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ABSTRACT

Kruskal [J. Math. Phys. **3**, 806 (1962)] showed that each nearly periodic dynamical system admits a formal $U(1)$ symmetry, generated by the so-called roto-rate. We prove that such systems also admit nearly invariant manifolds of each order, near which rapid oscillations are suppressed. We study the nonlinear normal stability of these slow manifolds for nearly periodic Hamiltonian systems on *barely symplectic manifolds*—manifolds equipped with closed, non-degenerate 2-forms that may be degenerate to leading order. In particular, we establish a sufficient condition for long-term normal stability based on second derivatives of the well-known adiabatic invariant. We use these results to investigate the problem of embedding guiding center dynamics of a magnetized charged particle as a slow manifold in a nearly periodic system. We prove that one previous embedding and two new embeddings enjoy long-term normal stability and thereby strengthen the theoretical justification for these models.

I. INTRODUCTION

Dynamical systems with multiple timescales often exhibit special slow trajectories that lie along almost invariant sets known as slow manifolds.[1–3] Because slow manifolds generally fail to be true invariant objects, analysis of their dynamical significance requires special care. The most well-understood case is normal hyperbolicity: the slow manifold attracts or repels nearby trajectories. As shown by Fenichel,[4] given a normally hyperbolic slow manifold, nearby there must be a true normally hyperbolic invariant manifold. Less well-understood are the normally elliptic slow manifolds: slow manifolds around which nearby trajectories oscillate. These objects may fail to approximate true invariant manifolds,[5] but they frequently form the basis for model reduction in dynamical systems with weak dissipation. For instance, the equations governing quasigeostrophic flow describe motion on an elliptic slow manifold inside of the rotating shallow water model;[6–8] normal oscillations correspond to fast gravity waves. Other examples include the incompressible Euler equations,[9] which arise as a slow manifold inside of the compressible Euler equations, and ideal magnetohydrodynamics, which may be understood as a slow manifold for a pair of charged ideal fluids.[10]

Reduction to an elliptic slow manifold is fraught with theoretical challenges. Principal among these is the question of normal stability: Do trajectories that begin near an elliptic slow manifold exhibit secular normal drifts? While ellipticity implies marginal linear stability on short timescales, normal instability may still arise at later times due to resonance phenomena. Thus, a dynamical model obtained by reduction to an elliptic slow manifold may spontaneously break down. Even worse, such breakdown may be undetectable from within the reduced model itself.

An important example of an elliptic slow manifold for which normal stability remains an open problem was constructed recently by Xiao and Qin,[11] who proposed a novel method for symplectic integration of the so-called guiding center equations for charged particles in a strong magnetic field. As shown by Littlejohn in Refs. 12–15, the guiding center equations comprise a Hamiltonian system with a non-canonical symplectic structure. Moreover, the natural Lagrangian for guiding centers is degenerate; its velocity Hessian is singular. This makes the formulation of symplectic integrators for guiding center dynamics extremely challenging because standard symplectic integration theory is intended for either canonical symplectic structures or non-degenerate Lagrangians. However, Xiao and Qin suggested a method for circumventing this difficulty: embed the guiding center system as an elliptic slow manifold in a larger system that does admit a regular Lagrangian. Applying conventional symplectic integration methods to this larger system then leads to a higher-dimension structure-preserving scheme with a slow manifold that formally recovers the guiding center dynamics of interest. (Their scheme is therefore a slow manifold integrator, as described in Ref. 3.) This idea provides an elegant solution to symplectic integration of guiding center dynamics, provided the integrator's slow manifold is normally stable. On the other hand, if a normal instability does exist, then the scheme will fail after the instability onset time. At present, normal stability remains an open question.

MacKay suggested in Ref. 2 that a useful method for establishing elliptic normal stability, in general, is identifying an adiabatic invariant whose set of critical points gives the slow manifold. Such an adiabatic invariant appears, for example, in the slow manifold analysis[16] of singularly perturbed Hamiltonian systems that lose a degree of freedom in the singular limit. Then, the sign-definiteness of the normal Hessian should imply normal stability by a Lyapunov-type argument. We say that the slow manifold satisfies a *free-action principle*. However, it is unclear, in general, how to identify such an adiabatic invariant or even if such an adiabatic invariant exists.

In this article, we will identify an important class of elliptic slow manifolds for which the free-action principle always applies. In particular, we will show that each nearly periodic system with an appropriate Hamiltonian structure admits elliptic slow manifolds of arbitrary order and that these slow manifolds coincide with critical sets for adiabatic invariants. Moreover, we will prove rigorously that sign-definiteness of the normal Hessian implies long-term normal stability. After establishing this theoretical result, we will apply it in the study of guiding center dynamics of individual charged particles in a strong magnetic field.

Using our general theory, we will show that Xiao and Qin's slow manifold embedding of guiding center dynamics enjoys normal stability in continuous time. This result leaves normal stability in discrete time an open question but motivates further study in that direction. We will also construct a pair of alternative finite-dimensional slow manifold embeddings of guiding center dynamics. One is a covariant relativistic generalization of the Xiao–Qin embedding. The other is a special case of a more general embedding that applies to *any* symplectic Hamiltonian system. Like Xiao and Qin's case, the larger systems into which we embed come equipped with a regular Lagrangian structure. We use our general theory to prove long-term normal stability for these new embeddings and thereby identify promising future extensions of Xiao and Qin's idea.

In order to ensure that our abstract theory is general enough to handle the guiding center system, we were forced to consider Hamiltonian systems on symplectic manifolds whose symplectic forms may be very nearly degenerate. We formalize this near-degeneracy by supposing that the symplectic form Ω_ϵ is a smooth function of the parameter ϵ that quantifies the timescale separation in a nearly periodic system and that Ω_0 may be degenerate. We call manifolds equipped with symplectic forms of this type *barely symplectic*. By working at this level of generality, our abstract results exhibit an interesting competition between stabilizing and destabilizing influences on the slow manifolds that we construct. In particular, stronger degeneracy of Ω_ϵ as $\epsilon \to 0$ appears to destabilize the slow manifolds, while vanishing of early terms in the adiabatic invariant series (see Ref. 17 for explicit formulas for the first few terms) has a stabilizing effect. From this perspective, the guiding center embeddings we study are remarkable because the stabilizing and destabilizing influences balance, leading to normal stability results that would be expected for ϵ-independent symplectic manifolds.

A. Notational conventions

In this article, smooth shall always mean C^∞. We reserve the symbol M for a smooth manifold equipped with a smooth auxiliary Riemannian metric g. We say $f_\epsilon : M_1 \to M_2$, $\epsilon \in \mathbb{R}$, is a smooth ϵ-dependent mapping between manifolds M_1, M_2 when the mapping $M_1 \times \mathbb{R} \to M_2 : (m,\epsilon) \mapsto f_\epsilon(m)$ is smooth. Similarly, $\boldsymbol{T}_\epsilon$ is a smooth ϵ-dependent tensor field on M when (a) $\boldsymbol{T}_\epsilon(m)$ is an element of the tensor algebra $\mathcal{T}_m(M)$ at m for each $m \in M$ and $\epsilon \in \mathbb{R}$ and (b) $\boldsymbol{T}_\epsilon$ is a smooth ϵ-dependent mapping between the manifolds M and $\mathcal{T}(M) = \cup_{m\in M}\mathcal{T}_m(M)$.

The symbol X_ϵ will always denote a smooth ϵ-dependent vector field on M. If $\boldsymbol{T}_\epsilon$ is a smooth ϵ-dependent section of either $TM \otimes TM$ or $T^*M \otimes T^*M$, then $\widehat{\boldsymbol{T}}_\epsilon$ is the corresponding smooth ϵ-dependent bundle map $T^*M \to TM : \alpha \mapsto \iota_\alpha \boldsymbol{T}_\epsilon$ or $TM \to T^*M : X \mapsto \iota_X \boldsymbol{T}_\epsilon$, respectively. Note that if Ω is a symplectic form on M with the associated Poisson bivector $\mathcal{J}$, then $\widehat{\Omega}^{-1} = -\widehat{\mathcal{J}}$.

II. KRUSKAL'S THEORY OF NEARLY PERIODIC SYSTEMS

In 1962, Kruskal presented an asymptotic theory[18] of averaging for dynamical systems whose trajectories are all periodic to leading order. Nowadays, Kruskal's method is termed one-phase averaging,[19] which suggests a contrast with the multi-phase averaging methods underlying, e.g., Kolmogorov–Arnol'd–Moser (KAM) theory. Since this theory provides the framework for the results in this article, we review its main ingredients here.

Definition 1. A nearly periodic system on a manifold M is a smooth ϵ-dependent vector field X_ϵ on M such that $X_0 = \omega_0 R_0$, where

- *$\omega_0 : M \to \mathbb{R}$ is strictly positive,*
- *R_0 is the infinitesimal generator for a circle action $\Phi^0_\theta : M \to M$, $\theta \in U(1)$, and*
- *$\mathcal{L}_{R_0}\omega_0 = 0$.*

The vector field R_0 is called the limiting roto-rate, and the set $S_0 = \{s \in M \mid R_0(s) = 0\}$ is called the limiting slow manifold.

Remark 2. *In addition to requiring ω_0 is sign-definite, Kruskal assumed that R_0 is nowhere vanishing. However, this assumption is not essential for one-phase averaging to work. In fact, the limiting slow manifold S_0 will play a crucial role in the rest of this article. Note that, by Lemma 16, $S_0 \subset M$ is indeed a submanifold.*

Kruskal's theory applies to both Hamiltonian and non-Hamiltonian systems. In the Hamiltonian setting, it leads to stronger conclusions. A general class of Hamiltonian systems for which the theory works nicely may be defined as follows:

Definition 3. *Let (M, Ω_ϵ) be a manifold equipped with a smooth ϵ-dependent presymplectic form Ω_ϵ. Assume that there is a smooth ϵ-dependent 1-form ϑ_ϵ such that $\Omega_\epsilon = -\mathbf{d}\vartheta_\epsilon$. A nearly periodic Hamiltonian system on (M, Ω_ϵ) is a nearly periodic system X_ϵ on M such that $\iota_{X_\epsilon}\Omega_\epsilon = \mathbf{d}H_\epsilon$ for some smooth ϵ-dependent function $H_\epsilon : M \to \mathbb{R}$.*

Kruskal showed that all nearly periodic systems admit an approximate $U(1)$-symmetry that is determined to leading order by the unperturbed periodic dynamics. He named the generator of this approximate symmetry the *roto-rate*. In the Hamiltonian setting, he showed that both the dynamics and the Hamiltonian structure are $U(1)$-invariant.

Definition 4. *A roto-rate for a nearly periodic system X_ϵ on a manifold M is a formal power series $R_\epsilon = R_0 + \epsilon R_1 + \epsilon^2 R_2 + \cdots$ with vector field coefficients such that*

- *R_0 is equal to the limiting roto-rate,*
- $\exp(2\pi\mathcal{L}_{R_\epsilon}) = 1$, and
- $[X_\epsilon, R_\epsilon] = 0$,

where the second and third conditions are understood in the sense of formal power series.

Proposition 5 (*Ref. 18*). *Every nearly periodic system admits a unique roto-rate R_ϵ. The roto-rate for a nearly periodic Hamiltonian system on an exact presymplectic manifold (M, Ω_ϵ) satisfies $L_{R_\epsilon}\Omega_\epsilon = 0$ in the sense of formal power series.*

Corollary 6. *The roto-rate R_ϵ for a nearly periodic Hamiltonian system X_ϵ on an exact presymplectic manifold (M, Ω_ϵ) with Hamiltonian H_ϵ satisfies $L_{R_\epsilon}H_\epsilon = 0$.*

Proof. Since $[R_\epsilon, X_\epsilon] = \mathcal{L}_{R_\epsilon}X_\epsilon = 0$ and $\mathcal{L}_{R_\epsilon}\Omega_\epsilon = 0$, we may apply the Lie derivative $\mathcal{L}_{R_\epsilon}$ to Hamilton's equation $\iota_{X_\epsilon}\Omega_\epsilon = \mathbf{d}H_\epsilon$ to obtain

$$\mathcal{L}_{R_\epsilon}(\mathbf{d}H_\epsilon) = \mathcal{L}_{R_\epsilon}(\iota_{X_\epsilon}\Omega_\epsilon) = \iota_{\mathcal{L}_{R_\epsilon}X_\epsilon}\Omega_\epsilon + \iota_{X_\epsilon}(\mathcal{L}_{R_\epsilon}\Omega_\epsilon) = 0.$$

Thus, $\mathcal{L}_{R_\epsilon}H_\epsilon$ is a constant function. However, by averaging over the $U(1)$-action, we conclude that the constant must be zero. □

To prove Proposition 5, Kruskal used a pair of technical results, each of which is interesting in its own right. The first establishes the existence of a non-unique normalizing transformation that asymptotically deforms the $U(1)$ action generated by R_ϵ into the simpler $U(1)$-action generated by R_0. The second is a subtle bootstrapping argument that upgrades leading-order $U(1)$-invariance to all-orders $U(1)$-invariance for integral invariants. We state these results here for future reference.

Definition 7. *Let $G_\epsilon = \epsilon G_1 + \epsilon^2 G_2 + \cdots$ be an $O(\epsilon)$ formal power series whose coefficients are vector fields on a manifold M. The Lie series with generator G_ϵ is the formal power series $\exp(\mathcal{L}_{G_\epsilon})$ whose coefficients are differential operators on the tensor algebra over M.*

Definition 8. *A normalizing transformation for a nearly periodic system X_ϵ with roto-rate R_ϵ is a Lie series $\exp(\mathcal{L}_{G_\epsilon})$ with generator G_ϵ such that $R_\epsilon = \exp(\mathcal{L}_{G_\epsilon})R_0$.*

Proposition 9 (*Ref. 18*). *Each nearly periodic system admits a normalizing transformation.*

Proposition 10. *Let α_ϵ be a smooth ϵ-dependent differential form on a manifold M. Suppose α_ϵ is an absolute integral invariant for a C^∞ nearly periodic system X_ϵ on M. If $\mathcal{L}_{R_0}\alpha_0 = 0$; then, $\mathcal{L}_{R_\epsilon}\alpha_\epsilon = 0$, where R_ϵ is the roto-rate for X_ϵ.*

According to Noether's celebrated theorem, a Hamiltonian system that admits a continuous family of symmetries also admits a corresponding conserved quantity. Therefore, one might expect that a Hamiltonian system that admits an approximate symmetry must also have an approximate conservation law. Kruskal showed that this is indeed the case for nearly periodic Hamiltonian systems, as the following generalization of his argument shows:

Proposition 11. *Let X_ϵ be a nearly periodic Hamiltonian system on the exact presymplectic manifold (M, Ω_ϵ). Let R_ϵ be the associated roto-rate. There is a formal power series $\overline{\vartheta}_\epsilon = \overline{\vartheta}_0 + \epsilon\overline{\vartheta}_1 + \cdots$ with coefficients in $\Omega^1(M)$ such that $\Omega_\epsilon = -\mathbf{d}\overline{\vartheta}_\epsilon$ and $\mathcal{L}_{R_\epsilon}\overline{\vartheta}_\epsilon = 0$. Moreover, the formal power series $\mu_\epsilon = \iota_{R_\epsilon}\overline{\vartheta}_\epsilon$ is a constant of motion for X_ϵ to all orders in perturbation theory. In other words,*

$$\mathcal{L}_{X_\epsilon}\mu_\epsilon = 0$$

in the sense of formal power series.

Proof. To construct the $U(1)$-invariant primitive $\overline{\vartheta}_\epsilon$, we select an arbitrary primitive ϑ_ϵ for Ω_ϵ and set

$$\overline{\vartheta}_\epsilon = \frac{1}{2\pi}\int_0^{2\pi} \exp(\theta\mathcal{L}_{R_\epsilon})\vartheta_\epsilon d\theta.$$

This formal power series satisfies $\mathcal{L}_{R_\epsilon}\overline{\vartheta}_\epsilon = 0$ because

$$\mathcal{L}_{R_\epsilon}\overline{\vartheta}_\epsilon = \frac{1}{2\pi}\int_0^{2\pi} \frac{d}{d\theta} \exp(\theta\mathcal{L}_{R_\epsilon})\vartheta_\epsilon d\theta = 0.$$

Moreover, since $\mathcal{L}_{R_\epsilon}\Omega_\epsilon = 0$ by Ref. 18, Proposition 5, we have

$$-\mathbf{d}\overline{\vartheta}_\epsilon = \frac{1}{2\pi}\int_0^{2\pi} \exp(\theta\mathcal{L}_{R_\epsilon})\Omega_\epsilon d\theta = \frac{1}{2\pi}\int_0^{2\pi}\Omega_\epsilon d\theta = \Omega_\epsilon,$$

whence $\overline{\vartheta}_\epsilon$ is a primitive for Ω_ϵ.

To establish all-orders time-independence of $\mu_\epsilon = \iota_{R_\epsilon}\overline{\vartheta}_\epsilon$, we apply Cartan's formula and Corollary 6 according to

$$\mathcal{L}_{X_\epsilon}\mu_\epsilon = \iota_{X_\epsilon}\mathbf{d}\iota_{R_\epsilon}\overline{\vartheta}_\epsilon = -\iota_{R_\epsilon}\iota_{X_\epsilon}\Omega_\epsilon = -\mathcal{L}_{R_\epsilon}H_\epsilon = 0.$$

We note that the adiabatic invariant constructed in this proof does not change if the primitive ϑ_ϵ is subject to the transformation $\vartheta_\epsilon \mapsto \vartheta_\epsilon + \alpha_\epsilon$, where α_ϵ is an exact ϵ-dependent 1-form. When α_ϵ is merely closed rather than exact, the value of μ_ϵ does change but only by a global constant. See the discussion surrounding Eq. (3.12) in Ref. 17. □

Definition 12. The formal constant of motion μ_ϵ provided by Proposition 11 is the adiabatic invariant associated with a nearly periodic Hamiltonian system.

III. SLOW MANIFOLDS FOR NEARLY PERIODIC SYSTEMS

Let X_ϵ be a smooth ϵ-dependent vector field on a manifold M equipped with an auxiliary Riemannian metric g. Without loss of generality, assume $X_\epsilon = O(1)$. An Nth-order *slow manifold* $S_\epsilon \subset M$ for X_ϵ is an ϵ-dependent submanifold such that the normal component of X_ϵ along S_ϵ is $O(\epsilon^{N+1})$ and the tangential component is $O(\epsilon)$. Note, in particular, that S_0 must be a manifold of fixed points for X_0. A slow manifold is *elliptic* if the normal linearized dynamics for X_0 along S_0 are purely oscillatory.

Intuitively, trajectories for X_ϵ that begin near an elliptic slow manifold S_ϵ should slowly drift along S_ϵ (while rapidly oscillating around it) for some large time interval before possibly wandering away. By dropping the normal component $X_\epsilon^\perp$ of X_ϵ along S_ϵ, one obtains a well-defined vector field $X_\epsilon^\parallel$ on S_ϵ that may be interpreted as a model for the slow drift dynamics. By way of this procedure, elliptic slow manifolds give rise to formal "reduced" models in various scientific disciplines, especially in plasma physics, with its numerous multiscale models. The purpose of this section is to prove that nearly periodic systems always admit elliptic slow manifolds of every order.

Before proceeding, we must resolve a technical issue. The "definition" of slow manifolds given above is somewhat imprecise since S_ϵ moves as $\epsilon \to 0$. In order to eliminate this ambiguity, we introduce parameterizations.

Definition 13. Let X_ϵ be a smooth ϵ-dependent vector field on a manifold M such that X_0 is not identically zero. An Nth-order parameterized slow manifold for X_ϵ is a smooth ϵ-dependent embedding $\mathcal{S}_\epsilon : S_0 \to M$ of some fixed manifold S_0 into M with the following two properties:

(1) *For each $s_0 \in S_0$, $|X_\epsilon^\perp(\mathcal{S}_\epsilon(s_0))| = O(\epsilon^{N+1})$ as $\epsilon \to 0$.*
(2) *For each $s_0 \in S_0$, $|X_\epsilon^\parallel(\mathcal{S}_\epsilon(s_0))| = O(\epsilon)$ as $\epsilon \to 0$.*

Suppose $\varphi_\epsilon : S_0 \to S_0$ is a smooth ϵ-dependent diffeomorphism. If $\mathcal{S}_\epsilon$ is a parameterized slow manifold, then $\mathcal{S}_\epsilon' = \mathcal{S}_\epsilon \circ \varphi_\epsilon$ is also a parameterized slow manifold with the same image and of the same order. We may therefore precisely define an Nth-order slow manifold as a smooth ϵ-dependent submanifold $S_\epsilon \subset M$ that is the image of some Nth-order parameterized slow manifold.

We will now suppose that X_ϵ is a nearly-periodic system, as defined in Sec. II, and proceed to construct slow manifolds. Our overarching strategy will be to find vector fields $X_\epsilon^{(N)}$ that agree with X_ϵ modulo terms that are $O(\epsilon^{N+1})$ and that possess genuine invariant submanifolds $S_\epsilon^{(N)}$. Then, we will prove that certain open subsets of $S_\epsilon^{(N)}$ are Nth-order slow manifolds for X_ϵ. Roberts discussed essentially the same backward analysis strategy in a more general setting in Refs. 20 and 21.

The motivation for this strategy comes from Kruskal's result (Proposition 9) on existence of normalizing transformations for nearly periodic systems. In a formal sense, normalizing transformations provide coordinates that expose a hidden $U(1)$-symmetry underlying each nearly periodic system. The connection between this observation and invariant manifold theory is that the set of $U(1)$-invariant points in phase space must be an invariant manifold for any dynamical system with $U(1)$-symmetry. The hitch in this argument, and the reason we will only find slow manifolds, rather than genuine invariant manifolds, is that Kruskal's normalizing transformations are only defined as the formal power series. We will be forced to truncate these series and in the process lose exact invariance. However, since we may truncate at any order, that loss of invariance can be made arbitrarily small.

Theorem 14. *Let X_ϵ be a nearly periodic system on a manifold M. The associated limiting slow manifold, S_0, is a zeroth-order slow manifold for X_ϵ. Moreover, for each $N > 0$ and each codimension-0 compact submanifold $C_0 \subset M$, with or without boundary, there exists an Nth-order parameterized slow manifold $\mathcal{S}_\epsilon^{(N)} : \Sigma_0 \to M$ for X_ϵ, where $\Sigma_0 = S_0 \cap \operatorname{int} C_0$.*

Remark 15. It is worth highlighting two limiting cases of the theorem. (1) If the set of equilibrium points for R_0 is empty, then the theorem is vacuously true; the slow manifolds are merely empty sets. (2) If M is compact, we may take $C_0 = M$ and conclude that X_ϵ admits a slow manifold diffeomorphic to S_0 at each order N. For non-compact M, the slow manifolds provided by the theorem may fail to be diffeomorphic to S_0.

To prove Theorem 14, we will use a pair of supporting lemmas.

Lemma 16 (see, e.g., Ref. 22). The set of fixed points $Z \subset M$ of a $U(1)$-action on a manifold M is an embedded submanifold.

Proof. Let $\Phi_\theta : M \to M, \theta \in U(1)$, be the $U(1)$-action, and let R be the corresponding infinitesimal generator. Assume that M is equipped with a metric tensor g that satisfies $\mathcal{L}_R g = 0$. Note that if g is an arbitrary metric on M, then $\langle g\rangle = \frac{1}{2\pi}\int_0^{2\pi}\Phi_\theta^* g\,d\theta$ is a Riemannian metric that satisfies $\mathcal{L}_R\langle g\rangle = 0$. Therefore, our assumption introduces no loss of generality.

Suppose $m \in Z$, and let $\exp_m : T_mM \to M$ be the Riemannian exponential map at m. Since g is $U(1)$-invariant, the geodesic flow on TM commutes with $T\Phi_\theta$. It follows that the exponential map at m intertwines the $U(1)$-actions Φ_θ on M and $L_\theta \equiv T_m\Phi_\theta$ on T_mM, i.e., $\exp_m \circ L_\theta = \Phi_\theta \circ \exp_m$ for each $\theta \in U(1)$. In light of this equivariance property and the inverse function theorem, we may therefore choose $U(1)$-invariant open subsets $U_m \subset M$ and $U_0 \subset T_mM$, containing $m \in M$ and $0 \in T_mM$, respectively, such that $\varphi_m = \exp_m \mid U_0 : U_0 \to U_m$ is a diffeomorphism. Since this diffeomorphism is equivariant, the preimage Z_0 of $Z_m \equiv Z \cap U_m$ under φ_m must be equal to the fixed point set for the $U(1)$ action L_θ on U_0. Since L_θ is linear, Z_0 must be a linear subspace of T_mM. By restricting φ_m^{-1} to Z_0, we therefore obtain a coordinate chart on Z near m. Since $m \in Z$ is arbitrary, this shows that Z is an embedded submanifold. □

Lemma 17. If Y_ϵ is a smooth ϵ-dependent vector field on M that commutes with the infinitesimal generator R of a $U(1)$-action, then the set of fixed points Z for R is an invariant submanifold for Y_ϵ for each ϵ.

Proof. Suppose $m \in Z$. We will show that the component of Y_ϵ normal to Z vanishes, i.e., $Y_\epsilon^\perp(m) = 0$. Let $w : M \to \mathbb{R}$ be a smooth $U(1)$-invariant bump function equal to 1 near m and 0 outside of a compact set containing m. Let $F_t = \exp(twY_\epsilon)$ denote the flow map for wY, and let $\Phi_\theta = \exp(\theta R)$ denote the $U(1)$-action generated by R. Since $[R, wY_\epsilon] = 0$, we have $F_t \circ \Phi_\theta = \Phi_\theta \circ F_t$ for each $t \in \mathbb{R}, \theta \in U(1)$. Since m is an equilibrium for R, we therefore have $F_t(m) = \Phi_\theta(F_t(m))$ for each t, θ. In other words, $F_t(m)$ is a an equilibrium for R for each t. By Lemma 17, we then see that the parameterized curve $\gamma(t) = F_t(m)$ defines a smooth mapping from $\mathbb{R}$ into the submanifold Z. The curve's velocity at $t = 0$ is therefore tangent to Z at $\gamma(0) = F_0(m) = m$. However, $d\gamma(0)/dt = w(m)Y_\epsilon(m) = Y_\epsilon(m)$, whence it follows that $Y_\epsilon(m)^\perp = 0$. □

Proof of Theorem 14. First, we prove that S_0 is a zeroth-order slow manifold for X_ϵ. By Lemma 16, we know S_0 is a submanifold. The mapping $\mathcal{S}_\epsilon^{(0)} : S_0 \to M : s_0 \mapsto s_0$ therefore defines a smooth embedding of S_0 into M. We claim that $\mathcal{S}_0$ is a parameterized zeroth-order slow manifold. To see this, note $R_0(\mathcal{S}_\epsilon^{(0)}(s_0)) = 0$ for each $s_0 \in S_0$, which implies

$$X_\epsilon(\mathcal{S}_\epsilon^{(0)}(s_0)) = \omega_0(\mathcal{S}_\epsilon^{(0)}(s_0))R_0(\mathcal{S}_\epsilon^{(0)}(s_0)) + O(\epsilon) = O(\epsilon),$$

as claimed. It follows that $S_0 = \mathcal{S}_\epsilon^{(0)}(S_0)$ is a zeroth-order slow manifold.

Next, we prove the existence of higher-order slow manifolds. Fix an integer $N > 0$ and a codimension-0 compact submanifold $C_0 \subset M$, with or without boundary. By Proposition 9, there exists a normalizing transformation $\exp(\mathcal{L}_{G_\epsilon})$ for X_ϵ with generator G_ϵ. We would like to construct a diffeomorphism $\Psi_\epsilon : M \to M$ that agrees with the formal diffeomorphism $\exp(G_\epsilon)$ to Nth-order at least on $\operatorname{int} C_0$. The simplest strategy for this construction would be to set $\Psi_\epsilon = \exp(\sum_{k=1}^N \epsilon^k G_k)$, but, when M is non-compact, integral curves of the vector field $G_\epsilon^{(N)} = \sum_{k=1}^N \epsilon^k G_k$ may not exist for all time, and so the exponential $\exp(G_\epsilon^{(N)})$ may not exist either. To overcome this difficulty, we introduce the vector field $\mathcal{G}_\epsilon^{(N)} = wG_\epsilon^{(N)}$, where $w : M \to \mathbb{R}$ is a smooth function defined as follows: If $\partial C_0 = \emptyset$ so that C_0 is a union of connected components of M, $w = 1$ on C_0 and $w = 0$ otherwise. If $\partial C_0 \neq \emptyset$, then we use the tubular neighborhood theorem to construct an increasing sequence of compact, codimension-0 submanifolds with boundaries $C_0 \subset C_0' \subset C_0''$ such that $C_0 \subset \operatorname{int} C_0'$ and $C_0' \subset \operatorname{int} C_0''$ and define w so that it satisfies $w = 1$ in C_0' and $w = 0$ outside of C_0''. Thus, w and, therefore $\mathcal{G}_\epsilon^{(N)}$, have compact support. Since smooth vector fields with compact support have well-defined flow maps, we thereby obtain a smooth diffeomorphism $\Psi_\epsilon = \exp(\mathcal{G}_\epsilon^{(N)})$. This diffeomorphism agrees with $\exp(G_\epsilon)$ to Nth-order on $\operatorname{int} C_0'$ in the following sense. If $\boldsymbol{T}_\epsilon$ is any smooth ϵ-dependent tensor field on M, then

$$\forall m\in \operatorname{int} C_0',\quad (\Psi_\epsilon^* \boldsymbol{T}_\epsilon)(m) = \left(\boldsymbol{T}_\epsilon + \mathcal{L}_{\mathcal{G}_\epsilon^{(N)}}\boldsymbol{T}_\epsilon + \frac{1}{2}\mathcal{L}^2_{\mathcal{G}_\epsilon^{(N)}}\boldsymbol{T}_\epsilon + \cdots\right)(m)$$
$$= \left(\boldsymbol{T}_\epsilon + \mathcal{L}_{G_\epsilon^{(N)}}\boldsymbol{T}_\epsilon + \frac{1}{2}\mathcal{L}^2_{G_\epsilon^{(N)}}\boldsymbol{T}_\epsilon + \cdots\right)(m)$$
$$= \left(\boldsymbol{T}_\epsilon + \mathcal{L}_{G_\epsilon}\boldsymbol{T}_\epsilon + \frac{1}{2}\mathcal{L}^2_{G_\epsilon}\boldsymbol{T}_\epsilon + \cdots\right)(m) + O(\epsilon^{N+1})$$
$$= (\exp(\mathcal{L}_{G_\epsilon})\boldsymbol{T}_\epsilon)(m) + O(\epsilon^{N+1}), \tag{1}$$

in the sense of formal power series. Here, we have used that $w = 1$ on an open neighborhood of any $m \in \operatorname{int} C_0'$ to replace the Lie derivatives $\mathcal{L}_{\mathcal{G}_\epsilon^{(N)}}$ with $\mathcal{L}_{G_\epsilon^{(N)}}$.

Using the diffeomorphism Ψ_ϵ, we now construct a vector field $X_\epsilon^{(N)}$ such that (a) $X_\epsilon^{(N)} = X_\epsilon + O(\epsilon^{N+1})$ in $\operatorname{int} C_0'$ and (b) $X_\epsilon^{(N)}$ admits an exact parameterized invariant manifold $\mathfrak{S}_\epsilon^{(N)} : S_0 \to M$. Let $\overline{X}_\epsilon = \exp(-\mathcal{L}_{G_\epsilon})X_\epsilon = \overline{X}_0 + \epsilon\overline{X}_1 + \epsilon^2\overline{X}_2 + \cdots$, and set $\overline{X}_\epsilon^{(N)} = \sum_{k=1}^N \epsilon^k \overline{X}_k$. Since $\exp(-\mathcal{L}_{G_\epsilon})R_\epsilon = R_0$, where R_ϵ is the roto-rate for X_ϵ and $[R_\epsilon, X_\epsilon] = 0$ to all orders, each of the $\overline{X}_k$, and therefore $\overline{X}_\epsilon^{(N)}$, commutes with R_0. By Lemma 17, it follows that the set of equilibrium points for R_0, i.e., S_0, is an invariant manifold for $\overline{X}_\epsilon^{(N)}$. We claim that the vector field $X_\epsilon^{(N)} = \Psi_\epsilon^* \overline{X}_\epsilon^{(N)}$ satisfies properties (a) and (b) above. For (a), we use Eq. (1) to obtain

$$\forall m\in \operatorname{int} C_0',\quad X_\epsilon^{(N)}(m) = (\exp(\mathcal{L}_{G_\epsilon})\overline{X}_\epsilon^{(N)})(m) + O(\epsilon^{N+1})$$
$$= (\exp(\mathcal{L}_{G_\epsilon})\overline{X}_\epsilon)(m) + O(\epsilon^{N+1})$$
$$= (\exp(\mathcal{L}_{G_\epsilon})\exp(-\mathcal{L}_{G_\epsilon})X_\epsilon)(m) + O(\epsilon^{N+1})$$
$$= X_\epsilon(m) + O(\epsilon^{N+1}),$$

in the sense of formal power series. For (b), we restrict the inverse of Ψ_ϵ to S_0 to obtain the embedding $\mathfrak{S}_\epsilon^{(N)} = \Psi_\epsilon^{-1} \mid S_0 : S_0 \to M$; since S_0 is an invariant manifold for $\overline{X}_\epsilon^{(N)}$, $\mathfrak{S}_\epsilon^{(N)}(S_0)$ is an invariant manifold for $X_\epsilon^{(N)}$.

To complete the proof, we first use $(X_\epsilon^{(N)})^\perp = 0$ along $\mathfrak{S}_\epsilon^{(N)}$ and $\mathfrak{S}_\epsilon^{(N)}(S_0 \cap \operatorname{int} C_0) \subset C_0'$ for sufficiently small ϵ to obtain

$$\forall s_0 \in S_0 \cap \operatorname{int} C_0,\ |X_\epsilon^\perp(\mathfrak{S}_\epsilon^{(N)}(s_0))| = |(X_\epsilon^{(N)})^\perp(\mathfrak{S}_\epsilon^{(N)}(s_0))| + O(\epsilon^{N+1})$$
$$= 0 + O(\epsilon^{N+1}). \tag{2}$$

Then, we note that $X_\epsilon^{(N)} = X_0 + O(\epsilon) = \omega_0 R_0 + O(\epsilon)$ since Ψ_ϵ is near-identity, which implies, in particular,

$$\forall s_0 \in S_0 \cap \operatorname{int} C_0,\ |(X_\epsilon^{(N)})^\parallel(\mathfrak{S}_\epsilon^{(N)}(s_0))| = O(\epsilon). \tag{3}$$

Equations (2) and (3) say that $\mathcal{S}_\epsilon^{(N)} = \mathfrak{S}_\epsilon^{(N)} \mid S_0 \cap \operatorname{int} C_0$ is an Nth-order parameterized slow manifold. □

IV. NORMAL STABILITY IN NEARLY PERIODIC HAMILTONIAN SYSTEMS

Given an invariant manifold, or more generally an almost invariant manifold like those provided by Theorem 14, it is important to understand the stability of nearby trajectories. In other words, if a trajectory begins near such an object in phase space, then how long will it remain nearby? The answer to this question sets limits on model reduction strategies based on projecting to the (almost) invariant object. When all nearby trajectories remain nearby on some time interval $\mathcal{I}$, projecting should provide a reasonable reduced model for dynamics near the manifold for times in $\mathcal{I}$. We say that the manifold is *normally stable* on $\mathcal{I}$. However, after a trajectory's transverse displacement becomes large, the projected dynamics may have very little to do with the true dynamics. The purpose of this section is to establish a useful tool for establishing normal stability of slow manifolds in nearly periodic systems that admit a particular kind of Hamiltonian structure.

A nearly periodic system X_ϵ can exhibit a Hamiltonian structure in various ways. In certain cases, X_ϵ may be the Hamiltonian with respect to an ϵ-independent symplectic form Ω on M, meaning there is a smooth ϵ-dependent function $H_\epsilon : M \to \mathbb{R}$ such that $\iota_{X_\epsilon}\Omega = \mathbf{d}H_\epsilon$. Examples include 2-degree-of-freedom canonical Hamiltonian systems with Hamiltonians of the form $H(q_1, p_1, q_2, p_2) = \frac{1}{2}(q_1^2 + p_1^2) + \epsilon U(q_1, p_1, q_2, p_2)$, where U is any smooth function. More generally, X_ϵ may be Hamiltonian with respect to a smooth ϵ-dependent symplectic form Ω_ϵ whose singular limit Ω_0 is merely pre-symplectic. In other words, Ω_0 is closed but may be degenerate as a 2-form. We call such ϵ-dependent 2-forms *barely symplectic forms*. The Lorentz force equations describing the motion of a charged particle in a strong magnetic field exhibit a barely symplectic structure. Since the Hamiltonian structure for a single charged particle frequently appears in the infinite-dimensional Hamiltonian structures underlying non-dissipative models of plasma dynamics,[23–26] barely symplectic Hamiltonian structures play an important role in plasma physics.

Motivated by the above remarks, we will focus our attention on the normal stability of slow manifolds arising in nearly periodic Hamiltonian system on barely symplectic manifolds (defined below). Since our methods involve perturbation theory, we will focus, in particular, on barely symplectic forms that exhibit at worst a *regular singularity* as $\epsilon \to 0$. The class of barely symplectic forms with regular singularities at $\epsilon = 0$ appears to be broad enough to cover many significant applications, including those discussed in Sec. V.

A. Barely symplectic manifolds

Regular barely symplectic manifolds provide the differential-geometric setting for our results on slow manifold normal stability. The purpose of this subsection is to precisely define and outline the basic properties of these manifolds. The notion of the barely symplectic manifold should be contrasted with the related notion of the *folded symplectic manifold* introduced in Ref. 27. Rather than exhibiting singularities as a parameter tends to a limiting value, folded symplectic forms degenerate on hypersurfaces in phase space.

Definition 18. A barely symplectic form on a manifold M is a smooth ϵ-dependent 2-form Ω_ϵ such that

- $\mathbf{d}\Omega_\epsilon = 0$ *for each* $\epsilon \in \mathbb{R}$ and
- Ω_ϵ *is symplectic whenever* $\epsilon \neq 0$.

The barely symplectic form Ω_ϵ is exact if $\Omega_\epsilon = -\mathbf{d}\vartheta_\epsilon$ for some smooth ϵ-dependent 1-form ϑ_ϵ. A(n) (exact) barely symplectic manifold is a pair (M, Ω_ϵ), where M is a manifold and Ω_ϵ is a(n) (exact) barely symplectic form on M. A Hamiltonian system on a barely symplectic manifold is a smooth ϵ-dependent vector field X_ϵ such that $\iota_{X_\epsilon}\Omega_\epsilon = \mathbf{d}H_\epsilon$ for some smooth ϵ-dependent function H_ϵ, referred to as the system's Hamiltonian.

Since a barely symplectic form is non-degenerate for non-zero ϵ, each barely symplectic form induces a ϵ-dependent Poisson structure with a possible singularity at $\epsilon = 0$. We may therefore classify barely symplectic forms according to the severity of that singularity.

Definition 19. Let Ω_ϵ be a barely symplectic form, and let $\mathcal{J}_\epsilon$ be the ϵ-dependent Poisson bivector defined for $\epsilon \neq 0$ by inverting Ω_ϵ. We say Ω_ϵ is regular if there is some non-negative integer d such that $\mathcal{J}_\epsilon = \epsilon^{-d} j_\epsilon$, where j_ϵ is a smooth ϵ-dependent bivector. The smallest d such that $\mathcal{J}_\epsilon = \epsilon^{-d} j_\epsilon$ is called the degeneracy index of Ω_ϵ. When no such d exists, Ω_ϵ is irregular.

Example 1. Let (M_1, Ω_1), (M_2, Ω_2) be a pair of symplectic manifolds, and let $f_1(\epsilon), f_2(\epsilon)$ be a pair of smooth functions, each with at most an isolated zero at $\epsilon = 0$. If we equip the product manifold $M = M_1 \times M_2$ with the ϵ-dependent 2-form $\Omega_\epsilon = f_1(\epsilon)\pi_1^*\Omega_1 + f_2(\epsilon)\pi_2^*\Omega_2$, where $\pi_k : M \to M_k$ denotes projection onto the kth factor, we obtain a barely symplectic manifold (M, Ω_ϵ).

When $f_1(\epsilon) = \epsilon^n$ and $f_2(\epsilon) = \epsilon^m$ with $n \geq m \geq 0$, Ω_ϵ is a regular barely symplectic form. Since the Poisson bivector associated with Ω_ϵ is given by $\mathcal{J}_\epsilon = \epsilon^{-n}\mathcal{J}_1 + \epsilon^{-m}\mathcal{J}_2 = \epsilon^{-n}(\mathcal{J}_1 + \epsilon^{n-m}\mathcal{J}_2)$, the degeneracy index of Ω_ϵ is $d = n$. To obtain an irregular barely symplectic form, we may set $f_1(\epsilon) = \exp(-1/\epsilon^2)$, $f_2(\epsilon) = 1$. Then, $\mathcal{J}_\epsilon = \exp(1/\epsilon^2)\mathcal{J}_1 + \mathcal{J}_2 = \exp(1/\epsilon^2)(\mathcal{J}_1 + \exp(-1/\epsilon^2)\mathcal{J}_2)$. Although $\mathcal{J}_\epsilon = j_\epsilon / f_1(\epsilon)$ with j_ϵ smooth, $1/f_1(\epsilon)$ tends to ∞ as $\epsilon \to 0$ so quickly that the singularity cannot be tamed by any power of ϵ.

Although a barely symplectic form Ω_ϵ may fail to be symplectic when $\epsilon = 0$, regular Ω_ϵ with degeneracy index d nevertheless behave much like ordinary, ϵ-independent symplectic forms. Heuristically, one just needs to include the first $d+1$ terms in the power series expansion $\Omega_\epsilon = \Omega_0 + \epsilon\Omega_1 + \cdots$, rather than Ω_0 by itself, to employ to symplectic methods. As an illustration of this heuristic, we have the following version of Darboux's theorem for regular barely symplectic manifolds:

Proposition 20 (Darboux theorem for regular barely symplectic manifolds). Let (M, Ω_ϵ) be a compact, regular, barely symplectic manifold with degeneracy index d. Assume that the de Rham cohomology class $[\Omega_\epsilon]$ defined by the barely symplectic form satisfies $[\Omega_\epsilon] = [\Omega_0]$ for each ϵ, and set $\Omega_\epsilon^{(d)} = \sum_{k=0}^{d} \epsilon^k \Omega_k$, where Ω_k is the k*th coefficient in the power series expansion $\Omega_\epsilon = \Omega_0 + \epsilon\Omega_1 + \cdots$. There exists an $\epsilon_0 > 0$ and a smooth ϵ-dependent diffeomorphism $\Psi_\epsilon : M \to M$, $\epsilon \in [-\epsilon_0, \epsilon_0]$, such that $\Psi_\epsilon^*\Omega_\epsilon = \Omega_\epsilon^{(d)}$.*

Proof. Define $\Omega_\epsilon^\lambda = [1-\lambda]\Omega_\epsilon + \lambda\Omega_\epsilon^{(d)}$, $\lambda \in [0,1]$. We would like to establish the non-degeneracy of Ω_ϵ^λ for sufficiently small ϵ. Note that $\Omega_\epsilon^\lambda = \Omega_\epsilon + \lambda[\Omega_\epsilon^{(d)} - \Omega_\epsilon] = \Omega_\epsilon + O(\lambda\epsilon^{d+1})$. Therefore, if $\mathcal{J}_\epsilon = \epsilon^{-d} j_\epsilon$ denotes the Poisson bivector induced by Ω_ϵ, we have

$$-\widehat{\mathcal{J}}_\epsilon \widehat{\Omega}_\epsilon^\lambda = \mathrm{id}_{TM} + \lambda\epsilon\widehat{\psi}_\epsilon,$$

where $\epsilon\widehat{\psi}_\epsilon = -\epsilon^{-d}\widehat{j}_\epsilon[\widehat{\Omega}_\epsilon^{(d)} - \widehat{\Omega}_\epsilon]$ is a smooth ϵ-dependent bundle map $TM \to TM$. By openness of the set of invertible matrices and compactness of M, there is therefore some $\epsilon_0 > 0$ such that $\mathrm{id}_{TM} + \lambda\epsilon\psi_\epsilon$ is invertible for $\epsilon \in [-\epsilon_0, \epsilon_0]$, with smooth ϵ-dependent inverse. This implies $\widehat{\Omega}_\epsilon^\lambda$ is invertible for non-zero $\epsilon \in [-\epsilon_0, \epsilon_0]$ with the inverse given by $(\widehat{\Omega}_\epsilon^\lambda)^{-1} = -\widehat{\chi}_\epsilon^\lambda \widehat{\mathcal{J}}_\epsilon = -\epsilon^{-d}\widehat{\chi}_\epsilon^\lambda \widehat{j}_\epsilon$, where $\widehat{\chi}_\epsilon^\lambda = [\mathrm{id}_{TM} + \lambda\epsilon\widehat{\psi}_\epsilon]^{-1}$.

Next, we will derive a useful formula for the difference $\Omega_\epsilon^{(d)} - \Omega_\epsilon$. By the cohomological condition $[\Omega_\epsilon] = [\Omega_0]$, we have $[\Omega_\epsilon - \Omega_0] = 0$, which implies that there is a smooth ϵ-dependent 1-form ϑ_ϵ such that $\Omega_\epsilon = \Omega_0 - \mathbf{d}\vartheta_\epsilon$. Let $\vartheta_\epsilon = \vartheta_0 + \epsilon\vartheta_1 + \cdots$ be that form's formal power series expansion, and set $\vartheta_\epsilon^{(d)} = \sum_{k=0}^{d}\epsilon^k\vartheta_k$. By the equality of Taylor series, we have $\mathbf{d}\vartheta_0 = 0$ and $\Omega_\epsilon^{(d)} = \Omega_0 - \mathbf{d}\vartheta_\epsilon^{(d)}$. In particular, we have the useful identity

$$\Omega_\epsilon^{(d)} - \Omega_\epsilon = \Omega_0 - \mathbf{d}\vartheta_\epsilon^{(d)} - \Omega_\epsilon = \mathbf{d}(\vartheta_\epsilon - \vartheta_\epsilon^{(d)}). \tag{4}$$

Finally, we will construct the diffeomorphism Ψ_ϵ. For $\epsilon \in [-\epsilon_0, \epsilon_0]$, define $G_\epsilon^\lambda = (\widehat{\Omega}_\epsilon^\lambda)^{-1}(\vartheta_\epsilon^{(d)} - \vartheta_\epsilon)$. Since $\vartheta_\epsilon^{(d)} - \vartheta_\epsilon = O(\epsilon^{d+1})$ and $(\Omega_\epsilon^\lambda)^{-1} = O(\epsilon^{-d})$, G_ϵ^λ depends smoothly on both ϵ and λ. If F_λ^ϵ denotes the $t = \lambda$ flow map for G_ϵ^λ, we therefore have

$$\begin{aligned}\frac{d}{d\lambda}(F_\lambda^\epsilon)^*\Omega_\epsilon^\lambda &= (F_\lambda^\epsilon)^*\mathbf{d}\Big(\iota_{G_\epsilon^\lambda}\Omega_\epsilon^\lambda + \vartheta_\epsilon - \vartheta_\epsilon^{(d)}\Big)\\ &= (F_\lambda^\epsilon)^*(\widehat{\Omega}_\epsilon^\lambda G_\epsilon^\lambda + \vartheta_\epsilon - \vartheta_\epsilon^{(d)})\\ &= 0,\end{aligned}$$

where we used formula (4) on the first line. This proves the theorem with $\Psi_\epsilon = (F_1^\epsilon)^{-1}$ since it implies $(F_1^\epsilon)^*\Omega_\epsilon^{(d)} = \Omega_\epsilon$. □

While we will not use the barely symplectic Darboux theorem directly in this article, it will be useful in what follows to distill the theorem's essential ingredient into the following lemma:

Lemma 21. Let M be a smooth manifold, and let j_ϵ and Ω_ϵ be the formal power series with bivector and 2-form coefficients, respectively. Assume there exists a non-negative integer d such that $-\widehat{j}_\epsilon\widehat{\Omega}_\epsilon = \epsilon^d \mathrm{id}_{TM}$, in the sense of formal power series. Then, the smooth ϵ-dependent 2-form $\Omega_\epsilon^{(d)} = \sum_{k=0}^d \epsilon^k\Omega_k$ has the following properties:

1. *Given a compact set $C \subset M$, there exists an $\epsilon_0 > 0$ such that, for each non-zero $\epsilon \in [-\epsilon_0, \epsilon_0]$, $\Omega_\epsilon^{(d)}$ is non-degenerate on C. Moreover, $(\widehat{\Omega}_\epsilon^{(d)})^{-1} = -\epsilon^{-d}\widehat{\kappa}_\epsilon$, where κ_ϵ is a zeroth-order approximation of j_ϵ.*
2. *There exists a formal power series κ_ϵ with bivector coefficients such that $-\epsilon^{-d}\widehat{\kappa}_\epsilon$ is a formal inverse for $\widehat{\Omega}_\epsilon^{(d)}$.*

Remark 22. The κ_ϵ in property (2) above is not, in general, equal to the κ_ϵ in property (1); the former is merely a formal power series with smooth bivector coefficients, while the latter is a smooth ϵ-dependent bivector defined on the compact set C. However, within C, the power series expansion of κ_ϵ from (1) agrees with κ_ϵ from (2), which justifies the abuse of notation.

Proof. Let $\Delta\Omega_\epsilon = \epsilon^{-(d+1)}(\Omega_\epsilon - \Omega_\epsilon^{(d)})$. By assumption, $\epsilon^d\mathrm{id}_{TM} = -\widehat{j}_\epsilon(\widehat{\Omega}_\epsilon^{(d)} + \epsilon^{d+1}\Delta\widehat{\Omega}_\epsilon)$, which may also be written as

$$\epsilon^d(\mathrm{id}_{TM} + \epsilon\widehat{j}_\epsilon\Delta\widehat{\Omega}_\epsilon) = -\widehat{j}_\epsilon\widehat{\Omega}_\epsilon^{(d)}.$$

It follows that the formal power series $\widehat{\kappa}_\epsilon$ with bundle-map coefficients ($T^*M \to TM$) defined by

$$\begin{aligned}\widehat{\kappa}_\epsilon &= (\mathrm{id}_{TM} + \epsilon\widehat{j}_\epsilon\Delta\widehat{\Omega}_\epsilon)^{-1}\widehat{j}_\epsilon\\ &= (\mathrm{id}_{TM} - \epsilon\widehat{j}_\epsilon\Delta\widehat{\Omega}_\epsilon + \epsilon^2[\widehat{j}_\epsilon\Delta\widehat{\Omega}_\epsilon]^2 + \cdots)\widehat{j}_\epsilon\end{aligned} \tag{5}$$

satisfies $-\widehat{\kappa}_\epsilon\widehat{\Omega}_\epsilon^{(d)} = \epsilon^d\mathrm{id}_{TM}$, in the sense of formal power series. We will therefore establish property (2) if we can demonstrate that $\widehat{\kappa}_\epsilon$ is the bundle map associated with some formal power series κ_ϵ with bivector coefficients. Equivalently, we must show $\widehat{\kappa}_\epsilon^* = -\widehat{\kappa}_\epsilon$, where $\widehat{\kappa}_\epsilon^*$ denotes the dual of $\widehat{\kappa}_\epsilon$. For this, we directly compute

$$\begin{aligned}\widehat{\kappa}_\epsilon^* &= -\widehat{j}_\epsilon(\mathrm{id}_{T^*M} + \epsilon\Delta\widehat{\Omega}_\epsilon\widehat{j}_\epsilon)^{-1}\\ &= -(\mathrm{id}_{TM} - \epsilon\widehat{j}_\epsilon\Delta\widehat{\Omega}_\epsilon + \epsilon^2[\widehat{j}_\epsilon\Delta\widehat{\Omega}_\epsilon]^2 + \cdots)\widehat{j}_\epsilon\\ &= -\widehat{\kappa}_\epsilon,\end{aligned}$$

where we have used the identity $\widehat{j}_\epsilon[\Delta\widehat{\Omega}_\epsilon\widehat{j}_\epsilon]^n = [\widehat{j}_\epsilon\Delta\widehat{\Omega}_\epsilon]^n\widehat{j}_\epsilon$ for each non-negative integer n. We conclude that $\Omega_\epsilon^{(d)}$ satisfies property (2).

Next, we establish property (1). Define the smooth ϵ-dependent bivector $\kappa_\epsilon^{(d)} = \sum_{k=0}^d \epsilon^k\kappa_k$. The formal power series identity $-\widehat{\kappa}_\epsilon\widehat{\Omega}_\epsilon^{(d)} = \epsilon^d\mathrm{id}_{TM}$ implies that the Taylor expansion of the smooth ϵ-dependent bundle map $-\widehat{\kappa}_\epsilon^{(d)}\widehat{\Omega}_\epsilon^{(d)}$ is given by $-\widehat{\kappa}_\epsilon^{(d)}\widehat{\Omega}_\epsilon^{(d)} = \epsilon^d\mathrm{id}_{TM} + O(\epsilon^{d+1})$. Taylor's theorem with remainder therefore implies that there is a smooth ϵ-dependent bundle map $\widehat{\psi}_\epsilon : TM \to TM$ such that $-\widehat{\kappa}_\epsilon^{(d)}\widehat{\Omega}_\epsilon^{(d)} = \epsilon^d(\mathrm{id}_{TM} + \epsilon\widehat{\psi}_\epsilon)$. Given a compact set $C \subset M$, we may choose ϵ_0 small enough so that $(\mathrm{id}_{TM} + \epsilon\widehat{\psi}_\epsilon)$ is invertible on C when $\epsilon \in [-\epsilon_0, \epsilon_0]$. Thus, in C and for non-zero $\epsilon \in [-\epsilon_0, \epsilon_0]$, we have

$$(\widehat{\Omega}_\epsilon^{(d)})^{-1} = -\epsilon^{-d}(\mathrm{id}_{TM} + \epsilon\widehat{\psi}_\epsilon)^{-1}\widehat{\kappa}_\epsilon^{(d)},$$

as claimed. □

As an immediate application of Lemma 21, we will prove the following refinement of Kruskal's result[9] for regular barely symplectic manifolds:

Proposition 23. Let X_ϵ be a nearly periodic Hamiltonian system on a barely symplectic manifold (M,Ω_ϵ). Assume that (M,Ω_ϵ) is exact and regular with degeneracy index d. There exists a normalizing transformation for X_ϵ with generator K_ϵ such that the formal power series $\overline{\Omega}_\epsilon = \exp(-\mathcal{L}_{K_\epsilon})\Omega_\epsilon$ truncates at $O(\epsilon^d)$,

$$\overline{\Omega}_\epsilon = -\mathbf{d}\overline{\theta}_0 - \epsilon\mathbf{d}\overline{\theta}_1 - \cdots - \epsilon^d\mathbf{d}\overline{\theta}_d,$$

where each $\overline{\theta}_k$ satisfies $\mathcal{L}_{R_0}\overline{\theta}_k = 0$.

Proof. Let R_ϵ be the roto-rate for X_ϵ. By Proposition 11, there is a formal power series $\overline{\vartheta}_\epsilon$ such that $\Omega_\epsilon = -\mathbf{d}\overline{\vartheta}_\epsilon$ and $\mathcal{L}_{R_\epsilon}\overline{\vartheta}_\epsilon = 0$. By Theorem 9, there exists a normalizing transformation for X_ϵ with generator G_ϵ. Applying $\exp(-\mathcal{L}_{G_\epsilon})$ to the identities $\mathcal{L}_{R_\epsilon}\overline{\vartheta}_\epsilon = 0$ and $\Omega_\epsilon = -\mathbf{d}\overline{\vartheta}_\epsilon$ therefore implies $\mathcal{L}_{R_0}\overline{\theta}_\epsilon = 0$ and $\omega_\epsilon = -\mathbf{d}\overline{\theta}_\epsilon$, where $\overline{\theta}_\epsilon = \exp(-\mathcal{L}_{G_\epsilon})\overline{\vartheta}_\epsilon$ and $\omega_\epsilon = \exp(-\mathcal{L}_{G_\epsilon})\Omega_\epsilon$.

We would like to construct a Lie transform with generator A_ϵ such that

$$[A_\epsilon, R_0] = 0, \tag{6}$$

$$\exp(-\mathcal{L}_{A_\epsilon})\omega_\epsilon = -\mathbf{d}(\overline{\theta}_0 + \epsilon\overline{\theta}_1 + \cdots + \epsilon^d\overline{\theta}_d). \tag{7}$$

If such an A_ϵ exists, then the Baker–Campbell–Hausdorff (BCH) formula implies $\exp(\mathcal{L}_{G_\epsilon})\exp(\mathcal{L}_{A_\epsilon})$ is a normalizing transformation with the desired properties. We will construct $\exp(\mathcal{L}_{A_\epsilon})$ as the composition of a sequence of Lie transforms $\exp(\mathcal{L}_{a_\epsilon^{(k)}})$ with generators $a_\epsilon^{(k)}$.

Let $\exp(\mathcal{L}_{a_\epsilon^{(1)}})$ be a Lie transform with generator $a_\epsilon^{(1)}$. Applying $\exp(-\mathcal{L}_{a_\epsilon^{(1)}})$ to $\overline{\theta}_\epsilon$ gives

$$\begin{aligned}\exp(-\mathcal{L}_{a_\epsilon^{(1)}})\overline{\theta}_\epsilon &= \overline{\theta}_\epsilon - \mathcal{L}_{a_\epsilon^{(1)}}\overline{\theta}_\epsilon + \cdots \\ &= \overline{\theta}_\epsilon - \iota_{a_\epsilon^{(1)}}\mathbf{d}\overline{\theta}_\epsilon - \mathbf{d}\iota_{a_\epsilon^{(1)}}\overline{\theta}_\epsilon + \cdots,\end{aligned}$$

where we have applied Cartan's formula for the Lie derivative. Suppose $a_\epsilon^{(1)}$ is chosen such that

$$\epsilon^{d+1}\overline{\theta}_{d+1} - \iota_{a_\epsilon^{(1)}}\mathbf{d}(\overline{\theta}_0 + \cdots + \epsilon^d\overline{\theta}_d) = 0. \tag{8}$$

Then, we would have $\exp(-\mathcal{L}_{a_\epsilon^{(1)}})\mathbf{d}\overline{\theta}_\epsilon = \mathbf{d}(\overline{\theta}_0 + \cdots + \epsilon^d\overline{\theta}_d) + O(\epsilon^{d+2})$, which is one step closer to our target (7). Hence, let us assess the solvability of Eq. (8).

With $\omega_\epsilon^{(d)} = -\mathbf{d}(\overline{\theta}_0 + \cdots + \epsilon^d\overline{\theta}_d)$, Eq. (8) reads

$$\widehat{\omega}_\epsilon^{(d)} a_\epsilon^{(1)} = -\epsilon^{d+1}\overline{\theta}_{d+1}. \tag{9}$$

Since Ω_ϵ is a regular barely symplectic form, there is a smooth ϵ-dependent bivector j_ϵ such that $\mathcal{J}_\epsilon = \epsilon^{-d}j_\epsilon$ inverts Ω_ϵ, i.e., $-\epsilon^{-d}\widehat{j}_\epsilon\widehat{\Omega}_\epsilon = \mathrm{id}_{TM}$. Applying the Lie transform $\exp(-\mathcal{L}_{G_\epsilon})$ to these identities gives

$$-\widehat{b}_\epsilon\widehat{\omega}_\epsilon = \epsilon^d\mathrm{id}_{TM}, \tag{10}$$

where $b_\epsilon = \exp(-\mathcal{L}_{G_\epsilon})j_\epsilon$. Lemma 21 therefore implies that there is a formal power series κ_ϵ with bivector coefficients such that $-\epsilon^{-d}\widehat{\kappa}_\epsilon$ is a formal inverse for $\widehat{\omega}_\epsilon^{(d)}$. Applying this formal inverse to both sides of Eq. (9) now gives a formula for $a_\epsilon^{(1)}$ with the desired properties,

$$a_\epsilon^{(1)} = \epsilon\widehat{\kappa}_\epsilon\overline{\theta}_{d+1}.$$

Note that since $\overline{\vartheta}_{d+1}$ and $\omega_\epsilon^{(d)}$ are each R_0-invariant, formula (5) implies that this $a_\epsilon^{(1)}$ satisfies $[a_\epsilon^{(1)}, R_0] = 0$.

Modulo exact 1-forms, we now have $\exp\left(-\mathcal{L}_{a_\epsilon^{(1)}}\right)\overline{\theta}_\epsilon = \overline{\theta}_\epsilon^{(d)} + O(\epsilon^{d+2})$, where the higher-order terms that not displayed are each R_0-invariant. Using the same procedure used to find $a_\epsilon^{(1)}$, we may now construct an $a_\epsilon^{(2)}$ such that $\exp(-\mathcal{L}_{a_\epsilon^{(2)}})\exp(-\mathcal{L}_{a_\epsilon^{(1)}})\overline{\theta}_\epsilon = \overline{\theta}_\epsilon^{(d)} + O(\epsilon^{d+3})$ modulo exact 1-forms, where again the higher-order terms are R_0-invariant, and $[a_\epsilon^{(2)}, R_0] = 0$. Repeating this construction *ad infinitum* produces a sequence of R_0-invariant $a_\epsilon^{(n)}$ such that $\ldots\exp(-\mathcal{L}_{a_\epsilon^{(3)}})\exp(-\mathcal{L}_{a_\epsilon^{(2)}})\exp(-\mathcal{L}_{a_\epsilon^{(1)}})\overline{\theta}_\epsilon = \overline{\theta}_\epsilon^{(d)} + O(\epsilon^\infty)$ modulo exact 1-forms. The BCH formula therefore gives us an A_ϵ defined by $\exp(-\mathcal{L}_{A_\epsilon}) = \cdots\exp(-\mathcal{L}_{a_\epsilon^{(3)}})\exp(-\mathcal{L}_{a_\epsilon^{(2)}})\exp(-\mathcal{L}_{a_\epsilon^{(1)}})$ with the desired properties. □

Example 2. Dynamics of a non-relativistic charged particle in a strong magnetic field are described by the Lorentz force system $\dot{\boldsymbol{v}} = \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{x})$, $\dot{\boldsymbol{x}} = \epsilon \boldsymbol{v}$, on $M = \mathbb{R}^3 \times \mathbb{R}^3 \ni (\boldsymbol{x}, \boldsymbol{v})$. Here $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ denotes the magnetic field. Assuming $|\boldsymbol{B}|$ is nowhere vanishing, this system comprises a Hamiltonian nearly periodic system on the regular, exact, barely symplectic manifold $(M, -\mathbf{d}\vartheta_\epsilon)$, where $\vartheta_\epsilon = \boldsymbol{A} \cdot d\boldsymbol{x} + \epsilon \boldsymbol{v} \cdot d\boldsymbol{x}$ and $H_\epsilon = \epsilon^2 \frac{1}{2}|\boldsymbol{v}|^2$.

Particles that move under the influence of the Lorentz force rapidly gyrate around magnetic field lines, while drifting relatively slowly along and across them. The slow drifts are described by the so-called guiding center theory. In Refs. 12 and 13, Littlejohn devised a method of computing normalizing transformations for the Lorentz force system that exposed the Hamiltonian structure underlying guiding center dynamics for the first time. As explained in Ref. 28, arbitrary choices inherent to Littlejohn's method of selecting the generator G_ϵ of the normalizing transformation may be performed to ensure

$$\begin{aligned}\exp(-\mathcal{L}_{G_\epsilon})\vartheta_\epsilon &= \boldsymbol{A} \cdot d\boldsymbol{x} + \epsilon(\boldsymbol{v} \cdot \boldsymbol{b})\boldsymbol{b} \cdot d\boldsymbol{x} \\ &\quad + \epsilon^2 \frac{1}{2|\boldsymbol{B}|}(\boldsymbol{v} \times \boldsymbol{b} \cdot d\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{b})[\nabla \boldsymbol{b} \cdot \boldsymbol{v} \times \boldsymbol{b}] \cdot d\boldsymbol{x}) + O(\epsilon^\infty)\end{aligned}$$

modulo exact 1-forms. Since the degeneracy index for $-\mathbf{d}\vartheta_\epsilon$ is $d = 2$, this truncation could have been predicted by Proposition 23.

We close this section by noting that irregular barely symplectic forms may degenerate so rapidly as $\epsilon \to 0$ that symplectic methods do not apply to them. We are currently unfamiliar with any physical examples of such forms. As such, we consider the task of developing tools to handle the irregular case beyond the scope of this article.

B. Variational characterization of slow manifolds in nearly periodic Hamiltonian systems

In Sec. III, we constructed slow manifolds for nearly periodic systems as fixed point sets for certain truncations of the roto-rate. To facilitate the study of normal stability in nearly periodic Hamiltonian systems, this subsection will enhance that construction through the use of the adiabatic invariant μ_ϵ described in Definition 12. In particular, for nearly periodic Hamiltonian systems on regular, exact, barely symplectic manifolds, we will construct slow manifolds that coincide with the set of critical points for certain truncations of μ_ϵ. These manifolds will also comprise fixed-point sets of truncations of the roto-rate and thereby represent special cases of the manifolds given by Theorem 14.

Our construction will make use of the well-known result[29] from symplectic geometry that any $U(1)$-momentum map on a symplectic manifold is Morse–Bott with the critical manifold equal to the set of fixed points for the underlying $U(1)$-action. For us, certain truncations of the adiabatic invariant will serve as the momentum map, and the corresponding critical manifold will provide us with our desired slow manifold. However, since regular barely symplectic manifolds are not the same as ordinary symplectic manifolds, we will need to resort to the heuristic mentioned in Sec. IV A in our analysis. In particular, if Ω_ϵ is a regular barely symplectic form with degeneracy index d, we should include at least the first $d + 1$ terms in the 2-form's ϵ-power series before proceeding with symplectic methods.

We begin with a precise definition of Morse–Bott functions.

Definition 24. Let $f : M \to \mathbb{R}$ be a smooth function on a manifold M, and denote the set of critical points for f as $S_f = \{s \in M \mid \mathbf{d}f_s = 0\}$. For each $s \in S_f$, the quadratic term in f's Taylor expansion at s defines a unique symmetric bilinear form $\boldsymbol{H}_s(f) \in T_s^*M \otimes T_s^*M$ called the Hessian form. The function f is Morse–Bott when

- *the set S_f is a smooth embedded submanifold and*
- *for each $s \in S_f$, the null space of $\widehat{\boldsymbol{H}}_s(f) : T_sM \to T_s^*M$ is precisely T_sS_f.*

When f is Morse–Bott, we say S_f is the critical manifold for f. Since $\operatorname{im}\widehat{\boldsymbol{H}}_s(f) = (T_sS_f)_0$, the space of 1-forms that annihilate T_sS_f, $\widehat{H}_s(f)$ induces a linear isomorphism $\widehat{\boldsymbol{H}}_s^\perp(f) : T_sM/T_sS_f \to (T_sS_f)_0$ called the transverse Hessian operator.

Next, we explain an important mechanism by which Morse–Bott functions arise in nature. In particular, we will show that Noether invariants are automatically Morse–Bott. This result is striking since generic smooth functions are not Morse–Bott.

Proposition 25 (Ref. 29). If (M, Ω) is a symplectic manifold equipped with a symplectic $U(1)$-action and corresponding momentum map μ, then μ is Morse–Bott.

Proof. Let $\Phi_\theta : M \to M$ denote the symplectic $U(1)$-action, and let R denote the corresponding infinitesimal generator. By the definition of momentum maps, we have $\iota_R\Omega = \mathbf{d}\mu$. It follows from the non-degeneracy of Ω that the set of critical points S_μ for μ coincides with the set of fixed points for R. Since the set of fixed points for any circle action is a smooth submanifold by Lemma 16, we see that $S_\mu \subset M$ is a smooth submanifold.

Now, consider a point $s \in S_\mu$ and the corresponding Hessian form $\boldsymbol{H}_s(\mu)$. Let $\widehat{r} : T_sM \to T_sM$ be the infinitesimal generator of the linearized $U(1)$-action $T_s\Phi_\theta : T_sM \to T_sM$. Taylor expanding $\iota_R\Omega = \mathbf{d}\mu$ to first order at s implies $\Omega_s(\widehat{r}_sX_1, X_2) = \boldsymbol{H}_s(\mu)(X_1, X_2)$ for each pair $X_1, X_2 \in T_sM$. In terms of the bundle maps corresponding to Ω and $\boldsymbol{H}(\mu)$, the last condition is equivalent to

$$\widehat{\Omega}_s\widehat{r}_s = \widehat{\boldsymbol{H}}_s(\mu). \tag{11}$$

By the non-degeneracy of Ω_s, (11) shows $\widehat{\boldsymbol{H}}_s(\mu)X = 0$ if and only if $\widehat{r}_s X = 0$. This completes the proof since $\ker \widehat{r}_s = T_s S_\mu$. □

To connect the preceding results with nearly periodic Hamiltonian systems on barely symplectic manifolds, we must reckon with the fact that the $\epsilon \to 0$ limit of a barely symplectic form Ω_ϵ may be degenerate. This implies that Proposition 25 must be applied with care, since the non-degeneracy of Ω is crucial to that result. We are therefore motivated to introduce the notion of a *barely Morse–Bott* function.

Definition 26. A smooth ϵ-dependent function $f_\epsilon : M \to \mathbb{R}$ is barely Morse–Bott if there is an $\epsilon_0 > 0$, a manifold Σ_0, and a smooth ϵ-dependent embedding $\mathcal{S}_\epsilon : \Sigma_0 \to M$, $\epsilon \in [-\epsilon_0, \epsilon_0]$, such that

- *f_ϵ is Morse–Bott for each non-zero $\epsilon \in [-\epsilon_0, \epsilon_0]$ and*
- *$\mathcal{S}_\epsilon(\Sigma_0)$ is the critical manifold for f_ϵ for each non-zero $\epsilon \in [-\epsilon_0, \epsilon_0]$.*

We say $\mathcal{S}_\epsilon$ is the critical embedding for f_ϵ. A barely Morse–Bott function f_ϵ is regular if for each non-zero $\epsilon \in [-\epsilon_0, \epsilon_0]$ and critical point $s \in S_{f_\epsilon}$ we have $[\widehat{\boldsymbol{H}}_s^\perp(f_\epsilon)]^{-1} = \epsilon^{-\ell}\widehat{\boldsymbol{L}}_s(f_\epsilon)$ for some non-negative integer ℓ and smooth ϵ-dependent bundle map $\widehat{\boldsymbol{L}}_s(f_\epsilon) : (T_s S_{f_\epsilon})_0 \to T_s M / T_s S_{f_\epsilon}$. The smallest such ℓ is the degeneracy index of f_ϵ.

As a final preparatory step, we introduce some useful terminology for discussing nearly periodic Hamiltonian systems with adiabatic invariants μ_ϵ that vanish at the first few orders in ϵ. While this vanishing phenomenon may seem like a technical curiosity, it occurs in important applications such as magnetized charged particle dynamics and also may be exploited to strengthen our eventual results on normal stability.

Definition 27. If X_ϵ is a nearly periodic Hamiltonian system with adiabatic invariant $\mu_\epsilon = \mu_0 + \epsilon\mu_1 + \cdots$, the vanishing index for μ_ϵ is the largest non-negative integer v such that $\mu_\epsilon = O(\epsilon^v)$. If μ_ϵ has vanishing index v so that $\mu_k = 0$ for $k < v$, the associated reduced adiabatic invariant $\mu_\epsilon^* = \mu_0^* + \epsilon\mu_1^* + \epsilon^2\mu_2^* + \cdots$ is the unique formal power series such that

$$\mu_\epsilon = \epsilon^v \mu_\epsilon^*. \tag{12}$$

Theorem 28. *Let X_ϵ be a nearly periodic Hamiltonian system on the barely symplectic manifold (M, Ω_ϵ). Assume Ω_ϵ is exact and regular with degeneracy index d. Also assume that the adiabatic invariant $\mu_\epsilon = \epsilon^v\mu_\epsilon^*$ has vanishing index $v \geq 0$. For each integer $N \geq 0$ and compact, codimension-0 submanifold $C_0 \subset M$, with or without boundary, there exists an increasing sequence of codimension-0 compact submanifolds, $C_0 \subset C_0' \subset C_0''$, with $\operatorname{int} C_0' \supset C_0$, $\operatorname{int} C_0'' \supset C_0'$, and a smooth ϵ-dependent function $\mu_\epsilon^{*(N)} : \operatorname{int} C_0'' \to \mathbb{R}$ such that we have the following:*

1. *$\mu_\epsilon^{*(N)} - \mu_\epsilon^* = O(\epsilon^{N+1})$ on $\operatorname{int} C_0'$;*
2. *$\mu_\epsilon^{*(N)}$ is barely Morse–Bott with critical embedding $\mathcal{S}_\epsilon^{(N)} : S_0 \cap \operatorname{int} C_0'' \to \operatorname{int} C_0''$;*
3. *$\mu_\epsilon^{*(N)}$ is regular with the degeneracy index at most $d - v$; and*
4. *$\mathcal{S}_\epsilon^{(N)} \mid S_0 \cap \operatorname{int} C_0$ is an Nth-order parameterized slow manifold for X_ϵ whose image is contained in $\operatorname{int} C_0'$ for small enough ϵ.*

Remark 29. The theorem does not say $\mu_\epsilon^* - \mu_\epsilon^{*(N)} = O(\epsilon^{N+1})$ on $\operatorname{int} C_0''$.

Proof. The proof begins by repeating the argument from the Proof of Theorem 14 but using the generator K_ϵ provided by Proposition 23 in place of the generator G_ϵ provided by Proposition 9. Recall that this argument constructs an increasing sequence of compact codimension-0 submanifolds $C_0 \subset C_0' \subset C_0''$ with the desired nesting property when $\partial C_0 \neq \emptyset$; when $\partial C_0 = \emptyset$, we now take $C_0'' = C_0' = C_0$. In this manner, for each integer $N \geq 0$, we obtain a smooth ϵ-dependent diffeomorphism $\Psi_\epsilon : M \to M$ that is equal to the exponential of the vector field $\mathcal{K}_\epsilon^{(N)} = wK_\epsilon^{(N)}$. Here, $w = 1$ in C_0' and $w = 0$ outside of C_0'', which ensures $\Psi_\epsilon(C_0'') = C_0''$. We also know that $\mathcal{S}_\epsilon^{(N)} = \Psi_\epsilon^{-1}|S_0 \cap \operatorname{int} C_0''$ gives a smooth ϵ-dependent embedding that restricts to an Nth-order parameterized slow manifold on $S_0 \cap \operatorname{int} C_0$.

Next, we construct the function $\mu_\epsilon^{*(N)}$. Let d denote the degeneracy index for Ω_ϵ, and let $\overline{\Theta}_k$, $k \in 0, \ldots, d$, be the 1-forms given by Proposition 23. We introduce the smooth ϵ-dependent function $\overline{\mu}_\epsilon = \iota_{R_0}(\overline{\Theta}_0 + \epsilon\overline{\Theta}_1 + \cdots + \epsilon^d\overline{\Theta}_d)$. By construction, this function satisfies the Hamilton equation $\mathbf{d}\overline{\mu}_\epsilon = \iota_{R_0}\overline{\Omega}_\epsilon$, where $\overline{\Omega}_\epsilon$ is defined in the statement of Proposition 23. Since $\exp(\mathcal{L}_{K_\epsilon})\overline{\Omega}_\epsilon = \Omega_\epsilon$ and $\exp(\mathcal{L}_{K_\epsilon})R_0 = R_\epsilon$, we also have the formal power series identity $\iota_{R_\epsilon}\Omega_\epsilon = \mathbf{d}[\exp(\mathcal{L}_{K_\epsilon})\overline{\mu}_\epsilon]$. However, because the same identity is satisfied with the adiabatic invariant μ_ϵ in place of $\exp(\mathcal{L}_{K_\epsilon})\overline{\mu}_\epsilon$, we must have $\exp(\mathcal{L}_{K_\epsilon})\overline{\mu}_\epsilon = \mu_\epsilon + c_\epsilon$, where c_ϵ is some formal power series with constant coefficients. Using the fact that μ_ϵ and $\exp(\mathcal{L}_{K_\epsilon})\overline{\mu}_\epsilon$ each vanish on the zero locus $R_\epsilon = 0$, we find $c_\epsilon = 0$, whence it follows $\exp(\mathcal{L}_{K_\epsilon})\overline{\mu}_\epsilon = \mu_\epsilon$. If v denotes the vanishing index for μ_ϵ, we therefore obtain $\overline{\mu}_\epsilon = \epsilon^v \exp(-\mathcal{L}_{K_\epsilon})\mu_\epsilon^*$, which can only be satisfied if $\iota_{R_0}\overline{\Theta}_k = 0$ for $k = 0, \ldots, v-1$, or $\overline{\mu}_\epsilon = \epsilon^v\iota_{R_0}(\overline{\Theta}_v + \cdots + \epsilon^d\overline{\Theta}_d) = \epsilon^v(\overline{\mu}_0^* + \epsilon\overline{\mu}_1^* + \cdots + \epsilon^{d-v}\overline{\mu}_{d-v}^*)$, where $\overline{\mu}_k^* = \overline{\mu}_{v+k}$, $k = 0, \ldots, d-v$. Finally, we define

$$\mu_\epsilon^{*(N)} = \Psi_\epsilon^*(\overline{\mu}_0^* + \epsilon\overline{\mu}_1^* + \cdots + \epsilon^{d-v}\overline{\mu}_{d-v}^*). \tag{13}$$

Now, we would like to show that $\mu_\epsilon^{*(N)}$ defined in (13) is a regular barely Morse–Bott function with the degeneracy index at most $d - v$ and critical embedding $\mathcal{S}_\epsilon^{(N)}$. We will argue by showing that $\overline{\mu}_\epsilon^* = \Phi_{\epsilon*}\mu_\epsilon^{*(N)}$ is a regular barely Morse–Bott function on $\operatorname{int} C_0''$. First observe that because Ω_ϵ is a regular barely symplectic form with degeneracy index d, the 2-form $\overline{\Omega}_\epsilon$ has a formal inverse of the form $\epsilon^{-d}\bar{j}_\epsilon$, where $\bar{j}_\epsilon$ is a formal power series with bivector coefficients. Lemma 21 therefore implies that there is an $\epsilon_0 > 0$ such that $\overline{\Omega}_\epsilon$ is symplectic when restricted to C_0'' for all non-zero $\epsilon \in [-\epsilon_0, \epsilon_0]$. Moreover, there is a smooth ϵ-dependent bivector $\overline{\kappa}_\epsilon$ such that $(\widehat{\overline{\Omega}}_\epsilon)^{-1} = -\epsilon^{-d}\widehat{\overline{\kappa}}_\epsilon$ for non-zero $\epsilon \in [-\epsilon_0, \epsilon_0]$. Because $\mathbf{d}\overline{\mu}_\epsilon = \iota_{R_0}\overline{\Omega}_\epsilon$, Lemma 25 implies $\overline{\mu}_\epsilon \mid C_0''$, and therefore, $\overline{\mu}_\epsilon^* \mid C_0'' = \epsilon^{-\nu}\mu_\epsilon \mid C_0''$ is Morse–Bott for non-zero $\epsilon \in [-\epsilon_0, \epsilon_0]$ with the critical manifold $S_0 \cap \operatorname{int} C_0''$. This shows that $\overline{\mu}_\epsilon^* \mid \operatorname{int} C_0''$ is barely Morse–Bott with critical embedding $s_0 \in S_0 \cap \operatorname{int} C_0'' \mapsto s_0 \in \operatorname{int} C_0''$. To see that $\overline{\mu}_\epsilon^*$ is regular, first note that for each $s \in S_0 \cap \operatorname{int} C_0''$, we have $\widehat{\overline{\Omega}}_{\epsilon s}\widehat{r}_s = \epsilon^\nu \widehat{\boldsymbol{H}}_s(\overline{\mu}_\epsilon^*)$, where $\widehat{r}_s$ denotes the linearization of R_0 at s. Then, observe that because $\ker \widehat{r}_s = T_s S_0$, the map $\widehat{r}_s$ induces a linear isomorphism $\widehat{r}_s^\perp : T_s M / T_s S_0 \to \widehat{\overline{\Omega}}_{\epsilon s}^{-1}(T_s S_0)_0$, where $(T_s S_0)_0 \subset T_s^* M$ comprises covectors at s that annihilate tangent vectors in $T_s S_0$. Indeed, if $u \in \operatorname{im}\widehat{r}_s$, then $u = \widehat{r}_s w$ for some $w \in T_s M$, which implies $u = \widehat{\overline{\Omega}}_{\epsilon s}^{-1}\widehat{\overline{\Omega}}_{\epsilon s}\widehat{r}_s w = \epsilon^\nu \widehat{\overline{\Omega}}_{\epsilon s}^{-1}\widehat{\boldsymbol{H}}_s(\overline{\mu_\epsilon^*})w \in \widehat{\overline{\Omega}}_{\epsilon s}^{-1}(T_s S_0)_0$, and therefore, $\operatorname{im}\widehat{r}_s = \widehat{\overline{\Omega}}_{\epsilon s}^{-1}(T_s S_0)_0$ by a dimension count. It follows that the transverse Hessian operator is given by $\widehat{\boldsymbol{H}}_s^\perp(\overline{\mu}_\epsilon^*) = \epsilon^{-\nu}\widehat{\overline{\Omega}}_{\epsilon s}\widehat{r}_s^\perp$, whose inverse is given by

$$
\begin{aligned}
(\widehat{\boldsymbol{H}}_s^\perp(\overline{\mu}_\epsilon^*))^{-1}(\alpha) &= \epsilon^\nu(\widehat{r}_s^\perp)^{-1}(\widehat{\overline{\Omega}}_{\epsilon s})^{-1}(\alpha) \\
&= -\epsilon^{\nu-d}(\widehat{r}_s^\perp)^{-1}\widehat{\overline{\kappa}}_{\epsilon s}(\alpha),
\end{aligned}
$$

for each $\alpha \in (T_s S_0)_0$. We conclude that $\overline{\mu}_\epsilon^*$ is regular with degeneracy index $\ell \le d - \nu$.

To complete the proof, we now recall that our previous remarks imply $\exp(\mathcal{L}_{K_\epsilon})\overline{\mu}_\epsilon^* = \mu_\epsilon^*$ in the sense of formal power series. This implies that $\mu_\epsilon^{*(N)} = \Psi_\epsilon^*\overline{\mu}_\epsilon^*$ agrees with μ_ϵ^* within $O(\epsilon^{N+1})$ on $\operatorname{int} C_0'$ because so does $\mathcal{K}_\epsilon = wK_\epsilon^{(N)}$ agree with K_ϵ on $\operatorname{int} C_0'$. □

C. Free-action stability principle

We now find ourselves in a good position to prove the free-action principle for the slow manifolds provided by Theorem 28. In our proof, we will bound the distance between a trajectory and a normally elliptic slow manifold using adiabatic invariance and the quadratic approximation of the adiabatic invariant along the slow manifold. To that end, we will need a pair of technical lemmas.

Lemma 30. Let (E, g) be a real inner-product space with inner product g. Let $D_\epsilon : E \to E$ be a smooth ϵ-dependent linear map. Suppose there exists a positive real number ϵ_0 such that D_ϵ is positive definite for all $\epsilon \in (0, \epsilon_0]$. Then, for all $e \in E$ and $\epsilon \in (0, \epsilon_0]$,

$$
g(e, e) \le \|[D_\epsilon]^{-1}\| g(e, D_\epsilon e),
$$

where $\|\cdot\|$ denotes the induced operator norm.

Proof. Let $\lambda(A)$ and $\Lambda(A)$ denote the smallest and largest eigenvalues of a linear map $A : E \to E$, respectively. Define the induced operator norm $\|A\| = \sup_{\|e\|=1}\|Ae\|$. Recall that whenever A is symmetric positive-definite, we have $\|A\| = \Lambda(A)$.

Since D_ϵ is symmetric positive-definite for $\epsilon \in (0, \epsilon_0]$, we have the simple inequality

$$
g(e, D_\epsilon e) \ge \lambda(D_\epsilon) g(e, e)
$$

for all $e \in E$ and $\epsilon \in (0, \epsilon_0]$. Since $\lambda(D_\epsilon) = 1/\Lambda([D_\epsilon]^{-1})$ and $\Lambda([D_\epsilon]^{-1}) = \|[D_\epsilon]^{-1}\|$, the desired result follows. □

Lemma 31. Let X_ϵ be a C^∞ nearly periodic Hamiltonian system on M with reduced adiabatic invariant μ_ϵ^*. Fix an $\epsilon_0 > 0$, a compact set $C \subset M$, a non-negative integer N, and a smooth ϵ-dependent function $\mu_\epsilon^{*(N)}$ with $\mu_\epsilon^* - \mu_\epsilon^{*(N)} = O(\epsilon^{N+1})$ on C. For each non-negative integer k, there is a k-dependent constant $\chi_k > 0$ such that

$$
\forall t \in [-\epsilon^{-k}, \epsilon^{-k}]\ |\mu_\epsilon^{*(N)}(z(t)) - \mu_\epsilon^{*(N)}(z(0))| \le \epsilon^{N+1}\chi_k \tag{14}
$$

for all X_ϵ-integral curves $z : \mathbb{R} \to M$ contained in C and all $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Proof. The result follows from two basic estimates.

First estimate: For each $n \ge N$, let $\mu_\epsilon^{*(n)}$ be a smooth, ϵ-dependent, nth-order approximation of μ_ϵ^*. (Such functions may be constructed by merely truncating the formal power series μ_ϵ^* at the appropriate order.) By all-orders invariance of μ_ϵ^*, the smooth ϵ-dependent function $\mathcal{L}_{X_\epsilon}\mu_\epsilon^{*(n)}$ has the formal power series expansion

$$
\mathcal{L}_{X_\epsilon}\mu_\epsilon^{*(n)} = \mathcal{L}_{X_\epsilon}(\mu_\epsilon^{*(n)} - \mu_\epsilon^*) = O(\epsilon^{n+1}).
$$

Taylor's theorem with remainder therefore implies the existence of a smooth ϵ-dependent function $f_\epsilon^{(n)}$ such that $\mathcal{L}_{X_\epsilon}\mu_\epsilon^{*(n)} = \epsilon^{n+1}f_\epsilon^{(n)}$. Thus, if $z : \mathbb{R} \to M$ is any X_ϵ-integral curve, we have

$$\mu_\epsilon^{*(n)}(z(t)) - \mu_\epsilon^{*(n)}(z(0)) = \epsilon^{n+1}\int_0^t f_\epsilon^{(n)}(z(\bar{t}))d\bar{t} \tag{15}$$

for each $t \in \mathbb{R}$. Let $F^{(n)}$ denote the maximum value of the continuous function $(\epsilon, z) \mapsto |f_\epsilon^{(n)}(z)|$ on the compact set $[-\epsilon_0, \epsilon_0] \times C$. Formula (15) implies, in particular,

$$|\mu_\epsilon^{*(n)}(z(t)) - \mu_\epsilon^{*(n)}(z(0))| \leq \epsilon^{n+1}|t|F^{(n)} \tag{16}$$

for each X_ϵ-integral curve z contained in C and each $t \in \mathbb{R}$. Equation (16) provides our first important estimate.

Second estimate: Because $n \geq N$, there must be a smooth ϵ-dependent function $\Delta\mu_\epsilon^{(n,N)}$ such that $\mu_\epsilon^{*(n)} - \mu_\epsilon^{*(N)} = \epsilon^{N+1}\Delta\mu_\epsilon^{*(n,N)}$ in C. If $\Delta\mu^{*(n,N)}$ denotes the maximum value of $(\epsilon, z) \mapsto |\Delta\mu_\epsilon^{*(n,N)}(z)|$ on $[-\epsilon_0, \epsilon_0] \times C$, we therefore have the following bound on the difference:

$$|\mu_\epsilon^{*(n)}(z) - \mu_\epsilon^{*(N)}(z)| \leq \epsilon^{N+1}\Delta\mu^{*(n,N)} \tag{17}$$

for $(\epsilon, z) \in [-\epsilon_0, \epsilon_0] \times C$.

Combining the estimates: Using estimates (16) and (17) with $n = N + k$, we now have

$$\begin{aligned}
|\mu_\epsilon^{*(N)}(z(t)) - \mu_\epsilon^{*(N)}(z(0))| &= \Big|\Big(\mu_\epsilon^{*(N)}(z(t)) - \mu_\epsilon^{*(N+k)}(z(t))\Big) \\
&\quad - \Big(\mu_\epsilon^{*(N)}(z(0)) - \mu_\epsilon^{*(N+k)}(z(0))\Big) \\
&\quad + \Big(\mu_\epsilon^{*(N+k)}(z(t)) - \mu_\epsilon^{*(N+k)}(z(0))\Big)\Big| \\
&\leq 2\epsilon^{N+1}\Delta\mu^{*(N+k,N)} + \epsilon^{N+1+k}|t|F^{(N+k)} \\
&\leq \epsilon^{N+1}\Big(2\Delta\mu^{*(N+k,N)} + F^{(N+k)}\Big)
\end{aligned}$$

for each X_ϵ-integral curve $z : \mathbb{R} \to M$ contained in C and $t \in [-\epsilon^{-k}, \epsilon^{-k}]$. This proves the theorem with $\chi_k = 2\Delta\mu^{*(N+k,N)} + F^{(N+k)}$. □

Theorem 32 *(free-action principle). Let X_ϵ be a nearly periodic Hamiltonian system on the barely symplectic manifold (M, Ω_ϵ). Assume Ω_ϵ is regular and exact, with degeneracy index d. Also assume the adiabatic invariant μ_ϵ has vanishing index $v \geq 0$. Fix a compact codimension-0 submanifold, with or without boundary, $C_0 \subset M$. Let $\mathcal{S}_\epsilon^{(N)} : S_0 \cap \operatorname{int} C_0 \to M$ and $\mu_\epsilon^{*(N)}$ denote the Nth-order parameterized slow manifold and approximate adiabatic invariant provided by Theorem 28, respectively. We require $N + 1 > 3(d - v)$.*

Assume that for all sufficiently-small ϵ, $\boldsymbol{H}_s(\mu_\epsilon^*)$ is positive or negative semi-definite for all s in the closure of the image of $\mathcal{S}_\epsilon^{(N)}$. There is an $\epsilon_0 > 0$ such that for all non-zero $\epsilon \in [-\epsilon_0, \epsilon_0]$ and all X_ϵ-integral curves $z : \mathbb{R} \to M$ contained in C_0 that begin within ϵ^{N+1} of $S_\epsilon^{(N)} = \mathcal{S}_\epsilon^{(N)}(S_0)$, z will either (a) remain within $\epsilon^{(N+1-d+v)/2}$ of $S_\epsilon^{(N)}$ for $t \in [-\epsilon^{-k}, \epsilon^{-k}]$ for each positive integer k or (b) eventually run off the edge of $S_\epsilon^{(N)}$.

Remark 33. Note that the bound on a trajectory's normal deviation becomes weaker as $d - v$ becomes larger. In particular, the theorem provides no bound at all for slow manifolds with order $N \leq 3(d - v) - 1$. This suggests that, in general, larger degeneracy indices for the barely symplectic form have destablizing effects on the slow manifolds constructed in this article. It also suggests that larger vanishing indices for the adiabatic invariant have a stablizing effect. We describe a particular way in which these effects manifest themselves in an example following the proof of the theorem.

Proof. Given a submanifold $S \subset M$, denote the normal bundle to S with respect to the auxiliary Riemannian metric g on M as NS. Denote the radius r tubular neighborhood of S with respect to g as $\mathcal{T}_r(S) = \{m \in M \mid \text{distance}(m, S) < r\}$; the closure of $\mathcal{T}_r(S)$ as $\overline{\mathcal{T}}_r(S)$; and the radius $\leq r$ restriction of the normal bundle as $N^r S$.

Choose ϵ_0 small enough to ensure $\overline{\mathcal{T}}_{\epsilon^{N+1}}(S_\epsilon^{(N)}) \subset C_0'$ and $\mathcal{T}_{r_0}(S_\epsilon^{(N)}) \approx N^{r_0} S_\epsilon^{(N)}$ by way of the Riemannian exponential map for some $O(1)$ positive constant r_0 and for each $\epsilon \in [-\epsilon_0, \epsilon_0]$. Note that the set $\overline{\mathcal{T}}_{\epsilon^{N+1}}(S_\epsilon^{(N)})$, being a closed subset of the compact set C_0', is itself compact. Also note that by way of the diffeomorphism $\mathcal{T}_{r_0}(S_\epsilon^{(N)}) \approx N^{r_0} S_\epsilon^{(N)}$, we may identify points in $\mathcal{T}_{\epsilon^{N+1}}(S_\epsilon^{(N)})$ with pairs (s, n), where $s \in S_\epsilon^{(N)}$ and $n \in N_s S_\epsilon^{(N)}$.

Since, for each sufficiently small ϵ, $\boldsymbol{H}_s(\mu_\epsilon^{*(N)})$ is sign-semi-definite for all $s \in \overline{S}_\epsilon^{(N)}$, we may shrink ϵ_0 in order to ensure sign semi-definiteness for each $s \in \overline{S}_\epsilon^{(N)}$ uniformly in $\epsilon \in [-\epsilon_0, \epsilon_0]$. If we introduce the symmetric linear map $\boldsymbol{D}_{\epsilon s} : N_s S_\epsilon^{(N)} \to N_s S_\epsilon^{(N)}$ by requiring $\boldsymbol{H}_s(\mu_\epsilon^{*(N)})(n, n) = g_s(n, \boldsymbol{D}_{\epsilon s} n)$ for each $n \in N_s S_\epsilon^{(N)}$, the barely Morse–Bott property for $\mu_\epsilon^{*(N)}$ therefore implies that $\boldsymbol{D}_{\epsilon s}$ is symmetric positive- or negative-definite for each $s \in \overline{S}_\epsilon^{(N)}$ and non-zero $\epsilon \in [-\epsilon_0, \epsilon_0]$.

Because $\mu_\epsilon^{*(N)}$ is regular with the degeneracy index at most $d-\nu$, Lemma 30 and compactness of $\overline{S}_\epsilon^{(N)}$ imply there is a positive constant D_0, depending only on $S_\epsilon^{(N)}$ and $\mu_\epsilon^{*(N)}$, such that

$$g_s(n,n) \le \frac{1}{\epsilon^{d-\nu}D_0}|g_s(n,\boldsymbol{D}_{\epsilon s}n)| \tag{18}$$

for each $s\in\overline{S}_\epsilon^{(N)}$, $n\in N_s\overline{S}_\epsilon^{(N)}$, and non-zero $\epsilon\in[-\epsilon_0,\epsilon_0]$. By Taylor's theorem with remainder and $\mu_\epsilon^{*(N)}=0$ on $S_\epsilon^{(N)}$, we also have the following inequality relating $\boldsymbol{D}_{\epsilon s}$ to $\mu_\epsilon^{*(N)}$:

$$|g_s(n,\boldsymbol{D}_{\epsilon s}n)| \le |\mu_\epsilon^{*(N)}(s,n)| + T_0[g_s(n,n)]^{3/2} \tag{19}$$

for each $s\in\overline{S}_\epsilon^{(N)}$, $n\in N_s\overline{S}_\epsilon^{(N)}$, and $\epsilon\in[-\epsilon_0,\epsilon_0]$. Here, T_0 is a positive constant that depends only on $S_\epsilon^{(N)}$ and $\mu_\epsilon^{*(N)}$; it bounds the third normal derivative of $\mu_\epsilon^{*(N)}(s,n)$ on $\overline{S}_\epsilon^{(N)}$. Combining the previous two estimates, we therefore obtain the key geometric inequality

$$g_s(n,n) \le \frac{1}{\epsilon^{d-\nu}D_0}\Big(|\mu_\epsilon^{*(N)}(s,n)| + T_0[g_s(n,n)]^{3/2}\Big) \tag{20}$$

for each $s\in\overline{S}_\epsilon^{(N)}$, $n\in N_s\overline{S}_\epsilon^{(N)}$, and non-zero $\epsilon\in[-\epsilon_0,\epsilon_0]$.

Now, suppose that $z:\mathbb{R}\to M$ is an X_ϵ-integral curve contained in C_0 that begins in the narrow tubular neighborhood $\mathcal{T}_{\epsilon^{N+1}}(S_\epsilon^{(N)})\subset C_0'$. Let $\mathcal{I}_0=(a,b)$ be the maximal time interval during which $z(t)$ is contained in $\mathcal{T}_{r_0}(S_\epsilon^{(N)})$. For $t\in\mathcal{I}_0$, we write $z(t)=(s(t),n(t))$. By the geometric inequality (20) we have

$$g_{s(t)}(n(t),n(t)) \le \frac{1}{\epsilon^{d-\nu}D_0}\Big(|\mu_\epsilon^{*(N)}(s(t),n(t))| + T_0[g_{s(t)}(n(t),n(t))]^{3/2}\Big) \tag{21}$$

for $t\in\mathcal{I}_0$. However. by the near-constancy of $\mu_\epsilon^{*(N)}(s(t),n(t))$ given by Lemma 31, we anticipate that this inequality should allow us to bound the distance $d(t)=[g_{s(t)}(n(t),n(t))]^{1/2}$ between $S_\epsilon^{(N)}$ and $z(t)$. The following analysis makes this intuition precise.

By Lemma 31, for each non-negative integer k, there is a non-negative constant χ_k such that inequality (14) is satisfied for any X_ϵ-integral curve contained in C_0. The inequality holds, in particular, for $z(t)$ introduced in the previous paragraph for $t\in\overline{\mathcal{I}}_k\equiv[-\epsilon^{-k},\epsilon^k]\cap\mathcal{I}_0$, giving

$$g_{s(t)}(n(t),n(t)) \le \frac{1}{\epsilon^{d-\nu}D_0}\Big(\epsilon^{N+1}\chi_k + |\mu_\epsilon^{*(N)}(s(0)),n(0))| + T_0[g_{s(t)}(n(t),n(t))]^{3/2}\Big), \tag{22}$$

for $t\in\overline{\mathcal{I}}_k$. Note that if we introduce the polynomial

$$P_\epsilon(d) = d^2 - \frac{T_0}{\epsilon^{d-\nu}D_0}d^3, \tag{23}$$

we may write (22) equivalently as

$$P_\epsilon(d(t)) \le \frac{1}{\epsilon^{d-\nu}D_0}\Big(\epsilon^{N+1}\chi_k + |\mu_\epsilon^{*(N)}(s(0)),n(0))|\Big). \tag{24}$$

Again using Taylor's theorem with remainder and $\mu_\epsilon^{*(N)}=0$ on $S_\epsilon^{(N)}$, we estimate the size of the initial reduced adiabatic invariant according to

$$\begin{aligned}|\mu_\epsilon^{*(N)}(s(0)),n(0))| &\le |g_{s(0)}(n(0),\boldsymbol{D}_{\epsilon s(0)}n(0))| + T_0[g_{s(0)}(n(0),n(0))]^{3/2}\\ &\le D_1 d^2(0) + T_0 d^3(0)\\ &\le \epsilon^{2(N+1)}D_1 + \epsilon^{3(N+1)}T_0,\end{aligned} \tag{25}$$

where D_1 is the uniform bound on $\boldsymbol{D}_{\epsilon s}$ for $(\epsilon,s)\in[-\epsilon_0,\epsilon_0]\times\overline{S}_\epsilon^{(N)}$. Inequality (24) then becomes

$$P_\epsilon(d(t)) \le \frac{1}{\epsilon^{d-\nu}D_0}\Big(\epsilon^{N+1}\chi_k + \epsilon^{2(N+1)}D_1 + \epsilon^{3(N+1)}T_0\Big) \tag{26}$$

for $t \in \overline{\mathcal{I}}_k$. Now, for $d \geq 0$, P_ϵ increases monotonically from 0 before reaching its maximum value of $P_{\max} = (4/27)\epsilon^{2(d-v)}(D_0/T_0)^2$ at $d_{\max} = (2/3)\epsilon^{d-v}(D_0/T_0)$. Thus, if $N+1-d+v > 2(d-v)$ and we shrink ϵ_0, if necessary, then $d(0) < \epsilon^{N+1} < d_{\max}$ and $P(d(t)) < P_{\max}$ for $t \in \overline{\mathcal{I}}_k$ by (26). It follows that the distance $d(t)$ is bounded by the smallest non-negative solution d^* of the polynomial equation

$$P_\epsilon(d^*) = \frac{1}{\epsilon^{d-v} D_0}\left(\epsilon^{N+1}\chi_k + \epsilon^{2(N+1)} D_1 + \epsilon^{3(N+1)} T_0\right) \tag{27}$$

for $t \in \overline{\mathcal{I}}_k$. Since $d^* \sim \epsilon^{(N+1-d+v)/2}\sqrt{\chi_k/D_0}$ as $\epsilon \to 0$, we can shrink ϵ_0 further to ensure

$$d(t) \leq \epsilon^{(N+1-d+v)/2}\sqrt{\frac{\chi_k}{D_0}} \tag{28}$$

for $t \in \overline{\mathcal{I}}_k$. We conclude that $z(t)$ either remains in the narrow tubular neighborhood $\mathcal{T}_{d^*}(S_\epsilon^{(N)})$ for $t \in [-\epsilon^{-k}, \epsilon^k]$ or leaves the $O(1)$ tubular neighborhood $\mathcal{T}_{r_0}(S_\epsilon^{(N)})$ at some $t \in [-\epsilon^{-k}, \epsilon^k]$. The latter possibility corresponds to $z(t)$ running "off the edge" of the slow manifold $S_\epsilon^{(N)}$. □

Example 3. Given a positive integer n and an exact symplectic manifold (W, ω), consider the product manifold $M = \mathbb{C}^n \times W$. Denote points $m \in M$ as $m = (z_1, \ldots, z_n, w)$ with $z_k = (x_k, y_k) \in \mathbb{C}$ and $w \in W$. Equip M with the regular, barely symplectic form

$$\Omega_\epsilon = dx_1 \wedge dy_1 + \epsilon dx_2 \wedge dy_2 + \cdots + \epsilon^{n-1} dx_n \wedge dy_n + \epsilon^{n-1}\omega.$$

Note that the degeneracy index for Ω_ϵ is $d = n-1$. Define the smooth ϵ-dependent function

$$H_\epsilon(z_1, \ldots, z_n, w) = \sum_{k=1}^n \epsilon^{k-1}\frac{1}{2} z_k \overline{z}_k + \epsilon^n U(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, w), \tag{29}$$

where $\gamma_k = z_k \overline{z}_{k+1}$ and $U : \mathbb{C}^{n-1} \times W \to \mathbb{R}$ is any smooth function.

Consider the Hamiltonian system X_ϵ on (M, Ω_ϵ) determined by the Hamilton equation $\iota_{X_\epsilon}\Omega_\epsilon = \mathbf{d}H_\epsilon$. Explicitly, X_ϵ is given by

$$\begin{aligned} X_\epsilon = &-i\left(z_1 + \epsilon^n z_2 \frac{\delta U}{\delta \gamma_1}\right)\partial_{z_1} \\ &- i\left(z_2 + \epsilon^{n-1} z_3 \frac{\delta U}{\delta \gamma_2} + \epsilon^{n-1}\overline{z}_1 \frac{\delta U}{\delta \gamma_1}\right)\partial_{z_2} \\ &- \cdots \\ &- i\left(z_n + \epsilon \overline{z}_{n-1}\frac{\delta U}{\delta \gamma_{n-1}}\right)\partial_{z_n} \\ &+ \epsilon\widehat{\omega}^{-1}\mathbf{d}_w U. \end{aligned}$$

It is simple to verify that X_ϵ is a nearly-periodic system, whose limiting dynamics corresponds to a collection of decoupled oscillators z_k with unit angular frequency. In contrast to general nearly periodic systems, the roto-rate R_ϵ for this system is equal to the limiting roto-rate to all-orders, $R_\epsilon = R_0 = \sum_k -iz_k\partial_{z_k}$. (This is a consequence of $\mathcal{L}_{R_0}H_\epsilon = 0$ and $\mathcal{L}_{R_0}\Omega_\epsilon = 0$.) It follows that the adiabatic invariant μ_ϵ (a true invariant in this case) is given by

$$\mu_\epsilon = \iota_{R_0}\sum_{k=1}^n \epsilon^{k-1}\frac{1}{2}(y_k dx_k - x_k dy_k) = \sum_{k=1}^n \epsilon^{k-1}\frac{1}{2} z_k \overline{z}_k. \tag{30}$$

Note that the vanishing index for μ_ϵ is $v = 0$. We also know that the limiting slow manifold S_0 (in this case an actual invariant manifold) is an Nth-order slow manifold for each N.

Given a trajectory for X_ϵ that begins within ϵ^{N+1} of S_0, the tightest bound on the normal deviation that Theorem 32 can provide is $O(\epsilon^{(N+1-d)/2})$. Let us compare this worst-case bound with the worst-case bound implied by μ_ϵ-conservation in this example. Along our trajectory, $\mu_\epsilon = O(\epsilon^{2(N+1)})$. However, since the various oscillators z_k may exchange action $\frac{1}{2}z_k\overline{z}_k$ by way of the interaction potential U while keeping the sum μ_ϵ fixed, the oscillator configuration with $z_k = 0$ for $k < n$ and $z_n = O(\epsilon^{(2[N+1]-d)/2})$ is consistent with μ_ϵ and H_ϵ conservation. This bound is similar, although not identical, to the bound from Theorem 32. The discrepancy is due entirely to the exact invariance of μ_ϵ;

if the $\mu_\epsilon^{(N)}$ in Theorem 32 were conserved exactly, then χ_k in (27) would vanish, and the bound implied by Theorem 32 would instead be $O(\epsilon^{(2[N+1]-d)/2})$, exactly as in this example. We therefore conjecture that the bound given by Theorem 32 cannot be improved, in general (although it certainly can be improved in specific cases).

V. APPLICATIONS TO SLOW MANIFOLD EMBEDDING OF GUIDING CENTER DYNAMICS

We will present several applications of the general theory developed in Secs. III and IV. For these examples, it will be useful to discuss slow manifolds in the context of fast–slow systems, which we now define and explain.

Definition 34. An ordinary differential equation $\dot{y} = f_\epsilon(x,y)$, $\dot{x} = \epsilon g_\epsilon(x,y)$, where $f_\epsilon(x,y), g_\epsilon(x,y)$ are smooth in (ϵ, x, y), is a fast–slow system if

$$D_y f_0(x,y) \text{ is invertible whenever } f_0(x,y) = 0. \tag{31}$$

We refer to x as the slow variable and y as the fast variable.

Definition 35. A fast–slow system $\dot{y} = f_\epsilon(x,y)$, $\dot{x} = \epsilon g_\epsilon(x,y)$ admits a formal slow manifold if there is a formal power series $y_\epsilon^*(x) = y_0^*(x) + \epsilon y_1^*(x) + \epsilon^2 y_2^*(x) + \cdots$ that satisfies the first-order system of nonlinear partial differential equations,

$$\epsilon D y_\epsilon^*(x)[g_\epsilon(x, y_\epsilon^*(x))] = f_\epsilon(x, y_\epsilon^*(x)), \tag{32}$$

to all orders in ϵ.

Proposition 36. Each fast–slow system admits a unique formal slow manifold $y_\epsilon^*(x) = y_0^*(x) + \epsilon y_1^*(x) + \epsilon^2 y_2^*(x) + \cdots$. The first two coefficients of y_ϵ^* are determined by

$$f_0(x, y_0^*(x)) = 0, \tag{33}$$

$$D y_0^*(x)[g_0(x, y_0^*(x))] = D_y f_0(x, y_0^*(x))[y_1^*(x)] + f_1(x, y_0^*(x)). \tag{34}$$

These results imply that fast–slow systems admit slow manifolds of each order, as defined in Definition 13. The reason these systems are so convenient is that their slow manifolds may be computed without resorting to near-identity coordinate transformations. We will use this feature of fast–slow systems to simplify computations in what follows.

A. The classical Pauli particle embedding

As a first application, we consider the slow manifold embedding of guiding center dynamics introduced by Xiao and Qin in Ref. 11. We will establish long-term normal stability of this embedding in continuous time.

With $\boldsymbol{x} \in M$ and $(\boldsymbol{x}, \boldsymbol{v}) \in T_{\boldsymbol{x}}M$, the classical Pauli system is described by the following ordinary differential equations:

$$\frac{d\boldsymbol{x}}{dt} = \epsilon \boldsymbol{v}, \quad \frac{d\boldsymbol{v}}{dt} = \omega_c \boldsymbol{v} \times \boldsymbol{b} - \epsilon \mathcal{M} \nabla |\boldsymbol{B}|. \tag{35}$$

Here, $\mathcal{M} = \mu_P/m \in \mathbb{R}$ is a parameter, the cyclotron frequency is $\omega_c = q|\boldsymbol{B}|/m$, the vector $\boldsymbol{B}$ is the magnetic field with $\boldsymbol{b} = \boldsymbol{B}/|\boldsymbol{B}|$ the corresponding unit vector, and ϵ is the ordering parameter placed to indicate the cyclotron frequency as the fastest time scale in the system.

The classical Pauli system is Hamiltonian with respect to the vector field $X_\epsilon = (d\boldsymbol{x}/dt, d\boldsymbol{v}/dt)$ on the exact, regular, barely symplectic manifold (TM, Ω_ϵ). The one-form ϑ_ϵ, from which the barely symplectic form is computed as $\Omega_\epsilon = -\mathbf{d}\vartheta_\epsilon$, and the Hamiltonian H_ϵ are given by

$$\vartheta_\epsilon = (q\boldsymbol{A} + \epsilon m \boldsymbol{v}) \cdot d\boldsymbol{x}, \tag{36}$$

$$H_\epsilon = \epsilon^2 m(|\boldsymbol{v}|^2/2 + \mathcal{M}|\boldsymbol{B}|). \tag{37}$$

The degeneracy index for Ω_ϵ is $d = 2$, and it is straightforward to confirm that the Hamilton's equation $\iota_{X_\epsilon}\Omega_\epsilon = \mathbf{d}H_\epsilon$ recovers (35) for all values of ϵ. The nearly periodic nature is confirmed by observing the limiting vector field to be $X_0 = \omega_c \boldsymbol{v} \times \boldsymbol{b} \cdot \partial_{\boldsymbol{v}}$, which provides the nowhere vanishing frequency function $\omega_0 = \omega_c$ and the 2π-periodic vector field $R_0 = \boldsymbol{v} \times \boldsymbol{b} \cdot \partial_{\boldsymbol{v}}$ generating the $U(1)$-action on TM and satisfying $\mathcal{L}_{R_0}\omega_0 = 0$. The 2π-periodicity can be verified by analytically solving the flow of R_0,

$$\Phi_\theta(\boldsymbol{x}, \boldsymbol{v}) = \exp(\theta R_0)(\boldsymbol{x}, \boldsymbol{v}) = (\boldsymbol{x}, \boldsymbol{v} \cdot \boldsymbol{b}\boldsymbol{b} + \sin\theta\, \boldsymbol{v} \times \boldsymbol{b} + \cos\theta\, \boldsymbol{b} \times (\boldsymbol{v} \times \boldsymbol{b})). \tag{38}$$

Next, we will show that the classical Pauli system is fast-slow in order to efficiently identify the system's slow manifold and the corresponding induced slow dynamics.

Lemma 37. There exists a smooth orthonormal right-handed triad $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{b})$ *on M. Regarding the existence of such triads, see Ref. 30.*

Lemma 38. In the coordinates $(\boldsymbol{x}, u, v^1, v^2)$ *on TM defined by*

$$\boldsymbol{v} = u\boldsymbol{b} + v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2,$$

where $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{b})$ *is the orthonormal triplet provided by Lemma 37, the classical Pauli system (35) is equivalent to*

$$\frac{d\boldsymbol{x}}{dt} = \epsilon\boldsymbol{v}, \tag{39}$$

$$\frac{du}{dt} = \epsilon\boldsymbol{v}\cdot\nabla\boldsymbol{b}\cdot\boldsymbol{v} - \epsilon\mathcal{M}\boldsymbol{b}\cdot\nabla|\boldsymbol{B}|, \tag{40}$$

$$\frac{dv^1}{dt} = \omega_c\boldsymbol{e}_1\cdot\boldsymbol{v}\times\boldsymbol{b} - \epsilon(\mathcal{M}\nabla|\boldsymbol{B}| + u\boldsymbol{v}\cdot\nabla\boldsymbol{b})\cdot\boldsymbol{e}_1 - \epsilon v^2\boldsymbol{v}\cdot\boldsymbol{R}, \tag{41}$$

$$\frac{dv^2}{dt} = \omega_c\boldsymbol{e}_2\cdot\boldsymbol{v}\times\boldsymbol{b} - \epsilon(\mathcal{M}\nabla|\boldsymbol{B}| + u\boldsymbol{v}\cdot\nabla\boldsymbol{b})\cdot\boldsymbol{e}_2 + \epsilon v^1\boldsymbol{v}\cdot\boldsymbol{R}, \tag{42}$$

where $\boldsymbol{R} = \nabla\boldsymbol{e}_2\cdot\boldsymbol{e}_1$ *is Littlejohn's*[15] *gyrogauge vector.*

Proposition 39. The system of ordinary differential equations (39)–(42) comprises a fast–slow system with slow variable $x = (\boldsymbol{x}, u)$ *and fast variable* $y = (v^1, v^2)$*. The function* $f_\epsilon(x,y) = (dv^1/dt, dv^2/dt)$ *is given by*

$$f_0(x,y) = \begin{pmatrix}\omega_c\boldsymbol{e}_1\cdot\boldsymbol{v}\times\boldsymbol{b}\\ \omega_c\boldsymbol{e}_2\cdot\boldsymbol{v}\times\boldsymbol{b}\end{pmatrix}, \tag{43}$$

$$f_1(x,y) = \begin{pmatrix}-(\mathcal{M}\nabla|\boldsymbol{B}| + u\boldsymbol{v}\cdot\nabla\boldsymbol{b})\cdot\boldsymbol{e}_1 - v^2\boldsymbol{v}\cdot\boldsymbol{R}\\ -(\mathcal{M}\nabla|\boldsymbol{B}| + u\boldsymbol{v}\cdot\nabla\boldsymbol{b})\cdot\boldsymbol{e}_2 + v^1\boldsymbol{v}\cdot\boldsymbol{R}\end{pmatrix}, \tag{44}$$

and the function $g_\epsilon(x,y) = (d\boldsymbol{x}/dt, du/dt)$ *is given by*

$$g_0(x,y) = \begin{pmatrix}\boldsymbol{v}\\ \boldsymbol{v}\cdot\nabla\boldsymbol{b}\cdot\boldsymbol{v} - \mathcal{M}\boldsymbol{b}\cdot\nabla|\boldsymbol{B}|\end{pmatrix}. \tag{45}$$

Proposition 40. The first two coefficients of the formal slow manifold $y^*_\epsilon = ((v^1)^*_\epsilon, (v^2)^*_\epsilon)$ *for the fast–slow system (39)–(42) are given by*

$$\begin{pmatrix}(v^1)^*_0\\ (v^2)^*_0\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}, \tag{46}$$

$$\begin{pmatrix}(v^1)^*_1\\ (v^2)^*_1\end{pmatrix} = \omega_c^{-1}\begin{pmatrix}-(\mathcal{M}\nabla|\boldsymbol{B}| + u^2\boldsymbol{\kappa})\cdot\boldsymbol{e}_2\\ (\mathcal{M}\nabla|\boldsymbol{B}| + u^2\boldsymbol{\kappa})\cdot\boldsymbol{e}_1\end{pmatrix}, \tag{47}$$

where $\boldsymbol{\kappa} = \boldsymbol{b}\cdot\nabla\boldsymbol{b}$ *is the magnetic field-line curvature. In particular, if* $(\boldsymbol{v}_\perp)^*_\epsilon = (v^1)^*_\epsilon\boldsymbol{e}_1 + (v^2)^*_\epsilon\boldsymbol{e}_2$*, we have that*

$$(\boldsymbol{v}_\perp)^*_\epsilon = \epsilon\omega_c^{-1}\boldsymbol{b}\times(\mathcal{M}\nabla|\boldsymbol{B}| + u^2\boldsymbol{\kappa}) + O(\epsilon^2). \tag{48}$$

Investigating then the equations of motion of the slow variable along the slow-manifold, i.e., $\dot{x} = f_\epsilon(x, y^*_\epsilon(x))$, we find that

$$\frac{d\boldsymbol{x}}{dt} = \epsilon u \boldsymbol{b} + \epsilon^2 \omega_c^{-1} \boldsymbol{b} \times (\mathcal{M}\nabla|\boldsymbol{B}| + u^2 \boldsymbol{\kappa}) + \mathcal{O}(\epsilon^3), \tag{49}$$

$$\frac{du}{dt} = -\epsilon(\boldsymbol{b} + \epsilon u \omega_c^{-1} \boldsymbol{b} \times \boldsymbol{\kappa}) \cdot \mathcal{M}\nabla|\boldsymbol{B}| + \mathcal{O}(\epsilon^3). \tag{50}$$

These equations match exactly with the standard guiding-center equations derived from Littlejohn's Lagrangian if the magnitude $|\boldsymbol{B}|$ in the cyclotron frequency ω_c is replaced by the so-called $B_\parallel^* = (\boldsymbol{B}(+mu/q)\nabla \times \boldsymbol{b}) \cdot \boldsymbol{b}$ and μ_P in $\mathcal{M}$ is interpreted as the magnetic moment of the guiding-center. The factor $B_\parallel^*$ is needed to guarantee that the slow vector field $X_\epsilon^* = (d\boldsymbol{x}/dt, du/dt)$ is divergence free, i.e., that $\nabla \cdot (d\boldsymbol{x}/dt) + \partial_u(du/dt) = 0$. Furthermore, as explained in Ref. 31, dynamics on the slow manifold is necessarily Hamiltonian. The corresponding symplectic form $\Omega_\epsilon^* = -\mathbf{d}\vartheta_\epsilon^*$ and the Hamiltonian H_ϵ^* are given by pulling back Ω_ϵ and H_ϵ along the mapping $(x, y) \mapsto (x, y_\epsilon^*(x))$. This first provides

$$\vartheta_\epsilon^* = (q\boldsymbol{A} + \epsilon m u \boldsymbol{b}) \cdot d\boldsymbol{x} + \mathcal{O}(\epsilon^2), \tag{51}$$

$$H_\epsilon^* = \epsilon^2 m(u^2/2 + \mathcal{M}|\boldsymbol{B}|) + \mathcal{O}(\epsilon^3), \tag{52}$$

from which the Hamilton's equations $\iota_{X_\epsilon^*}\Omega_\epsilon^* = H_\epsilon^*$ would provide exactly the standard guiding-center equations, with the $B_\parallel^*$ corrections included.

Finally, we will demonstrate that the slow manifold for the classical Pauli system enjoys long-term normal stability. For this, we will show that the first nontrivial term in the adiabatic invariant for this system has sign-definite second variation along the limiting slow manifold $v^1 = v^2 = 0$. Let $\mu_\epsilon = \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \cdots$ denote the adiabatic invariant series for the classical Pauli system. According to Eq. (3.14) in Ref. 17, $\mu_0 = \iota_{R_0}\langle\vartheta_0\rangle$, where we have $R_0 = \boldsymbol{v} \times \boldsymbol{b} \cdot \partial_{\boldsymbol{v}}$ and $\vartheta_0 = q\boldsymbol{A} \cdot d\boldsymbol{x}$, and the angle brackets denote averaging over the $U(1)$-action Φ_θ that is generated by R_0. Since Φ_θ in (38) leaves the $\boldsymbol{x}$-position fixed and ϑ_0 depends only on $\boldsymbol{x}$, we have that $\langle\vartheta_0\rangle = \vartheta_0$, and since R_0 has only velocity components, we conclude $\mu_0 = 0$. Given that $\mu_0 = 0$, Eq. (3.15) in Ref. 17 then provides $\mu_1 = \iota_{R_0}\langle\vartheta_1\rangle$, where we have $\vartheta_1 = m\boldsymbol{v} \cdot d\boldsymbol{x}$. Using again the fact that Φ_θ leaves $\boldsymbol{x}$ fixed, the average is simple to compute, giving $\langle\vartheta_1\rangle = m\boldsymbol{v} \cdot \mathbf{v}\boldsymbol{b}\boldsymbol{b} \cdot d\boldsymbol{x}$. As there again is no $d\boldsymbol{v}$ component in $\langle\vartheta_1\rangle$, the contraction $\iota_{R_0}\langle\vartheta_1\rangle$ vanishes, giving $\mu_1 = 0$. Finally using Eq. (3.16) in Ref. 17, we find that $\mu_2 = \frac{1}{2}\langle\mathbf{d}\vartheta_0(\mathcal{L}_{R_0} I_0 \widetilde{X}_1, I_0\widetilde{X}_1)\rangle$, where $X_1 = \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} - \mathcal{M}\nabla|\boldsymbol{B}| \cdot \partial_{\boldsymbol{v}}$ is the first-order term in X_ϵ, $\widetilde{X}_1 = X_1 - \langle X_1\rangle$ and $I_0 = \mathcal{L}_{\omega_0 R_0}^{-1}$. The pullback of X_1 along R_0 is given by

$$\begin{aligned}
X_1^\theta = \Phi_\theta^* X_1 &= \boldsymbol{v} \cdot \boldsymbol{b}\boldsymbol{b} \cdot \partial_{\boldsymbol{x}} + \sin\theta \boldsymbol{v} \times \boldsymbol{b} \cdot \partial_{\boldsymbol{x}} + \cos\theta \boldsymbol{b} \times (\boldsymbol{v} \times \boldsymbol{b}) \cdot \partial_{\boldsymbol{x}} \\
&\quad + \mathcal{M}(-\boldsymbol{b}\boldsymbol{b} \cdot \nabla|\boldsymbol{B}| + \sin\theta \boldsymbol{b} \times \nabla|\boldsymbol{B}| + \cos\theta \boldsymbol{b} \times (\boldsymbol{b} \times \nabla|\boldsymbol{B}|)) \cdot \partial_{\boldsymbol{v}} \\
&\quad + \left\{(\boldsymbol{v} \cdot \boldsymbol{b})(\boldsymbol{b} \times \boldsymbol{\kappa}) \times \boldsymbol{v} - \frac{1}{2}[\boldsymbol{b} \times (\boldsymbol{v}_\perp \cdot \nabla\boldsymbol{b})] \times \boldsymbol{v} + \frac{1}{2}[(\boldsymbol{v} \times \boldsymbol{b}) \cdot \nabla\boldsymbol{b}] \times \boldsymbol{v}\right\} \cdot \partial_{\boldsymbol{v}} \\
&\quad + \cos\theta\{[\boldsymbol{b} \times (\boldsymbol{v}_\perp \cdot \nabla\boldsymbol{b})] \times \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{b})(\boldsymbol{b} \times \boldsymbol{\kappa}) \times \boldsymbol{v}\} \cdot \partial_{\boldsymbol{v}} \\
&\quad + \sin\theta\{[\boldsymbol{b} \times ((\boldsymbol{v} \times \boldsymbol{b}) \cdot \nabla\boldsymbol{b})] \times \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{b})\boldsymbol{\kappa} \times \boldsymbol{v}\} \cdot \partial_{\boldsymbol{v}} \\
&\quad + \frac{1}{2}\sin(2\theta)\{(\boldsymbol{v}_\perp \cdot \nabla\boldsymbol{b}) \times \boldsymbol{v} - [\boldsymbol{b} \times ((\boldsymbol{v} \times \boldsymbol{b}) \cdot \nabla\boldsymbol{b})] \times \boldsymbol{v}\} \cdot \partial_{\boldsymbol{v}} \\
&\quad - \frac{1}{2}\cos(2\theta)\{[\boldsymbol{b} \times (\boldsymbol{v}_\perp \cdot \nabla\boldsymbol{b})] \times \boldsymbol{v} + [(\boldsymbol{v} \times \boldsymbol{b}) \cdot \nabla\boldsymbol{b}] \times \boldsymbol{v}\} \cdot \partial_{\boldsymbol{v}}.
\end{aligned} \tag{53}$$

This permits us to compute the inverse

$$\begin{aligned}
\omega_c I_0 \widetilde{X}_1^\theta &= -\cos\theta \boldsymbol{v} \times \boldsymbol{b} \cdot \partial_{\boldsymbol{x}} + \sin\theta \boldsymbol{b} \times (\boldsymbol{v} \times \boldsymbol{b}) \cdot \partial_{\boldsymbol{x}} \\
&\quad + \mathcal{M}(-\cos\theta \boldsymbol{b} \times \nabla|\boldsymbol{B}| + \sin\theta \boldsymbol{b} \times (\boldsymbol{b} \times \nabla|\boldsymbol{B}|)) \cdot \partial_{\boldsymbol{v}} \\
&\quad + \sin\theta\{[\boldsymbol{b} \times (\boldsymbol{v}_\perp \cdot \nabla\boldsymbol{b})] \times \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{b})(\boldsymbol{b} \times \boldsymbol{\kappa}) \times \boldsymbol{v}\} \cdot \partial_{\boldsymbol{v}} \\
&\quad - \cos\theta\{[\boldsymbol{b} \times ((\boldsymbol{v} \times \boldsymbol{b}) \cdot \nabla\boldsymbol{b})] \times \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{b})\boldsymbol{\kappa} \times \boldsymbol{v}\} \cdot \partial_{\boldsymbol{v}} \\
&\quad - \frac{1}{8}\cos(2\theta)\{(\boldsymbol{v}_\perp \cdot \nabla\boldsymbol{b}) \times \boldsymbol{v} - [\boldsymbol{b} \times ((\boldsymbol{v} \times \boldsymbol{b}) \cdot \nabla\boldsymbol{b})] \times \boldsymbol{v}\} \cdot \partial_{\boldsymbol{v}} \\
&\quad - \frac{1}{8}\sin(2\theta)\{[\boldsymbol{b} \times (\boldsymbol{v}_\perp \cdot \nabla\boldsymbol{b})] \times \boldsymbol{v} + [(\boldsymbol{v} \times \boldsymbol{b}) \cdot \nabla\boldsymbol{b}] \times \boldsymbol{v}\} \cdot \partial_{\boldsymbol{v}} \\
&\quad - (\sin\theta \boldsymbol{v} \times \boldsymbol{b} + \cos\theta \boldsymbol{b} \times (\boldsymbol{v} \times \boldsymbol{b})) \cdot \nabla \ln \omega_c \boldsymbol{v} \times \boldsymbol{b} \cdot \partial_{\boldsymbol{v}}
\end{aligned} \tag{54}$$

and, from this, trivially the expression

$$\omega_c \mathcal{L}_{R_0}(I_0 \widetilde{X}_1^\theta) = \omega_c \partial_\theta (I_0 \widetilde{X}_1^\theta) = \sin\theta \boldsymbol{v} \times \boldsymbol{b} \cdot \partial_{\boldsymbol{x}} + \cos\theta \boldsymbol{b} \times (\boldsymbol{v} \times \boldsymbol{b}) \cdot \partial_{\boldsymbol{x}}$$
$$+ \mathcal{M}(\sin\theta \boldsymbol{b} \times \nabla |\boldsymbol{B}| + \cos\theta \boldsymbol{b} \times (\boldsymbol{b} \times \nabla |\boldsymbol{B}|)) \cdot \partial_{\boldsymbol{v}}$$
$$+ \cos\theta \{[\boldsymbol{b} \times (\boldsymbol{v}_\perp \cdot \nabla \boldsymbol{b})] \times \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{b})(\boldsymbol{b} \times \boldsymbol{\kappa}) \times \boldsymbol{v}\} \cdot \partial_{\boldsymbol{v}}$$
$$+ \sin\theta \{[\boldsymbol{b} \times ((\boldsymbol{v} \times \boldsymbol{b}) \cdot \nabla \boldsymbol{b})] \times \boldsymbol{v} + (\boldsymbol{v} \cdot \boldsymbol{b}) \boldsymbol{\kappa} \times \boldsymbol{v}\} \cdot \partial_{\boldsymbol{v}}$$
$$+ \frac{1}{4} \sin(2\theta) \{(\boldsymbol{v}_\perp \cdot \nabla \boldsymbol{b}) \times \boldsymbol{v} - [\boldsymbol{b} \times ((\boldsymbol{v} \times \boldsymbol{b}) \cdot \nabla \boldsymbol{b})] \times \boldsymbol{v}\} \cdot \partial_{\boldsymbol{v}}$$
$$- \frac{1}{4} \cos(2\theta) \{[\boldsymbol{b} \times (\boldsymbol{v}_\perp \cdot \nabla \boldsymbol{b})] \times \boldsymbol{v} + [(\boldsymbol{v} \times \boldsymbol{b}) \cdot \nabla \boldsymbol{b}] \times \boldsymbol{v}\} \cdot \partial_{\boldsymbol{v}}$$
$$- (\cos\theta \boldsymbol{v} \times \boldsymbol{b} - \sin\theta \boldsymbol{b} \times (\boldsymbol{v} \times \boldsymbol{b})) \cdot \nabla \ln \omega_c \boldsymbol{v} \times \boldsymbol{b} \cdot \partial_{\boldsymbol{v}}. \tag{55}$$

Since $\mathbf{d}\vartheta_0$ is independent of $\boldsymbol{v}$, we only need the $\boldsymbol{x}$-components of the vector fields $I_0 \widetilde{X}_1^\theta$ and $\mathcal{L}_{R_0} I_0 \widetilde{X}_1^\theta$, which finally provide the expression for the first nonvanishing term in the adiabatic invariant series,

$$\mu_2 = \frac{m}{2} \frac{|\boldsymbol{v} \times \boldsymbol{b}|^2}{\omega_c}. \tag{56}$$

As is straightforward to verify, the Hessian along the normal direction (v^1, v^2) is sign-definite,

$$\mathbf{H}_\perp(\mu_2) = \frac{m}{\omega_c} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{57}$$

confirming the normal stability of the slow-manifold via Theorem 32.

B. The proper-time relativistic Pauli embedding

As a second application, we generalize the discussion from Sec. V A to the Lorentz-covariant relativistic setting. We begin by recalling the standard Lorentz-covariant Hamiltonian formulation of charged particle dynamics on a flat Minkowski spacetime $(M, \langle \cdot, \cdot \rangle)$, whose inner product $\langle \cdot, \cdot \rangle$ has mostly positive signature. We denote spacetime events using the symbol $R \in M$ and elements of the spacetime tangent bundle $TM \approx M \times M$ as $(R, V) \in T_R M$. The electromagnetic field is specified by a 1-form A on M, whose exterior derivative gives the Faraday 2-form $F = \mathbf{d}A$. By way of the Minkowski inner product, the Faraday 2-form induces a Faraday tensor $\boldsymbol{F} : TM \to TM$, defined so that $\langle V_1, \boldsymbol{F} V_2 \rangle = \iota_{V_2} \iota_{V_1} F$ for each pair of vector fields V_1, V_2 on M. An individual charged particle with four-position R, four-velocity V, and proper time τ moves through such a spacetime according to the relativistic Newton–Lorentz equations,

$$\frac{dV}{d\tau} = \zeta \boldsymbol{F}(R) V, \quad \frac{dR}{d\tau} = \epsilon V. \tag{58}$$

We have written (58) in dimensionless form. To recover dimensional results, introduce the particle charge q, the particle mass m, the speed of light c, a spacetime length scale L, and a characteristic magnetic field strength B_0 (physically interpreted as the characteristic size of the Lorentz scalar $\sqrt{|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2}$). The dimensional four-position, four-velocity, proper time, and Faraday tensor are then given by LR, cV, $(mc/qB_0)\tau$, and $B_0 L^2 F$, respectively. The correct physical interpretations of the constants ζ and ϵ are therefore $\zeta = q/|q|$ and

$$\epsilon = \frac{mc^2}{|q| B_0 L}, \tag{59}$$

where the latter represents the ratio of the so-called "light radius" $\rho_c = (mc^2)/(|q| B_0)$ to the characteristic field scale length L, $\epsilon = \rho_c / L$.

Going forward, we will assume that the electromagnetic potential A decomposes as the sum $A = A_0 + \epsilon A_1$, where the "$\boldsymbol{E} \cdot \boldsymbol{B}$" and "$|\boldsymbol{B}|^2 - |\boldsymbol{E}|^2$" Lorentz scalars associated with $F_0 = \mathbf{d}A_0$ are zero and positive, respectively. This is a Lorentz-covariant way of asserting the system is strongly magnetized. As such, we refer to this assumption as the magnetization assumption.

Under the magnetization assumption, we claim that the Newton–Lorentz equations (58) comprise a nearly periodic Hamiltonian system $X_\epsilon = (dR/d\tau, dV/d\tau)$ on the exact, regular, barely symplectic manifold (TM, Ω_ϵ). We argue as follows: The barely symplectic form is given by $\Omega_\epsilon = -\mathbf{d}\vartheta_\epsilon$, where

$$\vartheta_\epsilon = \zeta(A_0 + \epsilon A_1) + \epsilon \langle V, dR \rangle. \tag{60}$$

The degeneracy index for Ω_ϵ is $d = 2$, as in the non-relativistic case. The system Hamiltonian is $H_\epsilon(R, V) = \frac{1}{2}\epsilon^2 \langle V, V \rangle$. It is straightforward to confirm that the Hamilton equation $\iota_{X_\epsilon} \Omega_\epsilon = \mathbf{d}H_\epsilon$ recovers (58) for all values of ϵ, which confirms the Hamiltonian nature of the relativistic

Newton–Lorentz equations. To demonstrate that X_ϵ is nearly periodic, we must show $X_0 = \omega_0 R_0$, where ω_0 is some nowhere-vanishing smooth function and R_0 is the generator of a $U(1)$-action on TM that satisfies $\mathcal{L}_{R_0}\omega_0 = 0$. For this, we turn to the following lemma:

Lemma 41. Under the magnetization assumption, the smooth function $\omega_0 = \sqrt{-\mathrm{tr}(\boldsymbol{F}_0^2)/2}$ is nowhere-vanishing. In addition, the vector field

$$R_0 = \frac{\zeta}{\omega_0}\boldsymbol{F}_0 V\partial_V \tag{61}$$

is the generator of a $U(1)$-action on TM and $\mathcal{L}_{R_0}\omega_0 = 0$.

Proof. First, we note that $\mathrm{tr}(\boldsymbol{F}_0^2) = 2(|\boldsymbol{E}_0|^2 - |\boldsymbol{B}_0|^2)$. According to the magnetization assumption, we therefore have $-\mathrm{tr}(\boldsymbol{F}_0^2) > 0$ on M. It immediately follows that ω_0 is real-valued and nowhere-vanishing, as required. We also note that $\mathcal{L}_{R_0}\omega_0 = 0$ is obvious since ω_0 depends only on R, while R_0 has no R-component.

Next, we identify the dimension of the null space for $\boldsymbol{F}_0$. With respect to an orthonormal basis (q_0, q_1, q_2, q_3) for T_RM such that q_0 is time-like, the coefficient matrix for $\boldsymbol{F}_0(R)$, $[\boldsymbol{F}_0]$, is given by $[\boldsymbol{F}_0] = [g][F]$, where $[g]_{ij} = \langle q_i, q_j\rangle$ is diagonal symmetric and $[F]_{ij} = F(e_i, e_j)$ is antisymmetric. Because the $\boldsymbol{E}\cdot\boldsymbol{B}$ Lorentz scalar vanishes for $\boldsymbol{F}_0$, we must have $0 = \det[\boldsymbol{F}_0] = (\det[g])(\det[F]) = -\det[F]$. In other words, the antisymmetric matrix $[F]$ must have a non-trivial null space. Since, by hypothesis, $[F]$ does not vanish, the block normal form for antisymmetric matrices implies that the spectrum for $[F]$ must be of the form $(i\lambda, -i\lambda, 0, 0)$, where $\lambda > 0$. In particular, the null space $K_R \subset T_RM$ for $\boldsymbol{F}_0(R)$ must be two-dimensional.

Now, we will characterize the behavior of $\boldsymbol{F}_0$ on the subspace orthogonal to its null space. Let $K_R^\perp \subset T_RM$ be the orthogonal complement to the null space K_R. If $V \in K_R^\perp$ and $W \in K_R$, then $\langle W, \boldsymbol{F}_0(R)V\rangle = -\langle V, \boldsymbol{F}_0(R)W\rangle = 0$. Therefore, $K_R^\perp$ is a two-dimensional invariant subspace for $\boldsymbol{F}_0(R)$ complementary to K_R. Let $\boldsymbol{F}_0^\perp(R) : K_R^\perp \to K_R^\perp$ denote the restriction of $\boldsymbol{F}_0(R)$ to $K_R^\perp$. By the Cayley–Hamilton theorem for 2×2 matrices, we have

$$(\boldsymbol{F}_0^\perp(R))^2 + \det(\boldsymbol{F}_0^\perp(R))\mathbb{I}^\perp = 0, \tag{62}$$

where we have used $\mathrm{tr}(\boldsymbol{F}_0^\perp(R)) = \mathrm{tr}(\boldsymbol{F}_0(R)) = 0$ and introduced the identity map $\mathbb{I}^\perp : K_R^\perp \to K_R^\perp$. Taking the trace of (62), we also obtain

$$\mathrm{tr}((\boldsymbol{F}_0^\perp(R))^2) + 2\,\det(\boldsymbol{F}_0^\perp(R)) = 0. \tag{63}$$

Combining (62) and (63), we find

$$\left(\frac{\boldsymbol{F}_0^\perp(R)}{\sqrt{-\mathrm{tr}((\boldsymbol{F}_0^\perp(R))^2)/2}}\right)^2 = -\mathbb{I}^\perp, \tag{64}$$

where we have used $\mathrm{tr}((\boldsymbol{F}_0^\perp)^2) = 2(E^2 - B^2) < 0$ to ensure the square root is real. This identity says $\boldsymbol{F}_0^\perp(R)/\omega_0(R)$ is a complex structure on the vector space $K_R^\perp$ for each R.

Finally, we determine the integral curves of the vector field R_0. If $(R(\lambda), V(\lambda))$ is such an integral curve, then the component curves satisfy the system of ordinary differential equations

$$\frac{dV}{d\lambda} = \frac{\zeta}{\omega_0}\boldsymbol{F}_0 V, \quad \frac{dR}{d\lambda} = 0.$$

Clearly, $R(\lambda) = R(0)$. For the four-velocity, we write $V(\lambda) = V_\parallel(\lambda) + V_\perp(\lambda)$, where $V_\parallel$ denotes the orthogonal projection into the null space $K_{R(0)}$ and $V_\perp$ denotes the orthogonal projection into $K_R^\perp$. These projected curves satisfy the linear system

$$\frac{dV_\parallel}{d\lambda} = 0, \quad \frac{dV_\perp}{d\lambda} = \frac{\zeta}{\omega_0}\boldsymbol{F}_0^\perp V_\perp.$$

We obviously have $V_\parallel(\lambda) = V_\parallel(0)$. To solve the $V_\perp$ equation, we observe that (64) implies $[\frac{\zeta}{\omega_0}\boldsymbol{F}_0^\perp]^2 = -\mathbb{I}_\perp$ and then recognize that we can compute the matrix exponential $\exp(\lambda\frac{\zeta}{\omega_0}\boldsymbol{F}_0^\perp)$ exactly using Euler's formula $\exp(i\theta) = \cos\theta + i\sin\theta$, giving

$$\begin{aligned} V_\perp(\lambda) &= \exp\left(\lambda\frac{\zeta}{\omega_0}\boldsymbol{F}_0^\perp\right)V_\perp(0) \\ &= \left(\cos\lambda\mathbb{I}_\perp + \sin\lambda\frac{\zeta}{\omega_0}\boldsymbol{F}_0^\perp\right)V_\perp(0). \end{aligned}$$

By 2π-periodicity of the solutions thus obtained, we conclude that R_0 generates a $U(1)$-action given explicitly by

$$\Phi_\theta(R,V) = (R, P_\parallel V + [\cos\,\theta \mathbb{I}_\perp + \sin\,\theta \zeta \boldsymbol{F}_0^\perp/\omega_0]P_\perp V), \tag{65}$$

where $P_\perp$ and $P_\parallel$ denote orthogonal projections into $K_R^\perp$ and K_R, respectively. □

Note that in the process of proving the above lemma, we identified important structural properties of $\boldsymbol{F}_0$. These are summarized in the following definition:

Definition 42. The parallel flat is the subbundle $K \subset TM$ whose fiber at $R \in M$ is the two-dimensional null space of $\boldsymbol{F}_0(R)$. The perpendicular flat is the orthogonal complement bundle $K^\perp$. The orthogonal projections into K and $K^\perp$ are given by $P_\parallel : TM \to TM$ and $P_\perp : TM \to TM$, respectively, where

$$P_\perp = -\frac{\boldsymbol{F}_0^2}{\omega_0^2}, \quad P_\parallel = \mathbb{I} - P_\perp. \tag{66}$$

We may now define the equations of motion for a relativistic Pauli particle and study their properties. The relativistic Pauli Hamiltonian $\mathcal{H}_\epsilon : TM \to \mathbb{R}$ is given by

$$\mathcal{H}_\epsilon(R,V) = \frac{1}{2}\epsilon^2\langle V,V\rangle + \epsilon^2\mathcal{M}\omega_0, \tag{67}$$

where $\mathcal{M} \in \mathbb{R}$ is a parameter and ω_0 is defined in Lemma 41. The relativistic Pauli system is the vector field $\mathcal{X}_\epsilon$ defined by the Hamilton equation $\iota_{\mathcal{X}_\epsilon}\Omega_\epsilon = \mathbf{d}\mathcal{H}_\epsilon$, where $\Omega_\epsilon = -\mathbf{d}\vartheta_\epsilon$ with ϑ_ϵ given in (60). As in the non-relativistic case, in defining this Pauli system, we have left the Lorentz symplectic structure unchanged while adding a Pauli potential $\mathcal{M}\omega_0$ to the Lorentz Hamiltonian. Also is parallel with the non-relativistic case, the relativistic Pauli system admits a non-degenerate Lagrangian structure with the Lagrangian $L(R,\dot{R}) = \epsilon\frac{1}{2}\langle\dot{R},\dot{R}\rangle + \zeta\iota_{\dot{R}}A - \epsilon\mathcal{M}\omega_0(R)$. The following analysis will demonstrate that the relativistic guiding center equations, as derived originally by Boghosian,[32] are embedded within the relativistic Pauli system as a slow manifold and that this slow manifold enjoys long-term normal stability. In so doing, we will generalize the observations of Xiao and Qin[11] to allow for time-dependent electromagnetic fields, strong $E \times B$ drifts, and all special relativistic effects such as time dialation. In addition, we will generalize our result on the continuous-time normal stability of the Pauli embedding to the relativistic setting.

First, we observe that $\mathcal{X}_\epsilon$ is a nearly periodic system. To show this, we note that $\mathcal{X}_\epsilon$ is given explicitly by $\mathcal{X}_\epsilon = (dR/d\tau, dV/d\tau)$ with

$$\frac{dR}{d\tau} = \epsilon V, \quad \frac{dV}{d\tau} = \zeta(\boldsymbol{F}_0 + \epsilon\boldsymbol{F}_1)V - \epsilon\mathcal{M}\nabla\omega_0. \tag{68}$$

Here, ∇ denotes the gradient operator associated with the Minkowski inner product. These equations differ from the Newton–Lorentz equations by a single $O(\epsilon)$ term. Therefore, $\mathcal{X}_0 = \omega_0 R_0$, where ω_0 and R_0 are defined as they were for the Newton–Lorentz system. Lemma 41 therefore implies $\mathcal{X}_\epsilon$ is nearly periodic, as claimed.

Next, we will show that the relativistic Pauli system is fast-slow in order to efficient identify the system's slow manifold and the corresponding induced slow dynamics.

Lemma 43. There exists a smooth orthonormal tetrad (e_0, e_1, e_2, e_3) on M such that (e_0, e_3) frames the null-space bundle $K \subset TM$ and (e_1, e_2) frames $K^\perp \subset TM$. Moreover, e_0 is time-like and e_k is space-like for $k = 1, 2, 3$.

Lemma 44. In the coordinates (R, V^0, V^1, V^2, V^3) on TM defined by

$$V = V^0 e_0 + V^1 e_1 + V^2 e_2 + V^3 e_3,$$

where (e_0, e_1, e_2, e_3) is the orthonormal tetrad provided by Proposition 43, the relativistic Pauli system (68) is equivalent to

$$\frac{dV^0}{d\tau} = \epsilon\langle\zeta\boldsymbol{F}_1 e_0, V\rangle - \epsilon\left\langle V_\perp, \nabla_V\left[\frac{\boldsymbol{F}_0^2}{\omega_0^2}\right]e_0\right\rangle - \epsilon V^3\mathcal{Q}(V) + \epsilon\mathcal{M}\langle e_0, \nabla\omega_0\rangle, \tag{69}$$

$$\frac{dV^1}{d\tau} = -\langle\zeta(\boldsymbol{F}_0 + \epsilon\boldsymbol{F}_1)e_1, V\rangle - \epsilon\left\langle V_\parallel, \nabla_V\left[\frac{\boldsymbol{F}_0^2}{\omega_0^2}\right]e_1\right\rangle + \epsilon V^2\mathcal{R}(V) - \epsilon\mathcal{M}\langle e_1, \nabla\omega_0\rangle, \tag{70}$$

$$\frac{dV^2}{d\tau} = -\langle\zeta(\boldsymbol{F}_0 + \epsilon\boldsymbol{F}_1)e_2, V\rangle - \epsilon\left\langle V_\parallel, \nabla_V\left[\frac{\boldsymbol{F}_0^2}{\omega_0^2}\right]e_2\right\rangle - \epsilon V^1\mathcal{R}(V) - \epsilon\mathcal{M}\langle e_2, \nabla\omega_0\rangle, \tag{71}$$

$$\frac{dV^3}{d\tau} = -\epsilon\langle\zeta\boldsymbol{F}_1 e_3, V\rangle + \epsilon\left\langle V_\perp, \nabla_V\left[\frac{\boldsymbol{F}_0^2}{\omega_0^2}\right]e_3\right\rangle - \epsilon V^0\mathcal{Q}(V) - \epsilon\mathcal{M}\langle e_3, \nabla\omega_0\rangle, \tag{72}$$

$$\frac{dR}{d\tau} = \epsilon V. \tag{73}$$

Here, the 1-forms $\mathcal{Q}$ *and* $\mathcal{R}$ *are defined according to* $\mathcal{Q}(V) = \langle \nabla_V e_0, e_3\rangle$ *and* $\mathcal{R}(V) = \langle \nabla_V e_1, e_2\rangle$*, and we have introduced the shorthand* $V_\parallel = P_\parallel V \in K_R$, $V_\perp = P_\perp V \in K_R^\perp$.

Proposition 45. The system of ordinary differential equations (69)–(73) comprises a fast–slow system with slow variable $x = (R, V^0, V^3)$ *and fast variable* $y = (V^1, V^2)$*. The function* $f_\epsilon(x,y) = (dV^1/d\tau, dV^2/d\tau)$ *is given by*

$$f_0(x,y) = \begin{pmatrix} -\zeta V^2\langle \boldsymbol{F}_0 e_1, e_2\rangle \\ \zeta V^1\langle \boldsymbol{F}_0 e_1, e_2\rangle \end{pmatrix}, \tag{74}$$

$$f_1(x,y) = \begin{pmatrix} -\langle \zeta \boldsymbol{F}_1 e_1, V\rangle - \left\langle V_\parallel, \nabla_V\left[\frac{\boldsymbol{F}_0^2}{\omega_0^2}\right]e_1\right\rangle + V^2\mathcal{R}(V) - \mathcal{M}\langle e_1, \nabla\omega_0\rangle \\ -\langle \zeta \boldsymbol{F}_1 e_2, V\rangle - \left\langle V_\parallel, \nabla_V\left[\frac{\boldsymbol{F}_0^2}{\omega_0^2}\right]e_2\right\rangle - V^1\mathcal{R}(V) - \mathcal{M}\langle e_2, \nabla\omega_0\rangle \end{pmatrix}. \tag{75}$$

Proposition 46. The first two coefficients of the formal slow manifold $y_\epsilon^* = (V_\epsilon^{1*}, V_\epsilon^{2*})$ *for the fast–slow system (69)–(73) are given by*

$$\begin{pmatrix} V_0^{1*} \\ V_0^{2*} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{76}$$

$$\begin{pmatrix} V_1^{1*} \\ V_1^{2*} \end{pmatrix} = \frac{\zeta\langle e_2, \boldsymbol{F}_0 e_1\rangle}{\omega_0^2}\begin{pmatrix} \langle \zeta \boldsymbol{F}_1 e_2, V_\parallel\rangle + \left\langle V_\parallel, \nabla_{V_\parallel}\left[\frac{\boldsymbol{F}_0^2}{\omega_0^2}\right]e_2\right\rangle + \mathcal{M}\langle e_2, \nabla\omega_0\rangle \\ -\langle \zeta \boldsymbol{F}_1 e_1, V_\parallel\rangle - \left\langle V_\parallel, \nabla_{V_\parallel}\left[\frac{\boldsymbol{F}_0^2}{\omega_0^2}\right]e_1\right\rangle - \mathcal{M}\langle e_1, \nabla\omega_0\rangle \end{pmatrix}. \tag{77}$$

In particular, if $(V_\perp)_\epsilon^* = (V^1)_\epsilon^* e_1 + (V^2)_\epsilon^* e_2$*, we have*

$$(V_\perp)_\epsilon^* = \epsilon\frac{\zeta \boldsymbol{F}_0}{\omega_0^2}\left(\zeta \boldsymbol{F}_1 V_\parallel + \nabla_{V_\parallel}\left[\frac{\boldsymbol{F}_0^2}{\omega_0^2}\right]V_\parallel - \mathcal{M}\nabla\omega_0\right) + O(\epsilon^2). \tag{78}$$

Now, it is simple to demonstrate that the slow manifold dynamics relativistic Pauli system approximately agrees with the covariant guiding center theory developed by Boghosian.[32] As explained in Ref. 31, dynamics on the slow manifold is necessarily Hamiltonian. The corresponding symplectic form $\Omega^* = -\mathbf{d}\vartheta_\epsilon^*$ is given by pulling back Ω_ϵ along the mapping $x \mapsto (x, y_\epsilon^*)$, which leads to

$$\vartheta_\epsilon^* = \zeta A + \epsilon\langle V_\parallel, dR\rangle + O(\epsilon^2). \tag{79}$$

This 1-form agrees with the 1-form reported in Eq. (3.489) in Ref. 32 to the displayed order. For agreement, we use $V_\parallel = V^1 e_1 + V^2 e_2 = K\hat{\boldsymbol{t}}$, where K and $\hat{\boldsymbol{t}}$ are defined by Boghosian. The slow manifold Hamiltonian is given by pulling back the Pauli Hamiltonian along the same map, giving

$$\mathcal{H}_\epsilon^* = \epsilon^2\left(\frac{1}{2}\langle V_\parallel, V_\parallel\rangle + \mu\omega_0\right) + O(\epsilon^3). \tag{80}$$

Since $\langle V_\parallel, V_\parallel\rangle = -K$, where the right-hand side uses Boghosian's notation, this Hamiltonian agrees with Eq. (3.488) from Ref. 32 to the displayed order. We conclude that slow manifold dynamics for the relativistic Pauli system agree with relativistic guiding center theory to the same order as in the non-relativistic case. This implies, in particular, that the strategy underlying Xiao and Qin's numerical integration scheme[11] may be applied in the covariant relativistic setting as well.

Finally, we will demonstrate that the slow manifold for the relativistic Pauli system enjoys long-term normal stability. For this, we will show that the first nontrivial term in the adiabatic invariant for this system has sign-definite second variation along the limiting slow manifold $V^1 = V^2 = 0$. Let $\mu_\epsilon = \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \cdots$ denote the adiabatic invariant series for the Pauli system. According to Eq. (3.14) in Ref. 17, $\mu_0 = \iota_{R_0}\langle\vartheta_0\rangle$, where R_0 is defined in Lemma 41, $\vartheta_0 = \zeta A_0$, and the angle brackets denote averaging over the $U(1)$-action Φ_θ generated by R_0, i.e., that given in (65). Since Φ_θ leaves the four-position R fixed $\langle\vartheta_0\rangle = \vartheta_0$ and since R_0 has only velocity components, we conclude $\mu_0 = 0$. According to Eq. (3.15) in Ref. 17, $\mu_1 = \iota_{R_0}\langle\vartheta_1\rangle$, where $\vartheta_1 = \zeta A_1 + \langle V, dR\rangle$. Again using the fact that Φ_θ leaves R fixed, the average is simple to compute, giving $\langle\vartheta_1\rangle = \zeta A_1 + \langle V_\parallel, dR\rangle$. Therefore, the contraction $\iota_{R_0}\langle\vartheta_1\rangle$ vanishes again, giving $\mu_1 = 0$. Finally, using Eq. (3.16) in Ref. 17, we find that $\mu_2 = \frac{1}{2}\langle \mathbf{d}\vartheta_0(\mathcal{L}_{R_0} I_0 \widetilde{\mathcal{X}}_1, I_0\widetilde{\mathcal{X}}_1)\rangle$, where $\mathcal{X}_1$ is the first-order term in $\mathcal{X}_\epsilon$, $\widetilde{\mathcal{X}}_1 = \mathcal{X}_1 - \langle\mathcal{X}_1\rangle$, and $I_0 = \mathcal{L}^{-1}_{\omega_0 R_0}$. Since $\mathbf{d}\vartheta_0 = F_0$, we only need to compute the R-components of the vector fields $I_0\widetilde{\mathcal{X}}_1$ and $\mathcal{L}_{R_0} I_0\widetilde{\mathcal{X}}_1$. For this purpose, we observe that the R-component of the vector field $\mathcal{X}_1^\theta = \Phi_\theta^*\mathcal{X}_1$ is given by

$$(\mathcal{X}_1^\theta)^R = P_\parallel V + [\cos\theta\mathbb{I}_\perp + \sin\theta\zeta \boldsymbol{F}_0^\perp/\omega_0]P_\perp V, \tag{81}$$

from which we infer

$$(I_0\widetilde{\mathcal{X}}_1^\theta)^R = \frac{1}{\omega_0}[\sin\theta\mathbb{I}_\perp - \cos\theta\zeta\boldsymbol{F}_0^\perp/\omega_0]P_\perp V,$$

$$(\mathcal{L}_{R_0} I_0\widetilde{\mathcal{X}}_1^\theta)^R = \frac{1}{\omega_0}[\cos\theta\mathbb{I}_\perp + \sin\theta\zeta\boldsymbol{F}_0^\perp/\omega_0]P_\perp V.$$

The second-order adiabatic invariant is therefore

$$\begin{aligned}\mu_2 &= \frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi}\left\langle \frac{1}{\omega_0}[\cos\theta\mathbb{I}_\perp + \sin\theta\zeta\boldsymbol{F}_0^\perp/\omega_0]P_\perp V, \boldsymbol{F}_0\left(\frac{1}{\omega_0}[\sin\theta\mathbb{I}_\perp - \cos\theta\zeta\boldsymbol{F}_0^\perp/\omega_0]P_\perp V\right)\right\rangle d\theta \\ &= -\frac{\zeta}{\omega_0}\frac{1}{2\pi}\int_0^{2\pi}\langle V_\perp, [\boldsymbol{F}_0/\omega_0]^2 V_\perp\rangle \sin^2\theta d\theta \\ &= \frac{\zeta\langle V_\perp, V_\perp\rangle}{2\omega_0}.\end{aligned} \tag{82}$$

Since $K_R^\perp$ is space-like for each R, the Hessian of μ_2 along $V^1 = V^2 = 0$ is sign-semi-definite, much as in the relativistic case. By Theorem 32 with $d = \nu = 2$ and energy conservation, we conclude that if a trajectory for the relativistic Pauli system with a bounded spatial component $R(t)$ begins within ϵ of $V^1 = V^2 = 0$, then it will remain within $\epsilon^{1/2}$ over large time intervals. It is also not difficult to show using Theorem 32 that the normal deviation from an Nth-order slow manifold will be bound by $\epsilon^{(N+1)/2}$ for trajectories that begin within ϵ^{N+1}.

C. The symplectic Lorentz embedding

As a final application, we will study a general method for embedding symplectic Hamiltonian systems as normally stable elliptic slow manifolds in higher-dimension Lagrangian systems with regular Lagrangians. This method applies, in particular, to the non-canonical guiding center system, but differs from the Pauli embeddings studied in Secs. V A and V B in an essential manner; where the dimensionality of either the non-relativistic or relativistic Pauli systems is two greater than that of the corresponding guiding center system, the dimension of the embedding space studied in this section is twice that of the system being embedded. We emphasize, however, that this method of embedding is *not* equivalent to the method of formal Lagrangians.[33,34] The formal Lagrangian technique does not embed using slow manifolds and does not lead to regular Lagrangians, in contrast to the method described here.

The basic idea behind our construction may be described as follows: Let X be a Hamiltonian system with Hamiltonian H on an exact symplectic manifold (M, β) equipped with a Riemannian metric g and symplectic form $\beta = -\mathbf{d}\alpha$. We would like to embed X in a larger system with a regular Lagrangian structure. To do this, we consider the dynamics of a charged particle with small mass $\epsilon > 0$ and unit positive charge moving on M. The magnetic field this particle experiences is given by the symplectic 2-form β. The electric field is given by $-\nabla H$, i.e., the Hamiltonian serves as an electrostatic potential. As the particle mass ϵ becomes smaller, the timescale for gyration around the magnetic field shrinks. In fact, as the following detailed analysis will demonstrate, the metric g on M can always be chosen to ensure that the particle's gyration around the magnetic field becomes periodic with short period as $\epsilon \to 0$. Therefore, the equations of motion for this particle comprise a nearly periodic system and, according to the theory developed in this article, admit slow manifolds of each order near which the rapid gyrations are suppressed. Strikingly, the corresponding slow manifold dynamics recover the original dynamics defined by X to leading order in ϵ. From the perspective of the small-mass particle moving on M, the original dynamics is recovered as a generalized $E \times B$-drift. Moreover, normal stability of the slow manifold emerges, by way of Theorem 32, as a consequence of adiabatic invariance of a generalized magnetic moment.

As a first step in a detailed description of this embedding technique, we review the construction of a Riemannian metric "compatible" with a given symplectic form β on a manifold M.

Lemma 47. Given a symplectic manifold (M, β), there exists a Riemannian metric g on M and an almost complex structure $\mathbb{J} : TM \to TM$ such that $g(V, W) = \beta(V, \mathbb{J}W)$ for each pair of vector fields V, W on M.

Proof. The proof is well-known (see, for instance, Ref. 35), but we give a reproduction here for completeness and to emphasize its constructive character.

Let G be an arbitrary Riemannian metric on M. The symplectic form β induces an antisymmetric, non-singular bundle map $\boldsymbol{\beta}_G : TM \to TM$ by requiring $G(V, \boldsymbol{\beta}_G W) = \beta(V, W)$ for each pair of vector fields V, W. The associated bundle map $\boldsymbol{S}_G = -\boldsymbol{\beta}_G\boldsymbol{\beta}_G$ is therefore symmetric positive-definite with the symmetric positive-definite square root $\sqrt{\boldsymbol{S}_G}$. We define the desired almost complex structure $\mathbb{J}$ according to $\mathbb{J} = \boldsymbol{\beta}_G^{-1}\sqrt{\boldsymbol{S}_G}$. To check that this formula does indeed define an almost complex structure, we compute as follows:

$$\begin{aligned}\mathbb{J}^2 &= \boldsymbol{\beta}_G^{-1}\sqrt{\boldsymbol{S}_G}\boldsymbol{\beta}_G^{-1}\sqrt{\boldsymbol{S}_G}\\ &= -\boldsymbol{\beta}_G^{-1}\boldsymbol{\beta}_G^{-1}\boldsymbol{\beta}_G\boldsymbol{\beta}_G\\ &= -\mathbb{I}.\end{aligned}$$

Here, we have used the fact that $\boldsymbol{S}_G$, and therefore $\sqrt{\boldsymbol{S}_G}$, commutes with $\boldsymbol{\beta}_G$. We then define the Riemannian metric g by requiring that the desired identity $g(V, W) = \beta(V, \mathbb{J}W)$ holds for arbitrary vector fields V, W. To show that the tensor g, thus defined, is symmetric and positive definite, we observe

$$g(V, W) = \beta(V, \mathbb{J}W) = G(V, \boldsymbol{\beta}_G\boldsymbol{\beta}_G^{-1}\sqrt{\boldsymbol{S}_G}W) = G(V, \sqrt{\boldsymbol{S}_G}W) = g(W, V),$$

by symmetry of $\sqrt{\boldsymbol{S}_G}$, and for non-zero V,

$$g(V, V) = G(V, \sqrt{\boldsymbol{S}_G}V) > 0,$$

by positive-definiteness of $\sqrt{\boldsymbol{S}_G}$. □

We will now describe our embedding technique in detail. Let (M, β) be an exact symplectic manifold. Using Lemma 47, choose a Riemannian metric g on M and an almost-complex structure $\mathbb{J}$ such that $g(V, W) = \beta(V, \mathbb{J}W)$ for arbitrary vector fields V, W. Let X be a Hamiltonian system on M with Hamiltonian H. We would like to embed the dynamics defined by X as slow manifold dynamics in a larger system with a regular Lagrangian structure.

The phase space for this larger system will be the tangent bundle TM, points of which will be denoted $(R, V) \in T_R M$. For $\epsilon \in \mathbb{R}$, we introduce the exact, regular, barely symplectic form $\Omega_\epsilon = -\mathbf{d}\vartheta_\epsilon$ on TM with the primitive

$$\vartheta_\epsilon = \pi^*\alpha + \epsilon g_R(V, dR), \tag{83}$$

where $\pi : TM \to M$ denotes the tangent bundle projection and α is a primitive for $\beta = -\mathbf{d}\alpha$. We also introduce the ϵ-dependent Hamilton function $\mathcal{H}_\epsilon(R, V) = \epsilon H(R) + \epsilon^2 \frac{1}{2} g_R(V, V)$. The symplectic Lorentz system is the smooth ϵ-dependent vector field $\mathcal{X}_\epsilon$ on TM defined by the Hamilton equation $\iota_{\mathcal{X}_\epsilon}\Omega_\epsilon = \mathbf{d}\mathcal{H}_\epsilon$. Our goal is to prove that the symplectic Lorentz system contains a slow manifold whose slow dynamics agrees with those of X to leading order in ϵ. Moreover, we would like to establish normal stability of this slow manifold using Theorem 32. As in Secs. V A and V B, we will proceed by showing that $\mathcal{X}_\epsilon$ is a nearly periodic Hamiltonian system, identifying the associated slow manifold, and then showing that the adiabatic invariant has sign-semi-definite second variation along the slow manifold.

To see that $\mathcal{X}_\epsilon$ is nearly periodic, suppose that $(R(t), V(t))$ is an $\mathcal{X}_\epsilon$-integral curve. It is not difficult to show that this curve must satisfy the system of evolution equations

$$\frac{DV}{dt} = \mathbb{J}V - \nabla H, \quad \frac{dR}{dt} = \epsilon V, \tag{84}$$

where DV/dt denotes the covariant derivative of V along the curve R. Also the analogy to the charged particle is now transparent: $\mathbb{J}V$ is the "$\boldsymbol{v} \times \boldsymbol{B}$" term and $-\nabla H$ is the "electrostatic electric field." To see how (84) emerge, write the Lagrangian in component form $L = (\alpha_i + \epsilon V^j g_{ij})\dot{R}^i - \epsilon(H + \epsilon V^i V^j g_{ij}/2)$ and obtain the Euler–Lagrange equations $\dot{R}^i = \epsilon V^i$ and $g_{ij}\dot{V}^j + \epsilon V^j V^k \Gamma_{ijk} = -\beta_{ij}V^j - \partial_i H$, where Γ_{ijk} is the Christoffel symbol of the first kind. The vector form is then recovered after identifying $d(V^j \boldsymbol{e}_j)/dt \cdot \boldsymbol{e}_i = g_{ij}\dot{V}^j + \epsilon V^j V^k \Gamma_{ijk}$, $\boldsymbol{e}^i \partial_i H = \nabla H$, and $-\boldsymbol{e}^i \Omega_{ij} V^j = \boldsymbol{e}^i g_{ik} \mathbb{J}^k{}_j V^j = \mathbb{J}V$.

In particular, when $\epsilon = 0$, we must have $dR/dt = 0$ and $dV/dt = \mathbb{J}V - \nabla H$. The solution to this limit system is $(R(t), V(t)) = \Phi_\theta(R(0), V(0))$, where $\Phi_\theta : TM \to TM$ is given by

$$\Phi_\theta(R, V) = (R, -\mathbb{J}\nabla H + \exp(\theta\mathbb{J})[V + \mathbb{J}\nabla H]). \tag{85}$$

Note that since $\exp(\theta\mathbb{J}) = \cos\theta\mathbb{I} + \sin\theta\mathbb{J}$, Φ_θ defines a $U(1)$-action on TM. We may therefore infer that the symplectic Lorentz system is a nearly periodic system with angular frequency $\omega_0 = 1$ and limiting roto-rate

$$R_0 = (\mathbb{J}V - \nabla H)\partial_V. \tag{86}$$

Next, we will show that the symplectic Lorentz system is fast-slow in order to efficiently identify the system's slow manifold and the corresponding induced slow dynamics.

Lemma 48. In the coordinates (R^i, V^i) *on TM, the symplectic Lorentz system (84) is equivalent to*

$$\dot{R}^i = \epsilon V^i, \tag{87}$$

$$g_{ij}\dot{V}^j + \epsilon V^j V^k \Gamma_{ijk} = -\beta_{ij}V^j - \partial_i H, \tag{88}$$

where $\Gamma_{ijk} = \frac{1}{2}(\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk})$ *is the Christoffel symbol of the first kind.*

Proposition 49. The system of ordinary differential equations (87) and (88) comprises a fast–slow system with slow variable $x = (R^i)$ *and fast variable* $y = (V^i)$*. The function* $f_\epsilon(x,y) = (dV^i/dt)$ *is given by*

$$f_0^i(x,y) = -g^{ij}\beta_{jk}V^k - g^{ij}\partial_j H, \tag{89}$$

$$f_1^i(x,y) = -g^{ij}\Gamma_{jk\ell}V^k V^\ell, \tag{90}$$

and the function $g_\epsilon(x,y) = (dR^i/dt)$ *is given by*

$$g_0^i(x,y) = V^i. \tag{91}$$

Proposition 50. The first two coefficients of the formal slow manifold $y^*_\epsilon = ((V^i)^*_\epsilon)$ *for the fast–slow system (39)–(42) are given by*

$$(V^i)^*_0\beta_{ij} = \partial_j H, \tag{92}$$

$$(V^j)^*_0\partial_j(V^i)^*_0 = -g^{ij}\beta_{jk}(V^k)^*_1 - g^{ij}\Gamma_{jk\ell}(V^k)^*_0(V^\ell)^*_0. \tag{93}$$

In particular, to leading order, the slow dynamics are given by $\dot{R} = -\mathbb{J}\nabla H$*, which recovers the original Hamiltonian dynamics on M.*

Finally, we will demonstrate that the slow manifold for the symplectic Lorentz system enjoys long-term normal stability. Again, letting $\mu_\epsilon = \mu_0 + \epsilon\mu_1 + \epsilon^2\mu_2 + \cdots$ denote the adiabatic invariant series, we have $\mu_0 = \iota_{R_0}\langle\vartheta_0\rangle$, where R_0 is given by (86), $\vartheta_0 = \pi^*\alpha$, and the angle brackets denote averaging over the $U(1)$-action Φ_θ that is generated by R_0. R_0 only has a component along V, Φ_θ leaves the R-position fixed. Then, since ϑ_0 depends only on R, we have that $\langle\vartheta_0\rangle = \vartheta_0$, and consequently, $\mu_0 = 0$. Given that $\mu_0 = 0$, the next candidate becomes $\mu_1 = \iota_{R_0}\langle\vartheta_1\rangle$, where we have $\vartheta_1 = g_R(V, dR)$. Then, taking the average of the pullback $\Phi^*_\theta(g_R(V, dR))$ with respect to θ provides $\langle\vartheta_1\rangle = g(-\mathbb{J}\nabla H, dR)$, which, again, is independent of V, therefore providing $\mu_1 = 0$. We are finally left to compute $\mu_2 = \frac{1}{2}\langle \mathbf{d}\vartheta_0(\mathcal{L}_{R_0}I_0\widetilde{\mathcal{X}}_1, I_0\widetilde{\mathcal{X}}_1)\rangle$, where $\mathcal{X}_1 = V^i\partial_{R^i} - g^{ij}\Gamma_{jkl}V^kV^\ell\partial_{V^i}$ is the first-order term in $\mathcal{X}_\epsilon$, i.e., $\widetilde{\mathcal{X}}_1 = \mathcal{X}_1 - \langle\mathcal{X}_1\rangle$ and $I_0 = \mathcal{L}^{-1}_{\omega_0 R_0}$. Since ϑ_0 has only R component that depends only on R, we only need the R components of the vector field $\widetilde{\mathcal{X}}^\theta_1 = \Phi^*_\theta\widetilde{\mathcal{X}}_1$, which is given by

$$(\widetilde{\mathcal{X}}^\theta_1)^R = -\mathbb{J}\nabla H + (\cos\theta\mathbb{I} + \sin\theta\mathbb{J})(V + \mathbb{J}\nabla H). \tag{94}$$

From this, we infer

$$(I_0\widetilde{\mathcal{X}}^\theta_1)^R = (\sin\theta\mathbb{I} - \cos\theta\mathbb{J})(V + \mathbb{J}\nabla H), \tag{95}$$

$$(\mathcal{L}_{\omega_0 R_0}I_0\widetilde{\mathcal{X}}^\theta_1)^R = (\cos\theta\mathbb{I} + \sin\theta\mathbb{J})(V + \mathbb{J}\nabla H), \tag{96}$$

and the second-order adiabatic invariant is therefore

$$\mu_2 = -\frac{1}{2}\frac{1}{2\pi}\int_0^{2\pi}\beta((\cos\theta\mathbb{I}+\sin\theta\mathbb{J})(V+\mathbb{J}\nabla H),(\sin\theta\mathbb{I}-\cos\theta\mathbb{J})(V+\mathbb{J}\nabla H))d\theta$$
$$= \frac{1}{2}\beta(V+\mathbb{J}\nabla H,\mathbb{J}(V+\mathbb{J}\nabla H))$$
$$= \frac{1}{2}g(V+\mathbb{J}\nabla H, V+\mathbb{J}\nabla H). \quad (97)$$

The Hessian of μ along $V = -\mathbb{J}\nabla H$ is the metric g and thus sign-definite. Through Theorem 32, the slow-manifold is then stable.

VI. DISCUSSION

In this article, we established a free-action stability principle for a large and interesting class of elliptic slow manifolds, namely, those that arise in Hamiltonian nearly periodic systems. We applied this general theory to establish continuous-time normal stability of the slow manifold embedding of guiding center dynamics introduced by Xiao and Qin in Ref. 11. Moreover, we extended the Xiao–Qin embedding and its stability to the Lorentz covariant relativistic setting. Finally, we introduced a general method for embedding any Hamiltonian system on a symplectic manifold as a normally stable elliptic slow manifold in a larger system with a regular Lagrangian.

Our normal stability results are based on exploiting adiabatic invariants. This idea is not new. MacKay highlighted the method in his review article. However, it is not clear, in general, when a given slow manifold should satisfy a free-action principle. It is therefore striking that a large and interesting class of slow manifolds satisfies the free-action principle "automatically."

One might attempt to establish a free-action principle for any nearly periodic Hamiltonian system, in particular, for nearly periodic systems on symplectic, presymplectic, Poisson, or even Dirac manifolds. However, our results are not so general. Instead, we have assumed that phase space is equipped with a closed 2-form that is non-degenerate except in the limit of infinite timescale separation. Such singular symplectic structures arise frequently in applications, especially in plasma physics, where disparate timescales abound. The problem of extending our results to more general Hamiltonian structure deserves further attention.

Our method of embedding any symplectic Hamiltonian system as a slow manifold in a regular Lagrangian system suggests interesting further developments. At first glance, it suggests that the method proposed by Xiao and Qin for developing structure-preserving integrators for guiding center dynamics may be extended to any Hamiltonian system. However, in his thesis[36] and subsequent work,[37] Ellison showed unwittingly that integrators derived in this manner will generally suffer from parasitic instabilities. Thus, the continuous-time normal stability established in Sec. V C may be broken after discretizing time. In future work, we plan to investigate strategies for discretizing the symplectic Lorentz system that do not destroy normal stability of the slow manifold and that preserve structural properties of the underlying continuous-time dynamics.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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