In this Colloquium recent advances in the field of quantum heat transport are reviewed. This topic has been investigated theoretically for several decades, but only during the past 20 years have experiments on various mesoscopic systems become feasible. A summary of the theoretical basis for describing heat transport in one-dimensional channels is first provided. The main experimental investigations of quantized heat conductance due to phonons, photons, electrons, and anyons in such channels are then presented. These experiments are important for understanding the fundamental processes that underlie the concept of a heat conductance quantum for a single channel. An illustration of how one can control the quantum heat transport by means of electric and magnetic fields, and how such tunable heat currents can be useful in devices, is first given. This lays the basis for realizing various thermal device components such as quantum heat valves, rectifiers, heat engines, refrigerators, and calorimeters. Also of interest are fluctuations of quantum heat currents, both for fundamental reasons and for optimizing the most sensitive thermal detectors; at the end of the Colloquium the status of research on this topic is given.

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I. INTRODUCTION

In this Colloquium we present advances on fundamental aspects of thermal transport in the regime where quantum effects play an important role. Usually this means dealing with atomic scale structures or low temperatures, or a combination of the two. The seminal theoretical work by Pendry (1983) presented, almost 40 years ago, the important observation that a ballistic channel for any type of a carrier can transport heat at the rate given by the so-called quantum of thermal conductance $G_Q$. During this millennium the theoretical ideas have developed into a plethora of experiments in systems involving phonons, electrons, photons, and recently particles obeying fractional statistics. We give an overview of these experiments backed by the necessary theoretical framework. The question as to whether or not a channel is ballistic, and under what conditions, is interesting as such, but it also has more practical implications. If one can control the degree of ballisticity,
i.e., the transmission coefficient of the channel, one can turn the heat current on and off. Such quantum heat switches, or heat valves as they are often called, are discussed in this Colloquium as well. Furthermore, the heat current via a quantum element in an asymmetric structure can violate reciprocity in the sense that rectification of the heat current becomes possible. The bulk of the Colloquium deals with the time average (mean) of the heat current. Yet the fluctuations of this quantity are interesting, and they provide a yardstick for the minimal detectable power and for the ultimate energy resolution of a thermal detector. We discuss such a noise and its implications in ultrasensitive detection.

The Colloquium begins with a theoretical discussion of thermoelectric transport in one-dimensional channels in Sec. II. In Sec. III we present the concept and method of how to measure heat currents in general. Section IV reviews the central elements of the experimental setups. After these general sections, we move on to heat transport in different physical systems: phonons in Sec. V, electrons and fractional charges in Sec. VI, and photons in Sec. VII, including some detailed theoretical discussion within the sections. Section VIII presents the minimal detectable power and for the ultimate energy resolution of a thermal detector. We discuss such a noise and its implications in ultrasensitive detection.

The Colloquium concludes with a summary and outlook that we have taken the Fermi energy as the zero of energy and the velocity of the particles with wave vector $k$ is then independent modes in the conductor, and $\epsilon_n(k)$ and $v_n(k)$ represent the chemical potential and temperature of each reservoir on the left and right, respectively.

\[ I = \frac{q}{h} \sum_n \int_{\epsilon(0)}^{\infty} d\epsilon [\theta_L(\epsilon) - \theta_R(\epsilon)] T_n(\epsilon), \]

\[ \mathcal{J} = \frac{1}{h} \sum_n \int_{\epsilon(0)}^{\infty} d\epsilon [\theta_L(\epsilon) - \theta_R(\epsilon)] T_n(\epsilon), \]

where $q$ is the particle charge, $\sum_n$ presents the sum over independent modes in the conductor, and $e_n(k)$ and $v_n(k)$ indicate the energy and the velocity of the particles with wave vector $k$, respectively. $T_n(k)$ indicates the particle transmission probability through the conductor via the channel; for ballistic transport $T_n(k) \equiv 1$, and $\theta_{LR}$ represents the statistical distribution functions in each reservoir. Changing the variable from wave vector to energy via the definition of the velocity $v_n(k) = (1/h) \partial e_n(k)/\partial k$, we have

\[ I = \frac{q}{h} \epsilon(0) \sum_n \int_{\epsilon(0)}^{\infty} d\epsilon [\theta_L(\epsilon) - \theta_R(\epsilon)] T_n(\epsilon), \]

\[ \mathcal{J} = \frac{1}{h} \epsilon(0) \sum_n \int_{\epsilon(0)}^{\infty} d\epsilon [\theta_L(\epsilon) - \theta_R(\epsilon)] T_n(\epsilon), \]

FIG. 1. Artistic representation of a generic conductor between two reservoirs. Both particles and heat are transported through. Depending on the strength and type of scattering at the impurities (dots) and walls, one can have either ballistic or diffusive transport. Here $\mu_i$ and $T_i$, for $i = L, R$, are the chemical potential and temperature of each reservoir on the left and right, respectively.
thermal conductance at equilibrium (\(T_L = T_R \equiv T\)), i.e., on \(G_{\text{th}}(T) \equiv d\dot{Q}/dT\)\(_L\)|\(_R\). The thermal conductance is then
\[
G_{\text{th}}^{(f)} = N\frac{1}{\hbar k_B T^2} \int_{-\infty}^{\infty} de \varepsilon^2 f(\varepsilon)[1 - f(\varepsilon)]
\]
\[= N\frac{\pi^2 k_B^2}{3h} T \equiv NG_Q, \tag{6}
\]
where the superscript \((f)\) stands for fermions and
\[
G_Q \equiv \frac{\pi^2 k_B^2}{3h} T \tag{7}
\]
is the thermal conductance quantum. The ratio of the thermal and electrical conductances satisfies the Wiedemann-Franz law \(G_{\text{th}}/G = \mathcal{L} T\), where the Lorenz number is \(\mathcal{L} = \pi^2 k_B^2 / (3e^2)\) (Ashcroft and Mermin, 1976).

We obtain the following thermal conductance for bosons \(G_{\text{th}}^{(b)}\) with the same procedure but with the distribution function \(\delta_{R,L}(\varepsilon) \equiv n_{R,L}(\varepsilon) = 1/\left(e^{\varepsilon/k_BT} - 1\right)\) in Eq. (2):
\[
G_{\text{th}}^{(b)} = \frac{\hbar^2}{2\pi k_B T^2} \sum_n \int_{0}^{\infty} d\omega \frac{\omega^2 e^{\omega\hbar\omega}}{(e^{\omega\hbar\omega} - 1)^2} T_n(\omega). \tag{8}
\]
Here \(\varepsilon = \hbar\omega\) is the energy of each boson. For a single fully transmitting channel \(T_n(\omega) = 1\), we then again obtain
\[
G_{\text{th}}^{(b)} = G_Q. \tag{9}
\]

Fermions and bosons naturally form the playground for most experimental realizations in the quantum regime. Yet the previous result for a ballistic channel \(G_{\text{th}} = G_Q\) is far more general. As demonstrated by Rego and Kirczenow (1999) and Blencowe and Vitelli (2000), this expression is invariant even if one introduces carriers with arbitrary fractional exclusion statistics (Wu, 1994). Recently Banerjee et al. (2017) experimentally observed the universality of the thermal conductance quantum for anyons.

III. THERMAL CONDUCTANCE: MEASUREMENT ASPECTS

A. Principles of measuring heat currents

For determining thermal conductance one needs in general a measurement of local temperature. Suppose that an absorber like the one in Fig. 2(a) is heated at a constant power \(\dot{Q}\). By continuity, the relation between \(\dot{Q}\) and temperature \(T\) of the absorber with respect to the bath temperature \(T_0\) can be written as
\[
\dot{Q} = \mathcal{K}(T^n - T_0^n), \tag{10}
\]
where \(\mathcal{K}\) and \(n\) are constants characteristic of the absorber and the process of thermalization. For the most common process in metals, the coupling of absorber electrons to the phonon bath, the standard expression is \(\dot{Q} = \Sigma V(T^5 - T_0^5)\) (Gantmakher, 1974; Roukes et al., 1985; Wellstood, Urbina, and Clarke, 1994; Schwab et al., 2000; Wang et al., 2019), where \(\Sigma\) is a material specific parameter and \(V\) is the volume of the absorber. It is often the case that the temperature difference \(\delta T \equiv T - T_0\) is small (\(|\delta T/T| \ll 1\)), and we can linearize Eq. (10) into
\[
\dot{Q} = G_{\text{th}}\delta T, \tag{11}
\]
where \(G_{\text{th}} = n\kappa T_0^{n-1}\) is the thermal conductance between the absorber and the bath. For the previous electron-phonon coupling, we then have \(G_{\text{th}}^{(\text{ep})} = 5\Sigma \mathcal{V} T_0^4\). We point out that electron-electron relaxation in metals is fast enough to secure a well-defined electron temperature (Pothier et al., 1997).

For the ballistic channel discussed widely in this Colloquium, \(G_{\text{th}} \equiv G_Q = \pi^2 k_B^2 T_0/(3h)\), and we have for a general temperature difference
\[
\dot{Q} = \frac{\pi^2 k_B^2}{6h} (T^2 - T_0^2) = \frac{\pi^2 k_B^2}{3h} T_m \delta T, \tag{12}
\]
where \(T_m \equiv (T + T_0)/2\) is the mean temperature.

In some experiments a differential two-absorber setup is preferable; see Fig. 2(b). This allows one to measure the temperatures of the two absorbers \((T_1\) and \(T_2\), separately) and determine the heat flux between the two without extra physical wiring connections for thermometry across the object of interest. In this case equations in this section apply if we replace \(T\) and \(T_0\) with \(T_1\) and \(T_2\), respectively. Such a setup offers more flexible calibration and sanity check options for the system, and also for tests of reciprocity (thermal rectification) by inverting the roles of source and drain, i.e., by reversing the temperature bias.

B. Thermometry and temperature control

Here we comment briefly on thermometry and temperature control in the experiments to be reported in this review. The control of the local temperature is typically achieved by Joule heating applied to the electronic system. But depending on the type of reservoir this heat is acting on the quantum conductor.

---

![FIG. 2. Thermal models.](image-url)
either directly or indirectly, such as via the phonon bath. The simplest heating element is a resistive on-chip wire. For heating and local cooling and, in particular, for thermometry, a hybrid normal-metal–insulator–superconductor (N-I-S) tunnel junction is a common choice (Giazotto et al., 2006; Muhonen, Meschke, and Pekola, 2012; Courtois et al., 2014). We defer discussion of this technique to Sec. IV.B. In several experiments a simple resistive on-chip wire is used as a local heater. For thermometry one may use a similar wire and measure its thermal noise (Schwab et al., 2000). Another option used in some recent experiments is to measure the current noise of a quantum point contact (Jezouin et al., 2013; Banerjee et al., 2017).

IV. EXPERIMENTAL SETUPS AND BACKGROUND INFORMATION

A. Thermal conductance of a superconductor

A superconductor obeying Bardeen-Cooper-Schrieffer (BCS) theory (Bardeen, Cooper, and Schrieffer, 1957) forms an ideal building block for thermal experiments at low temperatures. A basic feature of a BCS superconductor is its zero resistance, but in our context an even more important property is its essentially vanishing thermal conductance (Bardeen, Rickayzen, and Tewordt, 1959). In bulk superconductors both electronic and nonvanishing lattice thermal conductances play a role.

In small structures the exponentially vanishing thermal conductance at low temperatures can be exploited effectively to form thermal insulators that can at the same time provide perfect electrical contacts. In quantitative terms, according to the theory (Bardeen, Rickayzen, and Tewordt, 1959) the ratio of the thermal conductivity \( \kappa_{e,s} \) in the superconducting state and \( \kappa_{,n} \) in the normal state of the same material is given by

\[
\frac{\kappa_{e,s}}{\kappa_{e,n}} = \frac{\int_{-\Delta}^{0} \text{d}e e^2 f(e)/(\int_{0}^{0} \text{d}e e^2 f(e))}{\int_{0}^{0} \text{d}e e^2 f(e)},
\]

where \( \Delta \approx 1.76k_{B}T_{C} \) is the gap of the superconductor with critical temperature \( T_{C} \). For temperatures well below \( T_{C} \), i.e., for \( \Delta/(k_{B}T) \gg 1 \), we obtain the following as an approximate answer for Eq. (13):

\[
\frac{\kappa_{e,s}}{\kappa_{e,n}} \approx \frac{6}{\pi^2} \left( \frac{\Delta}{k_{B}T} \right)^2 e^{-\Delta/k_{B}T}.
\]

Since the normal state thermal and electrical conductivities are related by the Wiedemann-Franz law, we obtain

\[
\kappa_{e,s} \approx \frac{2\Delta^2}{e^2\rho T} e^{-\Delta/k_{B}T},
\]

where \( \rho \) is the normal state resistivity of the conductor material. As usual, for the basic case of a uniform conductor with cross-sectional area \( A \) and length \( \ell \) we may then associate the thermal conductance \( G_{th} \) with thermal conductivity \( \kappa \) as \( G_{th} = (A/\ell)\kappa \).

Aluminum and niobium are the most common superconductors used in the experiments described here. In many respects, Al follows BCS theory accurately. In particular, it has been shown (Saira, Kemppinen et al., 2012) that the density of states (DOS) at energies inside the gap is suppressed at least by a factor of \( \sim 10^{-7} \) leading to the exponentially high thermal insulation discussed here. The measured thermal conductivity of Al closely follows Eq. (15), as shown by Peltonen et al. (2010) and Feschenko et al. (2017). At the same time Nb films suffer from a nonvanishing subgap DOS, leading to power-law thermal conductance in \( T \), i.e., poor thermal insulation in the low temperature regime. In conclusion of this section we emphasize that Al is a perfect thermal insulator at \( T \ll 0.3T_{C} \), except in immediate contact with a normal metal leading to the inverse proximity effect; this proximity induced thermal conductivity typically has an effect only within few hundred nanometers of a clean normal-metal contact (Peltonen et al., 2010).

B. Heat transport in tunneling

One central element of this Colloquium is a tunnel junction between two electrodes L and R. The charge and heat currents through the junction can be obtained using perturbation theory, where the coupling Hamiltonian between the electrodes is written as the tunnel Hamiltonian (Bruus and Flensberg, 2004)

\[
\hat{H}_{c} = \sum_{j=L,R} \left( t_{\alpha} \hat{a}_{\alpha}^{\dagger} \hat{a}_{\alpha} + t_{\alpha}^{*} \hat{a}_{\alpha}^{\dagger} \hat{a}_{\alpha}^{\dagger} \right).
\]

Here \( t_{\alpha} \) is the tunneling amplitude and \( \hat{a}_{\alpha}^{\dagger} \) and \( \hat{a}_{\alpha} \) are the creation and annihilation operators for electrons in the left (right) electrode, respectively.

To have the expression for number current from R to L one first obtains operator for it as \( \hat{N}_{L} = \langle i/h \rangle [\hat{H}_{c}, \hat{N}_{L}] \), where \( \hat{N}_{L} = \sum_{\alpha} \hat{n}_{\alpha} \) is the operator for the number of electrons in L. One can then write the charge current operator as \( \hat{I} = -e\hat{N}_{L} \). To obtain the expectation value of the current that is measured in an experiment (\( I \equiv \langle \hat{I} \rangle \)), we employ linear response theory [Kubo formula (Kubo, 1957)] on the corresponding current operator, where \( I = -\langle i/h \rangle \int_{0}^{\infty} dt \langle \hat{N}_{L}(t), \hat{N}_{L}(t) \rangle_{0} \) with \( \langle \cdot \rangle_{0} \) the expectation value in the unperturbed state. Assuming that the averages are given by the Fermi distributions in each lead, we have at voltage bias \( V \) such that

\[
I = \frac{1}{eR_{T}} \int_{0}^{\infty} dt \langle n_{L}(\tilde{e})n_{R}(\tilde{e})[f_{L}(\tilde{e}) - f_{R}(\tilde{e})] \rangle_{0},
\]

where \( \tilde{e} = e - eV \). Here the constant prefactor includes the inverse of the resistance \( R_{T} \) of the junction such that \( 1/R_{T} = 2\pi \langle |t_{L}|^2 \rho_{L}(0)\rho_{R}(0)e^2/h \rangle \), with \( |t_{L}|^2 = |t_{R}|^2 = \text{const} \) and \( \nu_{L}(0) \) and \( \nu_{R}(0) \) the DOSs in the normal state at Fermi energy in the left and right electrodes, respectively. Under the integral, \( n_{L}(\tilde{e}) \) and \( n_{R}(\tilde{e}) \) are the normalized [by \( \nu_{L}(0) \) and \( \nu_{R}(0) \), respectively] energy-dependent DOSs, and \( f_{L}(\tilde{e}) \) and \( f_{R}(\tilde{e}) \) are the corresponding energy distributions that are Fermi-Dirac distributions for equilibrium electrodes.

For heat current we use precisely the same procedure but now for the operator of energy of the left electrode
\[ \dot{Q}_L = \frac{1}{e^2 R_T} \int d\bar{e} \, \hat{n}_L(\bar{e}) n_R(\bar{e}) [f_L(\bar{e}) - f_R(\bar{e})]. \] (18)

Here we comment on the relation between the energy and heat currents \( J \) and \( \dot{Q} \) introduced in Sec. II. Inserting \( \bar{e} = e - eV \), we immediately find that \( \dot{Q}_L = J - IV \), where \( J \equiv \langle e^2 R_T \rangle^{-1} \int d\bar{e} \, \hat{n}_L(\bar{e}) n_R(\bar{e}) [f_L(\bar{e}) - f_R(\bar{e})] \). Writing the equation (18) for the heat from the right electrode in analogy with Eq. (18), we find that \( \dot{Q}_R = -J \). Thus, we have \( \dot{Q}_L + \dot{Q}_R = -IV \), which presents energy conservation: the total power taken from the source goes into heating the two electrodes. This is natural since in steady state work equals heat, as the temperature-dependent current-voltage characteristics. This feature probes the temperature of the normal side of the junction (Giazotto et al., 2006; Muhonen, Meschke, and Pekola, 2012; Courtois et al., 2014). References on the topic besides the previously mentioned reviews include Nahum, Eiles, and Martinis (1994), Leivo, Pekola, and Averin (1996), Clark et al. (2004), Kuzmin et al. (2004), Prance et al. (2009), Nguyen et al. (2013), and Feshchenko, Koski, and Pekola (2014).

C. Hamiltonian of a quantum circuit

Another key element in our context is a harmonic oscillator, and in some cases a nonlinear quantum oscillator, usually in the form of a Josephson junction (Tinkham, 2004). To avoid dissipation the linear harmonic oscillator in a circuit is commonly made of a superconductor, often in the form of a coplanar wave resonator (Krantz et al., 2019). The Hamiltonian of such an \( LC \) oscillator, shown in Fig. 4(a), is composed of the kinetic \( q^2/2C \) and potential \( \Phi^2/2L \) energies, respectively, where \( q \) is the charge on the capacitor and \( \Phi \) is the flux of the inductor. The charge is the conjugate momentum to flux as \( q = C\Phi \), and the total Hamiltonian is then

\[ \hat{H} = \frac{\hat{q}^2}{2C} + \frac{\Phi^2}{2L}. \] (19)

i.e., that of a harmonic oscillator, with \( \hat{q} \) and \( \hat{\Phi} \) the charge and flux operators, respectively. Introducing the creation \( \hat{c}^\dagger \) and annihilation \( \hat{c} \) operators such that \( [\hat{c}, \hat{c}^\dagger] = 1 \), we have

\[ \Phi = \sqrt{\frac{\hbar Z_0}{2}} (\hat{c} + \hat{c}^\dagger), \quad \hat{q} = -i \sqrt{\frac{\hbar}{2Z_0}} (\hat{c} - \hat{c}^\dagger). \] (20)

FIG. 3. Properties of a \( N-I-S \) tunnel junction. (a) Calculated current-voltage curves at different values of \( T/T_C = 0.05 - 0.5 \) from bottom to top (both panels). At these subgap voltages the junction provides a sensitive thermometer. Inset: energy diagram of a biased by voltage \( V \) junction between a normal-metal (N) and superconducting (S) electrode connected via an insulating (I) barrier. Because of the BCS gap \( \Delta \) in S, transport is blocked at \( eV < \Delta \). (b) Similarly calculated power \( \dot{Q}_L \) vs \( V \) curves, demonstrating cooling of \( N \) at \( eV \lesssim \Delta \). At higher voltages \( eV \gg \Delta \), \( \dot{Q}_L \) becomes negative, meaning that it serves as a Joule heater of \( N \).

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FIG. 4. Central elements of superconducting quantum devices. (a) \( LC \) circuit with flux \( \Phi \) and charge \( q \). (b) Josephson junction with phase difference \( \phi \) and Josephson energy \( E_J \).
which yield the standard harmonic oscillator Hamiltonian

\[ H = \hbar \omega_0 (\hat{\phi}^2 + \frac{1}{2}). \quad (21) \]

where \( \omega_0 = 1/\sqrt{LC} \) and \( Z_0 = \sqrt{L/C} \) are the angular frequency and impedance of the oscillator.

For a Josephson tunnel junction, shown in Fig. 4(b), the Josephson relations (Josephson, 1962) are

\[ \hbar \dot{\phi} = 2eV, \quad I = I_c \sin \phi. \quad (22) \]

where \( \phi \) is the phase difference across the junction related to flux by \( \phi = (2e/h)\Phi \). In the second Josephson relation, \( I \) is the current through the junction. The sinusoidal current-phase relation applies strictly to a tunnel junction with critical current \( I_c \). For different types of weak links, sinusoidal dependence does not necessarily hold (Tinkham, 2004).

The energy stored in the junction (which is equal to the work done by the source) is then obtained for a current biased case from \( E = \partial E/\partial \Phi \) as

\[ E = \int^{\Phi} \dot{\Phi} d\Phi = -E_J \cos \phi. \quad (23) \]

Equation (23) constitutes the Josephson part of the Hamiltonian, also called \( \hat{H}_J \). For small values of \( \phi \), ignoring the constant part we have

\[ E \approx \Phi^2/2L_J, \quad (24) \]

where \( L_J = h/(2eI_c) \) is the Josephson inductance. Therefore, in the “linear regime” a Josephson junction can be considered a harmonic oscillator such that Eqs. (19)–(21) apply with \( L \) replaced by \( L_J \). Yet the actual nonlinearity of a Josephson junction makes it an invaluable component in quantum information processing and in quantum thermodynamics. A magnetic flux tunable Josephson junction, for instance, in the form of two parallel junctions with a superconducting loop in between, is the superconducting quantum interference device (SQUID) discussed in Secs. VII–IX.

**D. Quantum noise of a resistor**

The quantum noise of a resistor is an important quantity, as it determines the heat emission and absorption in the form of thermal excitations. In Sec. VII it becomes obvious how this noise yields the Joule power in a circuit.

Consider that the resistor in the quantum circuit is formed from a collection of harmonic oscillators with ladder operators \( \hat{b}_i \) and \( \hat{b}_i^\dagger \) with frequencies \( \omega_i \). The phase operator in the interaction picture reads

\[ \phi(t) = \sum_i \lambda_i (\hat{b}_i e^{i\omega_i t} + \hat{b}_i^\dagger e^{-i\omega_i t}) \quad (25) \]

with coefficients \( \lambda_i \). The following voltage fluctuations are related to the phase as \( v(t) = (\hbar/e) [\dot{\phi}(t)] \):

\[ v(t) = i\hbar \sum_i \lambda_i \omega_i (\hat{b}_i e^{i\omega_i t} - \hat{b}_i^\dagger e^{-i\omega_i t}). \quad (26) \]

The spectral density of voltage noise \( S_V(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} \langle v(t) v(0) \rangle \) is then given by

\[ S_V(\omega) = \frac{2\pi \hbar^2}{e^2} \int_0^\infty d\Omega \nu(\Omega) \lambda(\Omega)^2 \Omega^2 \{ [1 + n(\Omega)] \delta(\omega - \Omega) \\
+ n(\Omega) \delta(\omega + \Omega) \}, \quad (27) \]

where \( \nu(\Omega) \) is the oscillator density of states. Now we consider both positive and negative frequencies, which correspond to the quantum emission and absorption processes. For positive frequencies only the first term survives as

\[ S_V(\omega) = \frac{2\pi \hbar^2}{e^2} \nu(\omega) \lambda(\omega) \omega^2 [1 + n(\omega)]. \quad (28) \]

Similarly considering the negative frequencies, we find that

\[ S_V(\omega) = e^{-\beta \hbar \omega} S_V(\omega), \quad (29) \]

which is the detailed balance condition.

We know that the classical Johnson-Nyquist noise (Johnson, 1928; Nyquist, 1928) of a resistor at \( k_B T \gg \hbar \omega \) reads

\[ S_V(\omega) = 2k_B T R. \quad (30) \]

This is the classical fluctuation-dissipation theorem (FDT) (Callen and Welton, 1951) applied to the resistor. In this limit, by using the Taylor expansion we have \( (1 - e^{-\beta \hbar \omega})^{-1} \propto (\beta \hbar \omega)^{-1} \), so using Eq. (28) we have the following connection between the oscillator properties and the physical resistance (Karimi and Pekola, 2021):

\[ \lambda_i^2 = \frac{R e^2}{\pi \hbar (\omega_t) \omega_i}. \quad (31) \]

Substituting this result into Eq. (27), we obtain at all frequencies

\[ S_V(\omega) = 2R \frac{\hbar \omega}{1 - e^{-\beta \hbar \omega}}. \quad (32) \]

**V. PHONONS**

Quantized thermal conductance was demonstrated experimentally for the first time by Schwab et al. (2000). In their setup, as shown in the inset of Fig. 5, the “phonon cavity” consists of a 4 × 4 μm² block of a silicon nitride membrane with 60 nm thickness suspended by four legs of equal thickness. Each leg has catenoid waveguide shape whose diameter at the narrowest point is less than 200 nm. This waveguide shape as a 1D channel is the ideal profile to achieve unit transmissivity between the suspended cavity and the bulk reservoir (Rego and Kirczenow, 1998). Two Au-film resistors with 25 nm thickness were patterned on the suspended central...
experiments were performed by applying Joule heating on a membrane has been completely removed in the dark regions. Adapted from Schwab et al., 2000.

block; one of them serves to apply the Joule heating to generate the temperature gradient along the legs, and the other one worked as a thermometer to measure the phonon cavity temperature. The electron temperature of the resistor was measured with a low noise amplifier (dc SQUID) operating with nearly quantum-limited energy sensitivity by measuring the electrical Johnson noise of the resistor.

The measurement of Schwab et al. (2000) probes the thermal conductance by phonons across the four silicon nitride bridges as a function of bath temperature. These data are shown in the main panel of Fig. 5. The result exhibits the usual phononic thermal conductance ($\propto T^3$) at temperatures above 1 K. Below this temperature there is a rather abrupt leveling off of $G_{th}$ to the value $16G_0$ (here the notation is such that $g_0 \equiv G_0$). Schwab et al. (2000) argue that the coefficient 16 arises from the trivial factor 4 due to four independent bridges in the structure and the less trivial factor 4 due to four possible acoustic vibration modes of each leg in the low temperature limit: one longitudinal, one torsional, and two transverse modes. In later theoretical works the somewhat meandering behavior of $G_{th}/G_0$ below the crossover temperature was explained to arise from the remaining scattering of phonons in the bridges, i.e., from nonballistic transport, whose effect is expected to get weaker in the low temperature limit (Santamore and Cross, 2001).

Over the years, there have been a few other experiments on thermal conductance by phonons in restricted geometries. The one by Leivo (Manninen, Leivo, and Pekola, 1997; Leivo and Pekola, 1998; Leivo, 1999) employed 200-nm-thick silicon nitride membranes in various geometries; see Fig. 6. The experiments were performed by applying Joule heating on a central membrane in a manner analogous to the experiment of Schwab et al. (2000), and the resulting temperature change to obtain the thermal conductance was then read out by measuring the temperature-dependent conductance of $N$-$I$-$S$ probes processed on top of the same membrane. In this case the wiring running along the bridges was made of aluminum, which is known to provide close to perfect thermal isolation at temperatures well below the superconducting transition at $T_C \approx 1.4$ K; see Sec. IV.A. In general, there are many conduction channels in the wide bridges, as demonstrated in the Fig. 6 caption. Yet this number for a single $w = 4 \mu$m wide bridge is $N = 14$ at $T = 100$ mK, which is already close to the prediction of $N = 4$ given by Rego and Kirczenow (1998). The ballisticity of these $15 \mu$m long bridges is unknown, though. Yet these experiments provide evidence of thermal conductance close to the quantum limit.

The experiment of Schwab et al. (2000) was followed by several measurements using different temperature ranges and materials. Experiments on GaAs phonon bridges of sub-$\mu$m lateral dimensions were previously performed at temperatures above 1 K (Tighe, Worlock, and Roukes, 1997) and later down to 25 mK bath temperature (Yung, Schmidt, and Cleland, 2002). The latter experiment measuring the temperature of the GaAs platform in the middle using $N$-$I$-$S$ tunnel junctions demonstrated Debye thermal conductance at $T \gg 100$ mK but tended to follow the expected quantum thermal conductance at the lowest temperatures. In the more recent experiments by Tavakoli et al. (2017, 2018) the measurement on submicron-wide silicon nitride bridges was made differential in the sense that there was no need to add superconducting leads on these phonon-conducting legs. The results at the lowest

FIG. 5. View of the suspended structure of Schwab et al. (2000) for measuring quantized thermal conductance. Main panel: temperature dependence of the measured thermal conductance normalized by $16G_0$ ($16g_0$). Inset: in the center, a $4 \times 4 \mu$m$^2$ phonon cavity is patterned from the membrane; the bright areas on the central membrane are Au-thin-film transducers connected to Nb-thin-film leads on top of phonon waveguides. The thermal conductance by phonons in restricted geometries. The latter experiment measuring the temperature of the phonon cavity is patterned from the membrane; the bright areas on the central membrane are Au-thin-film transducers connected to Nb-thin-film leads on top of phonon waveguides. The thermal conductance by phonons across the four silicon nitride bridges as a function of bath temperature. These data are shown in the main panel of Fig. 5. The result exhibits the usual phononic thermal conductance ($\propto T^3$) at temperatures above 1 K. Below this temperature there is a rather abrupt leveling off of $G_{th}$ to the value $16G_0$ (here the notation is such that $g_0 \equiv G_0$). Schwab et al. (2000) argue that the coefficient 16 arises from the trivial factor 4 due to four independent bridges in the structure and the less trivial factor 4 due to four possible acoustic vibration modes of each leg in the low temperature limit: one longitudinal, one torsional, and two transverse modes. In later theoretical works the somewhat meandering behavior of $G_{th}/G_0$ below the crossover temperature was explained to arise from the remaining scattering of phonons in the bridges, i.e., from nonballistic transport, whose effect is expected to get weaker in the low temperature limit (Santamore and Cross, 2001).

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temperatures of ≈0.1 K fall about 1 order of magnitude below the quantum value, and the temperature dependence of thermal conductance is close to $T^2$. Tavakoli et al. (2017, 2018) proposed nonballistic transmission in their bridges as the origin of their results. Finally, experiments by Zen et al. (2014) demonstrated that thermal conductance can be strongly suppressed even in two dimensions with proper patterning of the membranes into a nanostructured periodic phononic crystal.

VI. ELECTRONS AND FRACTIONAL CHARGES

Charged particles play a special role in assessing quantum transport properties since they provide straightforward access to both the particle number current and the heat current. For instance, in the case of electrons we can count the carriers by directly measuring the charge current and the associated conductance. When the mean free path of the carriers is much larger than the physical dimensions of the contact, transport can become ballistic. According to Eq. (4), the electrical conductance then assumes only integer multiple values of elementary conductance quantum. The first experiments on quantized conductance of a point contact in a GaAs-AlGaAs two-dimensional high mobility electron gas (2DEG) heterostructures were performed by van Wees et al. (1988) and Wharam et al. (1988). van Wees et al. (1988) formed the point contact using a top metallic gate with a width $W \approx 250$ nm opening in a tapered geometry to form a voltage-controlled narrow and short channel in the underlying electron gas. The layout of the gate electrode is shown in the inset of Fig. 7. At negative gate voltages electrons are repelled under the gate and the width of the channel for carriers is $\lesssim 100$ nm, which is well below the mean free path of $l \approx 8.5 \mu m$. The measured conductance of the point contact shown in Fig. 7 exhibits well-defined plateaus at the expected positions $N\alpha e^2/\hbar$ as a function of applied gate voltage (van Wees et al., 1988). The factor of 2 with respect to Eq. (4) arises from spin degeneracy.

Thirty years after the experiments on quantized electrical conductance by electrons (van Wees et al., 1988; Wharam et al., 1988), Jezouin et al. (2013) measured the quantum-limited heat conductance of electrons in a quantum point contact. The principle and practical implementation of this experiment and its setup are shown in Figs. 8(a) and 8(b). A micrometer-sized metal plate is connected to both a cold phonon bath and a large electronic reservoir via an adjustable number $n$ of ballistic quantum channels with both reservoirs at $T_0$, as shown in Fig. 8(a). By injecting Joule power $Q_{\text{ext}}$ to the metallic plate, the electrons were heated up to temperature $T$, which can be directly measured by a noise thermometer. This power is then transmitted via the $n$ quantum channels at the rate $nG_0(T - T_0)$ through two quantum point contacts (QPC$_1$ and QPC$_2$) and to the phonon bath at rate $Q_\text{sp}$, which is independent of $n$. The two QPCs display clear plateaux of the measured electrical conductance at $n_1e^2/\hbar$ and $n_2e^2/\hbar$, respectively, where $n_1$ and $n_2$ are integers. The sum $n = n_1 + n_2$ determines the number of quanta carrying the heat out of the plate electronically. The structure used in this experiment (Jezouin et al., 2013) satisfies the conditions of having sufficient electrical and thermal contact between the metal plate and the two-dimensional electron gas underneath. Moreover, the thermal coupling to the phonon bath and via the QPCs is weak enough that the central electronic system forms a uniform Fermi gas (fast electron-electron relaxation and diffusion across the plate) at temperature $T$. A perpendicular magnetic field was applied to the sample so as to be in the integer quantum Hall effect regime at filling factors $\nu = 3$ or 4. Figure 8(c) shows $\alpha_p$, the measured electronic heat conductance normalized by $\pi^2k_B^2/(6\hbar)$ as a

![FIG. 7. Measured quantized conductance of a point contact in a two-dimensional electron gas as a function of gate voltage. The conductance demonstrates plateaus at multiples of $2e^2/\hbar$. Inset: schematic layout of the point contact. Adapted from van Wees et al., 1988.](image)

![FIG. 8. Measuring quantized heat carried by electrons. (a) When Joule power $Q_{\text{ext}}$ is applied to a metal plate (brown disk), the electronic temperature increases up to $T$, and the heat then flows via $n$ ballistic quantum channels to the reservoir and the phonon bath $Q_\text{sp}$, which both have fixed temperature $T_0$. (b) Colored scanning electron micrograph of the measured sample. In the center, the metallic Ohmic contact in brown is connected to two quantum point contacts (QPC$_1$ and QPC$_2$) in yellow (lightest area) via a two-dimensional Ga(Al)As electron gas in light green (surrounding the point contacts). The red lines with arrows around the metal plate indicate the two propagating edge channels ($\nu = 3$ or 4). The Joule power is applied to the metallic plate through a QPC, and the two LC-tank circuits are for noise thermometry measurements. (c) The gray line shows the predictions for the quantum limit of the heat flow, while the symbols exhibit the extracted electronic heat current normalized by $\pi^2k_B^2/(6\hbar)$ as a function of the number of electronic channels $n$. Adapted from Jezouin et al., 2013.](image)
function of the number n of electronic channels as symbols that fall on a straight line with unit slope shown by the gray line, thus demonstrating the quantized thermal conductance at the expected level. Equivalently, this experiment demonstrates Wiedemann-Franz law on the current plateaus.

The work of Jezouin et al. (2013) was preceded by two experiments of some two decades earlier (Molenkamp et al., 1992; Chiatti et al., 2006), where $G_Q$ was tested with an order of magnitude accuracy. Both measurements were performed on GaAs-based 2DEGs, and in both of them, thermal conductance was obtained by measuring the Seebeck coefficient (thermopower) and extracting the corresponding temperature difference. Molenkamp et al. (1992) then determined $G_{Rb}$, which agrees within a factor of 2 with the assumption that the Wiedemann-Franz law applies to the conduction plateaus of the QPC. Chiatti et al. (2006) conducted a similar experiment with the same philosophy but with improved control of the structure and system parameters. With these assumptions there is good agreement between thermal conductance and electrical conductance via the Wiedemann-Franz law.

In recent years, it has become possible to measure quantized thermal conductance even at room temperature (Cui et al., 2017; Mosso et al., 2017). The experiments are performed on metallic contacts of atomic size with scanning thermal microscopy probes. The material of choice is typically Au, although experiments on Pt have also been reported (Cui et al., 2017). The setup and experimental observations of Cui et al. (2017) are presented in Fig. 9. The electrical conductance plateaus at multiples of $2e^2/h$ are typically seen when pulling the contact to the few conductance channel limit. The noteworthy feature in the data is that the simultaneous thermometric measurement confirms the Wiedemann-Franz law for electric transport within 5%–10% accuracy, thereby demonstrating quantized thermal conductance (Cui et al., 2017).

In the measurement performed by Banerjee et al. (2017), the value of the quantum of thermal conductance for different Hall states including integer and fractional states was verified. They first confirmed the observations of Jezouin et al. (2013) in a similar setup in the integer states with filling factors $\nu = 1$ and 2. Figure 10(a) demonstrates the validity of quantized heat conductance at $\Delta NG_Q$ for $\Delta N = 1, 2, ..., 6$ channels with about 3% accuracy (inset). The main result of the work is the observation of thermal conductance of strongly interacting fractional states. Figure 10(b) shows that the thermal conductance is again a multiple of $G_Q$, even for the (particelike) $\nu = 1/3$ fractional state, although the electrical conductance is normalized by the effective charge $e^* = e/3$. As a whole, the work covers both particelike and holelike fractional states, testing the predictions of Kane and Fisher (1997).

As a final point in this section we mention that there are a large number of further experiments on various heat transport effects performed in the quantum Hall regime. We do not cover these experiments in detail here; see Granger, Eisenstein, and Reno (2009), Altimiras et al. (2010), le Sueur (2010), Nam, Hwang, and Lee (2013), Halbertal et al. (2016, 2017), Banerjee et al. (2018), Sivre et al. (2018), and Srivastav et al. (2019).

VII. PHOTONS

In this section we discuss transport by thermal microwave photons, presenting another bosonic system to study in this context.

A. A ballistic photon channel

The concept of microwave photon heat transport becomes concrete when it is described on a circuit level (Schmidt,
Schoelkopf, and Cleland, 2004). We start with a setup familiar from the century-old discussion by Johnson (1928) and Nyquist (1928). Two resistors $R_1$ and $R_2$ are directly coupled there to each other as shown in Fig. 11(a). They are generally at different temperatures $T_1$ and $T_2$. Each resistor then produces thermal noise with the spectrum $S_v(\omega)$ of Eq. (32); i.e., they are thermal photon sources. We first consider the fact that $R_1$ generates noise current $i_1$ on resistor $R_2$ as $i_1 = v_1/(R_1 + R_2)$. The spectral density of current noise is then $S_{i_1}(\omega) = (R_1 + R_2)^2 S_v(\omega)$. The voltage noise produced by resistor $R_i$ ($i = 1, 2$) is $S_v(\omega) = 2 R_i \hbar \omega / (1 - e^{-\beta \hbar \omega})$ for $i = 1, 2$. The power density produced by the noise of $R_1$ and dissipated in resistor $R_2$ is then $S_{p_2}(\omega) = [R_2/(R_1 + R_2)]^2 S_{i_1}(\omega)$. The corresponding total power dissipated in resistor $R_2$ due to the noise of resistor $R_1$ is

$$P_2 = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_{p_2}(\omega) = \frac{4 R_1 R_2}{(R_1 + R_2)^2} \int_{0}^{\infty} \frac{d\omega}{2\pi} \hbar \omega \left[ n_1(\omega) + \frac{1}{2} \right].$$

The net heat flux from $1 \to 2$ ($P_{\text{net}}$) is the difference between $P_2$ and $P_1$, where $P_1$ is the corresponding power produced by $R_2$ on $R_1$ by the uncorrelated voltage (current) noise described similarly. Thus,

$$P_{\text{net}} = \frac{4 R_1 R_2}{(R_1 + R_2)^2} \frac{\pi k_B^2}{12 \hbar} (T_1^2 - T_2^2).$$

Note that the integrals for $P_1$ and $P_2$ separately [see Eq. (33)] would lead to a divergence due to the zero point fluctuation term, but since these fluctuations cannot transport energy this term cancels out in the physical net power [Eq. (34)]. We find that, for a small temperature difference with $T_1 = T_2 = T$,

$$G_{\nu} = \frac{d P_{\text{net}}}{dT} = \left. \frac{4 R_1 R_2}{(R_1 + R_2)^2} \frac{\pi k_B^2}{6 \hbar} T \right|_T,$$

which is equal to the quantum of heat conductance

$$G_{\nu} = G_Q$$

for $R_1 = R_2$. For a general combination of resistance values the factor

FIG. 10. Measurements in the (a) integer and (b) fractional quantum Hall regimes with filling factors $\nu = 2$ and $1/3$, respectively. (a) Normalized coefficient of the dissipated power $\lambda = \delta P / (G_Q/2 T)$ as a function of $T_m^2$ for different configurations of $\Delta N = N_i - N_j$, where $N$ is the number of channels. The difference is presented in order to eliminate the $N$-independent contribution of the phononic heat current. Here $\delta P$ is the difference between dissipated power at different $N$, $\delta P = \Delta P(N_i, T_m) - \Delta P(N_j, T_m)$, and $T_m$ is the calculated temperature of the floating contact. The circles show the measured data and the dashed lines are linear fits to them. The slope of each set is shown in the inset as a function of $\Delta N$. The linear dependence has approximately unit slope $(0.98 \pm 0.03)$, confirming the quantum of thermal conductance for this integer state ($\nu = 2$). (b) Case of the fractional state $\nu = 1/3$. It is the same as (a) except that the difference of $\lambda$ between $N = 4$ and 2 is normalized by $\Delta N$ as a function of $T_m^2$. The slope of the linear fit (dashed line) to the measured data (circles) is close to unity. Adapted from Banerjee et al., 2017.

FIG. 11. Setup of two resistors $R_1$ and $R_2$ at temperatures $T_1$ and $T_2$, respectively, interacting with each other via the respective thermal noises. We present the quantum version of the classical Johnson-Nyquist problem in the text with the associated radiative heat current. (a) The plain two-resistor heat exchange can be modeled using a circuit approach where each resistor is accompanied by a thermal voltage noise source. The two sources are uncorrelated. (b) A realistic circuit includes inevitably reactive elements as well, as discussed in the text. These are added in the figure to allow for an analysis of the crossover between the quantum and classical regimes upon varying the operating temperature and the physical system size.
\[ r = \frac{4R_1R_2}{(R_1 + R_2)^2} \]  

represents a transmission coefficient. The circuit model for heat transport can be generalized to essentially any linear circuit composed of reactive elements and resistors, as was done by Pascal, Courtois, and Hekking (2011) and Thomas, Pekola, and Golubev (2019).

### B. Circuit limitations of the ballistic picture

What are the physical conditions for the experiment in a circuit to yield thermal conductance that is governed by \( G_Q \)? The Johnson-Nyquist work (Johnson, 1928; Nyquist, 1928) was out of this domain, as was a more recent experiment by Ciliberto et al. (2013). The necessary key ingredients for “quantumness” are that the experiment combines low temperatures and physically small structures. More quantitatively, the realistic circuit is never presented fully by the simple combination of two resistors, but the full picture of it instead includes inevitable reactive elements. A way of describing a more realistic circuit (Golubev and Pekola, 2015) is to include a parallel capacitance and series inductance in the basic circuit, as shown in Fig. 11(b). The point is that electromagnetics tells us that an order of magnitude estimate for capacitance is given by \( 10^4 \text{mK} \), for the Johnson-Nyquist work (Johnson, 1928; Nyquist, 1928) i st o include a parallel capacitance and series inductance in the realistic circuit. If we take a mesoscopic circuit with \( \ell = 100 \mu m \) at a low temperature \( T = 100 \text{mK} \), we find that \( \nu_0\ell \text{k_BT}/(hR) \approx 0.01 \), which satisfies the conditions in Eqs. (38). On the other hand, an \( \ell = 0.1 \text{m} \) macroscopic circuit at room temperature \( T = 300 \text{K} \) yields \( \nu_0\ell \text{k_BT}/(hR) \approx 3 \times 10^6 \), which is far into the classical regime. Some of those conditions can be avoided in a low temperature transmission line circuit (Partanen et al., 2016), as we discuss later.

### C. Experiments on heat mediated by microwave photons

We modeled in Sec. VII the heat emitted by a resistor and absorbed by another one in an otherwise dissipationless circuit. It was shown (Schmidt, Schoelkopf, and Cleland, 2004) that this heat carried by microwave photons behaves as if the two resistors were coupled by a contact whose ballisticity is controlled by the impedances in the circuit. Ideally, two physically small and identical resistors at low temperatures can come close to the ballistic limit, with thermal conductance approaching \( G_Q \). Motivated by this observation, several experiments assessing this result were set up in the past two decades (Meschke, Guichard, and Pekola, 2006; Timofeev et al., 2009; Partanen et al., 2016). They were all performed essentially in the same scenario: the resistors are normal metallic thin-film strips with sufficiently small size that their temperature varies significantly in response to typical changes of power affecting them. The electrical connection between the resistors is provided by superconducting aluminum leads, whose electronic heat conductance is vanishingly small at the temperature of operation; see Sec. IV.A. In one of the experiments (Meschke, Guichard, and Pekola, 2006) the superconducting lines were interrupted by a SQUID that acts as a tunable inductor providing a magnetic-flux-controlled valve of photon mediated heat current. All these experiments were performed at \( T \approx 0.1 \text{K} \), far below \( T_c \approx 1.4 \text{K} \) of aluminum. Temperatures are controlled and monitored by biased \( N-I-S \) tunnel junctions.

The experiment of Timofeev et al. (2009) was designed to mimic as closely as possible the basic configuration of Fig. 11(a) with a superconducting Al loop. In this case the distance between the resistors was about 50 \( \mu m \), and the temperatures of both the heated (or cooled) source and the drain resistor were measured. The experiment [Figs. 12(a)–12(c)] demonstrates thermal transport via the electronic channel, i.e., the quasiparticle thermal transport (Bardeen, Rickayzen, and Tewordt, 1959) described in Sec. IV.A, at temperatures exceeding \(~250 \text{mK} \). The result in this regime is in line with the basic theory, given the dimensions and material parameters of the aluminum leads. Below about 200 mK the photon contribution kicks in. In the loop geometry it turns out that the temperatures of the two resistors follow each other closely at the lowest bath temperatures, yielding thermal conductance given by \( G_Q \). Some uncertainty remains about the absolute value of \( G_e \) since the precise magnitude of the competing electron-phonon heat transport coefficient \( \Sigma \) remained somewhat uncertain. The measurement was backed by a reference experiment, where a sample similar to that described previously was measured under the same conditions and fabricated in the same way. This reference sample intentionally lacked one arm of the loop leading to poor matching of the circuit in the spirit discussed in Sec. VII.B. In this case the quasiparticle heat transport prevails as in the matched sample, but the photon \( G_e \) is vanishingly small, confirming, one could say even quantitatively, the ideas presented about the heat transfer via a nonvanishing reactive impedance.

The previously described experiment was performed on a structure with physical dimensions not exceeding 100 \( \mu m \). A natural question arises: is it possible to transport heat over macroscopic distances by microwave photons, like radiating the heat away from the entire chip? This could be important in quantum information applications; for superconducting qubit realizations, see Kjaergaard et al. (2020). This question was addressed experimentally by Partanen et al. (2016)
[Figs. 12(d) and 12(e)], who placed the two resistors at a distance of about 10 mm, i.e., about 100 times farther away from each other than was done earlier. Furthermore, the connecting line between the two baths was a 1 m long meander made of a superconductor, which acted as a transmission line. Such a coplanar line typically has an impedance of about 50 Ω irrespective of its length, thus potentially supporting the heat transport even over large distances. The thermal conductance was measured as in the work of Timofeev et al. (2009), with similar results proving the hypothesis of photon transport over macroscopic distances. These experiments may open the way for practical heat transport schemes in microwave circuits.

VIII. TUNABLE QUANTUM HEAT TRANSPORT

In this section we describe quantum systems where heat transport is controlled by either the magnetic or electric field.
to achieve useful functional operation. These devices include heat valves, heat interferometers, thermal rectifiers, and circuit refrigerators. Mesoscopic structures provide an option for controlling currents using external fields. Concerning charge currents, SQUIDs (Tinkham, 2004) and single-electron transistors (Averin and Likharev, 1991) provide hallmark devices in this context, where the magnetic field (flux in a superconducting loop) and electric field (gate voltage), respectively, are the parameters that control the current.

The first experiment on heat transport by thermal microwave photons (Meschke, Guichard, and Pekola, 2006) was realized in a setup where a SQUID was used as a heat valve. The experiment depicted in Fig. 13 shows two metallic AuPd resistors at a distance of a few tens of micrometers from each other, connected by superconducting Al lines. The loop is interrupted in each arm by a SQUID, whose flux can be controlled by the common external field for both of them. The thermal model of Fig. 2(b) applies to this circuit. In the experiment only the heated resistor’s temperature $T_{e1}$ was measured. The panel on the bottom right of Fig. 13 displays the magnetic-flux-dependent variation of temperature $T_{e1}$ at different bath temperatures $T_0$ under a constant level of heating. At bath temperatures well above 100 mK, the flux dependence vanishes since the inter-resistance thermal conductance by photons $G$ is much weaker than the conductance to the phonon bath $T \propto T^4$. On the contrary, toward low temperatures below 100 mK, the electron temperature $T_{e1}$ varies with magnetic flux as the inter-resistor coupling becomes comparable to the bath coupling, demonstrating the photonic thermal conductance. Moreover, the magnitude of the thermal conductance was shown to quantitatively follow from the circuit model presented in Sec. III.A when applied to the current setup. Among other things, the data and this calculation predicted that at $T_0 = 60$ mK the maximum value of thermal conductance with zero flux in the SQUID (i.e., with minimum Josephson inductance) was $\approx 50\%$ of $G_0$.

The photonic heat current was controlled by the magnetic field in the previous example. A dual method is to apply the electric field as a control, as indicated in Fig. 14(a). This procedure was realized in a recent experiment (Maillet et al., 2020), where the superconducting loop is interrupted by a Cooper-pair transistor ["charge qubit" (Nakamura, Pashkin, and Tsai (1999)]. In this setup, the Josephson coupling is tuned by the gate voltage. The thermal model of the experiment is pretty much as before, except that the Josephson element with its control field is different. In general it is easier to apply the electric field using gate voltage, especially locally on the chip, than local magnetic flux. As shown in Fig. 14(a), the device demonstrates gate-dependent modulation of the heat current. Its overall magnitude is consistent with the modeling of the circuit: at maximum thermal conductance $G_e \approx 0.35G_0$ was achieved.

As to the heat currents in single-electron circuits, similar control principles apply in general. An early experiment to control heat flow using gate voltage in a single-electron transistor formed of $N$-$I$-$S$ junctions was performed by Saira et al. (2007). The results on temperature of the system were quantitatively confirmed by a model employing standard single-electron tunneling theory and a heat balance equation on the measured central island of the transistor. Heat transport via a fully normal metallic single-electron transistor was measured by Dutta et al. (2017). Results of this experiment are shown in Fig. 14(b). The heat current between the source and drain with a temperature bias applied across was carried by electrons and modulated by the gate voltage such that the observed thermal conductance and simultaneously measured

FIG. 14. (a) SEM (top panel) and the equivalent electrical circuit diagram (middle panel) of the device including a Cooper-pair transistor coupled to two normal-metal resistors $R_1$ and $R_2$ at temperatures $T_1$ and $T_2$, respectively. Bottom panel: measured thermal conductance (symbols) normalized by $G_0$ as a function of gate charge $n_g = C_g V_g/e$ at $T_m = (T_1 + T_2)/2$. The solid line indicates the theoretical expectation. Adapted from Maillet et al., 2020. (b) SEM image (top panel) and schematic realization (middle panel) of the device consisting of a single-electron transistor and the heat transport measurement setup. The black circuit in the top left corner displays the heat transport setup. Bottom panel: measured gate dependence of the electronic temperature $T_e$ of the source island when it is lower than the bath temperature $T_0$. Adapted from Dutta et al., 2017.
electrical conductance go hand in hand. Yet deviations from the Wiedemann-Franz law due to a Coulomb blockade and quantum tunneling were observed, in agreement with theory (Kubala, König, and Pekola, 2008; Rodionov, Burmistrov, and Chtchelkatchev, 2010).

A. Electronic quantum heat interferometer

Another quantum interference experiment on heat current by electrons was performed by Giazotto and Martínez-Pérez (2012) (shown in Fig. 15). They used a magnetic-field-controlled SQUID as an interferometer. They could independently measure the electrical and heat transport via the device. For the latter, the SQUID was placed between two mesoscopic heat baths and the heat current was measured with the principle depicted in Fig. 2(b). The measurement was performed in a temperature regime exceeding that described in Sec. IV.A such that $T$ is high enough for the superconductor to have a substantial equilibrium quasiparticle population (i.e., not all electrons are paired). In this regime the superconductor as such can support heat current, and heat interference across the Josephson junctions of the SQUID becomes possible. Giazotto and Martínez-Pérez (2012) addressed experimentally for the first time a half-century-old proposal and theory (Maki and Griffin, 1965); more recent work was given by Gutmann et al. (1997), Gutman, Ben-Jacob, and Bergman (1998), Zhao, Löfwander, and Sauls (2003), and Golubev, Faivre, and Pekola (2013). Giazotto and Martínez-Pérez (2012) also demonstrated the potential of electronic caloritronics in superconducting circuits.

B. Cooling a quantum circuit

In the experiment performed by Tan et al. (2017) photon-assisted tunneling serves the purpose of decreasing the number of microwave quanta in a superconducting quantum circuit, namely, a coplanar wave resonator (harmonic oscillator). The optical micrograph of the sample presented in Fig. 16(a) shows resistive elements inserted at the two ends of the resonator, acting as heat sinks for it. Figure 16(b) displays the temperature of one of these resistors [a quantum circuit refrigerator (QCR)], measured and controlled by $N-I-S$ tunneling, effectively lowering and elevating the electronic temperature of it depending on the biasing of the cooler junction; see Sec. IV.B. The other resistor (“probe”) is passive but its temperature is likewise monitored. This temperature reacts weakly to the QCR temperature changes. Tan et al. (2017) developed a thermal model based on which they extracted the average number of photons in the resonator...

![FIG. 15. Josephson heat interferometer. (a) SEM of the device. The source and drain electrodes made out of Cu are connected to an Al island ($S_1$) through two AlO$_x$ tunnel barriers. $S_1$ is connected sideways to a dc SQUID that terminates at a large-volume lead $S_2$ (Al) for thermalization and to an Al tunnel probe ($S_3$) for independent SQUID characterization. $N-I-S$ junctions in source and drain are used to heat and monitor the temperature of each island. Red crosses indicate the Josephson junctions. The core of the device (SQUID) is shown schematically in the inset. Two identical superconductors with different temperatures are connected with the two tunnel junctions of the SQUID. When the magnetic flux $\Phi$ is applied, the heat current $Q_{\text{SQUID}}(\Phi)$ from hot to cold varies. (b) Maximal charge current of the SQUID $I_c$ as a function of $\Phi$ at 240 mK bath temperature. The dashed line presents the theoretical result assuming 0.3% asymmetry in the junctions, and symbols represent experimental data. (c) Flux modulation of $T_{\text{drain}}$ related to heat current measured at different $T_{\text{source}}$ values. Here the bath temperature is fixed at 235 mK. Adapted from Giazotto and Martínez-Pérez, 2012.]

![FIG. 16. Quantum circuit refrigerator (QCR). (a) Optical micrograph of a sample where a superconducting coplanar-waveguide resonator is in the center coupled to a QCR and a probe resistor indicated by the arrows. (b) Measured changes in the electron temperature of the QCR $\Delta T_{\text{QCR}}$ (purple circles) and the probe resistor $\Delta T_{\text{probe}}$ (red circles) as functions of the refrigerator operation voltage $V_{\text{QCR}}$. The black dashed line, closely following the probe data, and green dashed line show the given theoretical results on $\Delta T_{\text{probe}}$, including and excluding photon-assisted tunneling, respectively. (c) Average number of photons $n$ and the corresponding effective temperature of the resonator $T_{\text{res}}$, shown as the solid lines in red and gray, based on the thermal model introduced by Tan et al. (2017). Adapted from Tan et al., 2017.]

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and the corresponding temperature $T_{\text{res}}$. In this experiment $T_{\text{res}} \approx 800$ mK far exceeds all other temperatures, most notably the electronic temperatures of the two resistors $T_{\text{QCR}} \approx T_{\text{probe}} \approx 150$ mK, even under no bias on the QCR. The model then predicts cooling of the resonator down to about 400 mK under optimal biasing conditions of the QCR [Fig. 16(c)]. Based on the parameters given by Tan et al. (2017), one would estimate the resonator to have $T_{\text{res}} \approx 200$ mK when the QCR is not biased. Indeed, in a later work (Masuda et al., 2018) resonator temperatures in the 200 mK regime were reported at zero bias. When biased, the N-I-S junctions operate as an incoherent microwave source. The effective temperature of the electron reservoirs of the system (Masuda et al., 2018). This phenomenon was theoretically modeled by Silveri et al. (2017) using photon-assisted tunneling of the biased N-I-S junctions as the environment. The effective temperature of the resonator is expected to be lifted to $\sim eV/(2k_B)$ at bias voltage $V$.

IX. QUANTUM HEAT TRANSPORT MEDIATED BY A SUPERCONDUCTING QUBIT

In this section we introduce a superconducting qubit as an element that mediates heat by microwave photons between two baths. Different types of superconducting quantum bits, such as flux, charge, and transmon qubits, are options in such devices (Clarke and Wilhelm, 2008). They feature different coupling options and strengths, as well as different degrees of anharmonicity in the Josephson potential, which is discussed later. In the experiments of Ronzani et al. (2018) transmon-type qubits, introduced by Koch et al. (2007), were employed. This kind of qubit has levels whose positions can be controlled by magnetic flux through the SQUID loop. A transmon qubit is only weakly anharmonic, meaning that one typically needs to consider not only the two lowest levels that form the actual qubit but also the higher levels in this nearly harmonic potential. We point out an important difference: although even weak anharmonicity is enough to address only the two lowest levels in a microwave-driven experiment, one needs, on the contrary, to consider higher levels as well when the qubit sees a thermal bath with a wide spectrum. Yet in a typical experiment described later, the separation of the levels is of the order of 0.5 K, meaning that the thermal population of the third level is already small at the low temperatures of the experiment, say, below 0.2 K ($\sim e^{-5} < 0.01$).

In this section we present thermal transport experiments under conditions in which the qubit is not driven. Coherent properties of the qubit do not then play an important role. In the future the same devices will be driven by rf fields, and the off-diagonal elements of the density matrix will evolve as well.

A. Quantum heat valve

Figure 17(a) (top panel) shows a typical experimental configuration of heat control with a qubit from Ronzani et al. (2018)). The energy separation of the transmon qubit [Fig. 17(a), center] can be controlled by the external magnetic flux $\Phi$. The qubit is coupled capacitively (coupling $g$) to two nominally identical superconducting coplanar wave resonators that act as LC resonators with resonant frequencies of $\sim 5$ GHz each. For thermal transport experiments the $\lambda/4$ resonators are terminated by on-chip resistors that form the controlled dissipative elements in the circuit (Chang et al., 2019). The dissipation is then given by the inverse of the quality factor of the resonator and can be quantified by another coupling parameter $\gamma$. In this circuit, which is called a quantum heat valve, heat is carried wirelessly (via capacitors) by thermal microwave photons over a distance of a few millimeters from one bath to another. A schematic model of such a coupled circuit is shown at the top of Figs. 17(b) and 17(c).

![Figure 17](image-url)
It turns out that the measurement of heat transport in such a circuit addresses some fundamental questions of open quantum systems (Rivas et al., 2010; Levy and Kosloff, 2014; Purkayastha, Dhar, and Kulkarni, 2016a; Hofer et al., 2017; De Chiara et al., 2018; Donvil et al., 2018; Aurell and Montana, 2019; Magazzù and Grifoni, 2019; Donvil, Muratore-Ginanneschi, and Golubev, 2020; Hewgill, De Chiara, and Imparato, 2020). There are at least two possible ways of viewing the circuit, namely, the local view [Fig. 17(b)] and the global view [Fig. 17(c)]. In the local picture as we define it, the environment of the qubit is formed from the dissipative LC resonator with Lorentzian noise spectrum centered around its resonance frequency. In this regime, which occurs when $\gamma \gg g$, the system indeed acts as a valve admitting heat current to pass through only when the qubit frequency matches (within the range determined by the quality factor) the frequency of the resonators. This results in Lorentzian peaks in power centered at flux positions corresponding to said matching condition, demonstrated by both experiment and theory, shown in the bottom panel of Fig. 17(b). In the opposite limit, in the global regime when $\gamma \ll g$, the situation is different. The combined system composed of the resonators and the qubit then makes up a hybrid that interacts with the bare environment formed of the two resistors. In this limit the hybrid quantum system has the energy spectrum shown in the bottom panel of Fig. 17(a) exhibiting a multilevel structure. This is shown by the basic calculated spectrum and the spectroscopic measurement on a structure similar to that in the top panel but in the absence of the resistive loads. The data in Fig. 17(c) demonstrate results in the global regime, with the experiment and theory developed by Ronzani et al. (2018) in agreement with each other. This experiment is the first one to assess local-global crossover in the spirit of locating the Heisenberg cut between the quantum and classical worlds. In a recent theoretical analysis, we analyzed the crossover behavior between the two limiting regimes with the help of a direct solution of the Schrödinger equation including an oscillator bath (Pekola and Karimi, 2020).

**B. Thermal rectifier**

In a symmetric structure, as in Fig. 17, there is naturally no directional dependence of heat transport between the two baths. However, heat current rectification becomes possible if one breaks the symmetry of the structure (Segal and Nitzan, 2005). Heat rectification (Ruokola, Ojanen, and Jauho, 2009; Sothmann et al., 2012; Sánchez, Sothmann, and Jordan, 2015; Purkayastha, Dhar, and Kulkarni, 2016b; Motz et al., 2018; Goury and Sánchez, 2019; Kargi et al., 2019; Riera-Campany et al., 2019; Bhandari et al., 2021; Iorio et al., 2021) can be quantified in different ways, but in general finite rectification means that the magnitudes of forward and reverse heat currents differ under identical but opposite temperature biasing conditions. There have been a few experiments on heat current rectification, including ones on phonons in carbon nanotubes (Chang et al., 2006), electrons in quantum dots (Scheibner et al., 2008), mesoscopic tunnel junctions (Martínez-Pérez, Fornier, and Giazotto, 2015), and suspended graphene (Wang et al., 2017). Senior et al. (2020) realized rectification in a structure similar to that in Fig. 17 but by making the two resonators unequal in length: the two resonators had in this case frequencies of 3 and 7 GHz. An additional feature necessary for heat rectification is the nonlinearity of the central element, which arises from the anharmonicity of the transmon Josephson potential. Figure 18 shows data from Senior et al. (2020) where heat current through the structure is measured in forward and reverse directions under the same but opposite temperature biasing, respectively. Complicated flux dependence can be observed, but the main feature is that one reaches 10% rectification at best and that it depends strongly on the flux position determining the coupling asymmetry to the two baths. A quantitative analysis of the flux dependence is challenging and experiments in simpler setups would be welcome.

![Heat rectifier](image)

**FIG. 18.** Heat rectification using a transmon qubit. (a) Schematic illustration of a photon diode composed of an anharmonic oscillator (qubit) coupled to two LC resonators with largely different resonance frequencies. Bottom panel: circuit view of the system where we assign the quantum heat rectifier in the middle with a red symbol. (b) Dependence of source-drain heat current (power) as a function of magnetic flux in forward (1 → 2, dark line, $P^+$) and reverse (2 → 1, light-colored line, $P^-$) directions under identical but opposite bias conditions at a few different source temperatures (420, 400, and 380 mK from top to bottom). (c) Rectification ratio $R = P^+/P^-$ (with its minimal value subtracted) as a function of magnetic flux for the data in (b) at the three temperatures of the source. Adapted from Senior et al., 2020.

**X. HEAT CURRENT NOISE**

In this section we focus on fluctuations of currents, which are generally considered to be harmful and something to get rid of. The synonym of fluctuations, noise, proposes this negative side of the concept of fluctuations. Noise typically determines the minimal detectable signal in a measurement, i.e., the resolution. Here we do not consider noise caused by the measurement apparatus or from other extrinsic sources, but we instead focus on intrinsic noise due to fundamental quantum and thermal fluctuations. This noise, for instance, in the form of electrical current or heat current fluctuations, determines the ultimate achievable measurement accuracy. But besides being a limiting factor of a measurement, noise can serve as a signal to build on in order to realize a sensor: for instance, measurement of thermal current noise of a conductor.
provides one of the most popular and fundamental thermometers in use (Fleischmann, Reiser, and Enss, 2020).

We already discussed current and voltage noise of a linear dissipative element in Sec. IV.D. Here we review the heat current noise, both classical and quantum; see Sánchez and Böttiker (2012), Moskalets (2014), Pekola and Karimi (2018), Miller et al. (2020), Crépieux (2021), Eriksson et al. (2021), and Karimi and Pekola (2021). For simplicity, we first consider the tunneling as an example. Besides presenting the classical fluctuation-dissipation theorem for heat current, which we review in a general case after the presentation, we observe the quantum expression of heat current noise, including the frequency-dependent component due to zero point fluctuations surviving down to \( T = 0 \). Next we focus on the temperature dynamics of a finite system coupled to a bath, which yields the experimentally accessible fundamental fluctuations of the effective temperature of this subsystem. Finally, we review the experimental situation, which currently consists of only a small number of examples, on fluctuations in heat transport of quantum and classical systems.

A. FDT for heat in tunneling

We consider tunneling where the average heat current out from lead L was given by Eq. (18). Taking for simplicity the normal conductors (\( N-I-N \) junction) with \( n_L(t) = n_R(t) = 1 \), we have the following average heat current at \( eV = 0 \):

\[
\dot{Q}_L = \frac{1}{eR_T} \int dt \, e[f_L(t) - f_R(t)].
\]

The thermal conductance for tunneling \( G_{th} = \frac{d\dot{Q}_L}{dT_L} \bigg|_{T_L = T} \) is then given by \( G_{th} = \mathcal{C} T G_T \), where \( G_T = 1/R_T \) is the conductance of the tunnel junction. Like the fully transmitting transmitting channels in Sec. II, the tunnel junction satisfies the Wiedemann-Franz law.

The heat current operator \( \hat{H}_s \) to obtain the average heat current of Eq. (18) was calculated using the tunnel coupling operator of Eq. (16) and commuting it with the Hamiltonian of the left lead. We may use this operator to find the two-time correlator of it and Fourier transform it to find the spectral density of noise of the heat current at finite angular frequency \( \omega \) (but at \( eV = 0 \)) as \( S_Q(\omega) = \int dt \langle \hat{H}_s(t) \hat{H}_s(0) \rangle_{\text{einst}} \), yielding (Averin and Pekola, 2010; Sergi, 2011; Zhan, Denisov, and Hänggi, 2013; Karimi and Pekola, 2021)

\[
S_Q(\omega) = \frac{G_T}{6e^2} \left( (2\pi k_B T)^2 + (\hbar \omega)^2 \right) \frac{\hbar \omega}{1 - e^{-\hbar \omega/k_B T}}.
\]

For the symmetrized noise \( S_Q^{(s)}(\omega) = (1/2)[S_Q(\omega) + S_Q(-\omega)] \), we then have

\[
S_Q^{(s)}(\omega) = \frac{G_T}{12e^2} \left( (2\pi k_B T)^2 + (\hbar \omega)^2 \right) \hbar \omega \coth \left( \frac{\hbar \omega}{2k_B T} \right).
\]

Now there are two important limits to consider. First, for \( \omega \to 0 \) we obtain the classical fluctuation-dissipation theorem for heat current as

\[
S_Q^{(s)}(0) = 2k_B T^2 G_{th}.
\]

Second, on the other hand, the finite frequency noise does not vanish at zero temperature, but

\[
S_Q^{(s)}(\omega) = \frac{G_T}{12e^2} |\hbar \omega|^3, \quad T = 0.
\]

B. FDT for heat for a general system

Section X.A serves as an illustration of how noise and dissipation are related. Here we extend the discussion to a general setup beyond the tunneling case. This allows us to treat other mechanisms as well, for instance, the phonons, photons, and electron-phonon coupling relevant to this Colloquium. In general the FDT for heat applies in the form introduced in Eq. (42) for low frequency noise. To see this we may write the Hamiltonian

\[
\hat{H} = \hat{H}_s + \hat{H}_b + \hat{H}_c = \hat{H}_0 + \hat{H}_c,
\]

where the unperturbed Hamiltonian \( \hat{H}_0 = \hat{H}_s + \hat{H}_b \) is composed of the system and bath and \( \hat{H}_c \) is again the coupling. In linear response, we then have the expectation value of the heat current to the system

\[
\dot{Q} = \langle \dot{\hat{H}}_s \rangle = -\frac{i}{\hbar} \int_0^T dt' \langle \Delta \hat{H}_s(t') \Delta \hat{H}_s(0) \rangle_0.
\]

The expectation value of a general operator \( \Omega \) in the non-interacting system is written as \( \langle \Omega \rangle_0 = \text{Tr}(e^{-i\hat{H}_s T}e^{-\beta \hat{H}_s} \Omega) / \text{Tr}(e^{-\beta \hat{H}_s}) \), where \( \beta = (k_B T)^{-1} \) and \( \beta = (k_B T)^{-1} \) are the corresponding inverse temperatures of the system and bath, respectively. By definition, the thermal conductance is given by

\[
G_{th} = \left. \frac{\partial \langle \dot{Q} \rangle}{\partial T} \right|_{T_L = T} = \frac{1}{k_B T^2} \langle \delta \hat{H}_s \dot{\hat{H}}_s \rangle_0.
\]

where we used \( \langle \hat{H}_s \rangle_0 = \delta \hat{H}_s \). On the other hand, the spectral density of noise for the heat current at zero frequency is given by

\[
S_Q(0) = \int_0^\infty dt' \langle \Delta \hat{H}_s(t') \Delta \hat{H}_s(0) \rangle_0.
\]

which is analogous to what was introduced in the tunneling case. After some algebra and a careful comparison of Eqs. (46) and (47), we find the FDT given in Eq. (42).

C. Effective temperature fluctuations

Here we consider a system with varying temperature \( T(t) \). This setup, shown in Fig. 2(a), presents an absorber of a calorimeter or bolometer coupled via thermal conductance \( G_{th} \) to a heat bath at fixed temperature \( T_0 \). If we further assume that the small system has the heat capacity \( \mathcal{C} \), the energy balance equation reads for the heat current \( \dot{Q}(t) \) between the bath and the absorber
\[ \dot{Q}(t) = C \delta \dot{T}(t) + G_{th} \delta T(t), \]  

where \( \delta T(t) \) is the difference between the absorber temperature and that of the bath. To calculate thermal noise, we again evaluate the two-time correlator as

\[ \langle \dot{Q}(t) \dot{Q}(0) \rangle = C^2 \langle \delta \dot{T}(t) \delta \dot{T}(0) \rangle + G_{th}^2 \langle \delta T(t) \delta T(0) \rangle. \]  

which leads to

\[ S_Q(\omega) = (\omega^2 C^2 + G_{th}^2) S_T(\omega). \]  

Since we typically consider frequencies well below the temperature, \( S_Q(\omega) \) is essentially frequency independent [which was shown in Eq. (41) for tunneling], and the classical FDT holds for \( S_Q(\omega) \) in the form of Eq. (42) in equilibrium. Thus, we have

\[ S_T(\omega) = \frac{2k_B T_0^2}{G_{th}} \frac{1}{1 + (\omega \tau)^2}, \]  

where \( \tau = C/G_{th} \) is the thermal relaxation time. This means that in the low frequency limit \( S_T(0) = 2k_B T^2/G_{th} \). The root-mean-square (rms) fluctuation of temperature is obtained as the inverse Fourier transform of the noise spectrum at \( t = 0 \) as

\[ \langle \delta T^2 \rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S_T(\omega) = \frac{k_B T_0^2}{C}. \]  

which is the well-known textbook result for temperature fluctuations (Lifshitz and Pitaevskii, 1980; Heikkilä and Nazarov, 2009; van den Berg, Brange, and Samuelsson, 2015). The results of Eqs. (51) and (52) are directly accessible in experiments.

### D. Progress on measuring fluctuations of heat current and entropy

The previous discussion applies for systems and processes in or near equilibrium. In recent decades relations that also hold far from equilibrium and for finite times have been developed (Bochkov and Kuzovlev, 1981; Jarzynski, 1997; Crooks, 1999; Seifert, 2012). During the past 20 years they have also become experimentally feasible thanks mainly to advances in the production and manipulation of nanostructures. The best known nonequilibrium fluctuation relations governing entropy production \( \Delta S \) are given by \( P(\Delta S)/P(-\Delta S) = e^{\Delta S/k_B} \) and its corollary \( (e^{-\Delta S/k_B}) = 1 \). Here \( \langle \cdot \rangle \) refers to the average over many experimental realizations or to the expectation value for the measurement. For macroscopic systems near equilibrium these relations simplify to the second law of thermodynamics.

Here we give a summary of such nonequilibrium experiments on electrical systems. Fluctuations of entropy production and heat currents have been actively studied experimentally for more than a decade in the classical regime, but mainly via indirect means of detection since entropy is a tricky quantity for a direct measurement (Kleedorin et al., 2019). Two main classes of systems under study have been those in seminal experiments on molecules (Collin et al., 2005) and electrical circuits (Küng et al., 2012; Saira, Yoon et al., 2012; Ciliberto et al., 2013; Pekola, 2015; Bérut et al., 2016; Cottet et al., 2017). Other works go beyond FDT by addressing far-from-equilibrium fluctuation relations (Bochkov and Kuzovlev, 1981; Jarzynski, 1997; Crooks, 1999; Campisi, Hänggi, and Talkner, 2011; Seifert, 2012; Pekola and Khaymovich, 2019). As shown in Fig. 19(a), Ciliberto et al. (2013) examined the setup of two macroscopic resistors at temperatures near the ambient. An indirect measurement of entropy was facilitated by Ciliberto et al. (2013) via the detection of instantaneous electrical power \( IV \), integrated over time and divided by the corresponding temperature of the macroscopic resistor. This way several fluctuation relations for entropy production under nonequilibrium conditions (Seifert, 2005, 2012) could be verified together with the standard FDT in the linear response regime. Similarly, in the setup of Saira, Yoon et al. (2012) shown in Fig. 19(b),
detecting single electrons making nonequilibrium transitions across a junction in a single-electron box provides indirect means of observing the dissipated energy and entropy production quantitatively (Averin and Pekola, 2011; Koski et al., 2013). These experiments were performed at temperatures 3 orders of magnitude lower than in the work of Ciliberto et al. (2013). The relations of Crooks (1999) and Jarzynski (1997) as well as generalized relations incorporating the role of information in the Maxwell’s demon setup (Sagawa and Ueda, 2010) could be tested accurately in these experiments (Pekola and Khaymovich, 2019).

The reason for using indirect measurement of heat by detailed electrical characterization is the fact that the powers are far too small to resolve with direct thermometry (Sec. III.B). Next we focus on progress related to the direct measurement of current fluctuations.

E. Energy sensitivity of a calorimeter

The ultimate energy resolution of a thermal detector (see Fig. 20) is determined by the coupling of it to the heat bath associated with the fluctuations of the heat current. Taking a wideband thermometer on a calorimeter, the rms fluctuations of the effective temperature due to this intrinsic noise are given by Eq. (52). To find the energy resolution of the detector one needs to compare this noise to the impact of the absorption of a photon at energy $E$ at the time instant $t = 0$, meaning that $\dot{Q}(t) = E \delta(t)$ with the solution $\delta T(t) = (E/C)e^{-t/\tau}\theta(t)$, where the time constant $\tau \equiv C/G_{\text{th}}$ and $\theta(t)$ is the Heaviside step function. Thus, at $t = 0+$ the immediate rise of $T$ is $\delta T(0) = E/C$. The signal-to-noise ratio $\delta N = \delta T(0)/\sqrt{\langle \delta T^2 \rangle}$ is

$$\delta N = E/\sqrt{k_B T_0^2 C},$$

meaning that the energy resolution of the detector in this regime is

$$\delta E = \sqrt{k_B T_0^2 C}. \tag{54}$$

For convenience, we write $C = \eta k_B$, where $\eta$ is a dimensionless constant that we assess later. We then find that $\delta E = \sqrt{\eta k_B T_0}$. As an example, related to the experiment of Karimi et al. (2020) we take a metallic calorimeter where $C = \gamma V T_0$ at low temperatures; see Figs. 20(a) and 20(b). Here $\gamma \sim 100$ J K$^{-2}$ m$^{-3}$ for copper and $\gamma < 10^{-21}$ m$^{-3}$ is the volume of the absorber, yielding $\eta \sim 100$ and the energy resolution $\delta E/k_B \sim 0.1$ K at $T_0 = 0.01$ K (Karimi and Pekola, 2020).

Fluctuations in power have a direct impact on the performance of the calorimeters and bolometers (Irwin, 1995; Gildemeister, Lee, and Richards, 2001), i.e., thermal detectors of radiation. This noise determines the energy resolution of a calorimeter, and also the noise-equivalent power under continuous irradiation, as in the measurement of the cosmic microwave background (Mather, 1982). Direct measurements of fluctuating temperature are rare (Chui et al., 1992; Karimi et al., 2020). The pioneering measurement of Chui et al. (1992) employed a macroscopic calorimeter working at the so-called lambda point of liquid helium, i.e., its superfluid transition temperature at $T = 2.17$ K. Chui et al. (1992) managed to verify Eqs. (51) and (52) thanks to a high resolution of the thermometer measuring the magnetization of a paramagnetic salt (copper ammonium bromide) with the

![FIG. 20. Quantum calorimeter.](image)
help of a SQUID down to a $10^{-10} \text{K}/\sqrt{\text{Hz}}$ noise-equivalent temperature.

The nanofabricated detector of Karimi et al. (2020) worked in the regime where the measurement cutoff frequency was 10 kHz, which falls somewhat below $1/r$. Furthermore, the metallic absorber was proximitized by a superconducting contact further decreasing $C$ and $G_0$ (Heikkinen and Giazotto, 2009; Nikolic, Basko, and Belzig, 2020) and thus improving its performance. The experiment [Fig. 20(b)], which utilized a superconductor–normal-metal–insulator–superconductor (S-N-I-S) thermometer (Karimi and Pekola, 2018), demonstrated noise of the effective temperature of the calorimeter that is close to the expected fundamental fluctuation limit of Eq. (51) at low frequencies, at the same time promising a $\text{SNR} \sim 10$ in measuring an absorption event with the photon energy $E/k_B = 1$ K. Figure 20(c) demonstrates by simulation the validity of the previous analysis. The wideband detector would present $T$ fluctuations that are an order of magnitude smaller than the temperature jump due to the 1 K photon absorption event. In summary, this measurement demonstrates the feasibility of a microwave photon measurement using a metallic calorimeter at $T_0 = 10$ mK.

There exist several other concepts of ultrasensitive thermal detectors, either metallic ones (Gouvenius et al., 2016; Kuzmin et al., 2019) or those utilizing graphene or semiconductors (Roukes, 1999; Kokkoniemi et al., 2019; Lara-Avila et al., 2019; Kokkoniemi et al., 2020; Lee et al., 2020), or those based on temperature-dependent magnetization (Christian, 2005; Kempf et al., 2018). The advantage of graphene is its supposedly low heat capacity, which could make the thermal response time shorter than in metal detectors. Yet none of the proposed detectors has demonstrated detection of quanta in said microwave regime to date.

**XI. SUMMARY AND OUTLOOK**

In this Colloquium we focused on the fundamental aspects of quantum heat transport, with the main emphasis on experiments carried out during the past 20 years. In many respects the physics of heat transport in quantum nanostructures is currently well understood, and experiments tend to confirm the theoretical predictions. In some systems clean experiments are, however, more difficult to realize than in others from a practical point of view, and more experiments are needed: one example is presented by one-dimensional phonon structures, where pioneering experiments were performed long ago (Schwab et al., 2000), but where precise conditions regarding how to realize ballistic contacts are still under debate. Experiments on quantum heat transport serve also as tools to understand quantum matter itself, as recent experiments in the fractional quantum Hall regime demonstrate (Banerjee et al., 2017; Dutta et al., 2021). On the other hand, they provide us with ways of realizing new kinds of devices and determining how to nail down and achieve their ultimate limits of performance. We discussed the latter issue in Sec. X on noise in heat current.

As to the potentially useful devices based on quantum heat transport, we now discuss two examples. The first one is a rather straightforward application of heat management on chip for quantum information processes. Microwave photons provide a means to transport quanta and energy in general over large distances, as we discussed in Sec. VII.C. It could thus serve as a way to reset quantum circuits rapidly. There is, however, a trade-off to be considered. Rapid thermalization is almost a synonym for a low quality factor and fast decoherence in a quantum system, which are not desirable properties. Therefore, tunable coupling is a possible way to go, to switch on and off the coupling to a heat bath on demand. Variations of the many heat valves presented in this Colloquium could in principle serve the purpose. Tests of such an idea were proposed and experimented on by Partanen et al. (2018).

Quantum heat engines and cyclic refrigerators are presently under intensive study; see Humphrey and Linke (2005), Quan et al. (2007), Defner, Jarzynski, and Campo (2014), Pilgram, Sánchez, and López (2015), Campisi and Fazio (2016), Benenti et al. (2017), Brandner, Bauer, and Seifert (2017), Alici and Kosloff (2018), Josefsson et al. (2018), Bhandari et al. (2020), Majidi et al. (2021), and Raja et al. (2021). Experiments that are fully in the quantum regime have thus far been practically nonexistent, although there have been proposals addressing realistic setups (Abah et al., 2012; Karimi and Pekola, 2016). For instance, a so-called quantum Otto cycle can be realized by alternately coupling a superconducting qubit to two different heat baths (Karimi and Pekola, 2016). If this is done by varying the energy level separation of the qubit, as was done in the photonic heat valve or rectifier earlier, but now cyclically at rf frequencies, one can extract heat from the cold bath and dump it into the hot one when system parameters are chosen properly. We expect devices of this type or analogous ones to work in the near future. Interesting questions arise as to whether one can boost the powers and/or efficiencies by exploiting quantum dynamics, and as to which kinds of protocols can speed up the cycles for higher powers in general (Funo et al., 2019; Menczel et al., 2019; Solfanelli, Falsetti, and Campisi, 2020).

We note here that topological matter (Hasan and Kane, 2010; Qi and Zhang, 2011), specifically topological superconductors and Josephson junctions, have been proposed as potential novel elements in quantum thermodynamics and heat transport experiments due to their unconventional physical properties, see the recent work of Rivas and Martin-Delgado (2017), Bauer and Sothmann (2019), Scharf et al. (2020), and Pan, Sau, and Das Sarma (2021). Owing to the focus of the current paper, mainly on experiments, we do not discuss this topic further.

In this Colloquium we alluded to the connections of heat transport and quantum thermodynamics, mainly regarding concrete device concepts, including thermal detectors, heat engines, and refrigerators. On a more fundamental level, quantum heat transport is at the heart of open quantum systems physics (Breuer and Petruccione, 2002), with the non-Hermitian dynamics governed by the quantum noise (Gardiner and Zoller, 2010) widely discussed in this Colloquium. True thermodynamics counts on observations of heat currents and temperatures, and power consumption of the sources. Adopting this view, one can pose many questions, such as how to measure work and heat in an open quantum system, for which the measurement apparatus cannot be viewed as an innocent witness of what is happening in the quantum system itself. The calorimeter can eventually become...
the microscope of quantum dynamics on the level of the exchange of energy by individual quanta emitted or absorbed by the quantum system. This would give us the optimal tool to investigate stochastic thermodynamics in the true quantum regime. Many other fundamentally and practically important questions will arise and can potentially be answered by heat transport experiments. For instance, how does a quantum system thermalize, and does it find an equilibrium thermal state even in the absence of a heat bath? To conclude, investigations and exploitation of quantum heat transport will play an important role in the currently active field of quantum thermodynamics and in future quantum technologies in general.

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