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Distributed Consensus Tracking of Multi-Agent Systems with Time-varying Input/Output Delays and Mismatched Disturbances

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Abstract: This paper investigates the distributed consensus tracking control problem for general linear multi-agent systems (MASs) with bounded time-varying input/output delays and mismatched disturbances under a directed communication graph containing a spanning tree. First, a part of the mismatched disturbances is transformed into the input channel as the matched disturbance, that allows the deployment of a predictive extended state observer (ESO) in order to estimate the consensus tracking error and transformed matched disturbance simultaneously. Subsequently, the ESO is used for designing a controller. The stability of the closed-loop system is guaranteed via a Lyapunov-Krasovskii functional with sufficient conditions in terms of delay-dependent linear matrix inequalities (LMIs). An additional LMI is proposed which, in conjunction with the rest of LMIs, results in a solution with a larger upper bound on delays than what would be feasible without it. The efficacy of the proposed scheme is demonstrated via simple numerical examples.

Keywords: Time-varying delays, predictive extended state observer, linear matrix inequalities, mismatched disturbances, multi-agent systems.

1. INTRODUCTION

Consensus tracking control of multi-agent systems (MASs), which means followers' states are controlled to track the leader's state, has been studied for a long time (see, for example, Olfati-Saber and Murray (2004), Scardovi and Sepulchre (2009) and references therein) due to its wide variety of applications such as vehicle formations, distributed estimation/filtering, sensor networks, distributed optimization, cooperative robotics, and social networks.

Time delay systems have been investigated extensively, since time-delays exist inherently in realistic feedback control systems, which constitute the integration of sensors, controllers, actuators, and communication networks. The existence of communication networks in the feedback control loop induces delays (sensor-to-controller delay and controller-to-actuator delay) that occurs while exchanging data among devices. The delays can degrade the performance of control systems designed without considering those delays and can even destabilize the system. To actively compensate for input delays, predictive controllers have been proposed in the literature, e.g., the Smith predictor (Smith, 1957), the finite spectrum assignment approach (Manitis and Olbrot, 1979), and Artstein’s model reduction technique (Artstein, 1982). More recently, the technique of linear matrix inequalities (LMIs) appeared providing researchers a very powerful tool to deal with time-varying delays. For more details about time delay systems, please refer to Fridman (2014b) and references therein.

To deal with the consensus tracking problem for MASs with constant input delays inside the general linear dynamics, Jiang et al. (2018) adopted the Artstein’s model reduction technique to design a modified state predictor based on a dynamic system with an observer-like structure to predictively estimate the state of input-delayed system. Then, the same idea is followed in Najafi et al. (2019) for constant input delays and in Léchappé et al. (2016) for time-varying input delays. The time-varying delay in Léchappé et al. (2016) is required to have a lower and upper positive bounds, where its upper bound is dependent on that lower bound. The idea in Besançon et al. (2007); Najafi et al. (2013); Léchappé et al. (2016) is...
also adopted in this paper to deal with time-varying delays, but without requiring to have a lower positive bound. The consensus tracking problem for MASs is also studied in Wang et al. (2019) for constant input/output delays and matched disturbances by adopting the same idea as in the above three works. Ramya et al. (2019) investigated the consensus problem for constant input delays by adopting Smith predictor under undirected graphs.

In this paper, time-varying input/output delays and mismatched disturbances are addressed for MASs under a directed graph topology. The contributions are as follows.

- A new predictive extended state observer (ESO) is proposed without integral terms to control the construction of the control input by solving the high dimensional nonlinear matrix inequality for calculating the ESO parameter. Towards this end, nonlinear terms inside the matrix inequality are linearized to form an LMI and one intermediate variable with a special structure is designed to reduce the computing load for the high dimensional LMI. The proposed approach does not require delays to have a lower positive bound as it is the case for Léchappé et al. (2016).

- A larger upper bound of delays is achieved by proposing a new LMI transformed from the objective function when optimizing existing LMIs (sufficient conditions, e.g., in Fridman and Dambrine (2009); Fridman (2014a); Sanz et al. (2018)). How to adjust this LMI to make unstable MASs become stable is also provided.

2. NOTATION AND PRELIMINARIES

2.1 Notation

Throughout this paper, $\mathbb{N}, \mathbb{R}^{m \times n}$ and $\mathbb{R}^n$ are respectively the set of positive integer numbers, the $m \times n$ real matrix space and $n$-dimensional Euclidean vector space. $\otimes$ is the Kronecker product and $\text{col}(\cdot)$ denotes a column vector. $\text{diag}(a_1, \ldots, a_n)$ represents a diagonal matrix with diagonal elements being $a_1, \ldots, a_n$. Matrices are assumed to have compatible dimensions if not explicitly stated. $\lambda_{min}(A)$ and $\lambda_{max}(A)$ represent the minimal and maximal eigenvalues of $A$, respectively. The square matrix $A \geq (>) 0$ means $A$ is symmetric and positive (nonnegative) definite. For a scalar $x$, denote $|x|$ as its absolute value. For a vector $x$, denote $\|x\|$ as its 2-norm. $L_p(a, b), p \in \mathbb{R}$ represents the space of functions $\phi : (a, b) \to \mathbb{R}$ with the norm $\|\phi\|_{L_p} = \left( \int_a^b |\phi(\theta)|^p \, d\theta \right)^{1/p}$. $L_\infty(a, b)$ is the space of essentially bounded functions $\phi : (a, b) \to \mathbb{R}$ with the norm $\|\phi\|_{L_\infty} = \text{esssup}_{\theta \in (a, b)} |\phi(\theta)|$. When there exists no confusion, the variable $t$ will be omitted, e.g., $x = x(t)$.

For any integer $a \leq b$, denote $I^a_b = \{a, a + 1, \ldots, b\}$. Symmetric terms in symmetric matrices are denoted by $*$, e.g., $\begin{bmatrix} A & B \\ * & C \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$.

2.2 System model

Consider a group of $N$ followers with single input dynamics as follows:

$$
\begin{align*}
\dot{x}_i(t) &= AX_i(t) + Bu_i(t - \tau_u(t)) + \beta_{kd_i}(t), \\
y_i(t) &= CX_i(t - \tau_y(t)), i \in \mathcal{I}_i^N.
\end{align*}
$$

where $X_i(t) = [X_{i1}(t), \ldots, X_{in}(t)]^T \in \mathbb{R}^n$, $u_i(t) \in \mathbb{R}$ and $y_i(t) \in \mathbb{R}^q$ are the state, input and output of the $i$-th follower, respectively. The constant matrices $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times q}$ and $C \in \mathbb{R}^{q \times n}$ are known. $\beta_k \in \mathbb{R}^n$ is a vector with $k$-th entry being one and the rest being zero. So the possible values of $k$ is $1, 2, \ldots, n$. Note that the value of $k$ can be different for different agent. $d_i(t) : \mathbb{R}_+ \to \mathbb{R}$ is an unknown external disturbance. $\tau_u(t)$ and $\tau_y(t)$ are the known time-varying input and output delays, respectively.

Assumption 1. Time-varying delays are upper bounded, i.e., $0 \leq \tau_u(t) \leq \bar{\tau}_u, 0 \leq \tau_y(t) \leq \bar{\tau}_y$. Denote $\bar{\tau} = \bar{\tau}_u + \bar{\tau}_y$. The delays are differentiable and their dynamics are bounded, i.e., $\dot{\tau}_u \leq \bar{\tau}_u, \dot{\tau}_y \leq \bar{\tau}_y, \tau_u + \bar{\tau}_u, \tau_y + \bar{\tau}_y \leq \bar{\tau} < 1$. The values of $\tau_u(t), \tau_y(t), \tau_u, \tau_y, \bar{\tau}$ are known.

Assumption 2. $d_i(t)$ can be described as $d_i(t) = \nu_i(t) + \varpi_i(t)$ with the modeled disturbance component $\nu_i(t) \in \mathbb{R}$ satisfying

$$
\nu_i(t) = Fw_i(t), \quad \varpi_i(t) = E_iw_i(t), i \in \mathcal{I}_N^N,
$$

where the exogenous system $(E_i \in \mathbb{R}^{n \times n}, F \in \mathbb{R}^{1 \times n})$ is known and observable, $w_i(t) \in \mathbb{R}^n$ with unknown initial condition $w_i(0) = 0$ when $t \in [-\tau, 0)$. The unmodeled disturbance component $\varpi_i(t) \in \mathbb{R}$ is an unknown signal which is supposed to be locally essentially bounded meaning that $\varpi_i(t) \in L_\infty[0, t], \forall t \geq 0$, i.e., $|\varpi_i(t)| \leq \Delta_i$, with $\Delta_i$ is a priori given and $\varpi_i(t) = 0, t \in [-\tau, 0)$.

Assumption 3. $(A, B)$ is controllable and $(A, C)$ is detectable.

Assumption 4. $\begin{bmatrix} A & BF \\ 0 & E_i \end{bmatrix}$ is detectable.

Assumption 1 does not require the delays having lower positive bound while Léchappé et al. (2016) does, which can be regarded as an improvement. Assumption 1 also means the time-varying delays are slowly-varying delays with $\bar{\tau} < 1$. In Assumption 3, $(A, B)$ can be stabilizable as long as the disturbance does not affect the uncontrollable states; see Sanz et al. (2018) for more details.

The dynamics of leader indexed by 0 is

$$
\begin{align*}
\dot{x}_0(t) &= Ax_0(t), y_0(t) = Cx_0(t - \tau_y(t)),
\end{align*}
$$

where $x_0(t) \in \mathbb{R}^n, y_0(t) \in \mathbb{R}^q$ and $x_0(t) = 0, t \in [-\tau, 0)$. It is reasonable to have the leader without neighbors or input (Wang et al., 2019; Jiang, 2018), i.e., $u_0(t) = 0$. Note that not all followers can receive the leader’s output information. Denote the state and output consensus tracking error for follower $i$ as

$$
\begin{align*}
\dot{x}_i(t) &= X_i(t) - x_0(t), \quad \dot{y}_i(t) = y_i(t) - y_0(t).
\end{align*}
$$

2.3 Graph theory

A brief background about the graph theory is given in the following. In a weighted graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A), \mathcal{V} = \{1, 2, \ldots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ are the nodes and edges, respectively. An edge $(i, j) \in \mathcal{E}$ means agent $j$ can get information from agent $i$ but not necessarily conversely. $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the weighted adjacency matrix, where $a_{ij} = 1, (i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix $L = [l_{ij}] \in \mathbb{R}^{N \times N}$ is defined as $l_{ij} = -a_{ij}, i \neq j$ and $l_{ii} = \sum_{j \neq i} a_{ij}$. A directed path from node $i$ to $j$ is a sequence of edges $(i, i_1), (i_1, i_2), \ldots, (i_h, j)$ in
the directed network via distinct nodes \(i_1, i_2, \ldots, i_n\). A directed graph contains a directed spanning tree if there is a node from which a directed path exists to each other node.

**Assumption 5.** Graph \(G\) contains a directed spanning tree in which the leader acts as the root node.

Then, the Laplacian matrix of \(G\) can be partitioned as

\[
L = \begin{bmatrix}
0 & 0_1 \times N \\
L_2 & L_1
\end{bmatrix}, \quad \text{where } L_2 \in \mathbb{R}^{N \times 1} \text{ and } L_1 \in \mathbb{R}^{N \times N}.
\]

### 2.4 Model transformation

Based on Assumption 3, without loss of generality, consider the pair \((A, B)\) in the canonical controllable form as

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\alpha_1 & \alpha_2 & \cdots & \alpha_n & 0 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
\vdots \\
0 \\
b
\end{bmatrix}.
\]

As the possible values of \(k\) in (1) is 1, 2, \ldots, \(n\), if \(k = n\), it means that \(d_i(t)\) is matched disturbance which influences the system through the input channel. On the contrary, if \(k \neq n\), \(d_i(t)\) is mismatched disturbance which affects an agent's state through channels in which the input has no direct influence (Sanz et al., 2018). Note that \(k\) is not a controller parameter. Different values of \(k\) mean different agents can have different dynamics of vector \(\beta_k\). Analyzing the components of \(d_i(t)\) in (2), the idea is to put \(v_i(t)\) in the input channel so as to design the corresponding input to compensate it.

Denote the differential operation in the form as \(v_i := \dot{v}_i^0, \dot{v}_i := \dot{v}_i^1, \ddot{v}_i := \ddot{v}_i^2, \dddot{v}_i := \dddot{v}_i^3\) and so on. So, the model transformation (Ding, 2003) is employed as:

\[
x_{ij}(t) = X_{ij}(t), \quad j \in \mathbb{I}_i^0,
\]

\[
x_{ij}(t) = X_{ij}(t) + \dot{v}_i^{j-(k+1)}(t), \quad j \in \mathbb{I}_i^{k+1},
\]

where \(x_{ij}(t) = [x_{i1}(t), \ldots, x_{in}(t)]^T \in \mathbb{R}^n\) and \(x_{ij}(t) = 0\) when \(i \in [-\tau, 0]\). Then, system (1) is transformed to

\[
\dot{x}_i(t) = Ax_i(t) + B[u_i(t) - v_i(t)] + \dot{v}_i(t) + \beta_k \varpi_i(t),
\]

\[
y_i(t) = Cx_i(t - \tau_p(t)), \quad i \in \mathbb{I}_i^N,
\]

with

\[
\dot{v}_i(t) = \frac{1}{b_i}(\varpi_i^{n-k}(t) - \sum_{j=k+1}^n a_j \varpi_i^{j-(k+1)}(t)),
\]

where \(a_j\) comes from (5). The calculation details are shown in (A.1) in the Appendix A.

Similar as the right-hand-side structure in (8), denote

\[
\dot{w}_i(t) = \frac{1}{b_i}(\varpi_i^{n-k}(t) - \sum_{j=k+1}^n a_j \varpi_i^{j-(k+1)}(t)),
\]

where \(w_i(t) \in \mathbb{R}^n\) is the internal variable of disturbance component \(u_i(t)\) in (2). From (8), (9), and (2), it is easy to deduce \(\dot{v}_i(t) = Fw_i(t)\) and \(\dot{w}_i(t) = (E_i \varpi_i^{n-k}(t) - \sum_{j=k+1}^n a_j E_i \varpi_i^{j-(k+1)}(t))/b = E_i \ddot{w}_i(t)\). Hence, one can see that the component \(v_i(t)\) of the external disturbance \(d_i(t)\) in (1) could be regarded as the generator of the matched input disturbance \(\bar{v}_i(t)\) in transformed system (7), which is summarized as

\[
\dot{v}_i(t) = F \bar{w}_i(t), \quad \dot{\bar{w}}_i(t) = E_i \bar{w}_i(t),
\]

where \(\bar{v}_i(t) \in \mathbb{R}\) and \(\tilde{w}_i(t) \in \mathbb{R}^n\). From (9) and (10), if the values of either \(k\) or \(E_i\) are different, the value of \(\bar{v}_i(t)\) is different. Also, based on the model transformation (6), consensus tracking error (4) is transformed to

\[
\dot{x}_i(t) = x_i(t) - x_0(t), \quad \dddot{y}_i(t) = y_i(t) - y_0(t).
\]

Then, the consensus tracking error dynamics is

\[
\dot{\tilde{x}}_i(t) = A \tilde{x}_i(t) + B[u_i(t) - \tau_p(t)] + \bar{v}_i(t) + \beta_k \varpi_i(t),
\]

\[
\dot{\tilde{y}}_i(t) = C\tilde{x}_i(t - \tau_p(t)), \quad i \in \mathbb{I}_i^N.
\]

**Remark 1.** The model transformation (6) is to transform \(v_i(t)\) (2) inside mismatched disturbance \(d_i(t)\) from (1) to matched disturbance \(\bar{v}_i(t)\) in (12) via the relationship (8).

**Assumption 6.** The output signal \(y_i(t)\) is measurable, i.e., \(y_i(t)\) is known for agent \(i\), \(i \in \mathbb{I}_i^N\).

### 2.5 Problem statement

Under Assumptions 1-6, for any given initial conditions, design a distributed controller to achieve the following objectives:

\(O_1\): the consensus tracking error \(\dot{z}_i(t)\) in (12) is exponentially stable if \(\varpi_i(t) \equiv 0, i \in \mathbb{I}_i^N\), and stays bounded if \(\varpi_i(t) \in L_{M}^{\infty}[0, \tau] \forall t \geq 0\);

\(O_2\): the systems can enjoy larger delays.

### 3. PREDICTIVE ESO DESIGN

For the transformed disturbance dynamics (10) and consensus tracking error dynamics (12), define an augmented state \(z_i(t) = [\tilde{x}_i^T(t), \tilde{w}_i^T(t)]^T\) and then, one has

\[
\dot{z}_i(t) = \begin{bmatrix}
A & BF \\
0 & E_i
\end{bmatrix} z_i(t) + \begin{bmatrix}
B \\
0
\end{bmatrix} u_i(t - \tau_p(t)) + \begin{bmatrix}
\beta_k \\
0
\end{bmatrix} \varpi_i(t),
\]

\[
y_i(t) = C_1 z_i(t - \tau_p(t)), \quad i \in \mathbb{I}_i^N,
\]

where \(A_1 \in \mathbb{R}^{(n+s) \times (n+s)}, B_2 \in \mathbb{R}^{n \times s}, B_{w_i} \in \mathbb{R}^{n \times s}\) and \(C_1 \in \mathbb{R}^{s \times (n+s)}\).

The idea is to design a predictive ESO as \(z_i(t) := [\tilde{x}_i^T(t), \tilde{w}_i^T(t)]^T \in \mathbb{R}^{n+s}\) with \(\dot{z}_i(t) = 0, \tilde{w}_i(t) = 0, \forall t \in [-\tau, 0]\) to estimate \(z_i(t)\) in (13). To achieve the prediction, both the classic finite spectrum assignment approach (Manitius and Olbrot, 1979) and Artstein’s model reduction technique (Artstein, 1982) involve the integral term whose calculation is very time-consuming. So, designing a predictive ESO without any integral term is preferred. Inspired by works of Besançon et al. (2007); Najafi et al. (2013); Léchappé et al. (2016), \(\tilde{z}_i(t)\) is proposed as:

\[
\dot{\tilde{z}}_i(t) = A_1 \tilde{z}_i(t) + B_2 u_i(t) + L \left\{ \sum_{j=1, j \neq i}^N a_j y_i(t) - y_j(t) - C_2 \tilde{x}_i(t - \tau(t)) + C_2 \tilde{x}_i(t - \tau(t)) \right\} + a_0[y_i(t) - y_0(t) - C_2 \tilde{x}_i(t - \tau(t))],
\]

where \(\tau(t) := \tau_n(t) + \tau_\varpi(t), L \in \mathbb{R}^{(n+s) \times q}\) will be designed in Section 4 and \(a_{ij}\) is the \(i\)-th entry of the adjacent
matrix $\mathcal{A}$ of the graph $\mathcal{G}$ satisfying Assumption 5. The term $a_{i0}y_0(t)$ means only the followers having the leader as their neighbor can know its output information as $a_{i0} = 1$, otherwise $a_{i0} = 0$. The neighbors’ current output $y_j(t)$ and historical values of observer state $\hat{x}_j(t) = (t - \tau(t))$ and agent’s own historical observer state $\hat{z}_i(t) = (t - \tau(t))$ are utilized to design (14).

Remark 2. Compared to $z_i(t)$ dynamics (13), there exists no input delay in predictive ESO (14). As there is no integral term inside (14), the ESO calculation is not time-consuming. Thanks to the adoption of stored historical values, this ESO $\hat{z}_i(t)$ can predict $z_i(t)$ with $\tau_u(t)$ unites of time in advance, i.e., $\hat{z}_i(t) \rightarrow z_i(t + \tau_u(t))$.

As a result, denote the ESO estimating error $\hat{z}(t) \in \mathbb{R}^{n+}$ as

$$\hat{z}_i(t) = z_i(t) - \hat{z}_i(t - \tau_u(t)).$$

By using $y_i - y_j = (y_i - y_0) - (y_j - y_0)$, $C \hat{z}_i = C_{z_i} \hat{z}_i$, $C \hat{z}_j = C_{z_j} \hat{z}_j$, and combining (13) and (14), the $\hat{z}_i(t)$ dynamics is

$$\dot{\hat{z}}_i(t) = A_{\hat{z}} \hat{z}_i(t) - (1 - \tau_u(t))LC_z \sum_{j=1}^{N} \tilde{L}_{ij} \hat{z}_j(t - \tau(t)) + B_{w_i} \hat{w}_i(t) + \tau_u(t) \hat{A}_z \hat{z}_i(t - \tau_u(t)).$$

In order to guarantee the stability of ESO estimating error $\hat{z}_i(t)$, when $\tau(t) = 0$ and $\hat{w}_i(t) = 0$, $\hat{z}_i(t)$ should be able to be controlled to converge to zero, i.e., $\{\hat{A}_z, \hat{C}_z\}$ can be Hurwitz. To guarantee that, Assumption 4 is required.

Now, the control input is designed as

$$u_i(t) = -K \hat{\dot{x}}_i(t) - F \hat{\dot{w}}_i(t) = -[K, F] \hat{z}_i(t), i \in \mathbf{I}_N,$$  

where parameter $K \in \mathbb{R}^{1 \times n}$ will be designed in Section 4. Then, (14) and (16) change to

$$\dot{\hat{z}}_i(t) = \left[ \begin{array}{cc} A - BK & 0 \\ 0 & L_{i1} \end{array} \right] \hat{z}_i(t) + LC_z \sum_{j=1}^{N} \tilde{L}_{ij} \hat{z}_j(t - \tau_g(t)),$$

$$\dot{\hat{z}}_i(t) = A_{\hat{z}} \hat{z}_i(t) - (1 - \tau_u(t))LC_z \sum_{j=1}^{N} \tilde{L}_{ij} \hat{z}_j(t - \tau(t)) + B_{w_i} \hat{w}_i(t) + \tau_u(t) \hat{A}_z \hat{z}_i(t - \tau_u(t)).$$

Remark 3. The proposed control input in Léchappé et al. (2016) has to satisfy the globally Lipschitz condition while input (17) does not this constraint.

In the rest of this section, another augmented state dynamics will be constructed for the convenience of designing controller parameters $L$ in (14) and $K$ in (17). Based on ESO estimating error definition (15), integrating input (17) into consensus tracking error dynamics (12) gives

$$\dot{\hat{x}}_i(t) = (A - BK) \hat{x}_i(t) + [BK, BF] \hat{z}_i(t) + \beta_k \hat{w}_i(t).$$

By a slight abuse of notation, here, let $\beta_k := \beta_k$ for each agent $i$. Then, denote $\hat{\beta}_k := \hat{\beta}_k$ and $\hat{\beta}_k := \hat{\beta}_k$ and $\hat{\beta}_k := \hat{\beta}_k$, and the similar structure as $\hat{\beta}_k$, $\hat{\beta}_k = \hat{\beta}_k$ and $\hat{\beta}_k = \hat{\beta}_k$. Denote $\hat{x}(t) = \hat{x}_i(t), \hat{x}_j(t), \hat{x}_j(t), \hat{x}_j(t), \hat{x}_j(t), \hat{x}_j(t)$, and $\hat{z}(t), \hat{w}(t), \hat{z}(t)$.

Therefore, the Kronecker product format of consensus tracking error dynamics (19), ESO and its estimating error dynamics (18) is

$$\dot{\hat{z}}(t) = [I_N \otimes (A - BK)] \hat{z}(t) + (I_N \otimes [BK, BF]) \hat{z}(t) + \hat{\beta}_k \hat{w}(t),$$

$$\dot{\hat{z}}(t) = \text{diag}(A_{\hat{z}}) \hat{z}(t) - \left[ L_{1} \otimes (1 - \tau_u(t))LC_z \right] \hat{z}(t - \tau(t)) + \text{diag}(B_{w}) \hat{w}(t) + \tau_u(t) \text{diag}(A_{\hat{z}}) \hat{z}(t - \tau_u(t)), $$

$$\dot{\hat{z}}(t) = \text{diag}(A_{\hat{z}}) \hat{z}(t) - \left[ L_{1} \otimes (1 - \tau_u(t))LC_z \right] \hat{z}(t - \tau(t)) + \text{diag}(B_{w}) \hat{w}(t) + \tau_u(t) \text{diag}(A_{\hat{z}}) \hat{z}(t - \tau_u(t)),$$

where $L_{1}$ is the Laplacian matrix under Assumption 5. Finally, based on (20), denote another augmented variable dynamical $\hat{z}(t) = \left[ \hat{z}^T(t), \hat{z}(t), \hat{z}(t - \tau)(t) \right] \in \mathbb{R}^{N(3n+2s)}$ and rewrite (20) as

$$\dot{\hat{z}}(t) = \text{diag}(A_{\hat{z}}) \hat{z}(t) + \text{diag}(A_{\hat{z}}) \hat{z}(t - \tau_u(t)) + \text{diag}(B_{w}) \hat{w}(t),$$

$$\dot{\hat{z}}(t) = \text{diag}(A_{\hat{z}}) \hat{z}(t) + \text{diag}(A_{\hat{z}}) \hat{z}(t - \tau_u(t)) + \text{diag}(B_{w}) \hat{w}(t),$$

$$\dot{\hat{z}}(t) = \text{diag}(A_{\hat{z}}) \hat{z}(t) + \text{diag}(A_{\hat{z}}) \hat{z}(t - \tau_u(t)) + \text{diag}(B_{w}) \hat{w}(t).$$

So the original consensus tracking problem is transformed into the input-to-state stability problem of (21) with initial condition (22). The Lyapunov-Krasovskii functional (LKF) with descriptor method (Fridman, 2001) will be adopted to design $L$ and $K$ in the next section.

For the convenience of reading, a picture of the signal flow is presented as follows:

$$\begin{array}{lllllllll} x_i(t) & \rightarrow & \hat{x}_i(t) & \rightarrow & \hat{z}_i(t) & \rightarrow & \hat{w}_i(t) & \rightarrow & \hat{z}_i(t) & \rightarrow & \hat{z}_i(t) \\
(1) & \rightarrow & (7) & \rightarrow & (11) & \rightarrow & (9) & \rightarrow & (13) & \rightarrow & (15) \\
& & & & & & & & & \end{array}$$

4. STABILITY ANALYSIS

Inspired by the work of Fridman and Dambrine (2009), design one type of LKF as

$$V = \zeta^T(t)P \zeta(t) + \int_{t-\tau}^{t} e^{2\delta(s-\tau)} \zeta^T(s)S \zeta(s) ds$$

$$+ \int_{t-\tau}^{t} e^{2\delta(s-\tau)} Q \zeta(t) ds$$

$$+ \tau \int_{t-\tau}^{t} e^{2\delta(s-\tau)} R \zeta(s) ds d\theta,$$

where $0 \leq \tau(t) = \tau_u(t) + \tau_u(t) = \tau, 0 < \delta \in \mathbb{R}$ and $\theta \in \{Q, R, S\} \in \mathbb{R}^{N(3n+2s) \times N(3n+2s)}$ will be decided later. So $V$ is positive definite. Define

$$\gamma = \text{diag}(\gamma_1, \ldots, \gamma_N), \gamma_i > 0, i \in \mathbf{I}_N.$$ 

Based on (21), the derivative of LKF (23) satisfies
\[ \dot{V} + 26V - \omega T \gamma \omega = 2\zeta^T P \zeta + 2\delta \zeta^T P \zeta - \omega T \gamma \omega \\
+ \zeta^T (S + Q) \zeta - e^{-2\delta t} \zeta^T (t - \tau) S \zeta (t - \tau) \\
- (1 - \tau(t)) e^{-2\delta t} \zeta^T (t - \tau(t)) Q \zeta(t - \tau(t)) \\
+ \tau^2 \zeta^T (t) R \zeta(t) - \tau \int_{t-\tau}^{t} e^{2\delta (s-t)} \zeta^T (s) R \zeta(s) ds \leq 2\zeta^T P \zeta + 2\delta \zeta^T P \zeta \\
- (1 - \tau) e^{-2\delta t} \zeta^T (t - \tau(t)) Q \zeta(t - \tau(t)) \\
+ \zeta^T (S + Q) \zeta - e^{-2\delta t} \zeta^T (t - \tau) S \zeta (t - \tau) \\
+ \tau^2 \zeta^T R \zeta - \tau e^{-2\delta t} \int_{t-\tau}^{t} \zeta^T (s) R \zeta(s) ds - \omega T \gamma \omega \\
+ 2[\zeta^T P_1^T + \zeta^T P_2^T] [A\zeta + A_1 \zeta(t - \tau) + B \zeta \omega - \dot{\zeta}] = \zeta^T \Phi_1 \zeta, \tag{26} \]

where \( \zeta(t) = \text{col}(\zeta(t), \dot{\zeta}(t), \zeta(t - \tau), \dot{\zeta}(t - \tau), \omega(t)) \), and

\[
\Phi_1 = \begin{bmatrix}
\Phi_1(1,1) & \Phi_1(1,2) & e^{-2\delta t} S_{12} & \Phi_1(1,4) & P_1^T B_1 \\
* & * & \Phi_1(2,2) & * & * \\
* & * & \Phi_1(3,3) & e^{-2\delta t}(R - S_{12}) & 0 \\
* & * & * & * & * \\
* & * & * & * & -\gamma
\end{bmatrix}, \tag{27}
\]

where \( \{P_1, P_2, S_{12}\} \in \mathbb{R}^{N(3n+2s) \times N(3n+2s)} \) will be decided later. The inequality (26) comes from \( \dot{\tau}(t) \leq \dot{\tau} \) in Assumption 1, the Jensens inequality (see Appendix A) and Lemma 1 in Fridman (2014a) where the matrix \( S_{12} \) is introduced. For more mathematical details about calculating (26) and (27), please refer to Eqs. (6.5)-(6.20) in Jiang (2018). The last term (called the descriptor method in Fridman (2001) where \( P_2, P_3 \) are introduced) in inequality (26) is identically zero, which comes directly from (21). Following the Proposition 1 in Fridman and Dambrine (2009) (given in the Appendix A), if \( \Phi_1 < 0 \) is feasible, solving (26) gives

\[ \zeta^T(t) P \zeta(t) \leq e^{-2\delta t} \zeta^T(0) P \zeta(0) + (1 - e^{-2\delta t}) \frac{\lambda_{\max}(\gamma)}{2\delta} \| \omega(0, t) \|_2^2, \quad t \geq 0, \]

with \( \omega(t), i \in \mathbb{I}_N \) are locally essentially bounded, \( \delta \) defined in (23) and \( \gamma \) defined in (25), which means the closed-loop system is exponentially stable in terms of variable \( \zeta(t) \). Recalling \( \zeta(t) = [\dot{z}^T(t), \bar{z}^T(t), \bar{z}^T(t - \tau_1(t))]^T \), due to \( \zeta^T(t) P(t) \zeta(t) \geq \lambda_{\min}(P) \| \zeta(t) \|^2 \geq \lambda_{\min}(P) \| \dot{\zeta}(t) \|^2 \) and Assumption 2, one can conclude that

\[ \| \dot{\tilde{x}}(t) \|^2 \leq \frac{\lambda_{\max}(\gamma)}{2\lambda_{\min}(P)} \sum_{i=1}^{N} \Delta_i^2, \quad t \to \infty. \tag{29} \]

Remark 4. The LKF (23) is used to deal with time-varying delays. Note that the proof of calculating (26) is different from that in Fridman and Dambrine (2009) by integrating the descriptor method and matrix \( S_{12} \) inside. The reason for adopting the descriptor method is that some comparison simulations in Section 6.1.3 of Jiang (2018) shows that the closed-loop system can endure larger delays with the descriptor method used. It also shows that there is a trade-off between the exponential convergence rate \( \delta \) and upper bound \( \tilde{\tau} \): the larger the rate \( \delta \), the smaller the upper bound \( \tilde{\tau} \).

Theorem 1. Under Assumptions 1-6, given \( \tilde{\tau} \geq 0, \tilde{\tau} \in [0, 1], \delta > 0 \), and tuning parameters \( \kappa > 0, \varepsilon > 0, \gamma > 0 \), if there exist matrices \( \{P, Q, R, S\} > 0, S_{12}, U \in \mathbb{R}^{N \times N} > 0 \), \( X \in \mathbb{R}^{N \times N}, Y \in \mathbb{R}^{(n+s) \times q} \) such that the following LMIs are feasible:

\[
\Phi_2 = \begin{bmatrix}
\Phi_1(1,1) & \Phi_1(1,2) & e^{-2\delta t} S_{12} & \Phi_1(1,4) & P_1^T B_1 \\
* & \Phi_1(2,2) & * & * & * \\
* & * & \Phi_1(3,3) & e^{-2\delta t}(R - S_{12}) & 0 \\
* & * & * & * & * \\
* & * & * & * & -\gamma
\end{bmatrix} < 0, \tag{30}
\]

\[ A U + U A^T - B X - X^T B^T + \kappa U < 0, \tag{31} \]

\[ \begin{bmatrix}
R S_{12} & \ast \\
\ast & R
\end{bmatrix} \geq 0, \tag{32} \]

where \( \Phi_1(1,2) = P - P^T + \varepsilon A_2^T P_2, \Phi_1(2,1) = \tilde{\tau}^2 R - \varepsilon^2 P_2 + P_2^T \) and \( \Phi_2(1,4) = \Pi + e^{-2\delta t} (R - S_{12}) \), \( \Pi \) is known. Recall \( \Phi_1 \) is the ESO parameter \( L \) in (21), i.e., \( A_2 \) is unknown. So, \( \Phi_1(1,2) \) contains two nonlinear terms: \( P_2^T A_1 \) and \( P_2^T A_1 \), since \( P_2, P_3 \) are also unknown. By setting \( P_3 = \varepsilon P_2, \varepsilon > 0 \), (34)

\[ \text{from descriptor method, nonlinear matrix inequality } \Phi_1(27) \text{ is then linearized. There exist two main problems:} \]

1) Since \( P_3 \in \mathbb{R}^{N \times (3n+2s) \times N \times (3n+2s)} \) is high dimensional, when the number \( N \) of agents is large, the computing load is very heavy. Also, with high probability, the LMI will be infeasible even for a small delay bound \( \tilde{\tau} \) because of the large number of elements in \( P_2 \).
2) It is still not clear how to calculate \( L \) out of the value of \( L_1 \otimes LC_2 \).

After analyzing the internal structure of \( A_{12} \), which is quite special (7 zero blocks), setting \( P_2 \) in the form of (33) is preferred such that \( P_2^T A_{12} = \Pi \) with \( Y = P_{22}^T L \). In this case, \( \Phi_1 (27) \) becomes LMI \( \Phi_2 (30) \) which is related to \( \tau_a (t) \). Note that \( P_2 \) does not need to be symmetric or positive, and only \( \| P_{22} \| \neq 0 \) is required. Additionally, the dimension of \( P_{22} \in \mathbb{R}^{(n+a) \times (n+a)} \) is quite low and independent of agents' number \( N \). The elements needed to be calculated in \( P_2 (33) \) is also much less as 5 zero blocks and an identity block is integrated inside. And the feasibility of LMI \( \Phi_2 (30) \) is much higher.

To prove that \( \Phi_2 \) is negative definite for all \( \tau_a (t) \), note that \( \Phi_2 \) is affine with respect to the variable \( \tau_a (t) \) in \( [\tau_a, \tau_a] \) which is defined in Assumption 1. Then, by convexity, it suffices to ensure that \( \Phi_2 \) is negative for \( \tau_a (t) = \hat{\tau}_a \) and \( \tau_a (t) = \hat{\tau}_a \) (see Naghshtabrizi et al. (2008); Seuret and Da Silva Jr (2012); Fridman (2014b) for more detail).

After \( Y \) is calculated through LMIs (30), (32) and (33), the ESO parameter is \( L = (P_{22}^T)^{-1} Y \). The proof is finished.

The application of Theorem 1 is summarized in the following algorithm.

**Algorithm 1.** Consensus controller for upper bound \( \tau \).

1. Initial setting: delay derivative bound \( \hat{\tau} \), decay rate \( \delta \), input delay lower and upper bound \( \hat{\tau}_a, \tau_a \).
2. Set parameter \( \kappa \) in (31) to get matrices \( X, U \), then calculate \( K = X U^{-1} \) for control input protocol (17).
3. Calculate \( A_{12} \) in (21) based on \( K \).
4. Set parameters \( \varepsilon \) in (34), define \( P, Q, R, S, S_{12} \) and \( \gamma, P_{21}, P_{22}, P_{23}, P_{24}, P_{25}, Y \) as variables and solve LMIs (24), (25), (30) by replacing \( \hat{\tau}_a (t) \) as \( \hat{\tau}_a \) and \( \tau_a \), (32) and (33) to get \( P_{22} \) and \( Y \).
5. Calculate \( L = (P_{22}^T)^{-1} Y \) for predictive ESO (14).

5. DELAY SIZE ANALYSIS

This section is for the problem \( O_2 \) in Section 2.5. In the proof of Theorem 1, the feasibility and parameter \( L \) calculation of LMI (30) related to construction of \( P_2 \) is demonstrated. In this section, we analyze how to get an improved upper bound \( \hat{\tau} \). As stated in Remark 4, the larger the rate \( \delta \), the smaller the upper bound \( \hat{\tau} \). Herein, \( \delta \) is predefined and fixed.

Authors in Wang et al. (2019); Besançon et al. (2007); Najafi et al. (2013); Sanz et al. (2018); Fridman (2014a); Fridman and Dombrine (2009) arrive at the delay-dependent LMI conditions (e.g., LMIs (30), (32), (33) in this paper) successfully to prove the stability of closed-loop system for different control scenarios. However, they did not analyze further on the delay size and therefore, they only considered relatively small delays in their simulation examples. Léchappé et al. (2016) give the explicit expressions of the upper bound \( \hat{\tau} \), which is also related to the lower positive bound of time-varying delays. Delays in this paper do not need the lower positive bound.

The main idea is as follows. If the upper bound \( \hat{\tau} \) is so large that LMI (30) is not feasible, it means that the tracking error will diverge, i.e., \( \| \hat{x}(t) \| \) in (29) will diverge. It also means in this situation, the value of \( \lambda_{\text{max}} (\gamma) \) will be very large. So the objective of minimizing \( \lambda_{\text{max}} (\gamma)/(\gamma^2 \lambda_{\text{min}} (P)) \) is preferred. However, the eigenvalue function of unknown LMI variables (e.g., \( P \), \( \gamma \)) is not available, thus some transformation is needed as follows. From (25), one has \( \lambda_{\text{max}} (\gamma) = \max_{\gamma \in \mathbb{R}} \gamma_i \) which gets rid of the eigenvalue calculation. In addition to the predefined \( \delta \), only \( \lambda_{\text{min}} (P) \) needs attention. On the contrary to the common setting of \( P > 0 \), design

\[
P \geq \mu I, 0 < \mu \in \mathbb{R}, \{Q, R, S\} > 0
\]

such that \( \lambda_{\text{min}} (P) \geq \mu \). Then \( \lambda_{\text{max}} (\gamma)/(\gamma^2 \lambda_{\text{min}} (P)) \) changes to \( \mu \). In this way, the minimizing objective can change to an optimization constraint. For instance, by asking \( \lambda_{\text{max}} (\gamma)/(\gamma^2 \lambda_{\text{min}} (P)) \) to be much less as 5 zero, the constraint becomes

\[
\gamma_i - 2\delta \mu < 0, i \in \{1^N \}
\]

with adding \( \mu \) as an LMI variable in step 4 of Algorithm 1. Therefore, Algorithm 2 is summarized to get larger upper bound \( \hat{\tau} \), which will be verified in the next section by simulation comparisons.

The intuition explanation of this new algorithm is that by adding an objective-function-transformed constraint (e.g., (36)) on the right-hand side of consensus tracking error \( \| \hat{x}(t) \| \) in (29), the value of \( \| \hat{x}(t) \| \) will be more difficult to diverge or become large, i.e., the closed-loop system could endure larger delay size.

**Remark 5.** Algorithms 1 and 2 can be also applied to multi-input-multi-output systems.

**Algorithm 2.** Consensus controller for larger upper bound \( \hat{\tau} \).

1. Same steps 1-3 in Algorithm 1.
2. Set parameters \( \varepsilon \) in (34), define \( P, Q, R, S, S_{12} \) and \( \gamma, P_{21}, P_{22}, P_{23}, P_{24}, P_{25}, Y, \mu \) as variables, solve LMIs (25), (30) by replacing \( \hat{\tau}_a (t) \) as \( \hat{\tau}_a \) and \( \tau_a \), (32), (33), (35) and (36) to get \( P_{22} \) and \( Y \), then calculate \( L = (P_{22}^T)^{-1} Y \).

6. SIMULATIONS

Set system matrices as

\[
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0.1 \\ -0.1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0.2 \\ -0.2 & 0 \end{bmatrix},
\]

\[
B = [0 0 -2]^T, \quad F = [1 0]^T, \quad C = [1 0 0]\text{ and } \beta_k = [0 1 0]^T \text{ for agent 1, } \beta_k = [1 0 0]^T \text{ for agent 2. Then, } (A, B) \text{ is controllable, } (A, C) \text{ and } (E_i, F), \in \mathbb{R}^1 \text{ are observable.}
\]

Graph \( \mathcal{G} \) is shown in Fig. 1.

**Remark 6.** Graph \( \mathcal{G} \) looks like a cascaded structure, but actually it can be a complicated structure with a spanning tree. One reason to only choose two followers is to reduce the LMI (e.g., (30)) computation load which is partly
related to the number of followers. The other reason is for the convenience of presenting comparison results in the following.

For Algorithms 1 and 2, choose \( \hat{\tau} = 0.8, \delta = 0.1 \). Initial conditions are \( x_0(0) = [7.4008, 5.4257, -2]^T, x_1(0) = [4.2989, 4.6232, 3]^T, x_2(0) = [1.6305, 4.8824, -8]^T, \) \( \bar{w}_1(0) = [-4.0129, -2.3813]^T, \) \( \bar{w}_2(0) = [2.1504, 4.0372]^T \) and \( u_i(t) = 0, t \in [-\bar{\tau}, 0], i \in I_2^e \). Set the unmodeled disturbance component as \( \varpi_1(t) = 0, t \in [0s, 200s], i \in I_2^e; \varpi_2(t) = 13 \sin(5t), \) \( \varpi_2(t) = \cos(5t), t \in (200s, 400s) \). Set parameter \( \kappa = 0.5 \), then solving LMI (31) gets \( K = [-0.6863, -1.7969, 0.2283] \). Set \( \varepsilon = 0.3 \).

Fig. 2 shows Algorithm 1 is available for MAS consensus tracking control as errors converge during \([0s, 200s]\) and are bounded with unmodeled disturbance attenuation during \((200s, 400s]\).

Fig. 3 describes the consensus tracking error performance that by adopting Algorithm 1, agents are becoming unstable with \( \tau_1(t) = 0.5 + 0.12 \cos(0.5t), \) \( \tau_2(t) = 0.1 + 0.05 \sin(0.5t), \) \( \tilde{\tau} = 0.8 \) while agents are still stable with \( \tau_1(t) = 0.55 + 0.25 \cos(0.5t), \) \( \tau_2(t) = 0.5 - 0.05 \sin(0.5t), \) \( \tilde{\tau} = 0.95 \) under Algorithm 2. It demonstrates clearly that Algorithm 2 can make MASs endure larger input/output time-varying delays with larger total upper bound.

Fig. 4 gives more details about analyzing the influence of newly added constraint (36) (i.e., \( \gamma_2/\delta \mu < 1, i \in I_2^e \)) to the MAS stability. For the consensus tracking error of agent 2 in Fig. 4, we take \( \gamma_1/\delta \mu < 1 \) unchanged and analyze the influence of different constraints (values) of \( \gamma_2/\delta \mu \) to the error convergence. Except that the input delay values are a little larger, Fig. 4 (a) has the same setting as Fig. 3 column (b). One can see in Fig. 4 (a), agent 2 become unstable with \( \gamma_2/\delta \mu < 1, \mu = -3.7728 \exp(-12) \). Note that constraint (35): \( P > 0 \) is guaranteed.

The negative sign of \( \mu \) deserves attention. It means the solver fails to calculate \( \mu \) with the positive sign, which may be due to the too strong constraint of \( \gamma_2/\delta \mu < 1 \) for agent 2. Thus, this constraint is relieved from (b) to (f) in Fig. 4. One can see in Fig. 4 (b)-(f) that the sign of \( \mu \) becomes positive and its value becomes larger and larger. The consensus tracking system becomes stable from Fig. 4 (c) with \( \gamma_2/\delta \mu < 16 \) and has better control performance in Fig. 4 (f). Note that further relieving the constraint is meaningfulness as one can verify it from (29).

Remark 7. Fig. 4 shows when the closed-loop system is on the edge of stable/unstable state with delays, adjusting the constraint of \( \gamma_2/\delta \mu \) can improve the system ability of keeping stable, i.e., make it endure larger delay. A fast way of finding a good bound for the delay is to use the bisection algorithm.

7. CONCLUSION

The distributed consensus tracking problem with time-varying input/output delays and mismatched disturbances under the directed graph is studied via model transformation, predictive extended state observer design, Lyapunov-Krasovskii functional calculation and parameter setting inside the linear matrix inequality. Detailed analysis on how to obtain a larger delay upper bound and better robust control performance are also provided. The pro-
posed controller and algorithms are without any integral term and can be easily implemented in real applications. A shortcoming is that it is nearly impossible to apply the proposed theory to large-scale systems because of the requirement of total graph information (i.e., $L_1$ in (30)) for each agent. Part of ongoing research is to develop a scheme to obtain this information in a distributed manner.

Appendix A

Mathematical calculation from (6) to (7) in Section 2.4 is:

\[ j = 1, \dot{x}_{i1} = X'_{i2} = x_{i2}, \]

\[ \vdots \]

\[ j = k - 1, \dot{x}_{i(k-1)} = X'_{ik} = x_{ik}, \]

\[ j = k, \dot{x}_{ik} = X'_{ik} = x_{i(k+1)} + d_i = x_{i(k+1)} + \bar{w}_i, \]

\[ j = k + 1, \]

\[ \dot{x}_{i(k+1)} = X'_{i(k+1)} + \bar{v}_{i}^{k+1-k} = x_{i(k+2)} + \bar{v}_i = x_{i(k+2)}, \]

\[ \vdots \]

\[ j = n - 1, \dot{x}_{i(n-1)} = X'_{in} + v_{i}^{n-k} = x_{in}, \]

\[ j = n, \]

\[ \dot{x}_{in} = X'_{in} + v_{i}^{n-k} + 1 = x_{in}, \]

where $\phi \in L_2[-\tau(t), 0]$, $\tau(t) > 0$ be a continuous function and square matrix $P \succ 0$, then

\[ \int_{-\tau(t)}^{0} \phi^T(s)P\phi(s)ds \geq \frac{1}{\tau(t)} \int_{-\tau(t)}^{0} \dot{V}(s)ds \int_{-\tau(t)}^{0} \phi(s)ds. \]

Proposition 1 in Fridman and Dambrine (2009): if there exist $\delta > 0$, $\gamma > 0$ and matrices $[P, S, Q, R] \succ 0$ such that the along the trajectories of (21), the LKF (23) satisfies

\[ \dot{V} + 2\delta V - \bar{w}^T \bar{w} < 0. \]

Then, the solution of (21) and (22) satisfies (28).

REFERENCES


