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Investigating global convective dynamos with mean-field models: full spectrum of turbulent effects required

Jörn Warnecke,1 Matthias Rheinhardt,2 Mariangela Viviani,3,1 Frederick Gent,2,4 Simo Tuomisto,5 and Maarit J. Käpylä2,1,5

1Max-Planck-Institut für Sonnensystemforschung, Justus-von-Liebig-Weg 3, D-37077 Göttingen, Germany
2Department of Computer Science, Aalto University, PO Box 15400, FI-00076 Espoo, Finland
3Department of Physics, University of Calabria, I-87036, Rende (CS), Italy
4School of Mathematics, Statistics and Physics, Newcastle University, NE1 7RU, UK
5Nordita, KTH Royal Institute of Technology & Stockholm University, Hannes Alfvén’s väg 12, SE-11419, Sweden

ABSTRACT

Turbulent effects are argued to be essential for dynamos in the Sun and stars. While inaccessible observationally, they can be directly studied using global convective dynamo (GCD) simulations. We measure these turbulent effects, in terms of turbulent transport coefficients, from an exemplary GCD simulation using the test-field method. These coefficients are then used as an input into a mean-field (MF) model. We find a good agreement between the MF and GCD solutions, which validates our theoretical approach. This agreement requires all turbulent effects to be included, even those which have been regarded unimportant so far. Our results suggest that simple dynamo models as are commonly used in the solar and stellar community, relying on very few, precisely fine-tuned turbulent effects, may not be representative of the full dynamics of GCDs in astronomical objects.

Keywords: Magnetohydrodynamics (1964), Solar dynamo (2001), Solar cycle (1487), Stellar activity (1580), Stellar magnetic fields (1610)

1. INTRODUCTION

The magnetic fields of the Sun and other cool stars are generated by a dynamo mechanism operating in their interiors. Despite plentiful observations over a long time span, including at high resolution, the nature of the solar dynamo is not yet fully understood. One of the difficulties lies in the poor knowledge of the turbulent effects which are expected to play an important part in the magnetic field generation. Hence, it is common in the solar context to simplify these effects such that they can be fine-tuned to fit some of the magnetic field observations (Karak et al. 2014). As an alternative approach, global convective dynamo (GCD) models can be used to self-consistently generate these turbulent effects. While these models have parameters differing far from real astrophysical objects, they currently represent the best laboratories to this end. In the recent years the test-field method (TFM) has become a well-established tool to measure the turbulent transport coefficients (TTC), quantifying the turbulent effects, in such convective dynamo models (Schrinner et al. 2005, 2007, 2011, 2012; Schrinner 2011; Warnecke et al. 2018; Warnecke 2018; Viviani et al. 2019; Warnecke & Käpylä 2020). Already in these studies, it was found that the turbulent effects play an important part in the magnetic field evolution. However, to show that the TTC measured by the TFM capture the most important details of the magnetic field evolution, the coefficients need to be employed in a mean-field (MF) model and compared with the GCD simulations. Furthermore, only with the use of a MF model, one will be able to pin-point, which of the turbulent effects are essential for the nature of the dynamo.

In this work, we use in a MF model the TTC of an exemplary GCD model the magnetic field evolution of which shows similarities to the Sun, to investigate whether or not this evolution can be reproduced. Furthermore, we will investigate which minimal set of the coefficients are essential to reproduce the dynamo solution. We will conclude by discussing the further implications of the results.

2. MODELS AND METHODS

We analyze Run M5 of Warnecke (2018) and Warnecke & Käpylä (2020), a GCD simulation in a spherical shell. It has a rotation rate roughly four times
higher than the Sun, in terms of the Coriolis number\(^1\) and a Rayleigh number two orders of magnitude larger than the critical one for the onset of convection (Warnecke et al. 2018). The axisymmetric (azimuthally averaged) part of the generated magnetic field shows rather regular oscillations with a magnetic cycle period \(P_{\text{cyc}} = P_{\text{GCD}}^\text{cyc} = 4.4 \pm 0.6\) years (Warnecke 2018), and exhibits both equatorward and poleward branches of field migration in the butterfly (time-latitude) diagram, thus capturing main solar cycle features, see Figure 1. A second, weaker dynamo mode with a much shorter period of about 0.11 years is present at low latitudes near the surface, see Figure 1 and Käpylä et al. (2016) for a detailed discussion.

Our MF approach employs azimuthal averaging, indicated by an overbar, which adheres to the Reynolds rules; fluctuating fields are indicated by primes. The MF induction equation reads

\[
\partial_t \overline{B} = \nabla \times (\overline{U} \times \overline{B} + \mathcal{E}) - \nabla \times \eta \nabla \times \overline{B},
\]

where \(\overline{U}\) and \(\overline{B}\) are mean flow and mean magnetic field, respectively, and \(\eta\) is the magnetic diffusivity. To establish the MF model, a parameterization of the mean electromotive force \(\mathcal{E} = \overline{u'} \times \overline{B}'\) in terms of the mean field itself is crucial. Employing Taylor expansion, leaving out time derivatives and restricting to first-order temporal derivatives, a commonly quoted ansatz reads (Krause & Rädler 1980)

\[
\mathcal{E} = \alpha \overline{B} + \gamma \nabla \times \overline{B} - \beta (\nabla \times \overline{B}) - \delta (\nabla \times \overline{B}) - \kappa \left( \nabla \overline{B} \right)^{(s)},
\]

where \((\nabla \overline{B})^{(s)}\) is the symmetric part of the (covariant) derivative tensor of \(\overline{B}\). The most general representation of \(\mathcal{E}\) at some position \((r,\theta)\) and time \(t\) would involve a convolution integral over a neighbourhood of \((r,\theta,t)\), thus covering non-local and memory effects. In contrast, Eq. (2) is completely instantaneous in time and only rudimentarily non-local in space.

\(\alpha\) and \(\beta\) are symmetric rank-two tensors, \(\gamma\) and \(\delta\) are vectors, and \(\kappa\) is a rank-three tensor, with symmetry \(\kappa_{ijk} = \kappa_{ikj}\). These five tensors can be associated with different turbulent effects important for the magnetic field evolution: the \(\alpha\) effect (Steenbeck et al. 1966) can lead to field amplification via helical flows, the \(\beta\) effect describes turbulent pumping of the mean magnetic field. \(\beta\) describes turbulent diffusion; and the \(\delta\), or Rädler effect (Rädler 1969), can lead to dynamo action in the presence of, e.g., \(\alpha\) effect or shear, but not alone (Brandenburg & Subramanian 2005). The physical interpretation of \(\kappa\) is yet unclear, but quite generally it may contribute to both amplification and diffusion of \(\overline{B}\). Accounting for all symmetries in Eq. (2) and \(\partial_t \overline{B}_{r,\theta,\phi} = 0\), a total of 27 coefficients must be identified to close Eq. (1). Note, that due to the axisymmetry of \(\overline{B}\), the representation Eq. (2) is non-unique. In particular, components of \(\kappa\) can be recast into components of \(\beta\). Here we have chosen a formulation of \(\beta\) and \(\kappa\) which maximises the number of vanishing entries in \(\kappa\), that is, allocates as much information on the diffusive aspects of turbulent transport as possible in \(\beta\), thus facilitating physical interpretation, see Viviani et al. (2019) for details.

To determine the required 27 coefficients we apply the quasi-kinematic TFM to the original GCD model\(^2\). The TFM, as utilized in Schrinner et al. (2005, 2007) and Warnecke et al. (2018), requires nine additional realizations of the induction equation for the fluctuating magnetic field \(B'\) to be solved simultaneously with the GCD simulation. They are obtained by replacing \(\overline{B}\) in \(\nabla \times (\overline{u'} \times \overline{B}')\) by one out of nine linearly independent test fields \(\overline{B}^{(i)}\), \(i = 1, \ldots, 9\), while the velocity \(\overline{u} = \overline{U} + \overline{u'}\) is taken directly from GCD (see Schrinner et al. 2007, for details). Employing the corresponding electromotive forces \(\mathcal{E}^{(i)} = \overline{u'} \times \overline{B}^{(i)}\) (see Schrinner et al. 2007, for details). Employing the corresponding electromotive forces \(\mathcal{E}^{(i)} = \overline{u'} \times \overline{B}^{(i)}\), a uniquely solvable linear equation system for the coefficients in Eq. (2) can be formed.

In the stationary, saturated state of the GCD simulation the magnetic field acquires dynamically significant strength. Hence, the velocity and TTC derived from it are already magnetically quenched, that is, they differ from their counterparts in the non-magnetic (or kinematic) state. Consequently, the MF model of Eq. (1) is strictly speaking valid only for the mean field that is observed in the GCD simulation (Brandenburg et al. 2010).

We subject the TTC to the following preprocessing steps: firstly, we exclude the noise arising from large variations at small time scales (Warnecke et al. 2018) by averaging over their full time series; secondly, to damp spatial fluctuations, we apply a Gaussian smoothing in \(\theta\) using a kernel with a standard deviation equal to the grid spacing; thirdly, we remove negative values from the diagonal \(\beta\) components and apply a lower threshold of \(-0.77\) for \(\kappa_{\phi \phi \phi}\) to avoid unphysical local instabilities. For consistency, \(\overline{U}\) is time-averaged, too. Additionally,

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\(^{1}\) The Coriolis number is defined as \(\Omega \Delta R / \nu_{\text{rms}}\) with the overall angular frequency \(\Omega\), the volume integrated root mean square velocity \(\nu_{\text{rms}}\) and the thickness of the convective shell \(\Delta R\). For the Coriolis number of the Sun, see e.g. Saar & Brandenburg (1999).

\(^{2}\) The orginal TTC are available under http://doi.org/10.5281/zenodo.3629665
Investigating dynamos with mean-fields

Figure 1. Time-latitude (butterfly) diagrams of mean radial, $\overline{B_r}$, and azimuthal, $\overline{B_\phi}$, magnetic field from GCD (top) and the MF model (bottom) at fractional radius 0.95. The right panel shows a zoom-in to the first five years of the middle panel. The TTC are symmetrized, and $\alpha$ is scaled by 1.5, while exponential growth has been compensated for clarity. The color range is cut to make the northern hemisphere more visible. Black lines: zero contours of $\overline{B_\phi}$ from the MF model at the same time. See Appendix A for butterfly diagrams of the corresponding pure parity solutions.

we reduce the original resolution of the coefficients from $180 \times 256$ (radial × latitudinal) to $40 \times 64$ for computational efficiency.

The set of equations for GCD admits solutions with “pure” equatorial symmetries, that is, equatorially symmetric velocity, density and entropy fields, combined with an either symmetric (S) or antisymmetric (A) magnetic field. The equatorial symmetry of a field is quantified by the parity $P$, which takes values between -1 for A and +1 for S. In the GCD simulation, the parity of the magnetic field continuously varies between +1 and -1, indicating that A and S dynamo modes of similar strength are competing for dominance (Käpylä et al. 2016). Each of the measured TTC components, however, has nearly one or other pure parity (Warnecke et al. 2018). Therefore, to study the competing pure modes in isolation in the MF model we employ TTC, properly symmetrized so as to restrict their parity to, respectively, $\pm 1$.

For the MF simulations we solve Eq. (1) using the pre-processed $\overline{U}$ and TTC and the same $\eta$ as in the GCD model, while the magnetic field is initialized by a weak random seed field. All simulations were performed using the Pencil Code (Pencil Code Collaboration et al. 2021), see Tuomisto (2019) for details of its MF module.

3 A scalar $F$ is said to be symmetrized for parity 1 by $(F(r, \theta) + F(r, \pi - \theta))/2$ and for parity -1 by $(F(r, \theta) - F(r, \pi - \theta))/2$, while for a vector field the $r$ and $\phi$ components have parity the same as the field as a whole, and the $\theta$ component opposite parity.

3. RESULTS

First we show that the MF solution matches the one of the GCD model, when employing the symmetrized TTC. If the $\alpha$ tensor is scaled by a factor $f_\alpha$ between 1.40 and 1.525 we find growing oscillatory solutions, which resemble the GCD solution very well, as shown in Figure 1. Their periods of 4.5 to 4.9 years are in close agreement with $P_{\text{GCD}}^{\text{cyc}} \approx 4.4 \pm 0.6$ years (Warnecke 2018). The butterfly diagram is well reproduced, too: The poleward migrating $\overline{B_r}$ pattern has the same shape and slope as in the GCD simulation. In $\overline{B_\phi}$, the agreement of the pattern shapes is also striking as signified by the zero contours of the MF solution plotted over the GCD one. Deviations are visible at the highest latitudes where the testfield measurements are likely to be contaminated by the unphysical latitudinal boundary. Like the GCD solution, the MF one is neither purely antisymmetric nor symmetric about the equator. See Appendix A for details of the corresponding pure parity solutions of the MF model. The zoom-in to the early phases of the MF model (see lower rightmost panel of Figure 1) shows the high-frequency, poleward migrating cycle with roughly 0.11 years length at low latitudes. Field migration and period of this mode match closely those of the GCD. While it is regular in the MF, it appears incoherent in GCD, agreeing with the findings of Käpylä et al. (2016) for a very similar run. Given the absence of non-linearities, this mode becomes sub-dominant in the MF model.
We have repeated some of the MF runs with the non-symmetrized TTCs. Their slight hemispheric asymmetries are sufficient to excite a strongly asymmetric eigenmode, such that one hemisphere exhibits a magnetic field up to two orders of magnitude stronger than the other. The GCD simulation does not show such strong disparities, which we attribute to the non-linearities in the GCD providing a self-regulation mechanism: whenever significant disparity occurs, the hemisphere showing the higher dynamo efficiency would also experience a stronger back-reaction of the magnetic field on the flow (quenching). Thus, dominance of one hemisphere likely cannot persist for long. Instead, both the TTC and the mean field will stay close to a state of nearly pure parity. The MF model, being linear, cannot provide such self-regulation. Applying symmetrized coefficients, however, is sufficient to maintain consistent growth between the hemispheres.

The consistencies between MF and GCD solutions prove that in our case $E$ is well described by Eq. (2), meaning that higher-order terms, and also scale dependence and memory effect can be neglected. Another surprising aspect is that employing only the time-averaged, smoothed and downsampled TTC, that is, ignoring their large temporal and small-scale spatial variations, is not detrimental to the agreement with the GCD solution, but might explain the necessity of a moderate upscaling of $\alpha$.

For values of $f_\alpha < 1.40$, the solutions are growing with a dominantly non-oscillatory field, see Figure 2. Interestingly, the oscillation period is closest to that of the GCD model, when the growth rates of the A and S solutions are nearly identical hence enabling a mixed solution. For $f_\alpha > 1.525$, the oscillation period of the S mode is strongly reduced, closely matching that of the high-frequency mode reported for the GCD model. Oscillation periods close to $P^\text{GCD}_{\text{cyc}}$ depend only weakly on $f_\alpha$, consistent with the expected scaling of the period of an $\alpha \Omega$ dynamo wave $P_{\text{cyc}} \sim \alpha^{-0.5}$ (Parker 1955; Yoshimura 1975); see Figure 2. This agrees with earlier findings that direction and period of dynamo waves in GCD models can be well explained by the Parker-Yoshimura rule (Warnecke et al. 2014, 2018; Warnecke 2018).

Next we analyse which of the TTC and mean flow components are essential for reproducing the GCD solution. For this we perform around 2000 MF simulations with $f_\alpha = 1.5$, where we set on/off the various components of a certain TTC tensor at a time, while fixing all the others at their nominal values, to investigate the changes in the resulting MF solution. To classify the solutions, we define three different classes based on their period, see Figure 3. The first class (C for “correct”) has a period in the interval $(3.0 \ldots 6.0)$ years around $P^\text{GCD}_{\text{cyc}}$, the second (N) has dominantly non-oscillatory solutions, the third (H for “high frequency”) has oscillatory solutions, close to the near-surface high-frequency dynamo mode with $P_{\text{cyc}} = 0.11$ years. We require that the S and A solutions fall both into class C for a set of coefficients to be considered essential to reproduce the GCD solution (details in Table 1 of Appendix C).

First, we consider the effect of $\overline{U}$. As illustrated in Figure 3, only solutions with a non-zero $\overline{U}_\varphi$ are oscillatory with a period close to $P^\text{GCD}_{\text{cyc}}$; all other cases have only non-oscillatory solutions. This means that the meridional circulation $(\overline{U}_\varphi, \overline{U}_\theta)$ is not at all important, while differential rotation $(\Omega$ effect) is crucial.

All solutions with the full $\alpha$ tensor (or at best missing $\alpha\theta_\phi$) fall into class C. Most of the other solutions fall either in the N or H class. Given the prime importance of $\alpha$ and $\overline{\U}_\varphi$, we conclude that an $\alpha^2 \Omega$-type dynamo is operating in our model.

Next, focussing on $\gamma$ and $\delta$, we find that for obtaining class-C solutions, $\gamma_\varphi$ and $\gamma_\theta$ are important as without one of them only class-N or class-H solutions (additionally one with $P_{\text{cyc}} \simeq 1.8$ years) arise. Likewise, class-C solutions require $\delta_\varphi$ and $\delta_\theta$, otherwise only class-N solutions, or periods in between class C and H, are possible, see Figure 3.

The plentiful spectrum of solutions obtained by varying the selection of $\beta$ components (Figure 3) includes a large fraction that is non-physically unstable. This is because arbitrarily dropping components of $\beta$ can in general destroy its positive definiteness. Interestingly, only when at least the diagonal components plus $\beta_{\varphi\phi}$ and $\beta_{\theta\phi}$ are active, a class-C solution can be reproduced. However, $P_{\text{cyc}} = P^\text{GCD}_{\text{cyc}}$ requires the full $\beta$ tensor.

4 This refers to extremely localised rapidly growing field structures on the grid scale, typically appearing as a checkerboard pattern and being characteristic for negative diffusivity.
Figure 3. S and A solutions for all combinations of turned on components for each coefficient tensor with $P_{\text{cyc}}$ coded by colors. Grey: non-oscillatory (N), red: correct (C), and dark blue: high frequency (H) solutions. For $\kappa$, “others” includes all coefficients other than $\kappa_{\phi r}$ and $\kappa_{\theta r}$. Empty boxes or dashed vertical lines: all components are turned off. Dotted lines in colorbar: interval (3.0 . . . 6.0) years

From approximately 1500 solutions with various selections of $\kappa$ components, we find that around a third fall into class C, requiring the components $\kappa_{\theta r}$ and $\kappa_{\phi r}$ to be turned on, see Figure 3. Other solutions populate the H class or show periods of $P_{\text{cyc}} \simeq 1.7$ years. Interestingly, many solutions have growth rates larger than that of the full MF model. Thus the $\kappa$ tensor provides additional diffusion to the system. Despite the fact that $\kappa$ is often discarded, we thus find that at least two components are essential for reproducing the GCD solution.

For testing our conclusions regarding the essential tensor components, we performed an MF run with the minimal set of coefficients and found an excellent match with the GCD simulation, see Table 1 and Figure 5.

4. CONCLUSION

We find that a minimum set of TTC components, capable of reproducing the GCD solution, requires $U_{\phi}(\Omega)$, the full $\alpha$ (except $\alpha_{\theta r}$) and $\beta$ tensors, $\gamma_r$, $\gamma_\theta$, $\delta_r$, and $\delta_{\phi}$, and $\kappa_{\theta r}$, and $\kappa_{\phi r}$. This has two noteworthy implications.

Firstly, all our findings agree with previous works (Warnecke et al. 2014, 2018; Warnecke 2018; Warnecke & Käpylä 2020) insofar as we found that the oscillation period is mostly controlled by the $\alpha$ and $\Omega$ effects and not by the advection due to meridional circulation. Furthermore, they support the concepts of turbulent pumping ($\gamma$) and Rädler effect ($\delta$) to be essential for the dynamo (Warnecke et al. 2018; Gent et al. 2017; Warnecke & Käpylä 2020).

Secondly and more importantly, our work shows that most probably, all turbulent effects modelled by the tensors considered here are required in the generation of the magnetic field, and therefore are all needed to understand the dynamo operating in GCD simulations.
Our results, hence, suggest that simple models based on a handful of fine-tuned coefficients, as commonly used in the solar and stellar context, miss crucial effects at play in GCD models. Given the severely limited observability of stellar interiors, even yet in the case of the Sun, such models are currently the only laboratories with which to quantify these effects. Furthermore, to simply measure the coefficients and evaluate the relative strengths of the corresponding effects is insufficient. Even though the GCD simulations do not attain realistic parameters yet, only by examining TTCs from them within a MF model, one can determine and explain the nature of the dynamo.

Simulations have been conducted on supercomputers at GWDG, the Max Planck supercomputer at RZG in Garching, and facilities hosted by the CSC–IT Center for Science in Espoo, Finland. J.W. acknowledges funding by the Max-Planck/Princeton Center for Plasma Physics. M.V. acknowledges support from the HPC-EUROPA3 project (INFRAIA- 2016-1-730897), supported by the EC Research Innovation Action under the H2020 Programme. This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement n:o 818665 “UniSDyn”), and has been supported from the Academy of Finland Centre of Excellence ReSoLVE (project number 307411).

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APPENDIX

A. TIME-LATITUDE DIAGRAM OF PURE A AND S SOLUTIONS

We show in Figure 4 the purely symmetric (S) and antisymmetric (A) solutions of the MF model together with the GCD simulation. The solution shown in Figure 1 should be regarded as a weighted sum of these two solutions. As the MF model is linear, the weight depends on the initial condition, and is, in this sense, arbitrary. In this particular MF run, the parity, computed over all depths, is $-0.6$, indicating a larger contribution from A than the S solution. This also closely matches the DNS, where the parity switches back and forth between $-1$ and $1$ with a cycle period of around 20 years; the average parity is $-0.31 \pm 0.3$. Note that, while the dominating modes of the A and S solutions have equal growth rates, the high-frequency mode has a clearly higher growth rate in the S than in the A solution.

![Figure 4](image_url)

Figure 4. Time-latitude (butterfly) diagrams of mean radial, $\overline{B}_r$, and azimuthal, $\overline{B}_\phi$, magnetic field from the GCD (top) together with the symmetric (S) and antisymmetric (A) solutions of the MF model (middle and bottom) at fractional radius 0.95 (similar to Figure 1). The TTC are symmetrized, and $\alpha$ is scaled by 1.5, while exponential growth has been compensated for clarity.

B. TIME-LATITUDE DIAGRAM OF MINIMAL SET OF COEFFICIENTS

To show that the minimal set of coefficients, $U_\phi$, $\alpha$, $\gamma_{r,\theta}$, $\beta$, $\delta_{r,\phi}$, $\kappa_{\theta r,\phi \theta r}$, is able to reproduce the main features and the dynamo mode of the GCD simulation, we show in Figure 5 for comparison time-latitude diagrams of the GCD simulation and the MF model. Similar to the MF model including all coefficients, we find a very good agreement, most pronounced at mid latitudes.

C. SOLUTIONS CLOSE TO CYCLE PERIOD OF GCD SIMULATION

In Table 1 we show growth rates and oscillation frequencies of those solutions which are close in cycle period to the GCD simulation with $P_{\text{GCD}}^{\text{cyc}} = 4.4 \pm 0.6$ years (Class C). The solutions are labeled by their TTC selection.
Figure 5. Time-latitude (butterfly) diagrams of mean radial, $B_r$, and azimuthal, $B_\phi$, magnetic field from the GCD simulation (top) and the MF model (bottom) at fractional radius 0.95 (similar to Figure 1). The TTC are symmetrized, and $\alpha$ is scaled by 1.5, while exponential growth has been compensated for clarity. The color range is cut to make the northern hemisphere more visible.
Table 1. Solutions with cycles between 3 and 6 yrs. The symmetry indicates symmetric (S) and antisymmetric (A) solutions. \( \lambda \) is the growth rate of the volume–integrated root mean square \( \overline{B} \), \( P_{cyc} \) is the period of the dominant oscillatory mode. For each set of TTC selections (separated by horizontal lines) all other tensors are fully active. Only the last two rows show all active TTC explicitly, for the minimal set of coefficients.

<table>
<thead>
<tr>
<th>TTC selection</th>
<th>symmetry</th>
<th>( \lambda ) [1/yr]</th>
<th>( P_{cyc} ) [yr]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_\phi )</td>
<td>S</td>
<td>0.85</td>
<td>4.86</td>
</tr>
<tr>
<td>( U_\theta \ U_\phi )</td>
<td>S</td>
<td>0.99</td>
<td>5.28</td>
</tr>
<tr>
<td>( U_\phi \ U_\theta \ U_\phi )</td>
<td>S</td>
<td>0.82</td>
<td>4.29</td>
</tr>
<tr>
<td>( U_\theta \ U_\phi )</td>
<td>A</td>
<td>0.85</td>
<td>4.86</td>
</tr>
<tr>
<td>( U_\phi \ U_\theta \ U_\phi )</td>
<td>A</td>
<td>0.99</td>
<td>5.28</td>
</tr>
<tr>
<td>( U_\phi \ U_\theta U_\phi )</td>
<td>A</td>
<td>0.82</td>
<td>4.29</td>
</tr>
</tbody>
</table>

| \( \alpha_{r\theta} \alpha_{r\phi} \alpha_{\theta\phi} \alpha_{\phi\phi} \) | S     | -0.12          | 5.69             |
| \( \alpha_{rr} \alpha_{r\theta} \alpha_{r\phi} \alpha_{\theta\phi} \alpha_{\phi\phi} \) | S     | 1.29           | 4.67             |
| \( \alpha_{rr} \alpha_{r\theta} \alpha_{r\phi} \alpha_{\theta\phi} \alpha_{\phi\phi} \alpha_{\phi\phi} \) | S     | 0.92           | 4.54             |
| \( \alpha_{rr} \alpha_{r\theta} \alpha_{r\phi} \alpha_{\theta\phi} \alpha_{\phi\phi} \) | A     | 1.29           | 4.60             |

| \( \gamma_r \gamma_\theta \gamma_\phi \) | S     | 1.16           | 5.55             |
| \( \gamma_r \gamma_\theta \gamma_\phi \) | S     | 0.92           | 4.54             |
| \( \gamma_r \gamma_\theta \gamma_\phi \) | A     | 1.30           | 3.25             |
| \( \gamma_r \gamma_\theta \gamma_\phi \) | A     | 1.15           | 5.54             |

| \( \beta_{rr} \beta_{\theta\phi} \beta_{\phi\phi} \) | S     | 0.34           | 4.00             |
| \( \beta_{rr} \beta_{r\theta} \beta_{\theta\phi} \beta_{\phi\phi} \) | S     | 0.37           | 3.78             |
| \( \beta_{rr} \beta_{r\theta} \beta_{\theta\phi} \beta_{\phi\phi} \) | S     | 0.56           | 4.00             |
| \( \beta_{rr} \beta_{r\theta} \beta_{\theta\phi} \beta_{\phi\phi} \beta_{\phi\phi} \) | S     | 0.16           | 4.89             |
| \( \beta_{rr} \beta_{r\theta} \beta_{\theta\phi} \beta_{\phi\phi} \beta_{\phi\phi} \) | S     | 0.92           | 4.54             |
| \( \beta_{rr} \beta_{r\theta} \beta_{\theta\phi} \beta_{\phi\phi} \beta_{\phi\phi} \) | A     | 1.13           | 4.01             |
| \( \beta_{rr} \beta_{r\theta} \beta_{\theta\phi} \beta_{\phi\phi} \beta_{\phi\phi} \beta_{\phi\phi} \) | A     | 0.92           | 4.54             |

| \( \delta_r \delta_\phi \) | S     | 0.94           | 4.86             |
| \( \delta_r \delta_\phi \delta_\phi \) | S     | 0.92           | 4.54             |
| \( \delta_\theta \delta_\phi \) | A     | 0.50           | 5.31             |
| \( \delta_r \delta_\phi \) | A     | 0.94           | 4.86             |
| \( \delta_r \delta_\phi \delta_\phi \) | A     | 0.92           | 4.54             |

| \( U_\phi \ \alpha \gamma_r \gamma_\theta \beta \delta_r \delta_\phi \kappa_{\theta\phi} \kappa_{\phi\theta} \) | S     | 1.38           | 5.58             |
| \( U_\phi \ \alpha \gamma_r \gamma_\theta \beta \delta_r \delta_\phi \kappa_{\theta\phi} \kappa_{\phi\theta} \) | A     | 1.38           | 5.56             |