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# Interior formulation of axisymmetric Levinson plate theory

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## Abstract

In this study, we show that the axisymmetric Levinson plate theory is exclusively an interior theory and we provide a consistent variational formulation for it. First, we discuss an annular Levinson plate according to a vectorial formulation. The boundary layer of the plate is not modeled and, thus, the interior stresses acting as surface tractions do work on the lateral edges of the plate. This feature is confirmed energetically by the Clapeyron's theorem. The variational formulation is carried out for the annular Levinson plate by employing the principle of virtual displacements. As a novel contribution, the formulation includes the external virtual work done by the tractions based on the interior stresses on the inner and outer lateral edges of the Levinson plate. The obtained plate equations are consistent with the vectorially derived Levinson equations. Finally, we develop an exact plate finite element both by a force-based method and from the total potential energy of the Levinson plate.

*Keywords:* Levinson theory, Saint Venant's principle, Interior plate, Clapeyron's theorem, Finite element

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## 1. Introduction

The plate theories by Levinson and Reddy are widely known in the literature [1, 2]. Although these theories are based on the same assumed displacement field, their governing equations are somewhat different. The Levinson theory is derived using a vectorial approach, whereas the Reddy theory is obtained by a variational formulation. The Reddy theory includes higher-order load resultants which are not present in the Levinson theory. Consequently, the Levinson theory is considered to be “variationally inconsistent” due to this difference between the two theories. In this study, we show that the Levinson theory is in fact variationally consistent with certain provisions. Our scope is limited to axisymmetric annular and circular plates.

An early investigation on a dynamic axisymmetric Levinson plate was conducted by Hutchinson [3]. An extensive study on static axisymmetric Levinson plates was carried out by Wang et al. [4], who noted the “variational inconsistency” of the Levinson plate theory but also acknowledged that the theory has some merits. The merit that really counts is the simplicity of the Levinson theory. Due to this highly appealing feature, annular and circular Levinson plates have been later studied also by other authors [5, 6].

In a recent paper, we showed in an exact interior context that the displacement field on which the Levinson and Reddy beam and plate theories are based is exclusively an interior field [7]. This

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issue is generally left undiscussed in relation to the Levinson and Reddy theories. The use of interior kinematics means that the edge effects of a plate that decay with distance from the lateral edges of the plate are neglected by virtue of the Saint Venant's principle. Note that, for example, the well-known exact elasticity solution for a uniformly loaded simply-supported axisymmetric plate is, in fact, an interior solution (see, e.g. [8]).

In another paper [9], we showed that the Levinson beam theory [10] is an interior theory and we provided a consistent variational formulation for it. The formulation was carried out by making use of the fact that, due to the interior nature of the assumed displacement field, the stresses of the beam act as surface tractions on the lateral surfaces of the Levinson beam. In the present study, we introduce a variational formulation for the Levinson plate theory which relies on similar reasoning.

The remainder of this study is organized as follows. In Section 2, the static axisymmetric annular Levinson plate theory and its consistency with the Clapeyron's theorem are considered. In Section 3, a consistent variational formulation for the annular Levinson plate is carried out. An exact Levinson plate finite element is developed in Section 4 and conclusions are drawn in Section 5.

## 2. Levinson plate theory

### 2.1. Boundary conditions and displacement field

Fig. 1 presents an axisymmetric annular plate subjected to a rotationally symmetric transverse load  $q(r)$ . The thickness of the plate is  $h$  and the outer and inner radii of the plate are  $a$  and  $b$ , respectively. For the Levinson theory, it is assumed that 1) the transverse normal stress  $\sigma_z$  is zero throughout the plate and 2) the Poisson effect (lateral contraction/extension) is not present. On the basis of these assumptions, and to satisfy the homogeneous boundary conditions  $\tau_{rz}(r, \pm h/2) = 0$  on the upper and lower surfaces of the plate, the displacement field can be found as [3]

$$U_r(r, z) = z\phi - \frac{4z^3}{3h^2} \left( \phi + \frac{\partial u_z}{\partial r} \right), \quad U_z(r, z) = u_z \quad (1)$$

where  $u_z(r)$  is the transverse deflection of the mid-surface of the plate and  $\phi(r)$  is the rotation of the normal of the mid-surface. The homogeneous stress boundary conditions are satisfied in a *strong* (pointwise) sense on the upper and lower surfaces of the plate. It is important to note that in the Levinson theory the tractions on the lateral inner and outer plate edges are not specified at each point but only through load resultants and, thus, the boundary conditions are imposed only in a *weak* sense [11]. The replacement of the true stress boundary conditions at the plate edges by the statically equivalent weak boundary conditions (load resultants) means that the exponentially decaying edge effects are neglected by virtue of the Saint Venant's principle. The cross-sectional load resultants per unit length are calculated from the equations

$$M_r(r) = \int_{-h/2}^{h/2} \sigma_r z dz, \quad M_\theta(r) = \int_{-h/2}^{h/2} \sigma_\theta z dz, \quad Q_r(r) = \int_{-h/2}^{h/2} \tau_{rz} dz. \quad (2)$$

The chosen positive directions of the load resultants  $M_r(r)$  and  $Q_r(r)$  are given in Fig. 1.

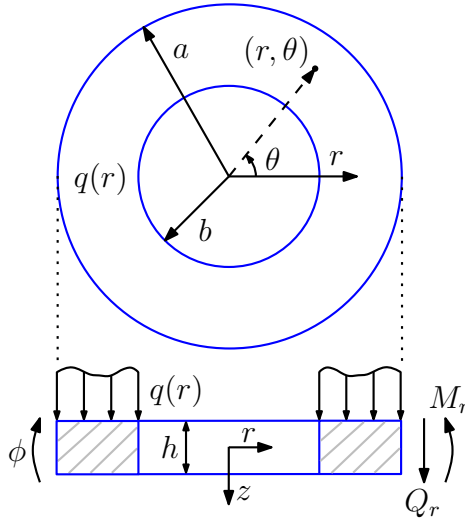


Figure 1: Axisymmetric annular Levinson plate under a rotationally symmetric transverse load  $q(r)$ . The positive directions of the load resultants and the rotation  $\phi$  on the outer edge are shown.

## 2.2. Implications of the interior definition

The displacement field (1) of the Levinson plate is an adequate interior field. The modeling of the boundary layer displacements (edge effects) would essentially require the use of Papkovitch–Fadle-type eigenfunctions [11]. To further elucidate the interior plate definition and its implications, let us consider the complete circular plate of radius  $a'$  shown in Fig. 2. The real stress or displacement boundary conditions are given at  $r = a'$ . The detailed distribution of the boundary stresses would bring about edge effects which decay exponentially towards the interior plate. In other words, the edge effects are significant only in the boundary layer, the radial thickness of which is typically of the same order with the thickness of the plate. Beyond that the fully-developed interior plate solution prevails. In engineering applications, instead of using a complete plate, an interior theory is usually applied and the conditions at the interior plate edge at  $r = a$  are chosen so as to imitate the true boundary conditions. This long-standing practice is well-suited especially for thin isotropic plates which are modeled using the Kirchhoff plate theory. The thinner the plate is, the weaker the edge effects are.

In the foregoing, a circular plate was discussed. In the case of an annular plate an analogous discussion may be extended to the inner boundary region. In terms of energetical considerations, the key feature of the interior plate definition is that the fully-developed interior stresses do work on the lateral edges of the plate. This has an effect on the total potential energy of the interior plate, which can be written as

$$\Pi = U - W_s \quad (3)$$

where the strain energy for an annular plate is

$$U = \pi \int_b^a \int_{-h/2}^{h/2} r(\sigma_r \epsilon_r + \sigma_\theta \epsilon_\theta + \tau_{rz} \gamma_{rz}) dz dr \quad (4)$$

and the work by surface tractions due to the interior stresses on the inner and outer lateral edges

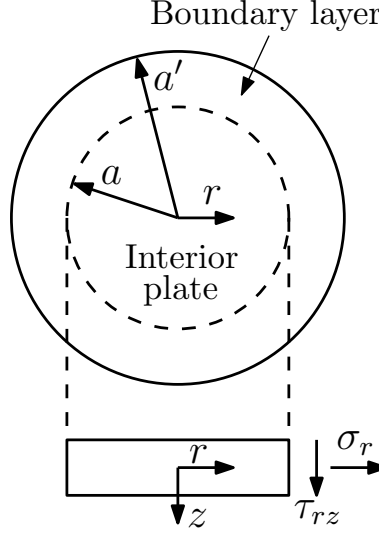


Figure 2: Circular plate consisting of an interior plate (Levinson plate) and a boundary layer. When only the interior plate is studied, the stresses  $\sigma_r$  and  $\tau_{rz}$  do work on the plate edge.

of the interior plate is given by

$$\begin{aligned} \frac{W_s}{2\pi} = & a \int_{-h/2}^{h/2} \sigma_r(a, z) U_r(a, z) dz - b \int_{-h/2}^{h/2} \sigma_r(b, z) U_r(b, z) dz \\ & + a \int_{-h/2}^{h/2} \tau_{rz}(a, z) U_z(a, z) dz - b \int_{-h/2}^{h/2} \tau_{rz}(b, z) U_z(b, z) dz . \end{aligned} \quad (5)$$

This feature will be exploited in the following sections in all energy-based developments related to the Levinson plate theory.

### 2.3. General static solution

The vectorially derived equilibrium equations for the annular Levinson plate are [4]

$$\frac{2Gh}{3} \frac{\partial}{\partial r} \left[ r \left( \phi + \frac{\partial u_z}{\partial r} \right) \right] = -rq(r) , \quad (6)$$

$$\frac{D}{5} \frac{\partial}{\partial r} \left\{ r \frac{\partial}{\partial r} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( 4r\phi - r \frac{\partial u_z}{\partial r} \right) \right] \right\} = -rq(r) . \quad (7)$$

where  $D = Eh^3/[12(1-\nu^2)]$  is the flexural rigidity and  $E$ ,  $G$  and  $\nu$  are the Young's modulus, shear modulus and Poisson ratio, respectively. The non-zero kinematic and constitutive relations for the plate read

$$\epsilon_r = \frac{\partial U_r}{\partial r} , \quad \epsilon_\theta = \frac{U_r}{r} , \quad \gamma_{rz} = \frac{\partial U_r}{\partial z} + \frac{\partial U_z}{\partial r} , \quad (8)$$

$$\sigma_r = \frac{E}{1-\nu^2} (\epsilon_r + \nu\epsilon_\theta) , \quad \sigma_\theta = \frac{E}{1-\nu^2} (\epsilon_\theta + \nu\epsilon_r) , \quad \tau_{rz} = G\gamma_{rz} , \quad (9)$$

respectively. The general solution to Eqs. (6) and (7) in the case of the load  $q(r) = q_n r^n$  can be written as

$$\begin{aligned}
u_z &= c_1 \left( \frac{r^2}{2} - \frac{12D}{5Gh} \right) + c_2 \ln r \\
&+ c_3 \left[ \frac{r^2}{4} (2 \ln r - 1) - \frac{6D}{5Gh} (2 \ln r + 1) \right] + c_4 \\
&+ \frac{q_n}{n+2} \left[ \frac{r^2}{D(n+2)(n+4)^2} - \frac{1}{n+2} \frac{6}{5Gh} \right] r^{n+2} , \tag{10}
\end{aligned}$$

$$\begin{aligned}
\phi &= -c_1 r - \frac{c_2}{r} - c_3 \left( \frac{3D}{5Ghr} + r \ln r \right) \\
&- \frac{q_n}{n+2} \left[ \frac{r^2}{D(n+2)(n+4)} + \frac{3}{10Gh} \right] r^{n+1} . \tag{11}
\end{aligned}$$

We calculate the load resultants (2) using the stresses (9) and the general solution (10) and (11). Then, we write the constants  $c_1$ ,  $c_2$  and  $c_3$  in terms of the load resultants and substitute them back into (9) to obtain

$$\sigma_r = \frac{12M_r z}{h^3} + \frac{z(3h^2 - 20z^2)}{5h^3 r} \left[ \frac{rq(r)}{\nu - 1} - Q_r \right] , \tag{12}$$

$$\sigma_\theta = \frac{12M_\theta z}{h^3} + \frac{z(3h^2 - 20z^2)}{5h^3 r} \left[ \frac{\nu rq(r)}{\nu - 1} + Q_r \right] , \tag{13}$$

$$\tau_{rz} = \frac{3Q_r}{2h^3} (h^2 - 4z^2) . \tag{14}$$

Due to the linearity of the problem, Eqs. (12)–(14) are valid for any load which can be expressed as a Maclaurin series  $q(r) = \sum_{n=0}^{\infty} q_n r^n$ .

Finally, to elucidate the interior problem definition further, we consider the strain energy of the Levinson plate and the external work done on it by the surface tractions. The strain energy  $U$  and the work  $W_s$  due to the interior stresses on the lateral edges of the Levinson plate are given by Eqs. (4) and (5), respectively. In addition, we consider the external work due to a constant uniform load ( $n = 0, q_n = q$ ) given by

$$W_q = 2\pi \int_b^a r q u_z dr . \tag{15}$$

By substituting the expressions for displacements, strains and stresses according to the general solution (10) and (11) into Eqs. (4), (5) and (15), we find that

$$2U - W_q - W_s = 0 . \tag{16}$$

The above calculation shows that in static equilibrium the strain energy of the plate is equal to one-half of the work done by the surface tractions if they were to move (slowly) through their respective displacements. This is exactly in line with both the *Clapeyron's theorem* (e.g. [12]) and the given interior definition. We conclude that the Levinson plate theory is an interior theory. The work (5) is an inseparable part of all energy-based developments related to the theory.

### 3. Variational formulation of the annular plate

The variational formulation is carried out for a constant uniform load  $q$  according to the system of Fig. 1. We re-write the interior kinematics of the annular Levinson plate in the form

$$U_r(r, z) = z\phi - \alpha z^3 (\phi + u'_z), \quad (17)$$

$$U_z(r, z) = u_z, \quad (18)$$

where  $\alpha = 4/3h^2$  and comma denotes differentiation with respect to coordinate  $r$ . To carry out the variational formulation, we assume that the kinematic central axis variables  $u_z(r)$  and  $\phi(r)$  are sufficiently smooth but otherwise arbitrary functions. The strains (8) calculated using the displacements (17) and (18) are

$$\epsilon_r = z\phi' - \alpha z^3 (\phi' + u''_z), \quad (19)$$

$$\epsilon_\theta = \frac{1}{r} [z\phi - \alpha z^3 (\phi + u'_z)], \quad (20)$$

$$\tau_{rz} = (1 - 3\alpha z^2) (\phi + u'_z). \quad (21)$$

The internal virtual work (virtual strain energy) of the plate is

$$\begin{aligned} \delta U &= \int_b^a \int_0^{2\pi} \int_{-h/2}^{h/2} r(\sigma_r \delta \epsilon_r + \sigma_\theta \delta \epsilon_\theta + \tau_{rz} \delta \gamma_{rz}) dz d\theta dr \\ &= 2\pi \int_b^a \left[ r M_r \delta \phi' - \alpha r P_r (\delta \phi' + \delta u''_z) + M_\theta \delta \phi \right. \\ &\quad \left. - \alpha P_\theta (\delta \phi + \delta u'_z) + r (Q_r - 3\alpha R_r) (\delta \phi + \delta u'_z) \right] dr, \end{aligned} \quad (22)$$

where

$$M_r = \int_{-h/2}^{h/2} \sigma_r z dz = \frac{D}{5} \left[ 4 \left( \phi' + \frac{\nu}{r} \phi \right) - u''_z - \frac{\nu}{r} u'_z \right], \quad (23)$$

$$M_\theta = \int_{-h/2}^{h/2} \sigma_\theta z dz = \frac{D}{5} \left[ 4 \left( \nu \phi' + \frac{\phi}{r} \right) - \nu u''_z - \frac{u'_z}{r} \right], \quad (24)$$

$$Q_r = \int_{-h/2}^{h/2} \tau_{rz} dz = \frac{2Gh}{3} (\phi + u'_z) \quad (25)$$

are the classical interior load resultants (2) and

$$P_r = \int_{-h/2}^{h/2} \sigma_r z^3 dz = \frac{Dh^2}{140} \left[ 16 \left( \phi' + \frac{\nu}{r} \phi \right) - 5 \left( u''_z + \frac{\nu}{r} u'_z \right) \right], \quad (26)$$

$$P_\theta = \int_{-h/2}^{h/2} \sigma_\theta z^3 dz = \frac{Dh^2}{140} \left[ 16 \left( \nu \phi' + \frac{\phi}{r} \right) - 5 \left( \nu u''_z + \frac{u'_z}{r} \right) \right], \quad (27)$$

$$R_r = \int_{-h/2}^{h/2} \tau_{rz} z^2 dz = \frac{Gh^3}{30} (\phi + u'_z) \quad (28)$$

are higher-order load resultants for which it is difficult to assign a physical meaning in the context of classical elasticity. Actually, in the end the higher-order load resultants will be eliminated from the equilibrium equations of the Levinson plate theory. The external virtual work due to the uniform load is given by

$$\delta W_q = 2\pi \int_b^a r q \delta u_z dr . \quad (29)$$

The external virtual work due to the interior stresses on the lateral edges is calculated using Eqs. (5), (17) and (18) yielding

$$\delta W_s = 2\pi \left[ r M_r \delta \phi - \alpha r P_r (\delta \phi + \delta u'_z) + r Q_r \delta u_z \right]_{r=b}^{r=a} . \quad (30)$$

By applying the principle of virtual displacements,  $\delta U = \delta W_q + \delta W_s$ , we obtain through integration by parts of  $\delta U$  in Eq. (22) the equilibrium equations

$$r Q_r - (r M_r)' + M_\theta = \alpha \left[ P_\theta - (r P_r)' + 3r R_r \right] , \quad (31)$$

$$(r Q_r)' + r q = \alpha \left[ P_\theta' - (r P_r)'' + 3(r R_r)' \right] \quad (32)$$

from the integrals and the boundary conditions

$$\left\{ \alpha \left[ P_\theta - (r P_r)' + 3r R_r \right] \delta u_z \right\}_{r=b}^{r=a} = 0 \quad (33)$$

from the remaining terms. Consequently, we must require that either the virtual displacement  $\delta u_z$  or the expression multiplying it must vanish at  $r = a$  and  $r = b$ . It follows from the interior problem definition of Section 2 that the virtual displacement  $\delta u_z$  is free in the whole interior plate region. Therefore, the natural interior boundary conditions become

$$f(a) = 0 , \quad (34)$$

$$f(b) = 0 , \quad (35)$$

where we have introduced the notation

$$f(r) \equiv \alpha \left[ P_\theta - (r P_r)' + 3r R_r \right] . \quad (36)$$

Using Eq. (32), we can write

$$f' = (r Q_r)' + r q , \quad (37)$$

$$f'' = (r Q_r)'' + q . \quad (38)$$

Furthermore, by Eqs. (25) and (38) we obtain

$$f'' - q - Q_r' = \frac{2}{3} G h \left[ \phi' + u_z'' + r(\phi'' + u_z''') \right] . \quad (39)$$

On the other hand, using the load resultants (23)–(28) and Eqs. (31) and (36) we arrive at

$$f + \frac{D}{105r} (\phi + u'_z) = \frac{D}{105} \left[ \phi' + u_z'' + r(\phi'' + u_z''') \right] . \quad (40)$$

It follows from Eqs. (39) and (40) that

$$f'' - \frac{1}{r} f' - \beta^2 f = 0 , \quad (41)$$



where  $\beta^2 = 70Gh/D$ . The solution of Eq. (41) is then

$$f(r) = d_1 r J_1(i\beta r) + d_2 r Y_1(-i\beta r) , \quad (42)$$

where  $J_1$  and  $Y_1$  are the Bessel functions of the first and second kind, respectively. The imaginary arguments of the Bessel functions imply that instead of being oscillatory, they are related to exponential behavior, that is, to artificial edge effects. The application of the boundary conditions (34) and (35) results in  $d_1 = d_2 = 0$ , i.e.,  $f(r) = 0$ . In other words, the spurious edge effects, which cannot exist in the interior context, vanish. Furthermore, the equilibrium equations (31) and (32) for a uniform constant load simplify to

$$(rM_r)' - M_\theta = rQ_r , \quad (43)$$

$$(rQ_r)' = -rq , \quad (44)$$

which are fully compatible with the vectorially derived Levinson plate equations (6) and (7). It is important to note that if  $f(r)$  does not vanish, the equilibrium equations (31) and (32) violate the fundamental equilibrium equations of an axisymmetric plate derived in the vectorial (Newtonian) way [13]. We can conclude that, the virtual work due to the interior stresses on the lateral edges of the Levinson plate is an intrinsic part of the variational formulation.

#### 4. Exact Levinson plate finite element

The general solution (10) and (11) can be used as the basis for the derivation of exact circular and annular Levinson plate finite elements. The finite elements constitute an alternative representation of the general solution. The circular and annular elements are derived in the same manner. For the sake of brevity, we consider here only the circular element under a constant uniform load ( $n = 0, q_n = q$ ). Fig. 3 presents the setting according to which the finite element is developed. The node at  $r = a$  in Fig. 3 has two degrees of freedom, the transverse displacement  $u_{z,a}$  and rotation  $\phi_a$ . Using Eqs. (10) and (11), we obtain the following two equations

$$u_{z,a} = u_z(a) , \quad \phi_a = \phi(a) . \quad (45)$$

For finite displacements at the center of the plate, it is required that  $c_2 = c_3 = 0$ . By solving Eqs. (45) for the two remaining integration constants  $c_1$  and  $c_4$  we obtain

$$c_1 = -\frac{\phi_a}{a} + \frac{3q}{40Gh^3} [5a^2(\nu - 1) - 2h^2] , \quad (46)$$

$$c_4 = u_{z,a} + \phi_a \left[ \frac{2h^2}{5a(\nu - 1)} + \frac{a}{2} \right] + \frac{9qa^2}{40Gh} + \frac{3q}{800Gh^3} \left[ \frac{16h^4}{\nu - 1} - 25a^4(\nu - 1) \right] . \quad (47)$$

By substituting Eqs. (46) and (47) into Eqs. (10) and (11) we can write the displacements (1) in the form

$$U_r(r, z) = \mathbf{N}_r \mathbf{u} + qL_r , \quad (48)$$

$$U_z(r, z) = \mathbf{N}_z \mathbf{u} + qL_z , \quad (49)$$

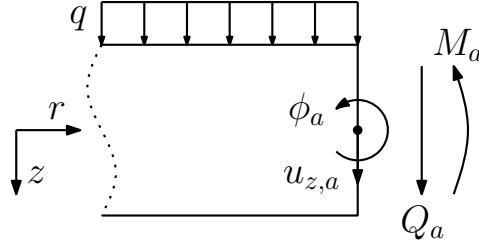


Figure 3: Set-up according to which the exact circular Levinson plate finite element is developed.

where the shape functions are

$$\mathbf{N}_r = \left\{ 0 \quad \frac{rz}{a} \right\}, \quad (50)$$

$$\mathbf{N}_z = \left\{ 1 \quad \frac{a^2 - r^2}{2a} \right\} \quad (51)$$

and

$$\mathbf{u} = \left\{ \begin{matrix} u_{z,a} \\ \phi_a \end{matrix} \right\} \quad (52)$$

is the displacement vector. In addition we have

$$L_r = \frac{rz}{8Gh^3} [8z^2 + 3(r^2 - a^2)(\nu - 1)], \quad (53)$$

$$L_z = \frac{3(a^2 - r^2)}{32Gh^3} [4h^2 + (r^2 - a^2)(\nu - 1)]. \quad (54)$$

Therefore, once the displacements  $u_{z,a}$  and  $\phi_a$  are known, the 2D displacement field can be calculated by substituting them into Eqs. (48) and (49), after which the calculation of the 2D interior strains and stresses is straightforward.

Next, we substitute Eqs. (46) and (47) into Eqs. (10) and (11). By using Eqs. (2) we obtain, with the nodal loads defined per unit length, the finite element equations

$$Q_a = Q_r(a) = -\frac{qa}{2}, \quad (55)$$

$$M_a = M_r(a) = \frac{Eh^3}{12a(1-\nu)}\phi_a - \frac{q}{40} \left[ 5a^2 + \frac{1+\nu}{\nu-1}h^2 \right]. \quad (56)$$

The force-based derivation presented above for the circular Levinson plate finite element is similar to that presented by Reddy and Wang for an annular Mindlin plate [14]. As the point of departure in terms of methodology, we consider next the derivation of the circular Levinson element also from the total potential energy

$$\Pi = U - W_q - W_s. \quad (57)$$

The stresses on the plate edge  $r = a$  in Eq. (5) are written as given by Eqs. (12) and (14), where the load resultants are expressed as nodal loads, that is,

$$\begin{aligned} \frac{W_s}{2\pi} &= a \int_{-h/2}^{h/2} \left[ \frac{12M_a z}{h^3} + \frac{z(3h^2 - 20z^2)}{5ah^3} \left( \frac{qa}{\nu-1} - Q_a \right) \right] U_r(a, z) dz \\ &+ a \int_{-h/2}^{h/2} \left[ \frac{3Q_a}{2h^3} (h^2 - 4z^2) \right] U_z(a, z) dz. \end{aligned} \quad (58)$$

Then, by using the general solution (10) and (11) in terms of Eqs. (46) and (47) to calculate the total potential energy (57) and by applying the principle of minimum total potential energy

$$\frac{\partial \Pi}{\partial u_{z,a}} = 0 , \quad (59)$$

$$\frac{\partial \Pi}{\partial \phi_a} = 0 \quad (60)$$

we obtain the finite element equations (55) and (56). We see that the work  $W_s$  done by the interior stresses is an integral part of the circular Levinson plate element. The finite element formulation for the annular Levinson plate follows the same recipe as given above for the circular plate, but leads to rather unwieldy expressions and, thus, is omitted here.

*Example – Simply-supported plate.* Finally, as an elementary example, let us consider a simply-supported axisymmetric circular Levinson plate under a uniform load. At the plate edge at  $r = a$ , we have  $u_{z,a} = M_a = 0$ . We acquire now from Eq. (56)

$$\phi_a = -\frac{3qa}{10Eh^3} [5a^2(\nu - 1) + h^2(1 + \nu)] . \quad (61)$$

The displacements are calculated from Eqs. (48) and (49) after which the strains and stresses are obtained from Eqs. (8) and (9), respectively. Finally, calculation of the load resultants (2) yields

$$M_r(r) = \frac{q}{16}(a^2 - r^2)(3 + \nu) , \quad (62)$$

$$M_\theta(r) = \frac{q}{16} [a^2(3 + \nu) - r^2(1 + 3\nu)] , \quad (63)$$

$$Q_r(r) = -\frac{rq}{2} , \quad (64)$$

which are the same as for the thin (Kirchhoff) plate theory. The purpose of this paper has been to study the vectorial and variational formulations of the Levinson plate theory. The accuracy of the theory can be studied by the aid of the detailed further examples on circular and annular Levinson plates given by Wang et al. [4]. All their solutions are completely valid since the Levinson plate theory is actually variationally consistent.

## 5. Conclusions

It was shown that the Levinson plate theory is an interior theory since the utilized assumed displacement field is based exclusively on interior kinematics. On the basis of this, a consistent variational formulation for an axisymmetric annular Levinson plate was presented. Initially, the variationally derived equilibrium equations were the same as those for the Reddy plate theory [14]. However, by taking into account the external virtual work done by surface tractions due to interior stresses on the lateral edges of the Levinson plate, the edge effects were eliminated from the plate and the variationally derived equilibrium equations were shown to reduce to the vectorially derived Levinson equations. The validity of this procedure was confirmed energetically by the Clapeyron's theorem. The total differential order of Levinson's equations is two units lower than Reddy's, making the Levinson theory appealing both in terms of analytical and computational developments. In fact, the governing equations of the Levinson theory are similar to those of

Mindlin's. However, the Levinson theory does not require the use of an extrinsic shear coefficient since the theory includes a parabolic shear stress distribution. In addition, the general interior stresses (12)–(14) also account for cubic normal stresses due to a distributed load in a manner similar to elasticity solutions (see, e.g. [8]).

Treatments analogous to the one presented in this paper may be carried out for plates of different shapes. The equilibrium equations for any interior plate are obtained by integrating the corresponding stress equilibrium equations with respect to the thickness coordinate of the plate. As an example, for a plate theory in cartesian coordinates this integration procedure can be found in the book by Vinson [15]. The integration ultimately leads to a load resultant based treatment in which the Saint Venant's principle is applied and, thus, detailed stress distributions on the plate edges and the edge effects are not included in the resulting theory. Furthermore, in such a case the interior stresses acting as surface tractions do work on the lateral edges of the plate, which should be remembered in all energy-based considerations.

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