Zonta, Filippo; Iorio, Riccardo; Burby, Joshua W.; Liu, Chang; Hirvijoki, Eero

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Published in:
Physics of Plasmas

DOI:
10.1063/5.0058118

Published: 01/09/2021

Document Version
Publisher's PDF, also known as Version of record

Please cite the original version:
Zonta, F., Iorio, R., Burby, J. W., Liu, C., & Hirvijoki, E. (2021). Dispersion relation for gauge-free electromagnetic drift kinetics. *Physics of Plasmas*, *28*(9), Article 092504. https://doi.org/10.1063/5.0058118

Physics of Plasmas

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Submitted: 27 May 2021 . Accepted: 18 August 2021 . Published Online: 07 September 2021

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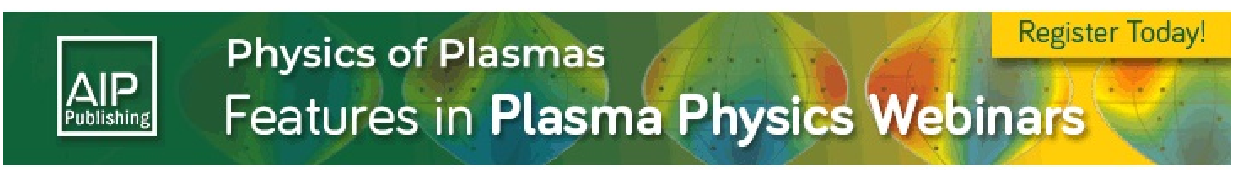

Physics of Plasmas ARTICLE scitation.org/journal/php

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ABSTRACT

Recently, a new approach to gyrokinetics, invariant under electromagnetic gauge transformations, was developed. The gyrocenter equations of motion are now expressed in terms of the perturbed fields instead of the potentials, in a form suitable for numerical simulations and analytic studies. In this paper, we verify that the long-wavelength limit, i.e., the drift-kinetic limit of the new gyrokinetic theory, is in line with existing work, providing a solid foundation for simulations. We compute the dispersion relation of the new drift-kinetic theory in slab geometry and find agreement with a long-wavelength limit of the full Vlasov–Maxwell model.

I. INTRODUCTION

The intrinsic multiscale nature of magnetically confined plasmas forces investigations of the associated kinetic dynamics to span a wide range of time and length scales. Such a formidable task has been greatly reduced by the introduction of gyrokinetic theory, which aims at removing the fast time scales while still providing an extensive description of the relevant dynamics. Without it, one would have to resolve a six-dimensional Vlasov equation coupled to the Maxwell equations. With the help of gyrokinetics, the original problem is reduced to five dimensions—three spatial coordinates, the parallel velocity, and the magnetic moment which serves as a parameter if Coulomb collisions are neglected. The nonlinear theory of gyrokinetics (see Refs. 1–3 for comprehensive reviews) focuses on low frequency electromagnetic perturbations and has been extensively used over the past decades to study plasma turbulence and transport, both theoretically and numerically.

One way to derive gyrokinetics is to make use of the Lie-transform perturbation theory to decouple the fast gyromotion from the gyrocenter motion of a single particle and to express the entire system in a variational setting, self-consistently coupling the electromagnetic potentials in the single-gyrocenter Lagrangian to the electromagnetic field Lagrangian. This method, having roots in the action principle of Low[4] for the Vlasov–Maxwell system, and pioneered in gyrokinetics by Sugama[5] and Brizard,[6] allows not only to find both the Maxwell equations and the evolution equation for particles but also to study the possible energy- and momentum-conservation laws in the system. All of the classic work in gyrokinetics utilizing the action principles, however, involves the perturbed electromagnetic potentials ϕ_1 and $\boldsymbol{A}_1$, generally leading to the loss of electromagnetic gauge invariance during the Lie-transform perturbation process. As a consequence, one either has to deal with the time derivative of the vector potential appearing in the equations of motion for a single gyrocenter or face the infamous cancelation problem in the Ampère equation.

Recently, a new gauge-free electromagnetic gyrokinetic theory has been developed[7] to counter these problems (see also Ref. 8 for the related conservation laws). The gyrocenter perturbation Lagrangian is expressed in terms of so-called gyrodisk averages of the perturbed electromagnetic fields $\boldsymbol{E}_1$ and $\boldsymbol{B}_1$ rather than in terms of the usual gyroangle averages of the potentials, ensuring gauge invariance. This new formulation is useful in many ways: It resolves the issues that have previously required the use of implicit schemes to deal with the temporal derivative of the vector potential in the particle equations of motion or lead to the cancelation problem. Second, it enables the formulation of the gyrokinetic system as a genuine infinite-dimensional ordinary differential equation in terms of three state variables—the distribution function and the electromagnetic fields—with constraints in the form of Gauss' laws, encouraging the search for structure-preserving discretization schemes analogous to the ones already developed for the full-particle Vlasov–Maxwell system;[9–16] it enables the development of hybrid kinetic–gyrokinetic or kinetic–drift-kinetic models and makes analytical comparisons with the existing theory more straightforward. Further discussions on this topic can be found, e.g., in Secs. 3 and 4 of Ref. 17.

The aim of the present paper is to provide evidence for the validity of this new gyrokinetic theory by testing it against a known result from the kinetic theory of plasmas, namely, the susceptibility tensor which describes the linear response of the system to external perturbations, commonly referred to as the dispersion relation of electromagnetic waves propagating in a plasma. In order to do this in an expository manner, we consider in the present discussion the drift-kinetic limit ($k_\perp \rho \ll 1$) of the new gyrokinetic theory, as it allows straightforward, analytical expressions for the polarization and magnetization. Given then the set of equations that describes the long-wavelength limit of the new gyrokinetic theory, we assume a slab geometry corresponding to a uniform magnetic background, linearize the system and apply Fourier analysis assuming the perturbations to behave like plane waves, and compute the susceptibility tensor explicitly. The result we obtain is exactly the same as the one reported in Ref. 18 in the long-wavelength limit with the wave vector oblique to the background magnetic field.

II. DRIFT-KINETIC EQUATIONS.

The gauge-free gyrokinetic theory and its drift-kinetic limit can be derived from a variational principle following the steps listed below:

$$\begin{gathered}\text{Gauge} - \text{free gyrocenter Lagrangian}\\ \Downarrow \\ \text{Drift} - \text{center Lagrangian}\\ \Downarrow \\ \text{Drift} - \text{center Vlasov} - \text{Maxwell Lagrangian}\\ \text{in Lagrangian frame}\\ \Downarrow \\ \text{Drift} - \text{center Vlasov} - \text{Maxwell Lagrangian}\\ \text{in the Eulerian frame}\end{gathered}$$

First, the single-particle gauge-free gyrocenter Lagrangian given in Ref. 7 is reduced to a drift-center Lagrangian by taking the limit of the Larmor radius to be small compared to the perpendicular wavelengths of the perturbed field fluctuations. As a second step, the Vlasov–Maxwell Lagrangian is constructed by adding the electromagnetic field contribution to the single-particle contribution, which includes the density function as well as the single-particle motion described by the drift-center Lagrangian. Finally, the Vlasov–Maxwell Lagrangian is written purely in Eulerian variables. In this way all the factors, rather than being evaluated at the single-particle position after the time t, are functions of fixed phase-space coordinates. Then, in order to derive the Vlasov equation, the Maxwell equations, and the evolution of the single-particle distribution function, the action is varied, bearing in mind that variations in the so-called Euler–Poincarè framework[19] are constrained to specific types. Then, the calculation of the polarization and magnetization terms is straightforward, becoming functional derivatives of the kinetic energy term with respect to the electromagnetic fields.[20] For the convenience of the reader, the more abstract steps involved in the derivation of the equations of motion in the Euler–Poincarè setting is provided in Appendix B. For more details on the Euler–Poincarè reduction for the ordinary Vlasov–Maxwell system, see, e.g., Ref. 21.

The equations describing the gyrokinetic Vlasov–Maxwell system are given by

$$\frac{\partial(J_s f_s)}{\partial t} = -\frac{\partial}{\partial z^\alpha}(J_s V_s^\alpha f_s), \tag{1}$$

$$\frac{\partial \boldsymbol{D}}{\partial t} = \nabla \times \boldsymbol{H} - \boldsymbol{j}_{\rm gy}, \tag{2}$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = -\nabla \times \boldsymbol{E}_1, \tag{3}$$

$$\nabla \cdot \boldsymbol{D} = \varrho^{\rm gy}, \tag{4}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{5}$$

where $\varrho^{\rm gy}$ and $\boldsymbol{j}_{\rm gy}$ are the charge and current densities computed from the gyrocenter location and velocity. The gyrocenter distribution function of species s is denoted by f_s, while J_s is the Jacobian of the gyrocenter phase-space volume element $d\boldsymbol{z}_s = J_s d\boldsymbol{x} dv_\parallel d\mu d\theta$, and $V_s^\alpha \equiv \dot{z}_s^\alpha$ are the components of the Hamiltonian phase-space flow, or the equations of motion for the gyrocenter coordinates z^α.

The equations of motion, or the components of the vector field $V_s = V_s^\alpha(\boldsymbol{z}, t)\partial/\partial z^\alpha$, are

$$V_s^\alpha = \{z^\alpha, K_s\}_s^{\rm gy} - \{z^\alpha, \boldsymbol{x}\}_s^{\rm gy} \cdot e_s \boldsymbol{E}_1, \tag{6}$$

where K_s is the single-gyrocenter kinetic energy and $\{\cdot,\cdot\}_s^{\rm gy}$ is the gyrocenter Poisson bracket. The bracket is a bilinear operator acting on two functions of the gyrocenter coordinates according to

$$\{f,g\}^{\rm gy} = \frac{e}{m}\left(\frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \mu} - \frac{\partial f}{\partial \mu}\frac{\partial g}{\partial \theta}\right) + \frac{\boldsymbol{B}^*}{mB_\parallel^*}\cdot\left(\nabla f \frac{\partial g}{\partial v_\parallel} - \frac{\partial f}{\partial v_\parallel}\nabla g\right) - \frac{\boldsymbol{b}_0}{eB_\parallel^*}\cdot(\nabla f \times \nabla g), \tag{7}$$

where $\boldsymbol{B}^* = \boldsymbol{B}_0 + \boldsymbol{B}_1 + e^{-1} m v_\parallel \nabla \times \boldsymbol{b}_0$ and $B_\parallel^* = \boldsymbol{B}^* \cdot \boldsymbol{b}_0$. The phase-space Jacobian is $J = mB_\parallel^*$.

In the field equations, $\boldsymbol{D}$ and $\boldsymbol{H}$ denote, respectively, the fluctuating auxiliary electric field and the total auxiliary magnetic field, which are defined in terms of the so-called gyrocenter kinetic energy functional $\mathcal{K}$ according to

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E}_1 - \frac{\delta \mathcal{K}}{\delta \boldsymbol{E}_1}, \tag{8}$$

$$\boldsymbol{H} = \mu_0^{-1}(\boldsymbol{B}_0 + \boldsymbol{B}_1) + \frac{\delta \mathcal{K}}{\delta \boldsymbol{B}_1}, \tag{9}$$

where $\delta/\delta \boldsymbol{E}_1$ and $\delta/\delta \boldsymbol{B}_1$ refer to functional derivatives with respect to the fluctuating electric and magnetic fields $\boldsymbol{E}_1$ and $\boldsymbol{B}_1$. The gyrocenter kinetic energy functional is given by

$$\mathcal{K}[\boldsymbol{E}_1, \boldsymbol{B}_1, f] = \sum_s \int f_s K_s(\boldsymbol{z}, \boldsymbol{E}_1, \boldsymbol{B}_1) d\boldsymbol{z}_s, \tag{10}$$

where $K_s(z, \boldsymbol{E}_1, \boldsymbol{B}_1)$ is the single-gyrocenter kinetic energy. The polarization and the magnetization are thus identified as

$$\boldsymbol{P} = -\frac{\delta \mathcal{K}}{\delta \boldsymbol{E}_1}, \tag{11}$$

$$\boldsymbol{M} = -\frac{\delta \mathcal{K}}{\delta \boldsymbol{B}_1}. \tag{12}$$

We limit the discussion to the drift-kinetic $k_\perp \rho \ll 1$ ordering. We have chosen this approach for practical purposes as the drift-kinetic expression for K_s then becomes a convenient local function of $\boldsymbol{E}_1$ and $\boldsymbol{B}_1$:

$$K(\boldsymbol{z}, \boldsymbol{E}_1, \boldsymbol{B}_1) = \frac{1}{2} m v_\parallel^2 + \mu B_0 \left(1 + \frac{\boldsymbol{b}_0 \cdot \boldsymbol{B}_1}{B_0} + \frac{|\boldsymbol{B}_{1\perp}|^2}{2B_0^2}\right) - \frac{m}{2B_0^2} |\boldsymbol{E}_{1\perp} + v_\parallel \boldsymbol{b}_0 \times \boldsymbol{B}_1|^2, \tag{13}$$

where $\boldsymbol{B}_{1\perp} = \mathbf{1}_\perp \cdot \boldsymbol{B}_1$, $\boldsymbol{E}_{1\perp} = \mathbf{1}_\perp \cdot \boldsymbol{E}_1$ and $\mathbf{1}_\perp = \mathbf{1} - \boldsymbol{b}_0\boldsymbol{b}_0$. The final expression of K contains the guiding center kinetic energy $K_0 = \frac{1}{2} m v_\parallel^2 + \mu B_0$ as well as the drift-kinetic limits of the gyrokinetic corrections. For details regarding the derivation of the gyrokinetic energy expression, please consult Ref. 7.

III. SUSCEPTIBILITY

Next, we provide the expression for the resulting electromagnetic plasma susceptibility tensor and verify that it matches exactly the long-wavelength limit of the general plasma susceptibility tensor. We also explicitly provide the expressions for the auxiliary fields $\boldsymbol{D}$ and $\boldsymbol{H}$. The derivation of susceptibility is documented in Appendix A.

A. Drift-kinetic auxiliary fields

The computation of the functional derivatives of the kinetic energy functional (10), needed for the auxiliary fields, is straightforward and one finds

$$\frac{\delta \mathcal{K}}{\delta \boldsymbol{E}_1} = \sum_s \int f_s \frac{\partial K_s}{\partial \boldsymbol{E}_1} J_s dv_\parallel d\mu d\theta, \tag{14}$$

$$\frac{\delta \mathcal{K}}{\delta \boldsymbol{B}_1} = \sum_s \int f_s \frac{\partial K_s}{\partial \boldsymbol{B}_1} J_s dv_\parallel d\mu d\theta, \tag{15}$$

with $J_s = m_s B_\parallel^*$. In (15), even though the phase-space Jacobian J_s depends explicitly on $\mathbf{B}_1$, it is not to be varied as explained in Appendix B.

Computing the $\boldsymbol{E}_1$ gradient of the single-gyrocenter kinetic energy, the expression for the auxiliary $\boldsymbol{D}$ becomes

$$\boldsymbol{D} = \epsilon_0 \left(\mathbf{1} + \frac{c^2}{v_A^2} \mathbf{1}_\perp\right) \cdot \boldsymbol{E}_1 + \left\langle \frac{p_\parallel}{B_0^2} \right\rangle \boldsymbol{b}_0 \times \boldsymbol{B}_1, \tag{16}$$

where we have introduced the local quantities

$$\frac{1}{v_A^2(\boldsymbol{x}, t)} = \sum_s \frac{m_s \mu_0}{B_0^2} \int f_s J_s dv_\parallel d\mu d\theta, \tag{17}$$

$$\left\langle \frac{p_\parallel}{B_0^2} \right\rangle (\boldsymbol{x}, t) = \sum_s \frac{m_s}{B_0^2} \int v_\parallel f_s J_s dv_\parallel d\mu d\theta. \tag{18}$$

Computing similarly the $\boldsymbol{B}_1$ gradient of the single-gyrocenter kinetic energy provides us the auxiliary field $\boldsymbol{H}$ that is needed in the Maxwell–Ampère equation

$$\boldsymbol{H} = \mu_0^{-1} \left(1 + \frac{\beta_\perp - \beta_\parallel}{2}\right) \boldsymbol{B}_1 + \frac{\beta_\parallel - \beta_\perp}{2\mu_0} \boldsymbol{b}_0 \boldsymbol{b}_0 \cdot \boldsymbol{B}_1 + \mu_0^{-1} \left(1 + \frac{\beta_\perp}{2}\right) \boldsymbol{B}_0 - \left\langle \frac{p_\parallel}{B_0^2} \right\rangle \boldsymbol{E}_1 \times \boldsymbol{b}_0. \tag{19}$$

Here we have defined the local quantities

$$\beta_\parallel(\boldsymbol{x}, t) = 2 \sum_s \int \frac{m_s v_\parallel^2 \mu_0}{B_0^2} f_s J_s dv_\parallel d\mu d\theta, \tag{20}$$

$$\beta_\perp(\boldsymbol{x}, t) = 2 \sum_s \int \frac{\mu B_0 \mu_0}{B_0^2} f_s J_s dv_\parallel d\mu d\theta, \tag{21}$$

denoting the parallel and perpendicular components of plasma β. The expressions for $\boldsymbol{D}$ in (16) and $\boldsymbol{H}$ in (19) are then utilized to derive a wave equation and finally the susceptibility tensor.

We also remark that, given $\boldsymbol{D}$ from (4), inverting (16) for $\boldsymbol{E}_1$ requires one to keep the "light wave" term that is usually dropped in quasineutral gyrokinetics. If the displacement current is not kept, the variational principle behind the gyrokinetic theory will not provide an equation for the parallel electric field, and one is forced to finding one via other means.

B. Susceptibility for a linear response

To derive an expression for the susceptibility tensor $\boldsymbol{\chi}$ for describing a linear response, one takes the time derivative of the Maxwell–Ampère equation (2), utilizes the expressions for $\boldsymbol{D}$ and $\boldsymbol{H}$, and replaces the time derivative of the magnetic field with the induction equation. Then, the time derivative of the total current is related to the electric field by splitting the distribution functions according to $f_s = f_{s0} + f_{s1}$, assuming $\boldsymbol{B}_0$ and f_{s0} to be constant with respect to $\boldsymbol{x}$ and t, and linearizing the Vlasov equation. Finally, the perturbation quantities $(\boldsymbol{B}_1, \boldsymbol{E}_1, f_{s1})$ are assumed to behave as plane waves and to be proportional to $\exp[i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)]$ with $\boldsymbol{k}$ the wave vector and ω the frequency, and the Maxwell–Ampère equation is transformed into the form

$$\omega^2 (\mathbf{1} + \boldsymbol{\chi}) \cdot \boldsymbol{E}_1 + c^2 \boldsymbol{k} \times \boldsymbol{k} \times \boldsymbol{E}_1 = 0. \tag{22}$$

Explicit calculation of the susceptibility tensor is provided in Appendix A and here we simply give the result,

$$\begin{aligned} \boldsymbol{\chi} = {} & \frac{1}{\epsilon_0 \omega} \left\langle \frac{p_\parallel}{B_0^2} \right\rangle_0 (\boldsymbol{k}\boldsymbol{b}_0 + \boldsymbol{b}_0\boldsymbol{k} - 2\boldsymbol{b}_0 \cdot \boldsymbol{k}\mathbf{1}) + \frac{c^2}{v_{A0}^2} \mathbf{1}_\perp \\ & - \frac{c^2}{\omega^2} \frac{\beta_{\parallel 0} - \beta_{\perp 0}}{2} (\boldsymbol{k}\boldsymbol{k} - k^2 \mathbf{1}) + \frac{I_1}{\epsilon_0 \omega} \boldsymbol{b}_0 \boldsymbol{b}_0 \\ & + \frac{c^2}{\omega^2} \left(\frac{\beta_{\parallel 0} + \beta_{\perp 0}}{2} - \frac{I_3}{c^2}\right) \boldsymbol{k} \times \boldsymbol{b}_0 \boldsymbol{b}_0 \times \boldsymbol{k} \\ & - i \frac{I_2}{\epsilon_0 \omega} (\boldsymbol{k} \times \boldsymbol{b}_0 \boldsymbol{b}_0 + \boldsymbol{b}_0 \boldsymbol{b}_0 \times \boldsymbol{k}) \\ & - i \frac{1}{\epsilon_0 \omega} \left(\frac{j_{\parallel 0}}{\omega B_0} \mathbb{k} - \frac{\varrho_0^{\rm gy}}{B_0} \mathbb{b}_0\right). \end{aligned} \tag{23}$$

In the above, the following definitions have been employed:

$$j_{\parallel 0}^{\rm gy} = \sum_s e_s \int v_\parallel f_{s0} J_0 dv_\parallel d\mu d\theta, \tag{24}$$

$$\beta_{\parallel 0} = 2 \sum_s \int \frac{m_s v_\parallel^2 \mu_0}{B_0^2} f_{s0} J_0 dv_\parallel d\mu d\theta, \tag{25}$$

$$\beta_{\perp 0} = 2 \sum_s \int \frac{\mu B_0 \mu_0}{B_0^2} f_{s0} J_0 dv_\parallel d\mu d\theta, \tag{26}$$

$$\varrho_0^{\mathrm{gy}} = \sum_s e_s \int f_{s0} J_0 dv_\| d\mu d\theta, \tag{27}$$

$$\left\langle \frac{p_\|}{B_0^2} \right\rangle_0 = \sum_s \frac{m_s}{B_0^2} \int v_\| f_{s0} J_{s0} dv_\| d\mu d\theta, \tag{28}$$

$$\frac{1}{v_{A0}^2} = \sum_s \frac{m_s \mu_0}{B_0^2} \int f_{s0} J_{s0} dv_\| d\mu d\theta, \tag{29}$$

with $J_0 = mB_0$. The integrals $I_1(\omega, \boldsymbol{k})$, $I_2(\omega, \boldsymbol{k})$ and $I_3(\omega, \boldsymbol{k})$ are defined as

$$I_1(\omega, \boldsymbol{k}) = \sum_s \frac{e_s^2}{m_s} \int \frac{v_\| \partial f_{s0}/\partial v_\|}{\omega - v_\| \boldsymbol{b}_0 \cdot \boldsymbol{k}} J_{s0} dv_\| d\mu d\theta, \tag{30}$$

$$I_2(\omega, \boldsymbol{k}) = \sum_s \frac{e_s}{m_s} \int \frac{\mu \partial f_{s0}/\partial v_\|}{\omega - v_\| \boldsymbol{b}_0 \cdot \boldsymbol{k}} J_{s0} dv_\| d\mu d\theta, \tag{31}$$

$$I_3(\omega, \boldsymbol{k}) = \sum_s \int \frac{\mu^2}{m_s} \frac{\boldsymbol{b}_0 \cdot \boldsymbol{k} \partial f_{s0}/\partial v_\|}{\omega - v_\| \boldsymbol{b}_0 \cdot \boldsymbol{k}} J_{s0} dv_\| d\mu d\theta, \tag{32}$$

and, in Cartesian coordinates, the matrix u is constructed from a vector $\boldsymbol{u}$ according to

$$\mathsf{u} = \begin{pmatrix} 0 & u_z & -u_y \\ -u_z & 0 & u_x \\ u_y & -u_x & 0 \end{pmatrix}.$$

If we choose a background magnetic field parallel to the z-direction according to $\boldsymbol{b}_0 = \hat{\boldsymbol{z}}$, and an oblique direction for the waves according to $\boldsymbol{k} = k_\perp \hat{\boldsymbol{x}} + k_\| \hat{\boldsymbol{z}}$, the susceptibility tensor $\boldsymbol{\chi}$ reduces exactly to the expression given by Hasegawa [Ref. 18, see Eq. (2.159)]. This verifies that the drift-kinetic system derived from the gauge-free gyrokinetic framework produces the correct dispersion relation relevant at the long-wavelength limit $k_\perp \rho \ll 1$.

IV. SUMMARY

A gauge-free gyrokinetic theory, in which the equations are expressed directly in terms of the electromagnetic fields instead of the potentials, has been recently developed. In this paper, we have tested the drift-kinetic limit of the theory by comparing the resulting susceptibility tensor of plane waves propagating in a homogenous plasma to the well-known results from literature.

As the resulting susceptibility tensor exactly matches the known results derived from the full plasma dispersion relation, we can expect the gauge-free drift-kinetic theory to be useful especially in constructing hybrid models that treat the reduced dynamics of electrons while retaining the full kinetic motion of ions.

ACKNOWLEDGMENTS

The work of F.Z., R.I., and E.H. was supported by the Academy of Finland Grant Nos. 320058, 316088, and 315278, respectively. The work of J.B. was supported by the Los Alamos National Laboratory LDRD program under Project No. 20180756PRD4. The work of C.L. was supported by the Office of Fusion Energy Science of U.S. Department of Energy, under Contract No. DE-AC02-09CH11466.

APPENDIX A: DERIVATION OF THE SUSCEPTIBILITY

To derive an expression for the susceptibility tensor $\boldsymbol{\chi}$, we start by taking the time derivative of the Maxwell–Ampère equation (2) and replace the time derivative of the magnetic field with the induction equation

$$\frac{\partial^2 \boldsymbol{E}_1}{\partial t^2} + c^2 \nabla \times \nabla \times \boldsymbol{E}_1 = \frac{1}{\epsilon_0} \frac{\partial^2}{\partial t^2} \frac{\delta \mathcal{K}}{\delta \boldsymbol{E}_1} + \frac{1}{\epsilon_0} \nabla \times \frac{\partial}{\partial t} \frac{\delta \mathcal{K}}{\delta \boldsymbol{B}_1} - \frac{1}{\epsilon_0} \frac{\partial \boldsymbol{j}^{\mathrm{gy}}}{\partial t}. \tag{A1}$$

The next task is to linearize the right side and to express the result in terms of $\boldsymbol{E}_1$. We introduce the split $f_s = f_{s0} + f_{s1}$, and assume $\boldsymbol{B}_0$ and f_{s0} to be constant with respect to $\boldsymbol{x}$ and t. Then the perturbation quantities $(\boldsymbol{B}_1, \boldsymbol{E}_1, f_{s1})$ are assumed to behave as plane waves and to be proportional to $\exp[i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)]$ with $\boldsymbol{k}$ the wave vector and ω the frequency. Finally, the susceptibility tensor $\boldsymbol{\chi}$ is obtained by transforming Eq. (A1) into

$$\omega^2 (\mathbf{1} + \boldsymbol{\chi}) \cdot \boldsymbol{E}_1 + c^2 \boldsymbol{k} \times \boldsymbol{k} \times \boldsymbol{E}_1 = 0. \tag{A2}$$

In a homogenous background magnetic field, the phase-space Jacobian splits into

$$J = mB_0 + m\boldsymbol{b}_0 \cdot \boldsymbol{B}_1 \equiv J_0 (1 + B_0^{-1} \boldsymbol{b}_0 \cdot \boldsymbol{B}_1). \tag{A3}$$

In addition to the expressions (24), (25), (26), (27), and (28) we need also the following quantities that will be needed shortly:

$$j_{\|1}^{\mathrm{gy}} = \sum_s e_s \int v_\| f_{s1} J_0 dv_\| d\mu d\theta, \tag{A4}$$

$$\beta_{\perp 1} = 2 \sum_s \int \frac{\mu B_0 \mu_0}{B_0^2} f_{s1} J_0 dv_\| d\mu d\theta, \tag{A5}$$

where $J_0 = mB_0$. Using the expression for $\boldsymbol{D}$ given in Eq. (16), the linearized time derivative of the polarization current needed for Eq. (A1) becomes

$$\frac{1}{\epsilon_0} \frac{\partial^2}{\partial t^2} \frac{\delta \mathcal{K}}{\delta \boldsymbol{E}_1} = \frac{1}{\epsilon_0} \left\langle \frac{p_\|}{B_0^2} \right\rangle_0 \boldsymbol{b}_0 \times \nabla \times \frac{\partial \boldsymbol{E}_1}{\partial t} - \frac{c^2 \mathbf{1}_\perp}{v_{A0}^2} \cdot \frac{\partial^2 \boldsymbol{E}_1}{\partial t^2}. \tag{A6}$$

Then using the expression for $\boldsymbol{H}$ given in Eq. (19), the linearized time derivative of the magnetization current becomes

$$\begin{aligned} \frac{1}{\epsilon_0} \nabla \times \frac{\partial}{\partial t} \frac{\delta \mathcal{K}}{\delta \boldsymbol{B}_1} = {} & c^2 \frac{\beta_{\|0} - \beta_{\perp 0}}{2} \nabla \times \nabla \times \boldsymbol{E}_1 \\ & - c^2 \frac{\beta_{\|0}}{2} \nabla \times (\boldsymbol{b}_0 \boldsymbol{b}_0 \cdot \nabla \times \boldsymbol{E}_1) \\ & - \frac{1}{\epsilon_0} \left\langle \frac{p_\|}{B_0^2} \right\rangle_0 \nabla \times \left(\frac{\partial \boldsymbol{E}_1}{\partial t} \times \boldsymbol{b}_0 \right) \\ & + \frac{c^2}{2} \nabla \frac{\partial \beta_{\perp 1}}{\partial t} \times \boldsymbol{B}_0. \end{aligned} \tag{A7}$$

For the time derivative of the gyrocenter current density, we need the gyrocenter velocity scaled with the Jacobian up to linear order. The result is

$$J\dot{\boldsymbol{x}} = J_0 v_\| \left(\boldsymbol{b}_0 + \frac{\boldsymbol{B}_1}{B_0} \right) + J_0 \frac{\boldsymbol{E}_1 \times \boldsymbol{b}_0}{B_0} + \frac{\mu J_0}{e B_0} \boldsymbol{b}_0 \times \nabla (\boldsymbol{b}_0 \cdot \boldsymbol{B}_1), \tag{A8}$$

and thus the time derivative of the linearized gyrocenter current density is given by

$$\frac{1}{\epsilon_0}\frac{\partial \boldsymbol{j}^{\rm gy}}{\partial t} = \frac{1}{\epsilon_0}\frac{\partial j_{\parallel 1}}{\partial t}\boldsymbol{b}_0 - \frac{1}{\epsilon_0}\frac{j_{\parallel 0}}{B_0}\nabla\times\boldsymbol{E}_1 + \frac{c^2\beta_{\perp 0}}{2}\nabla\times(\boldsymbol{b}_0\boldsymbol{b}_0\cdot\nabla\times\boldsymbol{E}_1) + \frac{1}{\epsilon_0}\varrho_0^{\rm gy}\frac{\partial \boldsymbol{E}_1}{\partial t}\times\frac{\boldsymbol{b}_0}{B_0}. \tag{A9}$$

To eliminate $j_{\parallel 1}$ and $\beta_{\perp 1}$, we need an equation for f_{s1} in terms of f_{s0}. Linearization of the gyrokinetic Vlasov equation provides

$$\frac{\partial f_{s1}}{\partial t} + v_\parallel \boldsymbol{b}_0\cdot\nabla f_{s1} = -\dot{v}_\parallel\frac{\partial f_{s0}}{\partial v_\parallel}, \tag{A10}$$

where the linear expression for the parallel acceleration is given by

$$\dot{v}_\parallel = -\frac{\mu}{m}\boldsymbol{b}_0\cdot\nabla(\boldsymbol{b}_0\cdot\boldsymbol{B}_1) + \frac{e}{m}\boldsymbol{b}_0\cdot\boldsymbol{E}_1. \tag{A11}$$

To solve for f_{s1}, and to consequently obtain the conductivity tensor, we invoke the plane wave assumption. This provides us with

$$f_{s1} = -\boldsymbol{b}_0\cdot\boldsymbol{k}(\boldsymbol{b}_0\cdot\boldsymbol{k}\times\boldsymbol{E}_1)\frac{\mu}{\omega m}\frac{\partial f_{s0}/\partial v_\parallel}{\omega - \boldsymbol{b}_0\cdot\boldsymbol{k}v_\parallel} - i\boldsymbol{b}_0\cdot\boldsymbol{E}_1\frac{e}{m}\frac{\partial f_{s0}/\partial v_\parallel}{\omega-\boldsymbol{b}_0\cdot\boldsymbol{k}v_\parallel}. \tag{A12}$$

Now the expression for the f_{s1}-dependent term needed in the time derivative of the magnetization current becomes

$$\frac{1}{2}\nabla\frac{\partial\beta_{\perp 1}}{\partial t}\times\boldsymbol{B}_0 = -\boldsymbol{k}\times\boldsymbol{b}_0\boldsymbol{b}_0\times\boldsymbol{k}\cdot\boldsymbol{E}_1\, I_3(\omega,\boldsymbol{k}) - i\omega\boldsymbol{k}\times\boldsymbol{b}_0\boldsymbol{b}_0\cdot\boldsymbol{E}_1\, I_2(\omega,\boldsymbol{k}), \tag{A13}$$

and the f_{s1}-dependent term needed in the time derivative of the gyrocenter current becomes

$$\frac{\partial j_{\parallel 1}}{\partial t}\boldsymbol{b}_0 = i\boldsymbol{b}_0\boldsymbol{b}_0\times\boldsymbol{k}\cdot\boldsymbol{E}_1\omega I_2(\omega,\boldsymbol{k}) - \boldsymbol{b}_0\boldsymbol{b}_0\cdot\boldsymbol{E}_1\omega I_1(\omega,\boldsymbol{k}). \tag{A14}$$

The integrals are the ones given in Eqs. (30), (31), and (32). Combining then the expressions (A6), (A7), and (A9) leads to

$$\begin{aligned}\boldsymbol{\chi} = {}& \frac{1}{\epsilon_0\omega}\left\langle\frac{p_\parallel}{B_0^2}\right\rangle_0(\boldsymbol{k}\boldsymbol{b}_0+\boldsymbol{b}_0\boldsymbol{k}-2\boldsymbol{b}_0\cdot\boldsymbol{k}\mathbf{1}) + \frac{c^2}{v_{A0}^2}\mathbf{1}_\perp \\ &- \frac{c^2}{\omega^2}\frac{\beta_{\parallel 0}-\beta_{\perp 0}}{2}(\boldsymbol{k}\boldsymbol{k}-k^2\mathbf{1}) + \frac{I_1}{\epsilon_0\omega}\boldsymbol{b}_0\boldsymbol{b}_0 \\ &+ \frac{c^2}{\omega^2}\left(\frac{\beta_{\parallel 0}+\beta_{\perp 0}}{2} - \frac{I_3}{c^2}\right)\boldsymbol{k}\times\boldsymbol{b}_0\boldsymbol{b}_0\times\boldsymbol{k} \\ &- i\frac{I_2}{\epsilon_0\omega}(\boldsymbol{k}\times\boldsymbol{b}_0\boldsymbol{b}_0 + \boldsymbol{b}_0\boldsymbol{b}_0\times\boldsymbol{k}) \\ &- i\frac{1}{\epsilon_0\omega}\left(\frac{j_{\parallel 0}}{\omega B_0}\mathbb{k} - \frac{\varrho_0^{\rm gy}}{B_0}\mathrm{b}_0\right).\end{aligned} \tag{A15}$$

The matrices $\mathbb{k}$ and b_0 are constructed from the vectors $\boldsymbol{k}$ and $\boldsymbol{b}_0$ according to rule (A1).

APPENDIX B: EULER–POINCARÉ FORMALISM

Here, we provide a brief overview of how the equations describing the gyrokinetic Vlasov–Maxwell system are derived. The gyrocenter contribution is constructed by taking the single-gyrocenter Lagrangian $L(t,\boldsymbol{z}(t),\dot{\boldsymbol{z}}(t)) = \Theta_i(\boldsymbol{z}(t),t)\dot{z}^i(t) - \mathcal{H}(\boldsymbol{z}(t),t)$; multiplying it by an arbitrary, fixed phase-space density of gyrocenter positions F_0; and then integrating over the density in the phase space. This is the idea that Low[4] used in deriving his action principle. Here the explicit time-dependency in Θ and $\mathcal{H}$ is to be understood as the time-dependency introduced by the electromagnetic potentials $\boldsymbol{A}_1$ and ϕ_1 present in Θ and $\mathcal{H}$, even if they are not explicitly shown for avoiding clutter. Adding also the electromagnetic Lagrangian, the system Lagrangian then becomes

$$\begin{aligned}\mathcal{L}_{F_0}(\boldsymbol{g}_t,\partial_t\boldsymbol{g}_t,\boldsymbol{A}_1,\phi_1) = {}& \int_{\mathbb{R}^6}\big(\Theta_i(\boldsymbol{g}_t(\boldsymbol{z}),t)\partial_t g_t^i(\boldsymbol{z}) \\ &- \mathcal{H}(\boldsymbol{g}_t(\boldsymbol{z}),t)\big)F_0(\boldsymbol{z})dz^1\wedge,\ldots,\wedge dz^6 \\ &+ \frac{1}{2}\int_{\mathbb{R}^3}\left(\epsilon_0|\boldsymbol{E}_1|^2 - \mu_0^{-1}|\boldsymbol{B}_0+\boldsymbol{B}_1|^2\right)d\boldsymbol{x},\end{aligned} \tag{B1}$$

where $\boldsymbol{g}_t(\boldsymbol{z})$ denotes the flow of particles starting from $\boldsymbol{z}$ along the yet undetermined equations of motion, effectively $\boldsymbol{z}\mapsto\boldsymbol{g}_t(\boldsymbol{z}) = \boldsymbol{z}(t)$. The fields $\boldsymbol{E}_1$ and $\boldsymbol{B}_1$ naturally relate to $\boldsymbol{A}_1$ and ϕ_1 in the standard manner.

This Lagrangian has a mixed presentation, and does not directly provide the Vlasov equation. One can, however, recover pure Eulerian variables by re-expressing everything with respect to the position $\boldsymbol{g}_t(\boldsymbol{z})$. This is achieved by defining a new time-dependent phase-space density $F(\boldsymbol{z},t)$ and a vector field $\boldsymbol{V}(\boldsymbol{z},t)$ via the relations

$$F(\boldsymbol{g}_t(\boldsymbol{z}),t)dg_t^1(\boldsymbol{z})\wedge,\ldots,\wedge dg_t^6(\boldsymbol{z}) = F_0(\boldsymbol{z})dz^1\wedge,\ldots,\wedge dz^6, \tag{B2}$$

$$\boldsymbol{V}(\boldsymbol{z},t) = (\partial_t\boldsymbol{g}_t\circ\boldsymbol{g}_t^{-1})(\boldsymbol{z}). \tag{B3}$$

Consequently, the mixed term $\Theta_i(\boldsymbol{g}_t(\boldsymbol{z}),t)\partial_t g_t^i(\boldsymbol{z})$ can be written as $(\Theta_i V^i)(\boldsymbol{g}_t(\boldsymbol{z}),t)$, and the system Lagrangian expressed in Eulerian variables according to

$$\begin{aligned}\mathcal{L}(F,\boldsymbol{V},\boldsymbol{A}_1,\phi_1) = {}& \int_{\mathbb{R}^6}\big(\Theta_i(\boldsymbol{z},t)V^i(\boldsymbol{z},t) - \mathcal{H}(\boldsymbol{z},t)\big)F(\boldsymbol{z},t)dz^1\wedge,\ldots,\wedge dz^6 \\ &+ \frac{1}{2}\int_{\mathbb{R}^3}\left(\epsilon_0|\boldsymbol{E}_1|^2 - \mu_0^{-1}|\boldsymbol{B}_0+\boldsymbol{B}_1|^2\right)d\boldsymbol{x}.\end{aligned} \tag{B4}$$

To apply Hamilton's principle of least action with respect to the new Lagrangian, one only has to remember that the underlying quantity to be perturbed is the flow $\boldsymbol{g}_t$. Keeping this in mind, one can derive the resulting variations for the new variables,

$$\delta F = -\partial_i(\eta^i F), \tag{B5}$$

$$\delta V^i = \partial_t\eta^i + V^j\partial_j\eta^i - \eta^j\partial_j V^i, \tag{B6}$$

where $\boldsymbol{\eta}(\boldsymbol{z},t) = \delta\boldsymbol{g}_t\circ\boldsymbol{g}_t^{-1}$ is another vector field resulting from the arbitrary variation of the flow $\delta\boldsymbol{g}_t$. Additionally, the Vlasov equation is obtained by differentiating (B2) with respect to time, providing

$$\partial_t F + \partial_i (V^i F) = 0. \tag{B7}$$

In deriving these variations and the Vlasov equation, it is best to use the machinery of differential geometry. For explicit details on this matter, see Appendix A in Ref. 22.

After this, the process of finding the equations is straightforward. Given the action

$$\mathcal{A} = \int_{t_1}^{t_2} \mathcal{L}(F, \boldsymbol{V}, \boldsymbol{A}_1, \phi_1) dt, \tag{B8}$$

varying the particle motion, i.e., using both $\delta \boldsymbol{V}$ and δF, leads to the familiar Hamiltonian equations for the components of the vector field $\boldsymbol{V}$,

$$(\partial_j \Theta_i - \partial_i \Theta_j) V^i = \partial_j \mathcal{H} + \partial_t \Theta_j. \tag{B9}$$

As the Hamiltonian in the gauge-free theory is written in the form of

$$\mathcal{H} = K[\boldsymbol{E}_1, \boldsymbol{B}_1] + e\phi_1, \tag{B10}$$

with K the transformed kinetic energy, and the one-form

$$\Theta = \vartheta^{gc} + eA_1 \cdot dx \tag{B11}$$

consists of the time-independent guiding-center one-form $\vartheta^{\rm gc}$ and the perturbed vector potential $\boldsymbol{A}_1$, the equation for $\boldsymbol{V}$ can be written simply as

$$V^i = \{z^i, K\} - \{z^i, \boldsymbol{x}\} \cdot e\boldsymbol{E}_1, \tag{B12}$$

where $\{f, g\} = \partial_i f \Pi^{ij} \partial_j g$ is the Poisson bracket with Π^{ij} the inverse of the Lagrange matrix $\partial_j \Theta_i - \partial_i \Theta_j$. The explicit form of the bracket is given in (7).

We make a remark that, in the main part of this text, the phase-space density is split into the phase-space distribution function f and the phase-space Jacobian according to $F = fJ$. Further, the phase-space density form is then expressed as $F(\boldsymbol{z}, t) dz^1 \wedge, \ldots, \wedge dz^6 = f\boldsymbol{dz}$ where $\boldsymbol{dz}$ contains the bare volume element and the Jacobian J, i.e., $\boldsymbol{dz} = J dz^1 \wedge, \ldots, \wedge dz^6$. In applying the Hamilton's principle, the underlying variable to be varied is the flow map $\boldsymbol{g}_t$. Although the variation of the flow map induces variations of the Eulerian velocity field $\boldsymbol{V}$ and the density field F, these variations need to be treated as independent of the variations of the electromagnetic fields. If the density in the kinetic energy functional is represented as $F = fJ$, as is done in the current work, one has to keep in mind that varying the kinetic energy functional with respect to the magnetic field $\boldsymbol{B}_1$ should not reach the Jacobian J.

The gyrokinetic Maxwell's equations are obtained by varying the action with respect to ϕ_1 and $\boldsymbol{A}_1$. Variation with respect to ϕ_1 provides the Gauss's law (4) where the polarization density results from the presence of the electric field in the gyrocenter kinetic energy term K. Similarly, varying the action with respect to $\boldsymbol{A}_1$ results in the Amperé–Maxwell equation (2) with the polarization and magnetization currents resulting from the presence of the electromagnetic fields in the kinetic energy term K. Since the derivation of the field equations does not require any specific form for the variations but proceeds in the standard way, we leave the explicit derivation of these equations as an exercise.

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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