or quadratic differentials. The condition is stronger than asymptotic conformality [7, 25]. Any such conformal map has a quasiconformal extension with an  $L^2$  Beltrami differential. H. Guo [8] generalized these results, giving  $L^p$  models of the universal Teichmüller space. L. Takhtajan and L.-P. Teo [25] independently gave the  $L^2$  models in their monograph. They furthermore computed the curvature tensor, showed that it is a Kähler-Einstein metric, gave explicit potentials for the Weil-Petersson metric, and showed that the Weil-Petersson class Teichmüller space embeds in the Segal-Wilson universal Grassmannian. Y. Shen [22] later characterized the class

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of quasisymmetries associated with this refined Teichmüller space. There has been growing interest in what has come to be called the Weil-Petersson class universal Teichmüller space.

Our results are necessarily technical. To place them in context and motivate them, we briefly review some facts of Teichmüller theory in the classical  $L^\infty$  setting. In the next section we describe our approach and what must be done in the  $L^2$  setting to demonstrate the finiteness of the Weil-Petersson metric.

Classically, there are two principle ways to construct a complex structure on Teichmüller space. These are: (1) using the Banach space structure of the set of  $L^\infty$  Beltrami differentials and (2) using the Banach space structure of the  $L^\infty$  Nehari-bounded quadratic differentials obtained as the image of the Teichmüller space under the Bers embedding. These quadratic differentials lie on the double of the surface. The compatibility of these two - which aside from holomorphicity issues must in particular include the continuity of the Bers embedding with respect to the Teichmüller metric - typically occupies the bulk of an introductory textbook in Teichmüller theory [12, 13]. To demonstrate equivalence of these structures, one must show that the Bers embedding is holomorphic and that it possess local holomorphic sections. In fact one cannot construct a complex structure from Beltrami differentials without showing the existence of holomorphic sections.

The existence of sections is closely related to the structure of the tangent space. In the  $L^\infty$  case, the tangent space is modelled on the so-called “harmonic” Beltrami differentials, which are those given by a holomorphic quadratic differential composed with a reflection and multiplied by a suitable power of the hyperbolic metric (Definition 4.3). The harmonic Beltrami differentials are a model for the tangent space to the Teichmüller space of equivalence classes of Beltrami differentials. The harmonic Beltrami differentials are also a model of the Teichmüller space itself (obtained in a sense by exponentiating), but this model is only local. Demonstrating that the tangent space is modelled by harmonic Beltrami differentials is non-trivial even in the classical case and is closely related to the existence of holomorphic sections of the Bers embedding.

The Weil-Petersson metric is the inner product on each tangent space obtained by the  $L^2$  pairing of quadratic differentials. In the classical  $L^\infty$  setting the Weil-Petersson metric does not exist, because the quadratic differentials modelling the tangent space need not be  $L^2$ . It was observed by Nag and Verjovsky that the metric does converge for sufficiently regular directions in the Teichmüller space corresponding to analytic parametrizations. To prove the existence of a finite Weil-Petersson pairing of Beltrami differentials, one must construct a refinement of Teichmüller space and demonstrate that the  $L^2$  harmonic Beltrami differentials are a model of the tangent space.

The model of the tangent space by  $L^2$  harmonic Beltrami differentials has been accomplished only by Takhtajan and Teo in their monograph [25] in the case of the disk, through a significant technical effort. Note that even for the case of the disk, one needs to describe the tangent space (and hence the metric) by taking the quotient by infinitesimally trivial Beltrami differentials and showing that one obtains the harmonic  $L^2$  Beltrami differentials.

To our knowledge the monograph of Takhtajan and Teo is the only place where this is addressed for a Riemann surface of any type.

We generalize these results to Riemann surfaces of genus  $g$  with  $n$  boundaries homeomorphic to the circle. Our results were first obtained in March 2014 [21].

To do so, we use a construction of the complex structure which is distinct from the two above, which involves a fibre structure of Teichmüller space discovered by the first two authors [17]. This structure is motivated by constructions in conformal field theory, and relates to a kind of generalization of the Bers embedding. We now give a brief summary of those results relating to the present paper.

For surfaces of genus  $g$  bordered by  $n$  curves homeomorphic to the circle, in [18] and [19] the authors demonstrated the existence of a Teichmüller space with a Hilbert manifold structure, and showed that its inclusion into the general Teichmüller space is holomorphic. This was accomplished using a refinement of a fibre structure on Teichmüller space given by two of the authors in [17]. Later, the authors showed that each element of this Teichmüller space has a representative with Beltrami differentials in  $L^2 \cap L^\infty$  [20].

After the completion of this paper, the paper [27] of M. Yanagishita came to our attention, which contains another approach to the complex structure. Yanagishita proved the existence of a complex structure on an  $L^p$  Teichmüller space for Fuchsian groups satisfying the so-called “Lehner condition”, which includes genus  $g$  surfaces bordered by  $n$  curves. The complex structure there was obtained from the image of the Bers embedding into an open subset of the  $L^p$  harmonic quadratic differentials. Yanagishita [27] also showed that the Douady-Earle extension yields a representative Beltrami differential in  $L^p \cap L^\infty$ , that the Teichmüller and Kobayashi metrics are equivalent in this setting, and that the Bers embedding from  $L^2 \cap L^\infty$  Beltrami differentials is continuous. By our results in [20], every element of the refined Teichmüller space we consider here has a representative Beltrami differential in  $L^2 \cap L^\infty$ , and thus must agree with that of Yanagishita as a set for  $p = 2$ . The two complex structures are also likely biholomorphic, although this has not yet been shown. Two of the authors have shown that the two approaches yield the same complex structure for the classical  $L^\infty$  Teichmüller space [17], which is already non-trivial. Holomorphicity of the Bers embedding in the  $L^2$  case has not yet been addressed in the literature for surfaces with boundary curves, other than the disk.

In this paper, we show that our complex structure in [18, 19] also is locally equivalent to the complex structure of the space of  $L^2$  harmonic Beltrami differentials (denoted  $H_{-1,1}(\Sigma)$ ; see Definition 4.3). This is a more tractable space than  $L^2 \cap L^\infty$ , since  $L^2$  harmonic differentials are  $L^\infty$  (Theorem 4.5 ahead). Furthermore, we prove that the tangent space at any point is given by  $L^2$  harmonic Beltrami differentials. This generalizes the classical setting, in which  $L^\infty$  harmonic differentials model the tangent space. In particular this demonstrates that the Weil-Petersson metric is finite for these  $L^2$  Teichmüller spaces.

**1.2. Outline of our approach.** In this section we describe the main obstacles, and our method of overcoming them. Finally we outline the structure of the paper.

One might think that the aforementioned results of Guo [8], Cui [2], and Takhtajan and Teo [25] on the universal Teichmüller space can be passed down to arbitrary quotients by Fuchsian groups to obtain convergence on arbitrary Teichmüller spaces. However, this is not the case. Given a bordered Riemann surface, the relevant class of quadratic differentials are  $L^2$  on the surface itself, and hence on the fundamental domain of the associated Fuchsian group. The lift of such a quadratic differential to the universal cover is not in  $L^2$  unless the  $L^2$  integral is zero on each fundamental domain (see Remark 4.4 ahead). Our approach circumvents this problem, since our complex structure is not obtained using Fuchsian groups acting on the universal cover. (However, it is likely compatible with the structure obtained from the lift; indeed, we have shown this in the  $L^\infty$  case in [17]).

Our complex structure was constructed using a fiber structure discovered by two of the authors [17]. It arises from an identification of the Teichmüller space of bordered surfaces with a moduli space of Friedan-Shenker-Vafa [15], which the authors used in [19] to construct a Hilbert manifold structure on the refined Teichmüller space of bordered surfaces, as described above. We then use our fiber structure, together with Gardiner's Schiffer-variational coordinates on Teichmüller space [5], which we will henceforth refer to as the Gardiner-Schiffer variation, and the results of Earle and Fowler on marked holomorphic families and universality [3]. In some sense we obtained a generalization the Bers embedding, in which rather than modelling the Teichmüller space by conformal maps with invariant Schwarzians, we model it through a collection of disjoint conformal maps into the surface (in the spirit of conformal field theory). Thus quotients by Fuchsian groups are circumvented altogether.

The second major obstacle is that the Teichmüller space is modelled by Beltrami differentials which are in  $L^2 \cap L^\infty$ , and thus one immediately confronts the following technical problem. Given an arbitrary tangent vector to the Teichmüller space of a surface  $\Sigma$ , is it tangent to a curve  $\gamma(t)$  say, so that for each  $t$  there is a representative Beltrami differential  $\mu_t$  in  $L^2 \cap L^\infty$  which is furthermore holomorphic in  $t$  with respect to both the  $L^2$  and  $L^\infty$  norms? We show that this is indeed the case (Theorem 3.25). We also show that the space of  $L^2$  harmonic  $(-1, 1)$  differentials is contained in the space of  $L^\infty$  harmonic  $(-1, 1)$  differentials (Theorem 4.5). This in particular implies that the intersection of the  $L^2$  and  $L^\infty$  spaces of  $(-1, 1)$  differentials decomposes into a direct sum of infinitesimally trivial differentials and  $L^2$  harmonic differentials (Theorem 4.11). We also conclude that any tangent vector to the Teichmüller space can be represented via a perturbation by  $L^2$  harmonic Beltrami differentials (Theorem 4.14). This completes the description of the tangent space, and in particular implies that the Weil-Petersson metric is finite.

Finally, we show that at any point in the Teichmüller space, there is a biholomorphism onto an open ball in the space of harmonic  $L^2 (-1, 1)$ -differentials into the Teichmüller space (Theorem 4.16). In particular this gives us a new atlas on our refined Teichmüller space in terms of  $L^2 (-1, 1)$  harmonic differentials.

The paper is organized as follows. In Section 2, we give definitions of relevant function spaces. In Section 3, we give a brief synopsis of the complex structure obtained in our earlier work, and the coordinate system obtained from Gardiner-Schiffer coordinates. This is followed by the “preparation” theorem guaranteeing for any holomorphic curve the existence

of representatives with Beltrami differentials simultaneously in  $L^2$  and  $L^\infty$ . In Section 4.1, we obtain the results for the tangent space to Teichmüller space and infinitesimally trivial differentials. In Section 4.2 we obtain the local biholomorphisms from  $L^2$  harmonic  $(-1, 1)$  differentials into the Teichmüller space, and the new atlas for the complex structure in terms of these biholomorphisms. Finally in Section 4.3 we write the Weil-Petersson metric in terms of our model of the tangent space, in particular obtaining its finiteness.

We will often abbreviate “Weil-Petersson class” as “WP-class”. We will also refer to our refined Teichmüller space as “WP-class Teichmüller space” in light of the results of this paper.

As a final remark, we observe that much of the history of Teichmüller theory involves the exploration of distinctly obtained complex structures, whose non-trivial equivalence has led to significant insight. Our paper continues this exploration.

## 2. BORDERED RIEMANN SURFACES AND COLLARS

**2.1. Differentials on bordered surfaces.** First we establish some notation for the various function spaces involved. Let  $\Sigma$  be a Riemann surface with a hyperbolic metric. Let  $\{\phi_U : U \rightarrow \mathbb{C}\}$  be an atlas of local biholomorphic parameters covering  $\Sigma$ . For  $k, l \in \mathbb{Z}$  a  $(k, l)$ -differential  $h$  is a collection of functions  $h_U : \phi_U(U) \rightarrow \mathbb{C}$  such that whenever  $U \cap V$  is non-empty the functions  $h_U$  and  $h_V$  satisfy the transformation rule

$$(2.1) \quad h_V(g(z))g'(z)^k \overline{g'(z)}^l = h_U(z)$$

where  $g(z) = \phi_V \circ \phi_U^{-1}(z)$  is the change of parameter. That is,  $h$  has the expression  $h_U(z)dz^k d\bar{z}^l$  in local coordinates.

The hyperbolic metric induces a norm on  $(k, l)$ -differentials at any point in the following way. Setting  $m = k + l$ , at a fixed point in  $\Sigma$  the quantity

$$|h_U(z)|^p \rho_U(z)^{2-mp}$$

is independent of choice of coordinates  $z$ , where  $\rho_U(z)^2 |dz|^2$  is the local expression for the hyperbolic metric. By integrating over any subset  $W \subset \Sigma$  and taking the  $p$ th root we obtain a well-defined hyperbolic  $L^p$  norm  $\|h\|_{p,W}$  on  $W$ .

**Definition 2.1.** Let  $W \subset \Sigma$  be an open set. For  $1 \leq p \leq \infty$ , let

$$L_{k,l}^p(\Sigma, W) = \{(k, l) - \text{differentials } h : \|h\|_{p,W} < \infty\}.$$

Let

$$A_k^p(\Sigma, W) = \{h \in L_{k,0}^p(\Sigma, W) : h \text{ holomorphic}\}.$$

Denote  $L_{k,l}^p(\Sigma, \Sigma)$  by  $L_{k,l}^p(\Sigma)$  and  $A_k^p(\Sigma, \Sigma)$  by  $A_k^p(\Sigma)$ .

*Remark 2.2.* We will not distinguish the norms  $\|\cdot\|_{p,W}$  notationally with respect to the order of the differential, since the type of differential uniquely determines the norm. If the subscript “ $W$ ” is omitted, it is assumed that  $W = \Sigma$ .

**2.2. WP-class quasisymmetries and quasiconformal maps.** In [19] the authors defined a Teichmüller space of bordered surfaces which possesses a Hilbert manifold structure. We briefly review some of the definitions and results, as well as introduce new definitions necessary in the next few sections.

Let

$$\mathbb{D} = \{z : |z| < 1\}, \quad \mathbb{D}^* = \{z : |z| > 1\} \cup \{\infty\}, \quad \text{and} \quad \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

**Definition 2.3.** Let  $\mathcal{O}_{\text{WP}}^{\text{qc}}$  denote the set of holomorphic one-to-one maps  $f : \mathbb{D} \rightarrow \mathbb{C}$  with quasiconformal extensions to  $\bar{\mathbb{C}}$  such that  $(f''(z)/f'(z))dz \in A_1^2(\mathbb{D})$  and  $f(0) = 0$ .

By results of Takhtajan and Teo, the image of  $\mathcal{O}_{\text{WP}}^{\text{qc}}$  under the map

$$(2.2) \quad f \mapsto \left( \frac{f''(z)}{f'(z)}, f'(0) \right)$$

is an open subset of the Hilbert space  $A_1^2(\mathbb{D}) \oplus \mathbb{C}$  with the direct sum inner product [18, Theorem 2.3].

Elements of  $\mathcal{O}_{\text{WP}}^{\text{qc}}$  arise as conformal maps associated to quasisymmetries in the following way. Given a quasisymmetry  $\phi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ , by the Ahlfors-Beurling extension theorem, there exists a quasiconformal map  $w : \mathbb{D}^* \rightarrow \mathbb{D}^*$  such that  $w|_{\mathbb{S}^1} = \phi$ . This quasiconformal map has complex dilatation

$$\mu = \frac{\bar{\partial}w}{\partial w} \in L_{-1,1}^\infty(\mathbb{D}^*).$$

Let  $f^\mu$  be the solution to the Beltrami equation

$$\frac{\bar{\partial}f}{\partial f} = \hat{\mu}$$

where  $\hat{\mu}$  is the Beltrami differential which equals  $\mu$  on  $\mathbb{D}^*$  and 0 on  $\mathbb{D}$ . We normalize  $f^\mu$  so that  $f^\mu(0) = 0$ ,  $f^\mu(\infty) = \infty$  and  $f^{\mu'}(\infty) = 1$  for definiteness. We define

$$f_\phi = f^\mu|_{\mathbb{D}}.$$

It is a standard result in Teichmüller theory that  $f_\phi$  is independent of the choice of quasiconformal extension  $w$ , and furthermore  $f_\phi = f_\psi$  if and only if  $\phi = \psi$  [12, 13].

**Definition 2.4.** Using the above construction we say that  $\psi : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is a WP-class quasisymmetry if and only if  $f_\psi \in \mathcal{O}_{\text{WP}}^{\text{qc}}$ . We denote the set of WP-class quasisymmetries by  $\text{QS}_{\text{WP}}(\mathbb{S}^1, \mathbb{S}^1)$ , or more briefly by  $\text{QS}_{\text{WP}}(\mathbb{S}^1)$ .

From now on, let  $\Sigma$  be a Riemann surface of genus  $g$  bordered by  $n$  curves homeomorphic to  $\mathbb{S}^1$  where  $n > 0$ ; here the Riemann surface is “bordered” in the sense of Ahlfors and Sario [1, II.1.3]. We will also assume that the boundary of  $\Sigma$  consists of  $n$  connected components homeomorphic to  $\mathbb{S}^1$  in the relative topology inherited from  $\bar{\Sigma}$  with respect to the border structure. When we say that  $\Sigma$  is of genus  $g$  we mean that  $\Sigma$  is biholomorphic to a subset  $\Sigma^B$  of a compact Riemann surface  $\tilde{\Sigma}$  of genus  $g$  in such a way that the complement of  $\Sigma^B$  in  $\tilde{\Sigma}$  consists of  $n$  disjoint open sets biholomorphic to  $\mathbb{D}$ . Equivalently, the double of  $\Sigma$  has genus  $2g + n - 1$ . We will refer to such a Riemann surface as a “bordered surface of type  $(g, n)$ ”.

We will also frequently make use of a kind of chart on “collars” of the boundary. Let

$$\mathbb{A}_r = \{z : 1 < |z| < r\}.$$

**Definition 2.5.** Let  $\Sigma$  be a bordered Riemann surface of genus  $g$  bordered by  $n$  curves  $\partial_i \Sigma$ ,  $i = 1, \dots, n$ , homeomorphic to  $\mathbb{S}^1$ . A collar chart  $(\zeta, A)$  of  $\partial_i \Sigma$  is an open set  $A \subset \Sigma$  and a map  $\zeta : A \rightarrow \mathbb{A}_r$  for some  $r > 1$  such that  $\partial_i \Sigma$  is contained in the closure of  $A$  and  $\partial A \cap (\partial_i \Sigma)^c$  is compactly contained in  $\Sigma$ .

For any such  $A$ ,  $\mathbb{A}_r$ , and  $\zeta$ ,  $\zeta$  has a homeomorphic extension to  $A \cup \partial_i \Sigma$ .

We call  $A$  a “collar” of  $\partial_i \Sigma$ . We will also allow collar charts into annuli  $r < |z| < 1$  when convenient.

The important property of bordered Riemann surfaces of type  $(g, n)$  for our purposes is that for any such surface and any  $i$  a collar chart exists [20]. Furthermore,  $A$ ,  $r$  and  $\zeta$  can be chosen so that  $\partial A \setminus \partial_i \Sigma$  is an analytic curve. In that case  $\zeta$  has a homeomorphic extension to the closure of  $A$ , which takes  $\overline{A}$  onto the closed annulus  $\overline{\mathbb{A}_r}$ .

We may now define WP-class quasisymmetries between boundary curves of bordered Riemann surfaces, as in [19].

**Definition 2.6.** Let  $\Sigma_1$  and  $\Sigma_2$  be bordered Riemann surfaces of type  $(g_i, n_i)$  respectively, and let  $C_1$  and  $C_2$  be boundary curves of  $\Sigma_1$  and  $\Sigma_2$  respectively. Let  $\text{QS}_{\text{WP}}(C_1, C_2)$  denote the set of orientation-preserving homeomorphisms  $\phi : C_1 \rightarrow C_2$  such that there are collared charts  $H_i$  of  $C_i$ ,  $i = 1, 2$  respectively, and such that  $H_2 \circ \phi \circ H_1^{-1}|_{\mathbb{S}^1} \in \text{QS}_{\text{WP}}(\mathbb{S}^1)$ .

Equivalently, for *any* pair of collar charts  $H_i$  of  $C_i$ ,  $i = 1, 2$  respectively,  $H_2 \circ \phi \circ H_1^{-1}|_{\mathbb{S}^1} \in \text{QS}_{\text{WP}}(\mathbb{S}^1)$  [19].

*Remark 2.7.* The notation  $\text{QS}_{\text{WP}}(\mathbb{S}^1, C_1)$  will always be understood to refer to  $\mathbb{S}^1$  as the boundary of an annulus  $\mathbb{A}_r$  for  $r > 1$ .

### 3. WP-CLASS TEICHMÜLLER SPACE

**3.1. Quasiconformal maps with Beltrami differentials in  $L^2_{-1,1}(\Sigma)$ .** First, we make the following definition.

**Definition 3.1.** Let

$$\text{TBD}(\Sigma) = L^\infty_{-1,1}(\Sigma) \cap L^2_{-1,1}(\Sigma)$$

and

$$\text{BD}(\Sigma) = \{\mu \in \text{TBD}(\Sigma) : \|\mu\|_{\infty, \Sigma} \leq K \text{ for some } K < 1\}.$$

The “BD” in the above notation stands for “Beltrami differentials”. The “T” stands for “tangent”. Analytically the notation “TBD” is slightly inaccurate, since  $\text{BD}(\Sigma)$  is not a Hilbert or Banach linear space or manifold, so that it does not have a tangent space in the standard sense. Nevertheless the notation distinguishes the spaces conveniently in terms of their upcoming roles.

Next we define the Weil-Petersson class Teichmüller space of bordered Riemann surfaces of type  $(g, n)$ .

**Definition 3.2.** Let  $\Sigma$  and  $\Sigma_1$  be bordered Riemann surfaces of type  $(g, n)$ . Let  $\text{QC}_r(\Sigma, \Sigma_1)$  denote the set of quasiconformal maps  $f : \Sigma \rightarrow \Sigma_1$  such that  $\mu(f) \in \text{BD}(\Sigma)$ .

**Definition 3.3** (WP-class Teichmüller space). Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$ . We define an equivalence relation on triples  $(\Sigma, f, \Sigma')$  for  $f : \Sigma \rightarrow \Sigma'$  quasiconformal as follows:  $(\Sigma, f_1, \Sigma_1) \sim (\Sigma, f_2, \Sigma_2)$  if and only if there is a biholomorphism  $\sigma : \Sigma_1 \rightarrow \Sigma_2$  such that  $f_2^{-1} \circ \sigma \circ f_1$  is homotopic to the identity rel boundary.



The WP-class Teichmüller space of  $\Sigma$  [19] is the set

$$T_{\text{WP}}(\Sigma) = \{(\Sigma, f, \Sigma_1) : f \in \text{QC}_r(\Sigma, \Sigma_1)\} / \sim .$$

The term “rel boundary” means that the homotopy is the identity on the boundary curve throughout.

*Remark 3.4.* In [19] the authors used a different definition of  $T_{\text{WP}}(\Sigma)$ ; in place of  $\text{QC}_r(\Sigma, \Sigma_1)$ , we use  $\text{QC}_0(\Sigma, \Sigma_1) = \{f \text{ quasiconformal} : f|_{\partial_i \Sigma} \in \text{QS}_{\text{WP}}(\partial_i \Sigma, \partial_i \Sigma_1)\}$ . The definitions are in fact equivalent: any element of  $\text{QC}_0(\Sigma, \Sigma_1)$  is homotopic rel boundary to an element of  $\text{QC}_r(\Sigma, \Sigma_1)$  [20]. Furthermore, any collection of maps  $\phi_i \in \text{QS}_{\text{WP}}(\partial_i \Sigma, \partial_i \Sigma_1)$  has a quasiconformal extension in  $\text{QC}_r(\Sigma, \Sigma_1)$  [20].

*Remark 3.5.* In [19, Proposition 2.19] we showed that  $\text{QC}_0$  is closed under composition. That is, if  $f \in \text{QC}_0(\Sigma_1, \Sigma_2)$  and  $g \in \text{QC}_0(\Sigma_2, \Sigma_3)$  then  $g \circ f \in \text{QC}_0(\Sigma_1, \Sigma_3)$ . It would be satisfactory if this were also true for  $\text{QC}_r$  (especially for the theory of mapping class groups), although we do not yet see any independent reason why this should be true. Note that we do not need that result in this paper.

The complex Hilbert manifold structure on  $T_{\text{WP}}(\Sigma)$  is constructed using a natural fiber structure. This fiber structure is apparent in a closely related “rigged Teichmüller space” which we now define. We will use this ahead to define the coordinates on  $T_{\text{WP}}(\Sigma)$ .

We say that  $\Sigma^P$  is a punctured Riemann surface of type  $(g, n)$  if it is a genus  $g$  Riemann surfaces with  $n$  points removed. The removed points are furthermore given a specific order  $p_1, \dots, p_n$ , so the term “punctured Riemann surface of type  $(g, n)$ ” refers to such a surface together with the ordering. For the purposes of this paper we could also think of the surfaces as compact Riemann surfaces with  $n$  distinguished points listed in a specific order; we will move between these two pictures freely without changing the notation.

**Definition 3.6** (WP-class riggings). Let  $\Sigma^P$  be a punctured Riemann surface of type  $(g, n)$  with punctures  $p_1, \dots, p_n$ . A WP-class rigging on  $\Sigma^P$  is an  $n$ -tuple of maps  $\phi = (\phi_1, \dots, \phi_n)$  such that for each  $i = 1, \dots, n$ ,

- (1)  $\phi_i : \mathbb{D} \rightarrow \Sigma^P$  is one-to-one and holomorphic,
- (2)  $\phi_i(0) = p_i$ ,
- (3)  $\phi_i$  has a quasiconformal extension to a neighbourhood of the closure  $\overline{\mathbb{D}}$  of  $\mathbb{D}$ ,
- (4) if  $\zeta_i : U_i \rightarrow \mathbb{C}$  is a local biholomorphic coordinate on an open set  $U_i \subset \Sigma^P$  containing  $f_i(\overline{\mathbb{D}})$  such that  $\zeta_i(p_i) = 0$ , then  $\zeta_i \circ \phi_i \in \mathcal{O}_{\text{WP}}^{\text{qc}}$ , and
- (5) Whenever  $i \neq j$ ,  $\phi_i(\overline{\mathbb{D}}) \cap \phi_j(\overline{\mathbb{D}})$  is empty.

Denote the set of WP-class riggings on  $\Sigma^P$  by  $\mathcal{O}_{\text{WP}}^{\text{qc}}(\Sigma^P)$ .

When  $\Sigma^P$  and  $\Sigma_1^P$  are punctured Riemann surfaces of type  $(g, n)$ , then if  $f : \Sigma^P \rightarrow \Sigma_1^P$  is quasiconformal, it must extend to the compactification in such a way that it takes punctures to punctures.

**Definition 3.7** (WP-class rigged Teichmüller space). Let  $\Sigma^P$  be a punctured Riemann surface of type  $(g, n)$ . The WP-class rigged Teichmüller space of  $\Sigma^P$  is the set

$$\tilde{T}_{\text{WP}}(\Sigma^P) = \{(\Sigma^P, f, \Sigma_1^P, \phi)\} / \sim$$

of equivalence classes  $[\Sigma^P, f, \Sigma_1^P, \phi]$  where  $f : \Sigma^P \rightarrow \Sigma_1^P$  is a quasiconformal map and  $\phi \in \mathcal{O}_{\text{WP}}^{\text{qc}}(\Sigma_1^P)$ . The equivalence relation  $\sim$  is defined as follows:  $(\Sigma^P, f_1, \Sigma_1^P, \phi) \sim (\Sigma^P, f_2, \Sigma_2^P, \psi)$  if and only if there is a biholomorphism  $\sigma : \Sigma^P \rightarrow \Sigma_1^P$  such that

- (1)  $f_2^{-1} \circ \sigma \circ f_1$  is homotopic to the identity rel boundary
- (2)  $\psi_i = \sigma \circ \phi_i$  for  $i = 1, \dots, n$ .

*Remark 3.8.* In this case, the requirement that  $f_2^{-1} \circ \sigma \circ f_1$  is homotopic to the identity rel boundary implies that  $f_2^{-1} \circ \sigma \circ f_1(p_i) = p_i$  (in the sense that the unique extensions of  $f_1$ ,  $f_2$  and  $\sigma$  satisfy this equality) and that the punctures are preserved throughout the homotopy.

*Remark 3.9.* In particular, if  $(\Sigma^P, f_1, \Sigma_1^P, \phi) \sim (\Sigma^P, f_2, \Sigma_2^P, \psi)$  in  $\tilde{T}_0(\Sigma^P)$  then  $(\Sigma^P, f_1, \Sigma_1^P) \sim (\Sigma^P, f_2, \Sigma_2^P)$  in the Teichmüller space  $T(\Sigma^P)$ , and  $\sigma : \Sigma_1^P \rightarrow \Sigma_2^P$  preserves the ordering of the punctures.

*Remark 3.10.* In [19] we showed that  $\tilde{T}_{\text{WP}}(\Sigma^P)$  has a complex Hilbert manifold structure in the case that  $2g - 2 + n > 0$ . In this paper we will only consider  $\tilde{T}_{\text{WP}}(\Sigma^P)$  for  $2g - 2 + n > 0$ .

Next, we show that the WP-class Teichmüller space of bordered surfaces and the WP-class rigged Teichmüller space of punctured Riemann surfaces are closely related. The relation is obtained by “sewing caps” onto a given bordered Riemann surface to obtain a punctured surface. We outline this procedure below.

**Definition 3.11.** A parametrization  $\tau_i : \mathbb{S}^1 \rightarrow \partial_i \Sigma$  of the  $i$ th boundary curve of a bordered Riemann surface is called analytic if there exists a collar chart  $\zeta_i$  such that  $\zeta_i \circ \tau_i$  is an analytic diffeomorphism of  $\mathbb{S}^1$ .

Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$ . Let  $\tau = (\tau_1, \dots, \tau_n)$  be a WP-class quasiasymmetric parameterization of the boundary; that is

$$\tau_i \in \text{QS}_{\text{WP}}(\mathbb{S}^1, \partial_i \Sigma), \quad i = 1, \dots, n.$$

The existence of such a  $\tau$  is not restrictive, as the following theorem shows. In fact, we show that every boundary curve can be analytically parameterized.

**Theorem 3.12.** *There is a parametrization  $\tau = (\tau_1, \dots, \tau_n)$  of  $\partial \Sigma$  such that  $\tau_i$  is analytic for each  $i$ . In particular,  $\tau \in \text{QS}_{\text{WP}}(\mathbb{S}^1, \partial \Sigma)$ .*

*Proof.* Let  $(\zeta_i, U_i)$  be a collar chart of  $\partial_i \Sigma$ . By the Schwarz reflection principle,  $\zeta_i$  extends to an open neighbourhood of the boundary  $\partial_i \Sigma$  in the double  $\Sigma^D$  of the Riemann surface  $\Sigma$ , which maps  $\partial_i \Sigma$  into the circle  $\mathbb{S}^1$ . Let  $\tau_i$  be the restriction to  $\mathbb{S}^1$  of the extension of  $\zeta_i^{-1}$ . Then  $\zeta_i \circ \tau_i(z) = z$  satisfies the requirement.  $\square$

Given such a  $\tau$ , the  $\tau_i$ 's are in particular quasiasymmetries. Thus one can form the Riemann surface  $\Sigma^P = \Sigma \#_{\tau} \cup_{i=1}^n \mathbb{D}$  which is defined as follows. We identify points  $w$  on the boundary of the  $n$ th disk with points  $z$  on the  $n$ th boundary curves,  $w \sim z$ , if  $z = \tau_i(w)$ . The resulting set

$$\Sigma^P = (\Sigma \sqcup \mathbb{D} \sqcup \dots \sqcup \mathbb{D}) / \sim$$

is a topological space with the quotient topology and has a unique complex structure which agrees with the complex structures on  $\Sigma$  and each of the discs  $\mathbb{D}$  [15, Theorems 3.2, 3.3]. We refer to  $\Sigma^P$  as being obtained from  $\Sigma$  by “sewing on caps via  $\tau$ ”.

The inclusion maps from each disk  $\mathbb{D}$  into  $\Sigma^P$  are holomorphic. Denote the resulting maps by  $\tilde{\tau}_i : \mathbb{D} \rightarrow \Sigma^P$  and  $\tilde{\tau} = (\tilde{\tau}_1, \dots, \tilde{\tau}_n)$ . By [19, Proposition 4.7]  $\tilde{\tau} \in \mathcal{O}_{\text{WP}}^{\text{qc}}(\Sigma^P)$ .

Now let  $f \in \text{QC}_r(\Sigma, \Sigma_1)$ . Let  $\Sigma_1^P$  be obtained from  $\Sigma_1$  by sewing on caps via  $f \circ \tau = (f \circ \tau_1, \dots, f \circ \tau_n)$ . Note that  $f \circ \tau_i \in \text{QS}_{\text{WP}}(\partial_i \Sigma, \partial_i \Sigma_1)$  for each  $i$  by [19, Proposition 4.8].

There is a natural extension of  $f$  to a map  $\tilde{f} : \Sigma^P \rightarrow \Sigma_1^P$  defined as follows:

$$(3.1) \quad \tilde{f}(z) = \begin{cases} f(z), & z \in \bar{\Sigma} \\ z, & z \in \mathbb{D} \cup \dots \cup \mathbb{D}. \end{cases}$$

We thus have a natural map from  $T_{\text{WP}}(\Sigma)$  into  $\tilde{T}_{\text{WP}}(\Sigma^P)$ .

$$(3.2) \quad \begin{aligned} \Pi : T_{\text{WP}}(\Sigma) &\longrightarrow \tilde{T}_{\text{WP}}(\Sigma^P) \\ (\Sigma, f, \Sigma_1) &\longmapsto (\Sigma^P, \tilde{f}, \Sigma_1^P, \tilde{f} \circ \tilde{\tau}). \end{aligned}$$

Note that  $\tilde{f} \circ \tilde{\tau}_i$  is holomorphic on each copy of the disk  $\mathbb{D}$ .

The map  $\Pi$  was used in [19] to construct an atlas on  $T_{\text{WP}}(\Sigma)$ , making it a complex Hilbert manifold. We will describe the atlas of charts in the next section.

*Remark 3.13.* Observe that  $\tilde{f}$  is the unique quasiconformal map (up to composition with a conformal map) solving the Beltrami equation on  $\Sigma^P$  with the Beltrami differential

$$\tilde{\mu}(z) = \begin{cases} \mu(f)(z), & z \in \Sigma \\ 0, & z \in \mathbb{D} \cup \dots \cup \mathbb{D}. \end{cases}$$

Thus we may see the map  $\Pi$  as generalizing the Bers trick associating a conformal map  $f$  of the disk with elements of the universal Teichmüller space.

Finally we will need the following lemma.

**Lemma 3.14.** *Let  $\tau \in \text{QS}_{\text{WP}}(\mathbb{S}^1, \partial\Sigma)$  be an analytic parametrization of  $\partial\Sigma$ . Let  $\Sigma^P$  be obtained from  $\Sigma$  by sewing on caps via  $\tau$ , and let  $\tilde{\tau} : \mathbb{D} \rightarrow \Sigma^P$  be the resulting extension of  $\tau$ . For each  $i = 1, \dots, n$  there is a chart  $(\zeta_i, U_i)$  such that  $U_i$  contains the closure of  $\tilde{\tau}_i(\mathbb{D})$  and  $\zeta_i \circ \tilde{\tau}_i$  is the identity map on  $\mathbb{S}^1$ .*

*Proof.* Let  $(\xi_i, W_i)$  be any coordinate chart such that  $\overline{\tilde{\tau}_i(\mathbb{D})} \subset W_i$ . By assumption  $\xi_i \circ \tilde{\tau}_i(\mathbb{S}^1)$  is an analytic curve in  $\mathbb{C}$ , so the Riemann map  $\eta_i : \mathbb{D} \rightarrow \xi_i(\tilde{\tau}_i(\mathbb{D}))$  has a one-to-one holomorphic extension to some disk  $\mathbb{D}_R = \{z : |z| < R\}$  for  $R > 1$  onto a neighbourhood of the closure of  $\xi_i(\tilde{\tau}_i(\mathbb{D}))$ . By composing  $\eta_i$  by a Möbius transformation we can assume that  $\eta_i^{-1} \circ \xi_i \circ \tilde{\tau}_i$  is the identity (since it maps the disk to the disk). Set  $\zeta_i = \eta_i^{-1} \circ \xi_i$ ; this is a chart on some domain  $U_i \subseteq W_i$  where  $U_i$  contains the closure of  $\tilde{\tau}_i(\mathbb{D})$ .  $\square$

**3.2. Gardiner-Schiffer coordinates and the complex structure.** In this section we define Gardiner-Schiffer coordinates and the complex structure on  $T_{\text{WP}}(\Sigma)$ . Although the geometric idea is straightforward, the construction and rigorous proofs are somewhat involved. We restrict ourselves here to summarizing the necessary facts and refer the reader to [19] for a full treatment. In order to make rigorous statements about holomorphicity, we require the framework of marked holomorphic families of Earle and Fowler [3]. The appendix ahead contains a brief summary of the necessary elements of this theory for readers not familiar with it.

Gardiner-Schiffer variation is a technique for constructing coordinates on the Teichmüller space of a compact surface with punctures. See [5] or [13].

Let  $\Sigma^P$  be a punctured Riemann surface of type  $(g, n)$  and  $[\Sigma^P, f, \Sigma_1^P] \in T(\Sigma^P)$ . Let  $d = 3g - 3 + n$  which is the dimension of the Teichmüller space  $T(\Sigma^P)$ . Let  $D = (D_1, \dots, D_d)$  be a  $d$ -tuple of disjoint open sets  $D_i \subset \Sigma_1^P$ , each of which is biholomorphic to the unit disk

$\mathbb{D}$  via a map  $\eta_i : D_i \rightarrow \mathbb{D}$ . Let  $w_\epsilon = z + \epsilon\bar{z}$ , where  $\epsilon \in \mathbb{C}$ . For  $|\epsilon|$  sufficiently small,  $w_\epsilon$  is a quasiconformal homeomorphism and we let  $D_\epsilon = w_\epsilon(\mathbb{D})$ .

Let  $\Omega$  be a small open connected neighbourhood of  $0 \in \mathbb{C}^d$ , and let  $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \Omega$ . By excising the disks  $D_i$ , and gluing in  $D_{\epsilon_i}$  using the boundary identification  $w_{\epsilon_i} \circ \eta_i$ , one obtains a new Riemann surface  $\Sigma_{1,\epsilon}^P$  and a quasiconformal map

$$\nu_\epsilon : \Sigma_1^P \longrightarrow \Sigma_{1,\epsilon}^P$$

defined by  $\nu_\epsilon(z) = w_{\epsilon_i} \circ \eta_i$  for  $z \in D_i$  and  $\nu_\epsilon(z) = z$  for  $z \in \Sigma_1^P \setminus D$ . Note that  $\nu_\epsilon$  is holomorphic outside of the disks  $D_i$ , and on  $D_i$  the Beltrami differential of  $\nu_\epsilon$  is  $\epsilon_i d\bar{z}/dz$  in the coordinate system  $\eta_i$ . We now define the Schiffer variation map

$$\begin{aligned} \mathfrak{S} : \Omega &\longrightarrow T(\Sigma^P) \\ \epsilon &\longmapsto [\Sigma^P, \nu_\epsilon \circ f, \Sigma_{1,\epsilon}^P] \end{aligned}$$

It follows directly from the construction that  $\mathfrak{S}$  is holomorphic, but it can in fact give analytic coordinates on  $T(\Sigma^P)$  as the following theorem from [5] or [13] shows.

**Theorem 3.15.** *Let  $[\Sigma^P, f, \Sigma_1^P] \in T(\Sigma^P)$ . Let  $D_i \subset \Sigma_1^P, i = 1, \dots, d$ , be any disjoint biholomorphic images of the unit disk  $\mathbb{D}$ . There exist coordinates  $\eta_i : D_i \rightarrow \mathbb{D}$  for  $i = 1, \dots, d$ , and an open connected neighbourhood  $\Omega$  of  $0 \in \mathbb{C}^d$ , such that  $\mathfrak{S}$  is a biholomorphism onto its image (i.e. the inverse is a local coordinate chart).*

The collection of Riemann surfaces  $\Sigma_{1,\epsilon}^P$  form a marked holomorphic family of Riemann surfaces (see Definition 5.3 ahead) as follows. By [19, Theorem 3.15] the set

$$S(\Omega, D) = \{(\epsilon, x) : \epsilon \in \Omega \text{ and } x \in \Sigma_{1,\epsilon}^P\}$$

is a complex manifold, with projection

$$\begin{aligned} \pi : S(\Omega, D) &\longrightarrow \Omega \\ (\epsilon, x) &\longmapsto \epsilon \end{aligned}$$

and strong global trivialization

$$(3.3) \quad \begin{aligned} \theta : \Omega \times \Sigma^P &\longrightarrow S(\Omega, D) \\ (\epsilon, q) &\longmapsto (\epsilon, \nu_\epsilon \circ f(q)) \end{aligned}$$

giving  $S(\Omega, D)$  the structure of a marked holomorphic family of Riemann surfaces. In particular, note that  $\Sigma_{1,\epsilon}^P$  is a complex submanifold of  $S(\Omega, D)$  and for each fixed  $\epsilon$ ,  $x \mapsto \theta(\epsilon, x)$  is a quasiconformal map from  $\Sigma^P$  to  $\pi^{-1}(\epsilon) = \Sigma_{1,\epsilon}^P$ .

Using Theorem 5.8 we can embed the marked Schiffer family into the universal Teichmüller curve as described in the following theorem. Let  $\mu(\epsilon) = \mu(\nu_\epsilon \circ f)$ . Then the fiber in the Teichmüller curve over  $\mathfrak{S}(\epsilon)$  is the canonical Riemann surface  $\Sigma_{\mu(\epsilon)}^P$  as defined in (5.1). Note that  $\Sigma_{1,\epsilon}^P$  and  $\Sigma_{\mu(\epsilon)}^P$  are biholomorphically equivalent.

**Theorem 3.16.** *Let  $\Sigma^P$  be a punctured Riemann surface of type  $(g, n)$  such that  $2g - 2 + n > 0$ . Let  $S(\Omega, D)$  be a marked Schiffer family as defined above. There is a morphism of marked holomorphic families  $(\alpha, \beta)$  from  $\pi : S(\Omega, D) \rightarrow \Omega$  to  $\pi_T : \mathcal{T}(\Sigma^P) \rightarrow T(\Sigma^P)$  and moreover*

- (1)  $\alpha(\epsilon) = \mathfrak{S}(\epsilon) = [\Sigma^P, \nu_\epsilon \circ f, \Sigma_{1,\epsilon}^P]$ , and
- (2)  $\alpha$  and  $\beta$  are injective.

Setting

$$\sigma_\epsilon(z) = \beta(\epsilon, z) : \Sigma_{1,\epsilon}^P \rightarrow \Sigma_{\mu(\epsilon)}^P$$

we have the biholomorphism

$$(3.4) \quad \Gamma : S(\Omega, D) \longrightarrow \pi_T^{-1}(\mathfrak{S}(\Omega)) \subseteq \mathcal{T}(\Sigma^P) \\ (\epsilon, p) \longmapsto (\alpha, \beta)(\epsilon, p) = ([\Sigma^P, \nu_\epsilon \circ f, \Sigma_{1,\epsilon}^P], \sigma_\epsilon(p)).$$

*Proof.* Since the spaces are finite-dimensional, injective holomorphic functions are necessarily biholomorphic. So the only part not following immediately from Theorem 5.8 is the injectivity of  $\alpha$  and  $\beta$ . Since  $\mathfrak{S}$  is injective by Theorem 3.15, it remains only to show that  $\beta$  is injective. This follows from the fact that  $\beta$  is injective fiberwise and  $\alpha \circ \pi = \pi_T \circ \beta$ .  $\square$

Next, we need to define a local trivialization of the elements of  $\mathcal{O}_{\text{WP}}^{\text{qc}}(\Sigma^P)$ .

**Definition 3.17.** Let  $\Sigma^P$  be a punctured Riemann surface of type  $(g, n)$ . An  $n$ -chart  $(\zeta, E)$  on  $\Sigma^P$  is a collection of open subsets  $E = (E_1, \dots, E_n)$  of the compactification of  $\Sigma^P$  and local biholomorphic parameters  $\zeta_i : E_i \rightarrow \mathbb{C}$  such that

- (1)  $E_i \cap E_j$  is empty whenever  $i \neq j$ ,
- (2)  $p_i \in E_i$  for  $i = 1, \dots, n$ , and
- (3)  $\zeta_i(p_i) = 0$  for  $i = 1, \dots, n$ .

**Definition 3.18.** Let  $\Sigma^P$  be a punctured Riemann surface of type  $(g, n)$ . Let  $(\zeta, E)$  be an  $n$ -chart on  $\Sigma^P$ . We say that a collection  $U_1, \dots, U_n$  of open sets  $U_i \subseteq \mathcal{O}_{\text{WP}}^{\text{qc}}$  is compatible with  $(\zeta, E)$  if for all  $f_i \in U_i$ ,  $\overline{f_i(\mathbb{D})} \subseteq \zeta_i(E_i)$ . In this case we will also say that  $U = U_1 \times \dots \times U_n \subseteq \mathcal{O}_{\text{WP}}^{\text{qc}} \times \dots \times \mathcal{O}_{\text{WP}}^{\text{qc}}$  is compatible with  $(\zeta, E)$ .

The existence of such open sets is not immediately obvious. By [18, Theorem 3.4], for any open subsets  $F_i$  of  $\zeta_i(E_i)$ ,  $i = 1, \dots, n$ ,

$$\{f_i : \overline{f_i(\mathbb{D})} \subseteq F_i\}$$

is open in  $\mathcal{O}_{\text{WP}}^{\text{qc}}$  and thus for example

$$\{(f_1, \dots, f_n) \in \mathcal{O}_{\text{WP}}^{\text{qc}} : \overline{f_i(\mathbb{D})} \subseteq F_i, i = 1, \dots, n\}$$

is open in  $\mathcal{O}_{\text{WP}}^{\text{qc}} \times \dots \times \mathcal{O}_{\text{WP}}^{\text{qc}}$ .

**Definition 3.19.** Let  $\mathfrak{S} : \Omega \rightarrow T(\Sigma^P)$  be Schiffer coordinates based at  $[\Sigma^P, f, \Sigma_1^P]$ . We say that an  $n$ -chart  $(\zeta, E)$  on  $\Sigma_1^P$  is compatible with  $\mathfrak{S}$  if the closure of each disk  $D_i$ ,  $i = 1, \dots, d$ , in the Schiffer variation is disjoint from the closure of each open set  $E_i$  in the  $n$ -chart.

This definition ensures that the Schiffer variation maps  $\nu_\epsilon$  are conformal on the closures of each  $n$ -chart; this is crucial for the construction of coordinates on  $\tilde{T}_{\text{WP}}(\Sigma)$ .

**Definition 3.20.** Let  $\mathfrak{S} : \Omega \rightarrow T(\Sigma^P)$  be Schiffer coordinates based at  $[\Sigma^P, f, \Sigma_1^P]$  corresponding to disks  $D = (D_1, \dots, D_d)$ , and let  $S = S(\Omega, D)$  be the corresponding Schiffer family. Let  $(\zeta, E)$  be an  $n$ -chart on  $\Sigma_1^P$  and assume that  $\mathfrak{S}$  is compatible with an  $n$ -chart  $(\zeta, E)$ . Let  $U$  be an open subset of  $\mathcal{O}_{\text{WP}}^{\text{qc}} \times \dots \times \mathcal{O}_{\text{WP}}^{\text{qc}}$  which is compatible with  $(\zeta, E)$ . Define  $F(U, S, \Omega)$  by

$$F(U, S, \Omega) = \{(\Sigma^P, \nu_\epsilon \circ f, \Sigma_{1,\epsilon}^P, \psi) : \psi = \nu_\epsilon \circ \zeta^{-1} \circ \phi, \phi \in U, \epsilon \in \Omega\}$$

where  $\zeta^{-1} \circ \phi = (\zeta_1^{-1} \circ \phi_1, \dots, \zeta_n^{-1} \circ \phi_n)$ .

It was shown in [19] that these define a base for a topology on  $\tilde{T}_{\text{WP}}(\Sigma^P)$ , and furthermore that the set of charts

$$(3.5) \quad \begin{aligned} \mathcal{G} : F(U, S, \Omega) &\longrightarrow \Omega \times \mathcal{O}_{\text{WP}}^{\text{qc}} \times \cdots \times \mathcal{O}_{\text{WP}}^{\text{qc}} \\ (\Sigma^P, \nu_\epsilon \circ f, \Sigma_{1,\epsilon}^P, \psi) &\longmapsto (\epsilon, \zeta \circ \nu_\epsilon^{-1} \circ \psi) \end{aligned}$$

is an atlas defining a complex Hilbert manifold structure on  $\tilde{T}_{\text{WP}}(\Sigma^P)$ .

*Remark 3.21.* In [19], we used the notation  $G$  for  $\mathcal{G}^{-1}$ .

The complex structure on  $\tilde{T}_{\text{WP}}(\Sigma^P)$  passes upwards to  $T_{\text{WP}}(\Sigma)$ , according to the following theorem which the authors proved in [19, Theorem 4.18].

**Theorem 3.22.** *If  $\Sigma$  is a bordered surface of type  $(g, n)$ , with  $2g - 2 + n > 0$ , then the collection of charts on  $T_{\text{WP}}(\Sigma)$  of the form  $\mathcal{G} \circ \Pi$  is an atlas for a complex structure. Thus  $T_{\text{WP}}(\Sigma)$  is a complex Hilbert manifold. The map  $\Pi$  defined by (3.2) is a local biholomorphism; that is, for any point  $p \in T_{\text{WP}}(\Sigma)$  there is an open neighbourhood  $U$  of  $p$  such that the restriction of  $\Pi$  to  $U$  is a biholomorphism onto its image.*

We also have a natural projection from  $\tilde{T}_{\text{WP}}(\Sigma^P)$  onto  $T(\Sigma^P)$ , given by

$$(3.6) \quad \begin{aligned} \mathcal{F} : \tilde{T}_{\text{WP}}(\Sigma^P) &\longrightarrow T(\Sigma^P) \\ ([\Sigma^P, f, \Sigma_1^P], \psi) &\longmapsto [\Sigma^P, f, \Sigma_1^P]. \end{aligned}$$

**3.3. A preparation theorem.** The purpose of this section is to use the sewing technology to prove the following fact: given a holomorphic curve through the identity in  $T_{\text{WP}}(\Sigma)$ , it is possible to choose a Teichmüller equivalent curve in  $\text{BD}(\Sigma)$ . That is, the curve can be chosen so that at each point the representative of the Teichmüller equivalence class has a Beltrami differential simultaneously in  $L_{-1,1}^2(\Sigma)$  and  $L_{-1,1}^\infty(\Sigma)$  (Theorem 3.25 ahead).

To prove this, we first need the following modification of a lemma of Takhtajan and Teo [25]. Denote by  $\mathcal{S}(\phi)$  the Schwarzian of  $\phi$  and by  $\mathcal{A}(\phi)$  the pre-Schwarzian.

**Lemma 3.23.** *There is an open neighbourhood of  $0 \in A_1^2(\mathbb{D})$  such that the map*

$$\begin{aligned} \Psi : A_1^2(\mathbb{D}) &\longrightarrow A_2^2(\mathbb{D}) \oplus \mathbb{C} \\ \psi &\longmapsto \left( \psi_z - \frac{1}{2}\psi^2, \psi(0) \right) \end{aligned}$$

*is a biholomorphism onto its image.*

*In particular, there exists an open ball  $A$  centred on 0 in  $\mathcal{O}_{\text{WP}}^{\text{qc}}$  on which*

$$(3.7) \quad \|\mathcal{S}(\phi)\|_{2,\mathbb{D}}^2 + |\mathcal{A}(\phi)(0)|^2 \approx \|\mathcal{A}(\phi)\|_{2,\mathbb{D}}^2.$$

The norms are determined by treating  $\mathcal{S}(\phi)$  as a  $(2, 0)$ -differential and  $\mathcal{A}(\phi)$  as a  $(1, 0)$ -differential.

*Proof.* The map  $\Psi$  is in fact holomorphic and injective on  $A_1^2(\mathbb{D})$  by [25, Lemma A.1]. In particular,  $D\Psi|_0$  is injective and bounded. We claim that  $D\Psi|_0$  is also surjective.

We will use the following approach. Let  $(h, \alpha) \in A_2^2(\mathbb{D}) \oplus \mathbb{C}$ . Let  $\phi_t$  be the unique holomorphic function on  $\mathbb{D}$  satisfying  $\phi_t(0) = 0$ ,  $\phi_t'(0) = 1$ ,  $\mathcal{A}(\phi_t)(0) = t\alpha$  and  $\mathcal{S}(\phi_t(z)) = th(z)$ . We will show that for some  $\delta > 0$ , if  $|t| < \delta$  then  $(\mathcal{S}(\phi_t)(z), \mathcal{A}(\phi_t)(0))$  is in the

image of  $\Psi$ . This will show that  $D\Psi|_0$  is surjective, and thus by the open mapping theorem for bounded linear maps it would follow that  $D\Psi|_0$  is a topological isomorphism. Thus by the inverse function theorem the inverse is holomorphic [10]. Now if  $t$  is small enough then  $\phi_t(z) \in \mathcal{O}^{\text{qc}}$ . To see this, observe first that by holomorphicity of inclusion  $A_2^2(\mathbb{D}) \hookrightarrow A_2^\infty(\mathbb{D})$  [25] if  $\|\mathcal{S}(\phi_t)\|_{2,\mathbb{D}}$  is small enough then by the classical criterion for quasiconformal extendibility,  $\phi_t(z)$  has a quasiconformal extension to  $\mathbb{C}$ . By [25, Theorem 1.12, Part II], this together with the bound on  $\|\mathcal{S}(\phi_t)\|_{2,\mathbb{D}}$  shows that  $\mathcal{A}(\phi_t) \in A_1^2(\mathbb{D})$ . This completes the proof of the first claim. The final claim follows from the scale invariance of  $\mathcal{S}(\phi)$  and  $\mathcal{A}(\phi)$ .  $\square$

We will also need the following lemma due to the authors, explicitly separating out the contribution of the collar to the  $L^p$  norm.

**Lemma 3.24** ([20]). *Let  $\Sigma$  be a bordered Riemann surface of genus  $g$  with  $n$  boundary curves. Fix  $p \in [1, \infty)$ . Let  $(\zeta, U)$  be a collection of collar charts  $(\zeta_i, U_i)$  into  $\mathbb{D}$  for each boundary  $i = 1, \dots, n$ . There exist annuli  $\mathbb{A}_{r_i,1} = \{z : r_i < |z| < 1\} \subset \zeta_i(U_i)$  such that  $|z| = r_i$  is compactly contained in  $\zeta_i(U_i)$ , a compact set  $M$  such that*

$$M \cup \zeta_1^{-1}(\mathbb{A}_{r_1,1}) \cup \dots \cup \zeta_n^{-1}(\mathbb{A}_{r_n,1}) = \Sigma,$$

and constants  $a$  and  $b_i$  such that for any  $\alpha \in L_{k,l}^p(\Sigma)$

$$\|\alpha\|_p \leq a\|\alpha\|_{\infty, M} + \sum_{i=1}^n b_i \left( \iint_{\mathbb{A}_{r_i,1}} \lambda_{\mathbb{D}}^{2-mp}(z) |\alpha_{U_i}(z)|^p \right)^{1/p}.$$

The constants  $b_i$  depend only on the collar charts  $(\zeta_i, U_i)$ ,  $r_i$ ,  $p$ ,  $k$  and  $l$  (not on  $\alpha$ ), and  $a^p$  is the hyperbolic area of  $M$ .

Now we prove the central result of this section.

**Theorem 3.25.** *Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$  such that  $2g - 2 + n > 0$ . Assume that  $t \mapsto [\Sigma, f_t, \Sigma_t]$  is a holomorphic one-parameter curve in  $T_{\text{WP}}(\Sigma)$  such that  $[\Sigma, f_0, \Sigma_0] = [\Sigma, \text{Id}, \Sigma]$ . Then there exists  $\delta > 0$  and representatives  $(\Sigma, f_t, \Sigma_t)$  so that the following properties are satisfied for  $|t| < \delta$ :*

- (1)  $\|\mu(f_t)\|_2$  is uniformly bounded in  $t$ , where  $\mu(f_t)$  is the Beltrami differential of  $f_t$  on  $\Sigma$ ,
- (2)  $t \mapsto \mu(f_t)$  is a holomorphic curve in  $L_{-1,1}^2(\Sigma)$ , and
- (3)  $t \mapsto \mu(f_t)$  is a holomorphic curve in  $L_{-1,1}^\infty(\Sigma)$ .

*Proof.* As before, sew on caps via some  $\tau \in \text{QS}_{\text{WP}}(\Sigma)$  to obtain the associated punctured surface  $\Sigma^P$ . By Theorem 3.22 every curve through  $[\Sigma, \text{Id}, \Sigma]$  is the inverse image under  $\Pi$  of a curve through  $[\Sigma^P, \text{Id}, \Sigma^P, \tilde{\tau}] \in \tilde{T}_{\text{WP}}(\Sigma)$ . Thus it suffices to describe curves through this point in  $\tilde{T}_{\text{WP}}(\Sigma)$ . By [19, Corollary 4.22] the complex structure on  $T_{\text{WP}}(\Sigma)$  is independent of the choice of rigging  $\tau$  used to sew on caps. Thus without loss of generality, we may assume that  $\tau$  is an analytic parametrization (the existence of such a  $\tau$  is guaranteed by Theorem 3.12). By Lemma 3.14 there is an  $n$ -chart  $(\zeta, E)$  such that  $\zeta_i \circ \tilde{\tau}_i$  is the identity on  $\mathbb{D}$ .

Fix a coordinate chart  $\mathcal{G}$  on a neighbourhood  $F(U, S, \Omega)$  of  $([\Sigma^P, \text{Id}, \Sigma^P], \tilde{\tau})$  in  $\tilde{T}_{\text{WP}}(\Sigma^P)$ , with associated compatible  $n$ -chart  $(\zeta, E)$  on  $\Sigma^P$ . Let  $K_i$  be open, connected, simply-connected neighbourhoods of each puncture for  $i = 1, \dots, n$  such that  $\overline{K_i} \subset E_i$  for  $i = 1, \dots, n$ . We may choose the chart  $F(U, S, \Omega)$  in Definition 3.20 above such that  $\phi_i(\mathbb{D}) \subset \zeta_i(K_i)$  for all  $\phi_i \in U$  and  $i = 1, \dots, n$ .

Given the holomorphic curve  $\lambda(t)$  in  $T_{WP}(\Sigma)$ , assume that it is in the image of some chart  $\mathcal{G} \circ \Pi$  on  $F(U, S, \Omega)$  of the above form (perhaps shrinking the  $t$ -domain of the curve). Let  $(\epsilon(t), w_1(t, z), \dots, w_n(t, z)) \in \Omega \times \mathcal{O}_{WP}^{\text{qc}} \times \dots \times \mathcal{O}_{WP}^{\text{qc}} = \mathcal{G} \circ \Pi(\lambda(t))$  with  $w_i(0, z) = z$ ,  $i = 1, \dots, n$ . By [25, Lemma 2.1, Page 22], one has

$$(3.8) \quad \sup_{\mathbb{D}} (1 - |z|^2)^2 |\mathcal{S}(w_i(t, \cdot))(z)| \leq \sqrt{\frac{12}{\pi}} \left\{ \iint_{\mathbb{D}} (1 - |z|^2)^2 |\mathcal{S}(w_i(t, \cdot))(z)|^2 dA \right\}^{\frac{1}{2}}.$$

Now using Lemma 3.23, there exists  $C > 0$  such that for all  $i = 1, \dots, n$  one has

$$(3.9) \quad \left\{ \iint_{\mathbb{D}} (1 - |z|^2)^2 |\mathcal{S}(w_i(t, \cdot))(z)|^2 dA \right\}^{\frac{1}{2}} \leq C (|\mathcal{A}(w_i(t, \cdot))(0)| + \left\{ \iint_{\mathbb{D}} |\mathcal{A}(w_i(t, \cdot))(z)|^2 dA \right\}^{\frac{1}{2}}).$$

Letting  $P_i(t) := |\mathcal{A}(w_i(t, \cdot))(0)| + \left\{ \iint_{\mathbb{D}} |\mathcal{A}(w_i(t, \cdot))(z)|^2 dA \right\}^{\frac{1}{2}}$  we know that  $\mathcal{A}(w_i(t, \cdot))(0)$  is continuous, and by the definition of the complex structure on  $T_{WP}$ ,  $\|\mathcal{A}(w_i(t, \cdot))\|_{L^2(\mathbb{D})}$  is also a continuous function of  $t$ . Therefore since  $\mathcal{A}(w_i(0, \cdot)) = 0$  we have  $P_i(0) = 0$  and hence there exists an  $s > 0$  such that  $P_i(t) \leq \sqrt{\frac{\pi}{12C^2}}$  for all  $i = 1, \dots, n$ , if  $|t| < s$ . This fact together with (3.8) yields

$$(3.10) \quad \sup_{i=1, \dots, n} \sup_{\mathbb{D}} (1 - |z|^2)^2 |\mathcal{S}(w_i(t, \cdot))(z)| \leq 1$$

for  $i = 1, \dots, n$  and  $|t| < s$ . Given (3.10), the Ahlfors-Weill reflection result [12, Theorem 5.1, Chpt II] yields that  $w_i(t, z)$  has a jointly continuous quasiconformal extension to  $\mathbb{C}$  with dilatation  $m_t^i$  satisfying

$$(3.11) \quad m_t^i(1/\bar{z}) = -\frac{(1 - |z|^2)^2 z^2}{2 \bar{z}^2} \mathcal{S}(w_i(t, \cdot))(z).$$

Thus for  $|t| < s$

$$(3.12) \quad \iint_{\mathbb{D}} \frac{|m_t^i(1/\bar{z})|^2}{(1 - |z|^2)^2} dA = \frac{1}{4} \iint_{\mathbb{D}} (1 - |z|^2)^2 |\mathcal{S}(w_i(t, \cdot))(z)|^2 dA.$$

Now using once again (3.9) and the bound on  $P_i(t)$  obtained above, it is readily seen that there is an  $M$  such that  $\|m_t^i\|_{2, \mathbb{D}^*}^2 \leq M$  for each  $i$  and small enough  $t$ . This yields the uniform boundedness of  $\|m_t^i\|_{2, \mathbb{D}^*}$  for  $|t| < s$ .

Choose simple closed analytic curves  $\gamma_i$  in  $\zeta_i(E_i)$  so that  $\gamma_i$  and  $\overline{K_i}$  are nested and non-intersecting; that is,  $\gamma_i$  encloses  $\overline{K_i}$ . Choose  $R_i > 1$  such that  $w_i(t, |z| = R_i)$  is enclosed by  $\gamma_i$  and does not intersect it; this can be done for all  $|t| < s' < s$  for some  $s'$  since the extension  $w_i(t, z)$  is jointly continuous. Let  $L_i$  be the doubly-connected region bounded by  $\zeta_i^{-1}(|z| = R_i)$  and  $\zeta_i^{-1}(\gamma_i)$ . Let  $Y$  denote the pre-compact subset of  $\Sigma$  bounded by the curves  $\zeta_i^{-1}(\gamma_i)$ .

We will construct a family of quasiconformal maps  $g_t : \Sigma^P \rightarrow \Sigma^P$  with the following properties.

- (a)  $g_t$  is the identity on  $\overline{Y}$ .
- (b)  $\zeta_i \circ g_t \circ \zeta_i^{-1}$  is the Ahlfors-Weill extension of  $w_i(t, z)$  on  $|z| \leq R$  with dilatation given by (3.11).
- (c) The Beltrami differential of  $\zeta_i \circ g_t \circ \zeta_i^{-1}$  is holomorphic as a map from  $t$  into  $L_{-1,1}^\infty$  on  $\zeta_i(L_i)$ .



(d)  $g_0$  is the identity on  $\Sigma^P$ .

Observe that if  $g_t$  has these properties, then setting

$$\begin{aligned} F_t &= \nu_{\epsilon(t)} \circ g_t : \Sigma^P \rightarrow \Sigma_t^P \\ f_t &= F_t|_{\Sigma} : \Sigma \rightarrow \Sigma_t := F_t(\Sigma) \end{aligned}$$

we have that

$$\mathcal{G} \circ \Pi([\Sigma, f_t, \Sigma_t]) = (\epsilon(t), w_1(t, \cdot), \dots, w_n(t, \cdot))$$

and  $(\Sigma, f_0, \Sigma_0) = (\Sigma, \text{Id}, \Sigma)$ . It follows that  $(\Sigma, f_t, \Sigma_t)$  is a representative of  $\lambda(t)$ . We will show momentarily that  $(\Sigma, f_t, \Sigma_t)$  has the properties (1), (2), and (3). However we must first establish the existence of  $g_t$ .

Denote the Ahlfors-Weill extension of  $w_i(t, z)$  by  $\hat{w}_i(t, z)$ . It is well-known see e.g. [12], that the Ahlfors-Weill extension is holomorphic in  $t$  for fixed  $z$  so in particular the restriction of  $t \mapsto \hat{w}_i(t, z)$  to  $|z| \leq R_i$  is a holomorphic motion. By the extended lambda lemma [23], denoting by  $W$  the region enclosed by  $\gamma_i$ , there is a holomorphic motion  $H_i : \Delta \times W \rightarrow W$  for some disc  $\Delta$  centred on 0, which equals  $\hat{w}_i(t, z)$  on  $|z| \leq R$  and equals the identity on  $\gamma_i$ . Setting

$$g_t(p) = \begin{cases} \zeta_i^{-1} \circ H_i(t, \zeta_i(p)) & p \in \Sigma \setminus \bar{Y} \\ p & p \in \bar{Y} \end{cases}$$

we have that  $g_t$  satisfies properties (a) through (d).

Next, we show that  $(\Sigma, f_t, \Sigma_t)$  has the claimed properties. The uniform  $L^2$  bound can be established easily as follows. Fix  $r_i$  such that  $0 < r_i < R_i$  and let  $\Gamma_i = \zeta_i^{-1}(|z| = r_i)$ . Denote by  $V_i$  the collar neighbourhood bounded by  $\partial_i \Sigma$  and  $\Gamma_i$ . The Beltrami differential of  $g_t$  satisfies  $|\mu(g_t)(\zeta_i^{-1}(z))| = |\mu(\zeta_i \circ g_t \circ \zeta_i^{-1})(z)| = |m_t^i(1/\bar{z})|$ , and we have shown that the  $L^2$  norm of  $m_t^i(1/\bar{z})$  is uniformly bounded on  $|t| < s'$  and  $z \in \mathbb{D}$ . Thus  $\|\mu(g_t)\|_{2, V_i}$  is uniformly bounded for  $|t| < s'$  for all  $i$ . Now using the fact that  $g_t$  is the identity on  $\bar{Y}$  and the Schiffer variation  $\nu_{\epsilon(t)}$  has zero Beltrami differential outside of disks  $D_k$  disjoint from  $E_i$ , we see that

$$\mu(f_t) = \mu(\nu_{\epsilon(t)} \circ g_t) = \begin{cases} \epsilon(t) d\bar{z}/dz & z \in D_1 \cup \dots \cup D_d \\ \mu(g_t) & z \in \Sigma \setminus Y \\ 0 & \text{otherwise.} \end{cases}$$

(Note that the expression on the Schiffer disks is in terms of the local parameter on each of those disks). Thus applying Lemma 3.24 with  $k = -1$ ,  $l = 1$  and  $p = 2$  and with the collar chart  $(\zeta_i|_{V_i}, V_i)$ , and using the fact that the  $L^\infty$  norm of any Beltrami differential is bounded by one, we have a uniform bound on  $\|\mu(f_t)\|_{2, \Sigma}$  for  $|t| < s'$ . This proves property (1) for  $|t| < s'$ .

We now prove the second and third claims. By assumption, each  $w_i(t, z)$  is a holomorphic curve in  $\mathcal{O}_{WP}^{\text{qc}}(\mathbb{D})$ . Therefore for any  $t_0$  in a sufficiently small neighbourhood  $|t| < s''$  of 0 there is a holomorphic function  $g_{t_0}$  on  $\mathbb{D}$  such that

$$(3.13) \quad \lim_{t \rightarrow t_0} \iint_{\mathbb{D}} (1 - |z|^2)^2 \left| \frac{\mathcal{S}(w(t, z)) - \mathcal{S}(w(t_0, z))}{t - t_0} - g_{t_0}(z) \right|^2 dA = 0.$$

For the Ahlfors-Weill extension of  $w(t, z)$  with dilatation (3.11), setting

$$\omega_{t_0}^i(1/\bar{z}) = -\frac{(1 - |z|^2)^2 z^2}{2 \bar{z}^2} \cdot g_{t_0}(z)$$

we then have that

$$(3.14) \quad \lim_{t \rightarrow t_0} \iint_{\mathbb{D}} \frac{1}{(1 - |z|^2)^2} \left| \frac{\mu_t^i(1/\bar{z}) - \mu_{t_0}^i(1/\bar{z})}{t - t_0} - \omega_{t_0}(1/\bar{z}) \right|^2 = 0.$$

Observe also that since  $A_2^2(\mathbb{D}) \hookrightarrow A_2^\infty(\mathbb{D})$  is a bounded inclusion [25], (3.13) also implies that

$$\lim_{t \rightarrow t_0} \left\| (1 - |z|^2)^2 \frac{\mathcal{S}(w_i(t, z)) - \mathcal{S}(w_i(t_0, z))}{t - t_0} - g_{t_0}(z) \right\|_\infty = 0$$

and hence again by (3.11)

$$(3.15) \quad \lim_{t \rightarrow t_0} \left\| \frac{\mu_t^i(1/\bar{z}) - \mu_{t_0}^i(1/\bar{z})}{t - t_0} - \omega_{t_0}(1/\bar{z}) \right\|_\infty = 0.$$

We first prove that  $(\Sigma, f_t, \Sigma_t)$  has property (3). We need to establish that there is a Beltrami differential  $\kappa_{t_0}$  on  $\Sigma$  such that

$$(3.16) \quad \lim_{t \rightarrow t_0} \left\| \frac{\mu(g_t) - \mu(g_{t_0})}{t - t_0} - \kappa_{t_0} \right\|_\infty = 0$$

holds almost everywhere. It is enough to show the existence of such a Beltrami differential on each portion of the Riemann surface individually. Using (3.15) on  $1 < |z| < R_i$ , and lifting  $\omega_{t_0}$  to  $\Sigma$  via  $\zeta_i^{-1}$  establishes the claim on the region bounded by  $\zeta_i^{-1}(|z| = R_i)$  and  $\partial_i \Sigma$ , since the Beltrami differential of  $\nu_{\epsilon(t)}$  is zero on this region. Property (c) of  $g_t$  establishes the claim on the region  $L_i$ , again using the fact that the Beltrami differential of  $\nu_{\epsilon(t)}$  is zero there. Finally, on  $Y$  the claim follows from the fact that the Beltrami differential of  $g_t$  is zero there, and that in coordinates the Beltrami differential of  $\nu_{\epsilon(t)}$  is just  $\epsilon(t)d\bar{z}/dz$  on the Schiffer disks  $D_k$  and zero otherwise; note that  $\epsilon(t)$  is a holomorphic function of  $t$ . This establishes property (3) for  $|t| < s''$ .

Next we establish property (2). We will again use Lemma 3.24 in the case that  $k = -1$ ,  $l = 1$  and  $p = 2$ . We again use the collar chart  $(\zeta_i|_{V_i}, V_i)$ . We then have the estimate (for some compact  $M$  and regions  $\mathbb{A}_{r'_i}$  with  $r'_i < r_i$ )

$$\begin{aligned} \left\| \frac{\mu(f_t) - \mu(f_{t_0})}{t - t_0} - \omega_{t_0} \right\|_{2, \Sigma} &\leq a \left\| \frac{\mu(f_t) - \mu(f_{t_0})}{t - t_0} - \omega_{t_0} \right\|_{\infty, M} \\ &\quad + \sum_{i=1}^n b_i \left( \iint_{\mathbb{A}_{r'_i, 1}} \lambda_{\mathbb{D}}^2(z) \left| \frac{\mu_t^i(1/\bar{z}) - \mu_{t_0}^i(1/\bar{z})}{t - t_0} - \omega_{t_0}(1/\bar{z}) \right|^2 \right)^{1/2}. \end{aligned}$$

The first term goes to zero by property (3), and the remaining terms go to zero by (3.14). This establishes property (2) on  $|t| < s''$ . Now taking  $\delta = \min(s', s'')$  proves the theorem.  $\square$

#### 4. TANGENT SPACE TO WP-CLASS TEICHMÜLLER SPACE AND THE WP METRIC

In this section, we demonstrate the convergence on refined Teichmüller space of the generalized Weil-Petersson metric.

**4.1. The tangent space to Teichmüller space.** First, we need some results on the function spaces which will serve as models of the tangent space. In all of the following,  $\Sigma$  will be a bordered Riemann surface of type  $(g, n)$ .

Consider the spaces  $L_{0,2}^2(\Sigma)$  and  $L_{0,2}^\infty(\Sigma)$  of 2-differentials. We have a well-defined mapping from these spaces into  $L_{-1,1}^2(\Sigma)$  and  $L_{-1,1}^\infty(\Sigma)$  as follows. Let  $\psi$  be a two-differential, given in local coordinates  $(\zeta, U)$  by  $\psi_U(z)d\bar{z}^2$ . Assume that the hyperbolic metric on  $\Sigma$  is given by  $\rho_U(z)^2|dz|^2$  in local coordinates. It is easily checked that the locally defined functions

$$\psi_U(z)\rho_U^{-2}(z)$$

transform under change of coordinates as a  $(-1, 1)$  differential, and hence define a global  $(-1, 1)$  differential. Denote this map from 2-differentials to  $(-1, 1)$ -differentials by  $\mathfrak{B}$ . It's not hard to check that  $\mathfrak{B}$  has an inverse (obtained by multiplying by  $\rho_U(z)^2$  in local coordinates). The following property of  $\mathfrak{B}$  is an immediate consequence of Definition 2.1 and the definition of the norm.

**Proposition 4.1.** *Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$  and let  $\mathfrak{B}$  be defined as above. For any  $p \in [1, \infty]$*

$$\mathfrak{B}(L_{0,2}^p(\Sigma)) = L_{-1,1}^p(\Sigma).$$

Furthermore,  $\|\mathfrak{B}(\alpha)\|_p = \|\alpha\|_p$  for any  $\alpha \in L_{0,2}^p(\Sigma)$ .

*Remark 4.2.* This Proposition generalizes to other  $k, l$  by dividing by other powers of the hyperbolic metric, and also to arbitrary hyperbolic surfaces. However we do not need this here.

We now define the model spaces for the tangent space.

**Definition 4.3.** Let  $\mathfrak{B}$  be as above. Let  $\overline{A_2^i(\Sigma)}$  denote the set of complex conjugates of elements of  $A_2^i(\Sigma)$  for  $i = 2, \infty$ . Let

$$\begin{aligned} H_{-1,1}(\Sigma) &= \mathfrak{B}\left(\overline{A_2^2(\Sigma)}\right) \\ \Omega_{-1,1}(\Sigma) &= \mathfrak{B}\left(\overline{A_2^\infty(\Sigma)}\right). \end{aligned}$$

Observe that  $\mathfrak{B}$  is a bounded linear isomorphism in both cases, if  $H_{-1,1}(\Sigma)$  and  $\Omega_{-1,1}(\Sigma)$  are endowed with the norms inherited from  $L_{-1,1}^2(\Sigma)$  and  $L_{-1,1}^\infty(\Sigma)$  respectively.

For example, we have that

$$\begin{aligned} H_{-1,1}(\mathbb{D}^*) &= \left\{ (1 - |z|^2)^2 \overline{\psi(z)} d\bar{z}/dz : \iint_{\mathbb{D}^*} (1 - |z|^2)^2 |\psi(z)|^2 dA < \infty \right\} \\ \Omega_{-1,1}(\mathbb{D}^*) &= \left\{ (1 - |z|^2)^2 \overline{\psi(z)} d\bar{z}/dz : \sup_{z \in \mathbb{D}^*} (1 - |z|^2)^2 |\psi(z)| < \infty \right\}. \end{aligned}$$

The space  $\Omega_{-1,1}(\Sigma)$  is well-known to be complementary to the so-called infinitesimally trivial Beltrami differentials. It was shown by Takhtajan and Teo [25] that  $H_{-1,1}(\mathbb{D}^*)$  is the tangent space to the WP-class universal Teichmüller space (although they do not use that term), and  $T_{\text{WP}}(\mathbb{D}^*)$  can be modelled by  $H_{-1,1}(\mathbb{D}^*)$ . Furthermore, the Weil-Petersson metric converges on this tangent space. We will show that this is true for  $H_{-1,1}(\Sigma)$ , if one uses the Hilbert manifold structure which the authors defined in [19].

*Remark 4.4.* Lifting to the cover  $\mathbb{D}^*$  of  $\Sigma$  and attacking this problem using differentials which are invariant under the group of deck transformations does not appear to provide any advantage. This is because the relevant lifted differentials are  $L^2$  on the fundamental domain but not on the cover  $\mathbb{D}^*$ . If the  $L^2$  norm is non-zero on any fundamental domain, then the  $L^2$  norm of the lift to  $\mathbb{D}^*$  must be infinite. Thus  $H_{-1,1}(\Sigma)$  cannot be expressed as a set of invariant differentials in  $H_{-1,1}(\mathbb{D}^*)$ . This is an unavoidable consequence of replacing the  $L^\infty$  norm with an  $L^2$  norm.

Before proceeding, we must establish some analytic results. The main result which we require is the following.

**Theorem 4.5.** *If  $\Sigma$  is a bordered Riemann surface of type  $(g, n)$ , then  $H_{-1,1}(\Sigma) \subset \Omega_{-1,1}(\Sigma)$ . Furthermore, the inclusion map is bounded.*

First, we require two lemmas.

**Lemma 4.6.** *Let  $f : \mathbb{A}_r \rightarrow \mathbb{C}$  be a holomorphic function on  $\mathbb{A}_r$ . Then for any  $t \in (1, r)$*

$$(4.1) \quad \sup_{z \in \mathbb{A}_t} (1 - |z|^2)^2 |f(z)| \leq C(r, t) \left( \iint_{\mathbb{A}_r} (1 - |z|^2)^2 |f(z)|^2 dA \right)^{1/2}$$

with

$$C(r, t) = \frac{4}{\sqrt{2\pi}} \frac{\sqrt{r^2 + t^2}}{r^2 - t^2} \frac{(1 - t^2)^2}{r^2 + 3} + 4 \frac{\sqrt{3}}{\sqrt{\pi}} t.$$

*Proof.* To prove this lemma, let  $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$  be the Laurent series of  $f$  in  $\mathbb{A}_r$ . An elementary calculation reveals that

$$(4.2) \quad \iint_{\mathbb{A}_r} |f(z)|^2 (1 - |z|^2)^2 dA = 2\pi \sum_{n=-\infty}^{\infty} |a_n|^2 I_n(r),$$

where  $I_n(r) := \frac{1}{2} \int_1^{r^2} \rho^n (1 - \rho)^2 d\rho$ . Now the Cauchy-Schwarz inequality yields

$$(4.3) \quad \begin{aligned} |f(z)| &\leq \left( \sum_{n=-\infty}^{\infty} |a_n|^2 I_n(r) \right)^{1/2} \left( \sum_{n=-\infty}^{\infty} \frac{|z|^{2n}}{I_n(r)} \right)^{1/2} \\ &= \frac{1}{\sqrt{2\pi}} \left( \iint_{\mathbb{A}_r} (1 - |z|^2)^2 |f(z)|^2 dA \right)^{1/2} \left( \sum_{n=-\infty}^{\infty} \frac{|z|^{2n}}{I_n(r)} \right)^{\frac{1}{2}}. \end{aligned}$$

Now let us estimate the quantity  $\sum_{n=-\infty}^{\infty} \frac{|z|^{2n}}{I_n(r)}$ . To this end we split the sum as follows

$$(4.4) \quad \sum_{n=0}^{\infty} \frac{|z|^{2n}}{I_n(r)} + \sum_{n=1}^{\infty} \frac{|z|^{-2n}}{I_{-n}(r)} := \mathbf{I} + \mathbf{J}.$$

We observe that since on  $\mathbb{A}_r$  we have  $1 < |z| < r$ , then  $0 < \frac{|z|^2 - r^2}{1 - r^2} < 1$  on  $\mathbb{A}_r$ . Bearing this fact in mind we proceed with the estimates of the above terms.

To estimate  $\mathbf{I}$ , take an  $s$  with  $0 < s < \frac{|z|^2 - r^2}{1 - r^2}$  and set  $r_s = s + (1 - s)r^2$ , for that choice of  $s$ . This yields that

(4.5)

$$I_n(r) \geq \frac{1}{2} \int_{r_s}^{r^2} \rho^n (\rho - 1)^2 d\rho \geq \frac{r_s^n}{2} (r_s - 1)^2 \int_{r_s}^{r^2} d\rho = \frac{r_s^n}{2} (r_s - 1)^2 (r^2 - r_s) = \frac{r_s^n}{2} s(1-s)^2 (r^2 - 1)^3.$$

Therefore since  $0 < s < \frac{|z|^2 - r^2}{1 - r^2}$  implies that  $\frac{|z|^2}{r_s} < 1$ , we have

$$(4.6) \quad \mathbf{I} \leq \frac{2}{s(1-s)^2} \frac{1}{(r^2 - 1)^3} \sum_{n=0}^{\infty} \left( \frac{|z|^2}{r_s} \right)^n = \frac{2}{s(1-s)^2} \frac{1}{(r^2 - 1)^3} \frac{r_s}{r_s - |z|^2}.$$

Now we turn to the estimate for  $\mathbf{J}$ , to this end take an  $s'$  with  $\frac{|z|^2 - r^2}{1 - r^2} < s' < 1$  and set  $r_{s'} := s' + (1 - s')r^2$  for that choice of  $s'$ . This yields that

$$(4.7) \quad I_{-n}(r) \geq \frac{1}{2} \int_1^{r_{s'}} \rho^{-n} (\rho - 1)^2 d\rho \geq \frac{r_{s'}^{-n}}{2} \int_1^{r_{s'}} (\rho - 1)^2 d\rho = \frac{r_{s'}^{-n}}{6} (1 - s')^3 (r^2 - 1)^3.$$

Hence since  $\frac{|z|^2 - r^2}{1 - r^2} < s' < 1$  implies that  $|z|^{-2} r_{s'} < 1$

$$(4.8) \quad \mathbf{J} \leq \frac{6}{(1 - s')^3} \frac{1}{(r^2 - 1)^3} \sum_{n=0}^{\infty} (|z|^{-2} r_{s'})^n = \frac{6}{(1 - s')^3} \frac{1}{(r^2 - 1)^3} \frac{|z|^2}{|z|^2 - r_{s'}}.$$

Now (4.6) and (4.8) yield that

$$(4.9) \quad (1 - |z|^2)^2 \left( \sum_{n=-\infty}^{\infty} \frac{|z|^{2n}}{I_n(r)} \right)^{\frac{1}{2}} \leq \frac{\sqrt{2}}{\sqrt{s}(1-s)} \frac{(1 - |z|^2)^2}{(r^2 - 1)^{3/2}} \frac{\sqrt{r_s}}{\sqrt{r_s - |z|^2}} + \frac{\sqrt{6}}{(1 - s')^{3/2}} \frac{(1 - |z|^2)^2}{(r^2 - 1)^{3/2}} \frac{|z|}{\sqrt{|z|^2 - r_{s'}}} := R_1 + R_2.$$

Now since the inequality (4.9) is valid for all  $s \in (0, \frac{|z|^2 - r^2}{1 - r^2})$  and  $s' \in (\frac{|z|^2 - r^2}{1 - r^2}, 1)$ , we take  $s = \frac{|z|^2 - r^2}{2(1 - r^2)}$ ,  $s' = \frac{1}{2} + \frac{|z|^2 - r^2}{2(1 - r^2)}$ . With these choices of  $s$  and  $s'$  we have,  $1 - s = \frac{|z|^2 + r^2 - 2}{2(r^2 - 1)}$ ,  $r_s = \frac{r^2 + |z|^2}{2}$ ,  $r_s - |z|^2 = \frac{r^2 - |z|^2}{2}$  and  $1 - s' = \frac{1}{2}(\frac{|z|^2 - 1}{r^2 - 1})$ ,  $r_{s'} = \frac{1}{2}(1 + |z|^2)$ ,  $|z|^2 - r_{s'} = \frac{1}{2}(|z|^2 - 1)$ .

Plugging in these values into  $R_1$  and  $R_2$  yields that  $R_1 = 4 \frac{\sqrt{r^2 + |z|^2}}{r^2 - |z|^2} \frac{(1 - |z|^2)^2}{|z|^2 + r^2 + 2}$  and  $R_2 = 4\sqrt{6}|z|$ . This and (4.3) yield for  $z \in \mathbb{A}_r$  the pointwise estimate

(4.10)

$$(1 - |z|^2)^2 |f(z)| \leq \left( \frac{4}{\sqrt{2\pi}} \frac{\sqrt{r^2 + |z|^2}}{r^2 - |z|^2} \frac{(1 - |z|^2)^2}{|z|^2 + r^2 + 2} + 4 \frac{\sqrt{3}}{\sqrt{\pi}} |z| \right) \left( \iint_{\mathbb{A}_r} (1 - |z|^2)^2 |f(z)|^2 dA \right)^{1/2}.$$

Now observing that for  $1 < |z| < t$  one has

$$\frac{4}{\sqrt{2\pi}} \frac{\sqrt{r^2 + |z|^2}}{r^2 - |z|^2} \frac{(1 - |z|^2)^2}{|z|^2 + r^2 + 2} + 4 \frac{\sqrt{3}}{\sqrt{\pi}} |z| \leq \frac{4}{\sqrt{2\pi}} \frac{\sqrt{r^2 + t^2}}{r^2 - t^2} \frac{(1 - t^2)^2}{r^2 + 3} + 4 \frac{\sqrt{3}}{\sqrt{\pi}} t,$$

taking the supremum in (4.10) over all  $z \in \mathbb{A}_t$  yields the desired estimate.  $\square$

**Lemma 4.7.** *Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$  and let  $M$  be compactly contained in  $\Sigma$ . There is a constant  $D_M$  depending only on  $M$  such that for any  $\alpha \in H_{-1,1}(\Sigma)$ ,*

$$\|\alpha\|_{\infty, M, \Sigma} \leq D_M \|\alpha\|_{2, \Sigma}.$$

*Proof.* Since  $M$  is compact in  $\Sigma$ , there is a compact subset  $N$  of  $\Sigma$  such that  $M$  is a subset of the interior of  $N$ . There is a finite collection of open sets  $W_k$  and  $V_k$ ,  $k = 1, \dots, m$  such that

- (1)  $\overline{W_k} \subseteq V_k \subseteq N$  for  $k = 1, \dots, m$  where  $\overline{W_k}$  denotes the closure of  $W_k$ ,
- (2) There are coordinate charts  $\eta_k : V_k \rightarrow G_k$ , where  $G_k$  are bounded, open connected subsets of  $\mathbb{C}$ , and
- (3)  $M \subseteq \cup_{k=1}^m W_k$ .

Since  $N$  is a compact subset of  $\Sigma$ , and there are only finitely many charts  $(\eta_k, V_k)$ , there is a constant  $C > 0$  such that if  $\rho_{V_k}$  denotes the hyperbolic metric in local coordinates then

$$(4.11) \quad \frac{1}{C} \leq \rho_{V_k}(z) \leq C$$

for each  $k = 1, \dots, m$ .

Now let  $\alpha \in H_{-1,1}(\Sigma)$ . There is a  $\psi \in A_2^2(\Sigma)$  such that in  $\eta_k$  coordinates,  $\alpha$  has the form  $\rho_{V_k}(z)^{-2} \overline{\psi_{V_k}(z)} d\bar{z}/dz$  where  $\psi_{V_k}(z) dz^2$  is the expression for  $\psi$  in  $\eta_k$  coordinates. For all  $z \in W_k$ , we have by an elementary estimate that there is a constant  $E_k$ , which is independent of  $\psi$  (depending only on  $\eta_k(W_k)$  and  $\eta_k(V_k)$ ) such that

$$|\psi_{V_k}(z)| \leq E_k \left( \iint_{\eta_k(V_k)} |\psi_{V_k}(z)|^2 dA \right)^{1/2}$$

where  $dA$  denotes the area element  $d\bar{z} \wedge dz/2i$ . This can be obtained by applying the mean value property of holomorphic functions. Thus, applying (4.11) twice, we see that

$$\begin{aligned} \rho_{V_k}^{-2}(z) |\psi_{V_k}(z)| &\leq C^2 |\psi_{V_k}(z)| \\ &\leq C^2 E_k \left( \iint_{\eta_k(V_k)} |\psi_{V_k}(z)|^2 dA \right)^{1/2} \\ &\leq C^3 E_k \left( \iint_{\eta_k(V_k)} \rho_{V_k}(z)^{-2} |\psi_{V_k}(z)|^2 dA \right)^{1/2} \\ &\leq C^3 E_k \|\alpha\|_{2, \Sigma}. \end{aligned}$$

Since  $W_k$  cover  $M$ , taking

$$D_M = C^3 \max\{E_1, \dots, E_m\}$$

the claim is proven. □

We will also need the following Lemma proven in [20].

**Lemma 4.8.** *Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$  and let  $(\zeta_i, U_i)$  be a collar chart of  $\partial_i \Sigma$ . Let  $\lambda_{\mathbb{D}}(z) = 1/(1 - |z|^2)$  and let  $\rho_{U_i}^2 |dz|^2$  be the expression for the hyperbolic metric in  $(\zeta_i, U_i)$  coordinates. There is an annulus  $\mathbb{A}_{r,1} \subseteq \zeta_i(U_i)$  with  $\mathbb{A}_{r,1} := \{z; r < |z| < 1\}$  such that*

$$\frac{1}{K} \leq \left| \frac{\rho_{U_i}(z)}{\lambda_{\mathbb{D}}(z)} \right| \leq K$$

for all  $z \in \mathbb{A}_{r,1}$ .

We may now proceed with the proof of the Theorem.

*Proof.* (of Theorem 4.5). Choose collar charts  $(\zeta_i, U_i)$  and annuli  $\mathbb{A}_{r_i}$  satisfying the conclusion of Lemma 4.8 with constants  $K_i$ . Since there are only finitely many boundary curves we may assume that  $K_i = K$  for some  $K$  for all  $i$ . Choose  $s_i$  such that  $1 < s_i < r_i$  for each  $i$ , and let  $M$  be the subset of  $\Sigma$  given by

$$M = \Sigma \setminus \cup_{i=1}^n \{\zeta_i^{-1}(\mathbb{A}_{s_i})\}.$$

Clearly  $M$  is compactly contained in  $\Sigma$ .

Let  $\alpha \in H_{-1,1}(\Sigma)$ . Applying Lemma 4.7 we have that

$$\|\alpha\|_{\infty, M, \Sigma} \leq D_M \|\alpha\|_{2, \Sigma}$$

where  $D$  depends only on  $M$ . On the other hand, given  $\alpha$  by definition there is a  $\psi \in A_2^2(\Sigma)$  such that  $\alpha$  locally has the expression  $\rho_U(z)^{-2} \overline{\psi_U(z)}$ , where  $\psi_U$  is the local expression for  $\psi$ . In particular  $\alpha$  has the form  $\rho_{U_i}(z)^{-2} \overline{\psi_{U_i}(z)}$  in  $\zeta_i$  coordinates.

For each  $i = 1, \dots, n$ , choose  $t_i$  such that  $1 < t_i < s_i$ . By Lemmas 4.6 and 4.8, we have the estimate

$$\begin{aligned} \sup_{z \in \mathbb{A}_{t_i}} \rho_{U_i}(z)^{-2} |\psi_{U_i}(z)| &\leq K^2 \sup_{z \in \mathbb{A}_{t_i}} \lambda_{\mathbb{D}}(z)^{-2} |\psi_{U_i}(z)| \\ &\leq K^2 C(r_i, t_i) \left( \iint_{\mathbb{A}_{r_i}} \lambda_{\mathbb{D}}(z)^{-2} |\psi_{U_i}(z)|^2 dA \right)^{1/2} \\ &\leq K^3 C(r_i, t_i) \left( \iint_{\mathbb{A}_{r_i}} \rho_{U_i}(z)^{-2} |\psi_{U_i}(z)|^2 dA \right)^{1/2} \\ &\leq K^3 C(r_i, t_i) \|\alpha\|_{2, \Sigma}. \end{aligned}$$

If we let  $C = \max\{D_M, K^3 C(r_1, t_1), \dots, K^3 C(r_n, t_n)\}$  we thus have proven that

$$\|\alpha\|_{\infty, \Sigma} \leq C \|\alpha\|_{2, \Sigma}.$$

□

Next, we observe an immediate but important corollary.

**Corollary 4.9.** *Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$ .*

$$\Omega_{-1,1}(\Sigma) \cap \text{TBD}(\Sigma) = H_{-1,1}(\Sigma).$$

The remainder of this section is dedicated to showing that  $H_{-1,1}(\Sigma)$  is complementary to the kernel of the restriction of the Bers embedding to  $\text{TBD}(\Sigma)$ . In the next section, we will show that it is a model for the tangent space to WP-class Teichmüller space.

First, we will recapitulate some of the known facts regarding the kernel of the Bers embedding in the case of the standard Teichmüller space. One has the standard decomposition

$$L_{-1,1}^{\infty}(\mathbb{D}^*) = \mathcal{N}(\mathbb{D}^*) \oplus \Omega^{-1,1}(\mathbb{D}^*)$$

where

$$\mathcal{N}(\mathbb{D}^*) = \left\{ \mu \in L_{-1,1}^{\infty}(\mathbb{D}^*) : \iint_{\mathbb{D}^*} \mu \phi = 0 \quad \forall \phi \in A_2^1(\mathbb{D}^*) \right\}$$

is the linear space of “infinitesimally trivial” Beltrami differentials, which are by classical results precisely the kernel of the derivative at the identity of the Bers embedding of the

universal Teichmüller space [13, Chpt 3], [12, V.7]. We now define similar spaces on the bordered Riemann surface  $\Sigma$  of type  $(g, n)$ .

$$\mathcal{N}(\Sigma) = \left\{ \mu \in L_{-1,1}^\infty(\Sigma) : \iint_{\Sigma} \mu \phi = 0 \quad \forall \phi \in A_2^1(\Sigma) \right\}.$$

The following theorem is standard, although often phrased in its equivalent form using Fuchsian groups.

**Theorem 4.10.** *Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$ . Then*

$$L_{-1,1}^\infty(\Sigma) = \mathcal{N}(\Sigma) \oplus \Omega_{-1,1}(\Sigma).$$

Furthermore  $\mathcal{N}(\Sigma)$  is the kernel of the Bers embedding.

We sketch the proof in order to establish the notation and concepts for the proof of Theorem 4.11 ahead. Full details can be found in the references. We have (up to biholomorphism) that  $\Sigma = \mathbb{D}^*/G$  for some Fuchsian group  $G$ . Define

$$L_{-1,1}^\infty(\mathbb{D}^*, G) = \left\{ \mu \in L_{-1,1}^\infty(\mathbb{D}^*) : \mu \circ g \frac{\bar{g}'}{g'} = \mu \quad \forall g \in G \right\}.$$

Also define

$$\mathcal{N}(\mathbb{D}^*, G) = \mathcal{N}(\mathbb{D}^*) \cap L_{-1,1}^\infty(\mathbb{D}^*, G)$$

and

$$\Omega_{-1,1}(\mathbb{D}^*, G) = \Omega_{-1,1}(\mathbb{D}^*) \cap L_{-1,1}^\infty(\mathbb{D}^*, G).$$

It is immediate that

$$(4.12) \quad L_{-1,1}^\infty(\mathbb{D}^*, G) = \mathcal{N}(\mathbb{D}^*, G) \oplus \Omega_{-1,1}(\mathbb{D}^*, G).$$

Clearly we may identify  $L_{-1,1}^\infty(\mathbb{D}^*, G)$  with  $L_{-1,1}^\infty(\Sigma)$  and  $\Omega_{-1,1}(\mathbb{D}^*, G)$  with  $\Omega_{-1,1}(\Sigma)$ . We must show that  $\mathcal{N}(\mathbb{D}^*, G)$  can be identified with  $\mathcal{N}(\Sigma)$ , and that  $\mathcal{N}(\Sigma)$  is the kernel of the derivative of the Bers embedding at the point  $[\Sigma, \text{Id}, \Sigma]$ . This latter fact is well known: let  $F$  be a fixed fundamental domain of the group  $G$  and temporarily let

$$A_2^1(F) = \left\{ \phi(z) \in L^2(\mathbb{D}^*) : \phi \text{ holo, } \phi(g(z))g'(z)^2 dz^2 = \phi(z) \quad \forall g \in G, \text{ and } \iint_F |\phi(z)| dA < \infty \right\}$$

and temporarily let

$$N(G) = \left\{ \mu \in L_{-1,1}^\infty(\mathbb{D}^*, G) : \iint_F \mu \phi = 0 \quad \forall \phi \in A_2^1(F) \right\}.$$

It is clear that  $N(G)$  can be identified with  $\mathcal{N}(\Sigma)$ . By [12, Chapter V, Theorem 7.2],  $N(G)$  is the kernel of the derivative of the Bers embedding at the base point. It remains to show that  $\mathcal{N}(\mathbb{D}^*, G)$  can be identified with  $\mathcal{N}(\Sigma)$ .

To do this we show that  $\mathcal{N}(\mathbb{D}^*, G) = N(G)$ . Let  $\Theta : A_2^1(\mathbb{D}^*) \rightarrow A_2^1(F)$  be the Poincaré projection operator [12, V.7.3]. Let  $\mu \in \mathcal{N}(\mathbb{D}^*) \cap L_{-1,1}^\infty(\mathbb{D}^*, G)$ . Let  $\phi \in A_2^1(F)$ . Since  $\Theta$  is surjective [12, Theorem V.7.1] there is a  $\psi \in A_2^1(\mathbb{D}^*)$  such that  $\Theta(\psi) = \phi$ . By [12, Chapter V, (7.3)]

$$\iint_F \mu \phi = \iint_{\mathbb{D}^*} \mu \psi = 0,$$

so  $\mu \in N(G)$ .



Conversely assume that  $\mu \in N(G)$ . Let  $\phi \in A_2^1(\mathbb{D}^*)$ . By [12, Equation (7.3) V.7.3]

$$\iint_{\mathbb{D}^*} \mu \phi = \iint_F \mu(\Theta \phi) = 0.$$

So  $\mu \in \mathcal{N}(\mathbb{D}^*) \cap L_{-1,1}^\infty(\mathbb{D}^*, G)$ . Thus  $\mathcal{N}(\mathbb{D}^*, G) = N(G)$ .

We conclude that

$$(4.13) \quad L_{-1,1}^\infty(\Sigma) = \mathcal{N}(\Sigma) \oplus \Omega_{-1,1}(\Sigma).$$

It now follows that  $H_{-1,1}(\Sigma)$  is complementary to the infinitesimally trivial differentials. Define

$$\mathcal{N}_r(\Sigma) = \mathcal{N}(\Sigma) \cap \text{TBD}(\Sigma) = \mathcal{N}(\Sigma) \cap L_{-1,1}^2(\Sigma).$$

Thanks to Corollary 4.9 of Theorem 4.5, we have the following theorem.

**Theorem 4.11.** *Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$ .*

$$\text{TBD}(\Sigma) = \mathcal{N}_r(\Sigma) \oplus H_{-1,1}(\Sigma).$$

Furthermore,  $\mathcal{N}_r(\Sigma)$  is the intersection of the kernel of the Bers embedding at the base point with  $L_{-1,1}^2(\Sigma)$ ; that is, it is the kernel of the restriction of the Bers embedding to  $\text{BD}(\Sigma)$ .

*Proof.* The final statement follows from the above discussion. The decomposition follows from Theorem 4.10 by intersecting with  $L_{-1,1}^2(\Sigma)$  and applying Corollary 4.9.  $\square$

We also require the following classical result. Let  $G$  be a Fuchsian group acting on  $\mathbb{D}$ , and let  $F$  be a fundamental domain for  $G$ . (We choose the cover  $\mathbb{D}$  rather than  $\mathbb{D}^*$  for the next few paragraphs in order to be consistent with the references and avoid minor convergence issues). For  $(-1, 1)$  differentials  $\nu$ -invariant under  $G$ , define the integral map

$$(4.14) \quad \mathbf{K}(\nu)(z) = \frac{3}{\pi} (1 - |z|^2)^2 \iint_{\mathbb{D}} \frac{1}{(1 - \bar{\zeta}z)^4} \overline{\nu(\zeta)} dA_\zeta.$$

We have not yet addressed convergence. We claim that

$$(4.15) \quad \mathbf{K} : L_{-1,1}^\infty(\mathbb{D}, G) \longrightarrow \Omega_{-1,1}(F)$$

and

$$(4.16) \quad \mathbf{K} : L_{-1,1}^2(\mathbb{D}, G) \longrightarrow H_{-1,1}(F)$$

are bounded, where  $L_{-1,1}^2(\mathbb{D}, G)$  is the space of  $G$ -invariant  $(-1, 1)$  differentials such that

$$\|\nu\|_{2,F} = \iint_F \frac{|\nu(z)|^2}{(1 - |z|^2)^2} dA < \infty.$$

This can be identified with  $L_{-1,1}^2(\Sigma)$  if  $\Sigma = \mathbb{D}/G$ . This follows from [11, Lemma 3.4.9] and Proposition 4.1. Furthermore, the kernel of  $\mathbf{K}|_{L_{-1,1}^\infty(\mathbb{D}, G)}$  is just the infinitesimally trivial differentials  $\mathcal{N}(F)$ .

It is clear that these results can be written on the Riemann surface  $\Sigma$  rather than the fundamental domain. Restating the above results on  $\Sigma$ , and applying Theorem 4.11, we have the following theorem.

**Theorem 4.12.** *Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$ . There is a bounded projection  $P : L_{-1,1}^\infty(\Sigma) \rightarrow \Omega_{-1,1}(\Sigma)$  such that the restriction*

$$P|_{\text{BD}(\Sigma)} : \text{BD}(\Sigma) \rightarrow H_{-1,1}(\Sigma)$$

*is bounded with respect to the  $L_{-1,1}^2(\Sigma)$  norm. The kernel of the restriction of  $P$  to  $\text{TBD}(\Sigma)$  is  $\mathcal{N}_r(\Sigma)$ .*

*Remark 4.13.* In the last statement, we make use of the fact that the derivative of  $P$  is  $P$  itself. Note that  $P$  is linear on both spaces  $L_{-1,1}^2(\Sigma)$  and  $L_{-1,1}^\infty(\Sigma)$ .

When combined with Theorem 3.25, we get the following crucial consequence.

**Theorem 4.14.** *Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$  such that  $2g - 2 + n > 0$ . Assume that  $\mathbf{v}$  is a tangent vector to  $T_{\text{WP}}(\Sigma)$  at  $[\Sigma, \text{Id}, \Sigma]$ . There is a holomorphic curve  $t \mapsto [\Sigma, f_t, \Sigma_t]$ ,  $|t| < \delta$ , in  $T_{\text{WP}}(\Sigma)$  through  $[\Sigma, \text{Id}, \Sigma]$  at  $t = 0$  such that the Beltrami differential  $\mu_t$  of  $f_t$  is in  $H_{-1,1}(\Sigma)$  for all  $|t| < \delta$ ,  $\mu_t$  is holomorphic in  $t$ , and such that the tangent vector to this curve at  $[\Sigma, \text{Id}, \Sigma]$  is  $\mathbf{v}$ .*

*Proof.* Let  $\mathbf{v}$  be a tangent vector to  $T_{\text{WP}}(\Sigma)$  at  $[\Sigma, \text{Id}, \Sigma]$ . Let  $[\Sigma, f_t, \Sigma_t]$  be a holomorphic curve through  $[\Sigma, \text{Id}, \Sigma]$  at  $t = 0$ . By Theorem 3.25 we can assume that the Beltrami differential of  $f_t$  is in  $\text{BD}(\Sigma)$  for  $|t| < \delta$  for some  $\delta > 0$ . By Theorem 4.12 if we set  $\mu_t = P(\mu(f_t))$  the resulting Beltrami differential is in  $H_{-1,1}(\Sigma)$  for all  $|t| < \delta$ . Furthermore solving the Beltrami equation to obtain  $[\Sigma, g_t, \Sigma_t]$ , the tangent vector to  $[\Sigma, g_t, \Sigma_t]$  at  $t = 0$  must be the same as  $[\Sigma, f_t, \Sigma_t]$  by Theorem 4.11, since infinitesimally trivial differentials are in the kernel of the Bers embedding.  $\square$

**4.2.  $H_{-1,1}(\Sigma)$  model of Weil-Petersson class Teichmüller space.** In the previous section we showed that the tangent vector at the identity of every differentiable curve through the base element of the WP-class Teichmüller space is in  $H_{-1,1}(\Sigma)$ . In this section, we show that the WP-class Teichmüller space is locally biholomorphic to  $H_{-1,1}(\Sigma)$ . This is the main result of the paper and it allows us to define a finite WP-metric in Section 4.3. This also gives an alternate description of the complex structure.

We proceed as follows. First, in Theorem 4.15 below, we show that the restriction to  $H_{-1,1}(\Sigma)$  of the map  $\Phi$  into  $T_{\text{WP}}(\Sigma)$  taking Beltrami differentials to the solution of the Beltrami equation is holomorphic on some open neighborhood of 0. This result uses the Schiffer variation coordinates of Section 3.2 along with Theorem 4.5 and Lemma 4.8. Once this is established, we apply the preparation Theorem 3.25 together with the inverse function theorem to establish that in fact  $\Phi$  is a biholomorphism on some open ball. In [19] we showed that change of base point is a biholomorphism. Using this fact allows us to show in Theorem 4.17 that any point has a neighbourhood biholomorphic to a ball in  $H_{-1,1}(\Sigma)$ .

Denoting the unit ball of  $\Omega_{-1,1}(\Sigma)$  by  $\Omega_{-1,1}(\Sigma)_1$ , define the map

$$\begin{aligned} \check{\Phi} : \Omega_{-1,1}(\Sigma)_1 &\longrightarrow T(\Sigma) \\ \mu &\longmapsto [\Sigma, f_\mu, \Sigma_1]. \end{aligned}$$

where  $f_\mu : \Sigma \rightarrow \Sigma_1$  is a solution to the Beltrami equation with differential  $\mu$ . Let

$$\Phi : H_{-1,1}(\Sigma) \longrightarrow T_{\text{WP}}(\Sigma)$$

be the restriction of  $\check{\Phi}$  to  $H_{-1,1}(\Sigma)$ . Note that since  $H_{-1,1}(\Sigma) \subseteq L_{-1,1}^2(\Sigma)$ , by Definition 3.3  $\Phi$  maps into  $T_{\text{WP}}(\Sigma)$ . We will keep the distinction between  $\Phi$  and  $\check{\Phi}$ , even though  $\Phi$  is the restriction of  $\check{\Phi}$ , in order to indicate the change in norm on the domain.

**Theorem 4.15.** *Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$  such that  $2g - 2 + n > 0$ . Then there is an open neighbourhood  $B$  of 0 in  $H_{-1,1}(\Sigma)$ , such that the map  $\Phi$  is holomorphic on  $B$ .*

*Proof.* Fix a  $\tau \in \text{QS}_{\text{WP}}(\Sigma)$  (which exists by Theorem 3.12) and sew caps on  $\Sigma$  via  $\tau$  to obtain a punctured Riemann surface  $\Sigma^P$ . We assume moreover that  $\tau$  is an analytic parametrization, so that  $\tilde{\tau}_i$  has an analytic extension to an open neighbourhood of  $\overline{\mathbb{D}}$  for each  $i = 1, \dots, n$ . Since the complex structure is independent of  $\tau$  [19, Corollary 4.22], there is no loss of generality in this assumption.

Let  $\hat{B}$  be the open unit ball in  $\Omega_{-1,1}(\Sigma)$  centred at 0. Since inclusion  $\iota : H_{-1,1}(\Sigma) \rightarrow \Omega_{-1,1}(\Sigma)$  is holomorphic by Theorem 4.5, we see that  $B = \iota^{-1}(\hat{B})$  is open in  $H_{-1,1}(\Sigma)$  and contains 0. Given a  $\mu \in B$ , we let  $\hat{\mu}$  be the Beltrami differential obtained from  $\mu$  by setting it to be zero on the caps. Define a map into the Teichmüller space  $T(\Sigma^P)$  by

$$\begin{aligned} \Xi : B &\longrightarrow T(\Sigma^P) \\ \mu &\longmapsto [\Sigma^P, f_{\hat{\mu}}, \Sigma_{\hat{\mu}}^P] \end{aligned}$$

where  $\Sigma_{\hat{\mu}}^P$  and  $f_{\hat{\mu}}$  are defined as in expressions (5.1) and (5.2). In particular,  $f_{\hat{\mu}} : \Sigma^P \rightarrow \Sigma_{\hat{\mu}}^P$  is a solution to the Beltrami equation on  $\Sigma^P$  with dilatation  $\hat{\mu}$ . We claim that  $\Xi$  is holomorphic. This is because it can be written as the composition of four holomorphic maps. That is,  $\Xi = \Psi \circ \text{ext} \circ i \circ \iota$  where (1) inclusion  $\iota : H_{-1,1}(\Sigma) \rightarrow \Omega_{-1,1}(\Sigma)$  is holomorphic by Theorem 4.5; (2) the inclusion  $i : \Omega_{-1,1}(\Sigma) \rightarrow L_{-1,1}^\infty(\Sigma)$  is obviously holomorphic; (3)  $\text{ext} : L_{-1,1}^\infty(\Sigma) \mapsto L_{-1,1}^\infty(\Sigma^P)$  is holomorphic since by direct computation it is Gâteaux holomorphic and locally (in fact, globally) bounded; finally (4) the solution to the Beltrami equation  $\Psi : L_{-1,1}^\infty(\Sigma^P) \rightarrow T(\Sigma^P)$  is holomorphic (see [13]). Observe that  $\Xi = \mathcal{F} \circ \Pi \circ \Phi$ ; however note that this factorization was not necessary in the forgoing proof of holomorphicity of  $\Xi$ .

A word on the proof may be helpful. In order to write  $\Phi$  in coordinates  $\mathcal{G} \circ \Phi$ , we must write points in the image of  $\Pi \circ \Phi$  as elements of  $F(U, S, \Omega)$ . However, a given point in the image of  $\Pi \circ \Phi$  will not be of the Schiffer variation form  $(\Sigma^P, \nu_\epsilon, \Sigma_\epsilon, \psi)$ . In order to reach this form, we need to compose by some biholomorphism  $\sigma_\epsilon$  of  $\Sigma_\epsilon^P$  and invoke the Teichmüller equivalence under homotopy. This explains the presence of some extra compositions. Furthermore, as mentioned earlier, in order to make rigorous statements regarding holomorphicity, we must treat the Schiffer variation as a marked holomorphic family. The reader should bear these two points in mind in what follows.

Fix an  $n$ -chart  $(\zeta, E)$  on  $\Sigma_1^P$  such that  $\overline{\tilde{\tau}_i(\mathbb{D})} \subseteq E_i$  for each  $i = 1, \dots, n$ . Choose an open set  $K_i \subset \zeta_i(E_i)$  containing  $\zeta_i(\overline{\tilde{\tau}_i(\mathbb{D})})$  for  $i = 1, \dots, n$  such that  $\overline{K_i}$  is compactly contained in  $\zeta_i(E_i)$ . Let  $U_i \subset \mathcal{O}_{\text{WP}}^{\text{qc}}$  be open sets chosen so that  $\overline{\phi_i(\mathbb{D})} \subseteq K_i$  for all  $\phi_i$  in  $U_i$ ,  $i = 1, \dots, n$ . This is possible by [18, Theorem 3.4]. Let  $\mathfrak{S} : \Omega \rightarrow T(\Sigma^P)$  be a Schiffer variation based at  $[\Sigma, Id, \Sigma]$  which is compatible with the  $n$ -chart, and let  $F(U, S, \Omega)$  be the corresponding open set in  $\tilde{T}_{\text{WP}}(\Sigma^P)$ . Let  $\pi : S(\Omega, D) \rightarrow \Omega$  be the marked Schiffer family corresponding to  $\mathfrak{S}$ , with strong global trivialization

$$\begin{aligned} \theta : \Omega \times \Sigma^P &\rightarrow S(\Omega, D) \\ (\epsilon, q) &\mapsto (\epsilon, \nu_\epsilon(q)), \end{aligned}$$

as defined in (3.3).

By Theorem 3.16 there is a biholomorphic map

$$\begin{aligned} \Gamma : S(\Omega, D) &\longrightarrow \pi_T^{-1}(\mathfrak{S}(\Omega)) \\ (\epsilon, p) &\longmapsto ([\Sigma^P, \nu_\epsilon, \Sigma_\epsilon], \sigma_\epsilon(p)) \end{aligned}$$

where  $\sigma_\epsilon : \Sigma_{1,\epsilon}^P \rightarrow \Sigma_{\hat{\mu}} = \pi_T^{-1}([\Sigma^P, \nu_\epsilon, \Sigma_\epsilon])$  is a biholomorphism depending holomorphically on  $\epsilon$ , and  $f_{\hat{\mu}}^{-1} \circ \sigma_\epsilon \circ \nu_{\epsilon(\mu)}$  is homotopic to the identity (see Remark 5.6). Thus  $[\Sigma^P, \nu_{\epsilon(\mu)}, \Sigma_\epsilon] = [\Sigma^P, \sigma_\epsilon \circ \nu_{\epsilon(\mu)}, \Sigma_{\hat{\mu}}^P] = [\Sigma^P, f_{\hat{\mu}}, \Sigma_{\hat{\mu}}^P]$ . By restricting  $B$  sufficiently (namely, to  $\Xi^{-1}(\mathfrak{S}(\Omega))$  - note that by holomorphicity of  $\Xi$  the inverse image of  $\mathfrak{S}(\Omega)$  is open), one obtains

$$(4.17) \quad \begin{aligned} \theta^{-1} \circ \Gamma^{-1} : \pi_T^{-1}(\Xi(B)) &\longrightarrow \Omega \times \Sigma^P \\ ([\Sigma^P, f_{\hat{\mu}}, \Sigma_{\hat{\mu}}^P], w) &\longmapsto (\epsilon(\mu), \nu_{\epsilon(\mu)}^{-1} \circ \sigma_{\epsilon(\mu)}^{-1}(w)) \end{aligned}$$

where  $\Gamma \circ \theta$  is a strong global trivialization. In particular  $\epsilon(\mu)$  is holomorphic in  $\mu$  (as a function on  $H_{-1,1}(\Sigma)$ ).

By Theorem 3.22 it suffices to show that

$$\begin{aligned} \Pi \circ \Phi : B &\longrightarrow \tilde{T}_0(\Sigma^P) \\ \mu &\longmapsto [[\Sigma^P, f_{\hat{\mu}}, \Sigma_{\hat{\mu}}^P], f_{\hat{\mu}} \circ \tilde{\tau}_1, \dots, f_{\hat{\mu}} \circ \tilde{\tau}_n] \end{aligned}$$

is a biholomorphism. Since  $\sigma_{\epsilon(\mu)}^{-1} \circ f_{\hat{\mu}}$  is homotopic to  $\nu_{\epsilon(\mu)}$  we have by the equivalence relation of Definition 3.7

$$[[\Sigma^P, f_{\hat{\mu}}, \Sigma_{\hat{\mu}}^P], f_{\hat{\mu}} \circ \tilde{\tau}_1, \dots, f_{\hat{\mu}} \circ \tilde{\tau}_n] = [[\Sigma^P, \nu_\epsilon, \Sigma_\epsilon], \sigma_\epsilon^{-1} \circ f_{\hat{\mu}} \circ \tilde{\tau}_1, \dots, \sigma_\epsilon^{-1} \circ f_{\hat{\mu}} \circ \tilde{\tau}_n].$$

Therefore, in the coordinates  $\mathcal{G}$  defined in (3.5),  $\Pi \circ \Phi$  is written

$$\begin{aligned} \mathcal{G} \circ \Pi \circ \Phi : W &\longrightarrow \mathbb{C}^d \times \mathcal{O}_{\text{WP}}^{\text{qc}} \times \dots \times \mathcal{O}_{\text{WP}}^{\text{qc}} \\ \mu &\longmapsto \left( \epsilon(\mu), \left( \zeta_1 \circ \nu_{\epsilon(\mu)}^{-1} \circ \sigma_{\epsilon(\mu)}^{-1} \circ f_{\hat{\mu}} \circ \tilde{\tau}_1, \dots, \zeta_n \circ \nu_{\epsilon(\mu)}^{-1} \circ \sigma_{\epsilon(\mu)}^{-1} \circ f_{\hat{\mu}} \circ \tilde{\tau}_n \right) \right), \end{aligned}$$

where  $d$  is the dimension of  $T(\Sigma^P)$ . It was shown above that the first component is holomorphic in  $\mu$ . Thus we only need to show that the second component is holomorphic in  $\mu$ , as a map into  $\mathcal{O}_{\text{WP}}^{\text{qc}} \times \dots \times \mathcal{O}_{\text{WP}}^{\text{qc}}$ .

To this end, it is enough to show that the second component is Gâteaux holomorphic and locally bounded (see e.g. [6]). Thus we fix  $\omega \in B$  and consider the maps

$$(4.18) \quad t \longmapsto \zeta_i \circ \nu_{\epsilon(\omega+t\mu)}^{-1} \circ \sigma_{\epsilon(\omega+t\mu)}^{-1} \circ \widehat{f_{\omega+t\mu}} \circ \tilde{\tau}_i.$$

where  $t$  is restricted to some open neighbourhood of  $0 \in \mathbb{C}$  so that  $\omega + t\mu \in B$ . Since  $\Gamma^{-1}$  is holomorphic, and  $\theta^{-1}$  is holomorphic in  $\epsilon$  for fixed  $z \in E_i$ , we can conclude that  $\nu_{\epsilon(\omega+t\mu)}^{-1} \circ \sigma_{\epsilon(\omega+t\mu)}^{-1}$  depends holomorphically on  $t$ . Furthermore  $f_{\hat{\mu}}$  depends holomorphically on  $\hat{\mu}$  (because it is a strong local trivialization for the Teichmüller curve as defined in (5.2)), and hence  $\widehat{f_{\omega+t\mu}}$  depends holomorphically on  $t$  for fixed  $z$ . Since

$$\zeta_i \circ \nu_{\epsilon(\omega+t\mu)}^{-1} \circ \sigma_{\epsilon(\omega+t\mu)}^{-1} \circ \widehat{f_{\omega+t\mu}} \circ \tilde{\tau}_i$$

is also holomorphic in  $z$  on  $\mathbb{D}$ , by Hartogs' theorem it is jointly holomorphic, and thus all  $z$  derivatives are holomorphic in  $t$ . So

$$t \mapsto (\zeta_i \circ \nu_{\epsilon(\omega+t\mu)}^{-1} \circ \sigma_{\epsilon(\omega+t\mu)}^{-1} \circ \widehat{f_{\omega+t\mu}} \circ \tilde{\tau}_i)'(0)$$

is holomorphic in  $t$ . We now need to show that

$$t \mapsto \mathcal{A} \circ \zeta_i \circ \nu_{\epsilon(\omega+t\mu)}^{-1} \circ \sigma_{\epsilon(\omega+t\mu)}^{-1} \circ \widehat{f_{\omega+t\mu}}$$

is holomorphic in  $t$  (as a map into  $A_2^1(\mathbb{D})$ ) where

$$\mathcal{A}(h) = \frac{h''(z)}{h'(z)}.$$

Let  $e_z : \mathcal{O}_{\text{WP}}^{\text{qc}} \rightarrow \mathbb{C}$  denote point evaluation at  $z \in E_i$ . Since the point evaluations are a separating set of continuous linear functionals, to show that  $\mathcal{G}$  is Gâteaux holomorphic it is enough (see [6]) to prove that

$$e_z \circ \mathcal{A} \circ \zeta_i \circ \nu_{\epsilon(\omega+t\mu)}^{-1} \circ \sigma_{\epsilon(\omega+t\mu)}^{-1} \circ \widehat{f_{\omega+t\mu}}$$

is holomorphic in  $t$  for each  $z$ , and is locally bounded. The holomorphicity in  $t$  for fixed  $z$  follows from the same argument as above.

Local boundedness will follow from the local boundedness of  $\mathcal{G} \circ \Pi \circ \Phi$ . Thus the local boundedness of  $\mathcal{G} \circ \Pi \circ \Phi$  is the only remaining step. Recall that by construction, for every  $\mu$  in  $B$ ,  $\Pi \circ \Phi(\mu) \in F(U, S, \Omega)$ ; in particular, the closure of  $\zeta_i \circ \nu_{\epsilon(\mu)}^{-1} \circ \sigma_{\epsilon(\mu)}^{-1} (f_{\tilde{\mu}} \circ \tilde{\tau}_i(\mathbb{D}))$  is in  $K_i$ . Thus, since  $\theta$  is continuous, if we further restrict  $B$  so that  $\epsilon(\mu)$  is a subset of a compact set  $\Omega' \subseteq \Omega$  containing 0, we can guarantee that  $\Gamma^{-1} \circ \pi_T^{-1} \Xi(B)$  is contained in the compact set

$$\{(\epsilon, \nu_{\epsilon}(\overline{K}_i)) : \epsilon \in \Omega'\} \subseteq S(\Omega, D).$$

This takes care of the local boundedness of  $\epsilon(\mu)$ .

Fix an analytic extension of  $\tilde{\tau}$  to a disk  $D_s = \{z : |z| < s\}$  for some  $s > 1$ . Since  $f_{\tilde{\mu}}(z)$  is a continuous function of both  $z$  and  $\mu$  (again, because it is a strong local trivialization for the Teichmüller curve), and the same thing also holds for  $\nu_{\epsilon(\mu)}^{-1} \circ \sigma_{\epsilon(\mu)}^{-1}$ , there is a uniform  $R > 1$  such that for every  $\mu \in B$  in this ball, we have that

$$\nu_{\epsilon(\mu)}^{-1} \circ \sigma_{\epsilon(\mu)}^{-1} \circ f_{\tilde{\mu}} \circ \tilde{\tau}_i(D_R) \subseteq E_i$$

where  $E_i$  is the domain of the  $n$ -chart  $(\zeta_i, E_i)$  and  $D_R = \{z : |z| < R\}$ . Fix  $1 < r < R$  and let  $F_{\tilde{\mu}}$  be any quasiconformal extension of  $\zeta_i \circ \nu_{\epsilon(\mu)}^{-1} \circ \sigma_{\epsilon(\mu)}^{-1} \circ f_{\tilde{\mu}} \circ \tilde{\tau}_i$  to  $\mathbb{C}$  agreeing with the original map on  $\{z : |z| \leq r\}$ . This quasiconformal map represents the same element of  $\mathcal{O}_{\text{WP}}^{\text{qc}}$ . The  $L^2$  norm of the extension satisfies

$$(4.19) \quad \iint_{\mathbb{D}^*} \left| \frac{\bar{\partial} F_{\tilde{\mu}}(z)}{\partial F_{\tilde{\mu}}(z)} \right|^2 \frac{1}{(1 - |z|^2)^2} dA \leq C \|\mu\|_{2, \Sigma} + \text{Hyperbolic Area}(\{z : |z| > r\} \cup \{\infty\})$$

where the first term is a bound on the dilatation in  $|z| < r$  arising from Lemma 4.8, and the second term bounds the dilatation on  $|z| > r$  using only the fact that the complex dilatation of  $\tilde{f}_{\tilde{\mu}}$  is less than one. It is clear that both terms on the right hand side are finite and bounded by a fixed constant.

Referring to (2.2) we see that we need to show that  $|F'_{\tilde{\mu}}(0)|$  and

$$\iint_{\mathbb{D}} \left| \frac{F''_{\tilde{\mu}}(z)}{F'_{\tilde{\mu}}(z)} \right|^2 dA$$

are bounded. Since  $\overline{F_{\tilde{\mu}}(\mathbb{D})}$  is a subset of  $K_i$  and in particular bounded in some disk, by the Schwarz lemma  $|F'_{\tilde{\mu}}(0)| \leq M$  for some fixed  $M > 0$ . By Lemma 3.23, we have for some  $\delta > 0$

and  $|t| < \delta$ ,

$$\iint_{\mathbb{D}} \left| \frac{F''_{\hat{\mu}}(z)}{F'_{\hat{\mu}}(z)} \right|^2 dA \approx \iint_{\mathbb{D}} |SF_{\hat{\mu}}(z)|^2 (1 - |z|^2)^2 dA + \left| \frac{F''_{\hat{\mu}}(0)}{F'_{\hat{\mu}}(0)} \right|^2$$

where  $S$  denotes the Schwarzian derivative. By the classical second Taylor coefficient estimate for a univalent map of  $\mathbb{D}$  the second term on the right hand side is bounded by  $4|F'_{\hat{\mu}}(0)| \leq 4M$ . Finally, by [8, Theorem 2] we have the estimate

$$\iint_{\mathbb{D}} |SF_{\hat{\mu}}(z)|^2 (1 - |z|^2)^2 dA \leq C \iint_{\mathbb{D}^*} \frac{|\hat{\mu}(z)|^2}{(1 - |z|^2)^2} dA.$$

(Note that in [8, Theorem 2] the roles of  $\mathbb{D}$  and  $\mathbb{D}^*$  are reversed; however the left and right side are invariant under reflection in the circle  $z \mapsto 1/z$  so it immediately implies the estimate above). Hence by (4.19)  $\Phi$  is locally bounded. This concludes the proof of the theorem.  $\square$

**Theorem 4.16.** *Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$  such that  $2g - 2 + n > 0$ . There is an open neighbourhood  $U$  of  $0 \in H_{-1,1}(\Sigma)$  on which  $\Phi$  is a biholomorphism.*

*Proof.* We will show first that  $D\Phi(0)$  is a topological isomorphism. Since we have already shown that  $\Phi$  is holomorphic, it follows that  $D\Phi(0)$  is bounded, and thus by the open mapping theorem it suffices to show that  $D\Phi(0)$  is bijective.

We first show that  $D\Phi(0)$  is surjective. Let  $\mathbf{v}$  be a tangent vector to  $T_{\text{WP}}(\Sigma)$  at  $[\Sigma, \text{Id}, \Sigma]$ . By Theorem 4.14 we may find a holomorphic curve  $\alpha(t) = [\Sigma, g_t, \Sigma_t]$  such that the Beltrami differential  $\mu_t$  of  $g_t$  is in  $H_{-1,1}(\Sigma)$ , has tangent vector  $\mathbf{v}$  at 0 and is holomorphic in  $t$ . In that case  $\alpha(t) = \Phi(\mu_t)$  and so

$$\mathbf{v} = \frac{d}{dt} \alpha|_{t=0} = D\Phi \left( \left. \frac{d\mu_t}{dt} \right|_{t=0} \right)$$

which proves the claim.

Next we show that  $D\Phi(0)$  is injective. The kernel is trivial since

$$\ker D\Phi(0) = \ker D\check{\Phi}(0) \cap H_{-1,1}(\Sigma) = \{0\}$$

since  $\ker D\check{\Phi}(0)$  is trivial by Theorem 4.10.

Since  $\Phi$  is holomorphic by Theorem 4.15, by the inverse function theorem [10]  $\Phi$  has a  $C^1$  inverse in a neighbourhood of 0. Thus there is a  $U$  on which  $\Phi$  is a biholomorphism.  $\square$

Finally, we show that  $T_{\text{WP}}(\Sigma)$  possesses an atlas of charts modelled on  $H_{-1,1}(\Sigma)$ . We require a change of base point. Let  $[\Sigma, f, \Sigma_0] \in T_{\text{WP}}(\Sigma)$ . Define the change of base point map by

$$\begin{aligned} f^* : T_{\text{WP}}(\Sigma_0) &\longrightarrow T_{\text{WP}}(\Sigma) \\ [\Sigma_0, \rho, \Sigma_1] &\longmapsto [\Sigma, \rho \circ f, \Sigma_1]. \end{aligned}$$

It was shown in [19, Section 4.3] that this map is a biholomorphism. Thus we may conclude that

**Theorem 4.17.** *Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$   $2g - 2 + n > 0$ . For each point  $[\Sigma, f, \Sigma_0]$ , there is an open neighbourhood  $B$  of 0 in  $H_{-1,1}(\Sigma)$  such that*

$$\begin{aligned} \Phi_{(\Sigma, f, \Sigma_0)} : B &\longrightarrow T_{\text{WP}}(\Sigma) \\ \mu &\longmapsto [\Sigma, f_{\mu} \circ f, f_{\mu}(\Sigma_0)] \end{aligned}$$

is a biholomorphism onto an open neighbourhood of  $[\Sigma, f, \Sigma_0]$ . In particular, the collection of such biholomorphisms forms an atlas on  $T_{\text{WP}}(\Sigma)$ .

*Proof.* This follows immediately from Theorem 4.16 and the fact that  $f^*$  is a biholomorphism.  $\square$

Note that the map  $f^*$  is independent of the choice of representative of  $[\Sigma, f, \Sigma_0]$ .

**4.3. The explicit Weil-Petersson metric.** We are now ready to define the Weil-Petersson metric on the tangent space at the identity, which is done as follows. Any pair of tangent vectors can be represented by a pair of Beltrami differentials  $\mu, \nu \in H_{-1,1}(\Sigma)$ . Let  $\mu, \nu \in H_{-1,1}(\Sigma)$  be two representatives of elements of the tangent space at the identity of the refined Teichmüller space of  $\Sigma^B$ . For coordinates on an open set  $U$  containing the set  $W$  we define the local integral as in Section 2.1. Assuming that  $\mu = \mu_U d\bar{z}/dz$  and  $\nu = \nu_U d\bar{z}/dz$  in local coordinates, if  $W$  is a measurable set contained in  $U$  we can define the integral

$$\iint_W \mu_U(z) \bar{\nu}_U(z) \rho_U(z)^2 dA$$

where  $\rho_U(z)^2 |dz|^2$  is the local expression for the hyperbolic metric on  $\Sigma$  in the chart  $U$ . It is easily checked that this expression is invariant under change of coordinates. Using a partition of unity subordinate to a cover we can extend this to an integral over  $\Sigma$ , which we will denote by

$$(4.20) \quad \langle \mu, \nu \rangle_{[\Sigma, \text{Id}, \Sigma]} = \iint_{\Sigma} \mu \bar{\nu} dA_{\Sigma}$$

where  $dA_{\Sigma}$  is the hyperbolic area measure on  $\Sigma$ .

One may also represent tangent space elements as lying in the space of quadratic differentials  $A_2^2(\Sigma)$ ; that is for quadratic differentials  $\alpha, \beta \in A_2^2(\Sigma)$  given by

$$\alpha = \mathfrak{B}^{-1}(\mu) \quad \beta = \mathfrak{B}^{-1}(\nu)$$

we can define the integral

$$\langle \alpha, \beta \rangle_{[\Sigma, \text{Id}, \Sigma]} = \langle \mathfrak{B}(\alpha), \mathfrak{B}(\beta) \rangle_{[\Sigma, \text{Id}, \Sigma]}.$$

Finally, we observe that by changing the base point using  $f^*$ , we may define the Weil-Petersson metric at arbitrary points as follows. For a change of base point map  $f^*$  denote its derivative by  $Df^*$ .

**Definition 4.18.** Let  $\Sigma$  be a bordered Riemann surface of type  $(g, n)$  and let  $[\Sigma, f, \Sigma_0] \in T_{\text{WP}}(\Sigma)$ . For  $\mathbf{v}, \mathbf{w} \in T_{[\Sigma, f, \Sigma_0]} T_{\text{WP}}(\Sigma)$  define the generalized Weil-Petersson metric by

$$\langle \mathbf{v}, \mathbf{w} \rangle_{[\Sigma, f, \Sigma_0]} = \langle D(f^{-1})^* \mathbf{v}, D(f^{-1})^* \mathbf{w} \rangle_{[\Sigma_0, \text{Id}, \Sigma_0]}.$$

Since  $f^*$  is independent of the representative  $[\Sigma, f, \Sigma_0]$ , this is well-defined.

One may define a similar expression in terms of the quadratic differentials.

*Remark 4.19.* There is no transitive group action on  $T_{\text{WP}}(\Sigma)$  to make use of, and we cannot lift the picture to the cover. Thus we cannot hope to say what ‘‘right invariance’’ even means, as we can in the case of the universal Teichmüller space. This is the unique metric on  $T_{\text{WP}}(\Sigma)$  which is invariant under change of base point and agrees with (4.20) for a single choice of base point.

## 5. APPENDIX: MARKED HOLOMORPHIC FAMILIES

In this appendix we give a very brief account of the aspects of the theory of marked holomorphic families and the universality of the Teichmüller curve, which we needed in Section 3.2. These results are due to Earle and Fowler [3], and we include this account only for the convenience of the reader. A full treatment appears in [3], and also in the books [9, 13].

**Definition 5.1.** A holomorphic family of complex manifolds is a pair of connected complex manifolds  $(E, B)$  together with a surjective holomorphic map  $\pi : E \rightarrow B$  such that (1)  $\pi$  is topologically a locally trivial fiber bundle, and (2)  $\pi$  is a split submersion (that is, the derivative is a surjective map whose kernel is a direct summand).

**Definition 5.2.** A *morphism of holomorphic families* from  $(E', B')$  and  $(E, B)$  is a pair of holomorphic maps  $(\alpha, \beta)$  with  $\alpha : B' \rightarrow B$  and  $\beta : E' \rightarrow E$  such that

$$\begin{array}{ccc} E' & \xrightarrow{\beta} & E \\ \pi' \downarrow & & \downarrow \pi \\ B' & \xrightarrow{\alpha} & B \end{array}$$

commutes, and for each fixed  $t \in B'$ , the restriction of  $\beta$  to the fiber  $\pi'^{-1}(t)$  is a biholomorphism onto  $\pi^{-1}(\alpha(t))$ .

Throughout,  $(E, B)$  will be a holomorphic family of Riemann surfaces; that is, each fiber  $\pi^{-1}(t)$  is a Riemann surface.

Let  $\Sigma$  be a punctured Riemann surface of type  $(g, n)$ . This fixed surface  $\Sigma$  will serve as a model of the fiber. Let  $U$  be an open subset of  $B$ .

**Definition 5.3.**

- (1) A *local trivialization* of  $\pi^{-1}(U)$  is a homeomorphism  $\theta : U \times \Sigma \rightarrow E$  such that  $\pi(\theta(t, x)) = t$  for all  $(t, x) \in U \times \Sigma$ .
- (2) A local trivialization  $\theta$  is a *strong local trivialization* if for fixed  $x \in \Sigma$ ,  $t \mapsto \theta(t, x)$  is holomorphic, and for each  $t \in U$ ,  $x \mapsto \theta(t, x)$  is a quasiconformal map from  $\Sigma$  onto  $\pi^{-1}(t)$ .
- (3)  $\theta : U \times \Sigma \rightarrow E$  and  $\theta' : U \times \Sigma \rightarrow E$  are *compatible* if and only if  $\theta'(t, x) = \theta(t, \phi(t, x))$  where for each fixed  $t$ ,  $\phi(t, x) : \Sigma \rightarrow \Sigma$  is a quasiconformal homeomorphism that is homotopic to the identity rel boundary.
- (4) A *marking*  $\mathcal{M}$  for  $\pi : E \rightarrow B$  is a set of equivalence classes of compatible strong local trivializations that cover  $B$ .
- (5) A *marked holomorphic family of Riemann surfaces* is a holomorphic family of Riemann surfaces with a specified marking.

*Remark 5.4.* Let  $\theta$  and  $\theta'$  be compatible strong local trivializations. For each fixed  $t \in U$ ,  $[\Sigma, \theta(t, \cdot), \pi^{-1}(t)] = [\Sigma, \theta'(t, \cdot), \pi^{-1}(t)]$  in  $T(\Sigma)$ , so a marking specifies a Teichmüller equivalence class for each  $t$ .

We now define the equivalence of marked families.

**Definition 5.5.** A *morphism of marked holomorphic families* from  $\pi' : E' \rightarrow B'$  to  $\pi : E \rightarrow B$  is a pair of holomorphic maps  $(\alpha, \beta)$  with  $\beta : E' \rightarrow E$  and  $\alpha : B' \rightarrow B$  such that



- (1)  $(\alpha, \beta)$  is a morphism of holomorphic families, and
- (2) the markings  $B' \times \Sigma \rightarrow E$  given by  $\beta(\theta'(t, x))$  and  $\theta(\alpha(t), x)$  are compatible.

The second condition says that  $(\alpha, \beta)$  preserves the marking.

*Remark 5.6* (relation to Teichmüller equivalence). Define  $E = \{(s, Y_s)\}_{s \in B}$  and  $E' = \{(t, X_t)\}_{t \in B'}$  to be marked families of Riemann surfaces with markings  $\theta(s, x) = (s, g_s(x))$  and  $\theta'(t, x) = (t, f_t(x))$  respectively. Assume that  $(\alpha, \beta)$  is a morphism of marked families, and define  $\sigma_t$  by  $\beta(t, y) = (\alpha(t), \sigma_t(y))$ . Then  $\beta(\theta'(t, x)) = (\alpha(t), \sigma_t(f_t(x)))$  and  $\theta(\alpha(t), x) = (\alpha(t), g_{\alpha(t)}(x))$ . The condition that  $(\alpha, \beta)$  is a morphism of marked families is simply that  $\sigma_t \circ f_t$  is homotopic rel boundary to  $g_{\alpha(t)}$ . That is, when  $s = \alpha(t)$ ,  $[\Sigma, f_t, X_t] = [\Sigma, g_s, Y_s]$  via the biholomorphism  $\sigma_t : X_t \rightarrow Y_s$ .

Let  $\mathbb{H}$  be the upper-half plane and  $G$  be a Fuchsian group such that  $\Sigma = \mathbb{H}/G$  is a punctured Riemann surface (thus  $2g - 2 + n > 0$ ). Let  $T(G)$  be the “ $\mu$ -model” of the Teichmüller space of  $\Sigma$ . Let  $\mu$  be a Beltrami differential on  $\Sigma$  and let  $\tilde{\mu}$  be the lift of  $\mu$  to  $\mathbb{H}$  and extended by 0 to the lower half plane. Let  $w^\mu$  be the normalized solution of the Beltrami equation on  $\mathbb{C}$  with dilatation  $\tilde{\mu}$ . The *Bers fiber space* is the subset  $F(G) \subset T(G) \times \mathbb{C}$  given by

$$F(G) = \{([\mu], z) \mid [\mu] \in T(G), z \in w^\mu(\mathbb{H})\}.$$

Let  $\mathcal{T}(G) = F(G)/G$ . The group  $G$  acts freely and properly discontinuously by biholomorphisms on  $F(G)$  and the quotient is a marked holomorphic family. We will define the marking below.

**Definition 5.7.** The marked holomorphic family of Riemann surfaces

$$\begin{aligned} \pi_T : \mathcal{T}(G) &\longrightarrow T(G) \\ ([\mu], z) &\longmapsto [\mu] \end{aligned}$$

is called the *Teichmüller curve*.

Let  $G^\mu = w^\mu \circ G \circ (w^\mu)^{-1}$ . The fiber above a point  $[\mu] \in T(G)$  is the canonical Riemann surface

$$(5.1) \quad \Sigma_\mu = w^\mu(\mathbb{H})/G^\mu$$

which is independent of the Teichmüller equivalence class representative. The map  $w^\mu$  uniquely defines a map

$$(5.2) \quad f_\mu : \Sigma \rightarrow \Sigma_\mu.$$

Note that while the boundary values of these maps are independent of the Teichmüller equivalence class representative, the maps themselves are not.

Let  $L_{-1,1}^\infty(\Sigma)_1$  denote the open unit ball in  $L_{-1,1}^\infty(\Sigma)$  and let  $U$  be an open subset of  $T(\Sigma)$  for which a holomorphic section of the fundamental projection  $L_{-1,1}^\infty(\Sigma)_1 \rightarrow T(\Sigma)$  exists. The strong local trivialization

$$\theta : U \rightarrow \pi_T^{-1}(U)$$

is defined by  $\theta([\mu], z) = f_\mu(z)$

The following universal property of  $\mathcal{T}(\Sigma)$  (see [3, 9, 13]) is all that we need for our purposes.

**Theorem 5.8** (Universality of the Teichmüller curve). *Let  $\pi : E \rightarrow B$  be a marked holomorphic family of Riemann surfaces with fibre model  $\Sigma$  of type  $(g, n)$  with  $2g - 2 + n > 0$ , and trivialization  $\theta$ . Then there exists a unique morphism  $(\alpha, \beta)$  of marked families from  $\pi : E \rightarrow B$  to  $\pi_T : \mathcal{T}(\Sigma) \rightarrow T(\Sigma)$ . Moreover, the canonical “classifying” map  $\alpha : B \rightarrow T(\Sigma)$  is given by  $\alpha(t) = [\Sigma, \theta(t, \cdot), \pi^{-1}(t)]$ .*

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