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Brief paper

Multi-agent consensus with heterogeneous time-varying input and communication delays in digraphs[☆]

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ABSTRACT

This paper investigates the distributed consensus tracking control problem for general linear multi-agent systems with external disturbances and heterogeneous time-varying input and communication delays under a directed communication graph topology, containing a spanning tree. First, for all agents whose state matrix has no eigenvalues with positive real parts, a communication-delay-related observer, which is used to construct the controller, is designed for followers to estimate the leader's state information. Second, by means of the output regulation theory, the results are relaxed to the case that only the leader's state matrix needs to be asymptotically stable or marginally stable and, under these relaxed conditions, the controller is redesigned. Both cases lead to a closed-loop error system of which the stability is guaranteed via a Lyapunov–Krasovskii functional with sufficient conditions in terms of input-delay-dependent linear matrix inequalities (LMIs). An extended LMI is proposed which, in conjunction with the rest of LMIs, results in a solution with a larger upper bound on delays than what would be feasible without it. It is highlighted that the integration of communication-delay-related observer and input-delay-related LMI to construct a fully distributed controller (which requires no global information) is scalable to arbitrarily large networks. The efficacy of the proposed scheme is demonstrated via an illustrative numerical example.

1. Introduction

The design of algorithms for distributed coordination in network systems has attracted significant attention by many disciplines over the last few decades, such as control, communication, physics, biology, and computer science. The emergence of this type of network systems, stretching from smart grids, social, robotic, and traffic networks of various sorts to embedded electronic devices, has sparked immense interest in distributed coordination problems. One such coordination problem is *consensus tracking control of multi-agent systems (MASs)* in which followers are designed to track their leader; see, e.g., Olfati-Saber and Murray (2004).

The design of feedback control systems in MASs connected over a communication network inherits two types of delays: *input* and *communication* delays. Input delays (IDs) are related to the existence of communication links in the feedback control loop (sensor-to-controller delay and controller-to-actuator delay) inducing delays (due to, e.g., limited bandwidth, retransmissions, and slow processing times) while exchanging data among devices. Communication delays (CDs) are attributed to the delayed information from neighboring agents via the underlying communication network (due to retransmissions, congestion, limited bandwidth, etc.). Both types of delays affect the stability of the whole system.

Many consensus controllers have been proposed to tackle *homogeneous* CDs, e.g., in Zhou and Lin (2014). One key advantage of addressing the problem of having homogeneous delays is the easiness to put the MAS dynamics into a compact mathematical form related to the Laplacian matrix of the communication graph. For *heterogeneous* delays, however, the above advantage disappears and linear matrix inequality (LMI) conditions are often proposed, e.g., in Sun and Wang (2009), to deal with the heterogeneous nature of CDs. However, these LMI conditions are not scalable to arbitrarily large networks as the dimension of the LMI increases with the number of agents. Alternatively, heterogeneous fixed delays can be transformed into the Laplace domain and approaches in the frequency domain (e.g., generalized Nyquist criterion) can be utilized to design controllers for specific dynamics of MASs, i.e., single-input-single-output (Münz, Papachristodoulou, & Allgöwer, 2010), first order (Ahmed, Khan,

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Saeed, & Zhang, 2020) and general linear dynamics (Jiang, Chen, & Charalambous, 2021). However, time-varying delays cannot be transformed and analyzed in the frequency domain. The difficulty of standard techniques to deal with time-varying heterogeneous CDs (HCDs) motivates this work.

Different from CDs, IDs have been investigated for decades for the single-agent system. To actively compensate for IDs, predictive controllers have been proposed in the literature, e.g., the Smith predictor, the finite spectrum assignment approach, Artstein's model reduction technique and the transport partial differential equation technique; see Fridman (2014) and references therein. Apart from the Smith predictor in the frequency domain, above predictive controllers always have an integral term. As it is claimed in Van Assche, Dambrine, Lafay, and Richard (1999), *the integral term discretization should be carefully executed in real applications, especially for open-loop unstable systems, because the bad discretization may make systems become unstable*. Therefore, it is beneficial to drop the integral term, not only for improving the computational efficiency, but also for not compromising the stability of the system. Besançon, Georges, and Benayache (2007) proposed the predictive observer approach without any integral term, which is followed in Najafi, Hosseinnia, Sheikholeslam, and Karimadini (2013) for constant IDs and in Léchappé, Moulay, and Plestan (2016) for time-varying IDs. For MASs, the problem considering heterogeneous IDs (HIDs) emerges and is challenging as Kronecker format dynamics for MASs cannot be constructed like in the case of homogeneous ones. To deal with this problem, there are mainly two methods: *(i)* One is the frequency-domain approach for constant delays which is, e.g., utilized in De, Sahoo, and Wahi (2018) with single/double integrator dynamics. *(ii)* The other, instead of dealing with HIDs for the whole MAS using the Kronecker product method, is transforming the consensus problem into studying a single agent with its own ID, see, e.g., in Xu, Liu, and Feng (2018) where only the constant ID is handled and a sum term related to historical input information is needed in the discrete-time dynamics which echoes the integral term in the continuous-time dynamics. To the authors' best knowledge, to deal with time-varying HIDs for general linear MASs is still an open challenge, which is the other motivation of this work.

There are also some works considering IDs and CDs simultaneously. For example, for constant IDs and CDs, see Ahmed et al. (2020), Jiang et al. (2021) and Tian and Liu (2008); for constant IDs but time-varying CDs, see Xu et al. (2018) and Zhou and Lin (2014). To the authors' best knowledge, there is no work dealing with time-varying IDs and CDs simultaneously, which is more realistic in real applications. In this paper, we aim to close this gap.

In this paper, time-varying HIDs and HCDs and external disturbances are addressed for multi-agent consensus tracking under a directed graph topology. Instead of designing an observer related to IDs and CDs together as in Jiang et al. (2021), the idea is to decouple IDs and CDs during controller construction by designing an only-CD-related observer such that the ID can be dealt with inside each agent separately using the LMI technique. The main contributions are as follows.

- This work can deal with time-varying HIDs and time-varying HCDs simultaneously.
- A larger upper bound of IDs is achieved by proposing a new objective-function-transformed LMI when optimizing existing LMIs. How to adjust this LMI to make unstable MASs become stable is also provided.
- The proposed controller is *fully* distributed (no global information is needed) compared to aforementioned works in the literature dealing with delays, such as, Ahmed et al. (2020), De et al. (2018), Sun and Wang (2009) and Zhou and Lin (2014) and without integral terms (which is computationally efficient and does not compromise stability).

2. Preliminaries and problem formulation

2.1. Notations and graph theory

Throughout this paper, $\mathbb{R}, \mathbb{R}^{m\times n}$ and $\mathbb{R}^n$ are the real number space, the $m\times n$ real matrix space and the n-dimensional Euclidean vector space, respectively. $\otimes$ is the Kronecker product and $\mathrm{diag}(a_1,\ldots,a_n)$ represents a diagonal matrix with diagonal elements $a_1,\ldots,a_n$. Matrices are assumed to have compatible dimensions if not explicitly stated. A matrix $A\in\mathbb{R}^{n\times n}$ is called Metzler if every off-diagonal entry of A is non-negative. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ represent the minimum and maximum eigenvalues of A, respectively. The square matrix $A\succ 0$ ($A\succeq 0$) means A is symmetric and positive (semi) definite. $L_\infty(a,b)$ is the space of essentially bounded functions $\phi:(a,b)\to\mathbb{R}^n$ with the norm $\|\phi\|_\infty=\operatorname{ess\,sup}_{\theta\in(a,b)}|\phi(\theta)|$. For a vector x, denote $\|x\|$ as its 2-norm. For any integer $a\le b$, denote $\mathbf{I}_a^b=\{a,a+1,\ldots,b\}$. Symmetric terms in symmetric matrices are denoted by $*$, e.g.,

$$\begin{bmatrix} A & B \\ * & C\end{bmatrix}=\begin{bmatrix} A & B \\ B^T & C\end{bmatrix}.$$

In a weighted graph $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$, $\mathcal{V}=\{1,2,\ldots,N\}$ and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ are respectively the nodes and edges. $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ is the weighted adjacency matrix with $a_{ij}=1,(i,j)\in\mathcal{E}$ and $a_{ij}=0$ otherwise. An edge $(i,j)\in\mathcal{E}$ means agent i can get information from j but not necessarily conversely. The Laplacian matrix $\mathcal{L}=\left[l_{ij}\right]\in\mathbb{R}^{N\times N}$ is defined as $l_{ij}=-a_{ij}, i\neq j$ and $l_{ii}=\sum_{j\neq i}a_{ij}$. A directed path from node i to j is a sequence of edges $(i,i_1),(i_1,i_2),\ldots,(i_h,j)$ in the directed network with distinct nodes $i_1,i_2,\ldots,i_h$. A digraph (i.e., directed graph) contains a directed spanning tree if there is a node from which a directed path exists to each other node.

2.2. System model

Consider a group of homogeneous MASs with N followers and the leader indexed by 0 as

$$\dot{x}_i(t)=Ax_i(t)+Bu_i(t-\tau_{u_i}(t))+\upsilon_i(t), i\in\mathbf{I}_1^N, \tag{1}$$

$$\dot{x}_0(t)=Ax_0(t), \tag{2}$$

where $x_i(t)=[x_{i1}(t),\ldots,x_{in}(t)]^T\in\mathbb{R}^n$ and $u_i(t)\in\mathbb{R}^p$ are respectively the state and input of the i-th follower and $x_0(t)\in\mathbb{R}^n$. The impact of an uncertain environment on each agent's dynamics is modeled by the exogenous disturbance $\upsilon_i(t)\in\mathbb{R}^n$ which is supposed to be locally essentially bounded, meaning that $\upsilon_i(t)\in L_\infty[0,t),\forall t>0$, i.e., $\|\upsilon_i\|_\infty\le\Delta_i$ with Δ_i being a priori given. (A,B) is controllable. Not all followers can receive the leader's state information. $\tau_{u_i}(t)$ is the unknown time-varying HID. Denote the CD from agent j to agent i as $\tau_{c_{ij}}(t)$ which can be heterogeneous and time-varying. $\tau_{u_i}(t)$ and $\tau_{c_{ij}}(t)$ satisfy the following assumptions.

Assumption 1. Input delays are upper bounded ($0\le\tau_{u_i}(t)\le\bar{\tau}_i$, $\bar{\tau}=\max_{i\in\mathbf{I}_1^N}\bar{\tau}_i$) and differentiable with their derivatives upper bounded ($\dot{\tau}_{u_i}(t)\le\hat{\tau}_i$, $\hat{\tau}=\max_{i\in\mathbf{I}_1^N}\hat{\tau}_i$).

Assumption 2. Each agent i knows the value of $\tau_{c_{ij}}(t)$ when it receives information from its neighbor agent j.

In several real-world applications, devices use timestamps at the transmitted packets. As a result, the receiving node i is able to measure the delay $\tau_{c_{ij}}(t)$ for a packet arriving from node j. Note that the assumption of known CDs appears in several works in the literature (see, e.g., Hou, Fu, Zhang, and Wu (2017), Jiang et al. (2021) and Zhou and Lin (2014) and references therein).

Assumption 3. Graph $\mathcal{G}$ contains a directed spanning tree in which the leader acts as the root node.

Then, the Laplacian matrix $\mathcal{L}$ of $\mathcal{G}$ can be partitioned as $\mathcal{L} = \begin{bmatrix} 0 & 0_{1\times N} \\ \mathcal{L}_2 & \mathcal{L}_1 \end{bmatrix}$, where $\mathcal{L}_2 \in \mathbb{R}^{N\times 1}$ and $\mathcal{L}_1 \in \mathbb{R}^{N\times N}$. We denote the multi-agent set with and without the leader as $\mathcal{N}$ and $\bar{\mathcal{N}}$, respectively. Based on (1) and (2), denote the consensus tracking error for follower i as $\tilde{x}_i(t) = x_i(t) - x_0(t)$ and we have

$$\dot{\tilde{x}}_i(t) = A\tilde{x}_i(t) + Bu_i(t - \tau_{u_i}(t)) + \upsilon_i(t), i \in \mathbf{I}_1^N. \tag{3}$$

In addition to homogeneous MASs, we also consider the heterogeneous ones as

$$\begin{aligned} \dot{x}_i(t) =& A_i x_i(t) + B_i u_i(t - \tau_{u_i}(t)) + \upsilon_i(t), \\ y_i(t) =& C_i x_i(t), i \in \mathbf{I}_1^N, \\ \dot{x}_0(t) =& A_0 x_0(t), y_0(t) = C_0 x_0(t), \end{aligned} \tag{4}$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $\upsilon_i(t) \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{p_i}$, $y_i(t) \in \mathbb{R}^q$ and $x_0(t) \in \mathbb{R}^n$, $y_0(t) \in \mathbb{R}^q$. (A_i, B_i) are controllable. C_i and C_0 are output matrices. Here, the reason to choose A_0 for the leader instead of A is for the presentation convenience. Other variables are the same as the ones in homogeneous MASs. We change $\tilde{x}_i(t) = x_i(t) - x_0(t)$ for homogeneous MASs as the output consensus error $\tilde{x}_i(t) = y_i(t) - y_0(t)$ for heterogeneous MASs.

Problem 1. Considering time-varying HIDs and HCDs, for any given initial states $x_i(0) \cup x_0(0)$, design a distributed controller to achieve the following objectives:

I: the tracking error $\tilde{x}_i(t)$ for homogeneous MASs (1) and (2) is exponentially stable if $\upsilon_i(t) \equiv 0, i \in \mathbf{I}_1^N$, and stays bounded if $\upsilon_i(t) \in L_\infty[0, t), \forall t > 0$;

II: the output consensus error $\tilde{x}_i(t)$ for heterogeneous MASs (4) stays bounded;

III: MASs can endure larger delays.

3. Communication-delay-related observer

In this section, the CD-related observer is the first step (also the key step), to address multi-agent consensus when the time-varying IDs and CDs are considered simultaneously. In the rest of this paper, for the convenience of presentation, we will omit the term (t) in $\tau_{c_{ij}}(t)$ or $\tau_{u_i}(t)$. When there exists no confusion, the variable t will be omitted, e.g., $x = x(t)$.

3.1. Observer & controller for homogeneous MASs

Assumption 4. The state matrix A for MASs has no eigenvalues with positive real parts.

In order to achieve consensus tracking, each follower should have knowledge about the leader's state information. Thus, design a distributed observer $\xi_i(t) \in \mathbb{R}^n$ to estimate $x_0(t)$ as

$$\begin{aligned} \dot{\xi}_i(t) =& A\xi_i(t) + \epsilon \sum_{j\in\bar{\mathcal{N}}, j\neq i} a_{ij}[e^{A\tau_{c_{ij}}}\xi_j(t - \tau_{c_{ij}}) - \xi_i(t)] \\ &+ \epsilon a_{i0}[e^{A\tau_{c_{i0}}} x_0(t - \tau_{c_{i0}}) - \xi_i(t)], i \in \mathbf{I}_1^N, \end{aligned} \tag{5}$$

where $0 < \epsilon \in \mathbb{R}$ is a constant and $\xi_i(t) = 0, t \leq 0$. $\xi_j(t - \tau_{c_{ij}})$ denotes the communication-delayed observer information from agent j to agent i, i.e., $\xi_j(t - \tau_{c_{ij}})$ means agent j sends its observer information $\xi_j(t)$ to the neighboring agent i via communication topology edge (i, j) which has communication delay $\tau_{c_{ij}}$. The same holds for the leading agent $x_0(t - \tau_{c_{i0}})$. Denote the observer estimating error as $\tilde{\xi}_i = \xi_i - x_0$.

Remark 1. From the construction of observer (5), agent i does not need to use a delayed value of its state, unlike, e.g., Hou et al. (2017), Jiang et al. (2021) and Zhou and Lin (2014) in which their results would not be feasible if an agent does not use a delayed value of its state. Since the receiving node i is able to measure the delay $\tau_{c_{ij}}$ for a packet arriving from node j (Assumption 2), it is able to calculate observer (5).

Lemma 1. *Under Assumptions 2–4 and $\epsilon > 0$, the observer estimating error yields $\lim_{t\to\infty} \tilde{\xi}_i(t) = 0$ exponentially.*

Proof. See Appendix A.[1] ■

Now, the control input is chosen to be of the form as

$$u_i(t) = K(x_i(t) - \xi_i(t)), i \in \mathbf{I}_1^N, \tag{6}$$

where the controller gain matrix $K \in \mathbb{R}^{p\times n}$ will be designed later. Based on $u_i = K(x_i - \xi_i - x_0 + x_0) = K(\tilde{x}_i - \tilde{\xi}_i)$, integrating the above equation into (3) gives

$$\dot{\tilde{x}}_i = A\tilde{x}_i + BK\tilde{x}_i(t - \tau_{u_i}) + \upsilon_i - BK\tilde{\xi}_i(t - \tau_{u_i}). \tag{7}$$

We regard the term $\upsilon_i(t) - BK\tilde{\xi}_i(t - \tau_{u_i})$ as the disturbance to the error dynamics (7). As $\upsilon_i(t) \in L_\infty[0, \infty)$ in (1) and $\lim_{t\to\infty} \tilde{\xi}_i(t) = 0$ in Lemma 1, $(\upsilon_i(t) - BK\tilde{\xi}_i(t - \tau_{u_i})) \in L_\infty[0, \infty)$. Since (7) is only related to agent index i, thus, i will be omitted in the following. Therefore, denote $\zeta := \tilde{x}_i$, $\tau(t) := \tau_{u_i}(t)$, $\varpi := \upsilon_i - BK\tilde{\xi}_i(t - \tau_{u_i})$ and $\hat{\tau} := \hat{\tau}_i$ such that $\dot{\tau}(t) \leq \hat{\tau}$ from Assumption 1. Then, the transformed error dynamics is

$$\dot{\zeta}(t) = A\zeta(t) + BK\zeta(t - \tau(t)) + \varpi(t). \tag{8}$$

3.2. Observer & controller for heterogeneous MASs

Results in Assumption 4 is restrictive for all followers and the leader. However, for relaxing this assumption, one way is that followers and the leader should have different state matrix A, i.e., the system transforms to heterogeneous MAS as in (4). Then, the following assumption based on output regulation theory in Huang (2004) is needed.

Assumption 5. There exist solutions (X_i, U_i) for each follower i to the following linear matrix equations:

$$X_i A_0 = A_i X_i + B_i U_i, C_0 = C_i X_i, i \in \mathbf{I}_1^N. \tag{9}$$

Assumption 6. Eigenvalues of the leader's state matrix A_0 have one of the following properties: (i) negative real parts; (ii) zero real part but are simple, i.e., eigenvalues on the imaginary axis are all distinct from one another.

Based on Assumption 5, Assumption 4 can be relaxed to Assumption 6 in which the leader dynamics is asymptotically stable or marginally stable [Theorem 8.1, Hespanha (2018)]. The motivation behind this is that several real-world scenarios may involve follower state dynamics A_i that are open-loop unstable (e.g., fight aircrafts). The distributed observer in (5) is thus changed to

$$\begin{aligned} \dot{\xi}_i(t) =& A_0\xi_i(t) + \epsilon \sum_{j\in\bar{\mathcal{N}}, j\neq i} a_{ij}[e^{A_0\tau_{c_{ij}}}\xi_j(t - \tau_{c_{ij}}) - \xi_i(t)] \\ &+ \epsilon a_{i0}[e^{A_0\tau_{c_{i0}}} x_0(t - \tau_{c_{i0}}) - \xi_i(t)], i \in \mathbf{I}_1^N \end{aligned} \tag{10}$$

with $\xi_i(t) \in \mathbb{R}^n$. The difference is the replacement of A in (5) to A_0 in (10). Therefore, $\lim_{t\to\infty} \tilde{\xi}_i(t) = 0$ is still valid under Assumptions 2, 3 and 6 with the necessary condition for the

[1] Details of proofs and simulation setting can be found in our technical report (Jiang, Liu, & Charalambous, 2021).

positive parameter ϵ as $\mathrm{Re}(\lambda(I_N \otimes A_0 - \mathcal{L}_1 \otimes (\epsilon I_n))) < 0$. We redesign the control input as

$$u_i(t) = K_i^1 x_i(t) - K_i^2 \xi_i(t), i \in \mathbf{I}_1^N, \tag{11}$$

where the controller gain matrices $K_i^2 = U_i - K_i^1 X_i$ and $K_i^1 \in \mathbb{R}^{p_i \times n_i}$ will be designed later. Denote $\bar{x}_i = x_i - X_i x_0$. Based on (9), we have

$$\tilde{x}_i = y_i - y_0 = C_i(\bar{x}_i + X_i x_0) - C_0 x_0 = C_i \bar{x}_i, \tag{12}$$

which means the output consensus error $\tilde{x}_i$ is dependent on the term $\bar{x}_i$. Based on Eqs. (4), (9) and (11), the derivative of $\bar{x}_i$ is calculated as

$$\begin{aligned}\dot{\bar{x}}_i =& A_i \bar{x}_i + B_i K_i^1 \bar{x}_i(t - \tau_{u_i}) + v_i - B_i K_i^2 \tilde{\xi}_i(t - \tau_{u_i}) \\ &- B_i U_i(x_0(t) - x_0(t - \tau_{u_i})).\end{aligned} \tag{13}$$

One can see (13) has a similar math format as (7). Similarly, denote $\zeta := \bar{x}_i$, $\tau(t) := \tau_{u_i}(t)$, $\varpi := v_i - B_i K_i^2 \tilde{\xi}_i(t - \tau_{u_i}) - B_i U_i(x_0(t) - x_0(t - \tau_{u_i}))$, $\hat{\tau} := \hat{\tau}_i$ and $A := A_i, B := B_i, K := K_i^1$. Then, (13) transforms to (8). We should verify $\varpi \in L_\infty[0, \infty)$.

Denote $\varrho_i(t) = x_0(t) - x_0(t - \tau_{u_i})$; then, from (4) it is easy to get $\dot{\varrho}_i = A_0 \varrho_i + \dot{\tau}_{u_i} A_0 x_0(t - \tau_{u_i})$. Denote another augmented variable $\bar{\varrho}_i(t) = [\varrho_i^T(t), x_0^T(t - \tau_{u_i})]^T$ and we have

$$\dot{\bar{\varrho}}_i = \begin{bmatrix} A_0 & \dot{\tau}_{u_i} A_0 \\ 0 & A_0 \end{bmatrix} \bar{\varrho}_i = (\begin{bmatrix} 1 & \dot{\tau}_{u_i} \\ 0 & 1 \end{bmatrix} \otimes A_0) \bar{\varrho}_i.$$

Based on the fact that given the eigenvalues of $S \in \mathbb{R}^{n \times n}$ and $T \in \mathbb{R}^{m \times m}$ are $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$, respectively, then the eigenvalues of $S \otimes T$ are $\lambda_i \mu_j, i = 1, \ldots, n, j = 1, \ldots, m$, one can verify that the stability of $\bar{\varrho}_i(t)$ is determined by the eigenvalues of A_0. Thus, based on Assumption 6, $\bar{\varrho}_i(t)$ is asymptotically or marginally stable, i.e., $\varrho_i(t)$ is bounded. Therefore, $\varpi(t) \in L_\infty[0, \infty)$ is still valid here. Finally, K_i^1 will be designed as K in (8) such that $\bar{x}_i(t)$ is bounded. Based on (12), $\tilde{x}_i(t)$ for heterogeneous MASs (4) will be accordingly also bounded.

4. Stability analysis

From the previous section, one can see that by taking advantage of designing the only CD-related observer $\xi_i(t)$ for either homogeneous or heterogeneous MASs, both the original cooperative consensus tracking problem is transformed into the input-to-state stability problem of "single agent system" (8) involving only the time-varying ID. The LKF with descriptor method (Fridman, 2014) will be adopted to design K. Inspired by the work of Fridman (2014), design one type of LKF as

$$\begin{aligned}V =& \zeta^T P \zeta + \int_{t-\bar{\tau}}^{t} e^{2\delta(s-t)} \zeta^T(s) S \zeta(s) \mathrm{d}s \\ &+ \int_{t-\tau(t)}^{t} e^{2\delta(s-t)} \zeta^T(s) Q \zeta(s) \mathrm{d}s \\ &+ \bar{\tau} \int_{-\bar{\tau}}^{0} \int_{t+\theta}^{t} e^{2\delta(s-t)} \dot{\zeta}^T(s) R \dot{\zeta}(s) \mathrm{d}s \mathrm{d}\theta,\end{aligned}$$

where $0 \leq \tau(t) := \tau_{u_i} \leq \bar{\tau}$ from Assumption 1, $\delta > 0$ is a constant and matrices $P \succ 0, Q \succeq 0, R \succ 0, S \succ 0$. Denote a scaler $\gamma > 0$ and

$$W = \dot{V} + 2\delta V - \varpi^T \gamma \varpi. \tag{14}$$

The calculation of W is presented in Appendix B. Following the Proposition 1 in Fridman and Dambrine (2009), if there exist $\delta > 0, \gamma > 0$ and matrices $\{P, S, R\} \succ 0, Q \succeq 0$ such that along the trajectories of (8), the LKF satisfies the condition $W < 0$ (i.e., the matrix inequality Φ_1 in (B.2) satisfies $\Phi_1 \prec 0$ and (B.3) is feasible), then, the solution of error dynamics (8) satisfies

$$\begin{aligned}\zeta^T(t) P \zeta(t) \leq& e^{-2\delta t} \zeta^T(0) P \zeta(0) \\ &+ (1 - e^{-2\delta t}) \frac{\gamma}{2\delta} \|\varpi[0, t]\|_\infty^2, t > 0.\end{aligned} \tag{15}$$

Remark 2. The reason for adopting the descriptor method (P_2, P_3 in (B.1)) is that the controller parameter K can be designed conveniently and that some comparison simulations in Section 6.1.3 of Jiang (2018) shows that the closed-loop system can endure larger delays with the descriptor method used. It also shows that there is a trade-off between the exponential convergence rate δ and upper bound $\bar{\tau}$: the larger the rate δ, the smaller the upper bound $\bar{\tau}$. $Q = 0$ means the system can endure the fast-varying delay (i.e., $\tau_{u_i}(t)$ does not have any constraints on the delay derivative, e.g., $\dot{\tau}_{u_i}(t) \geq 1$) as the derivative upper bound $\hat{\tau}$ will disappear in $\Phi_1(4, 4)$ (Fridman, 2014).

Lemma 2. *Under Assumptions 1–4 (or Assumptions 1–3, 5–6), given $\bar{\tau} \geq 0, \hat{\tau} \in [0, 1), \delta > 0, \gamma > 0$ and $\varepsilon \in \mathbb{R}$, if there exist $n \times n$ matrices $\{\bar{P}, \bar{Q}, \bar{R}, \bar{S}\} \succ 0$ ($\bar{Q} = 0$ for including the case of $\hat{\tau} \geq 1$), $\{\bar{S}_{12}, M\} \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{p \times n}$ such that the following LMIs are feasible:*

$$\Phi_2 = \begin{bmatrix} \Phi_2(1,1) & \Phi_2(1,2) & e^{-2\delta\bar{\tau}} \bar{S}_{12} & \Phi_2(1,4) & I_n \\ * & \Phi_2(2,2) & 0 & \varepsilon BY & \varepsilon I_n \\ * & * & \Phi_2(3,3) & \Phi_2(3,4) & 0 \\ * & * & * & \Phi_2(4,4) & 0 \\ * & * & * & * & -\gamma I_n \end{bmatrix} \prec 0, \tag{16}$$

$$\Phi_2' = \begin{bmatrix} \bar{R} & \bar{S}_{12} \\ * & \bar{R} \end{bmatrix} \succeq 0, \tag{17}$$

$$\begin{aligned}\Phi_2(1,1) =& 2\delta\bar{P} + \bar{S} + \bar{Q} - e^{-2\delta\bar{\tau}} \bar{R} + AM + M^T A^T, \\ \Phi_2(1,2) =& \bar{P} - M + \varepsilon M^T A^T, \Phi_2(3,4) = e^{-2\delta\bar{\tau}}(\bar{R} - \bar{S}_{12}^T), \\ \Phi_2(1,4) =& BY + e^{-2\delta\bar{\tau}}(\bar{R} - \bar{S}_{12}), \\ \Phi_2(2,2) =& \bar{\tau}^2 \bar{R} - \varepsilon M^T - \varepsilon M, \Phi_2(3,3) = -e^{-2\delta\bar{\tau}}(\bar{S} + \bar{R}), \\ \Phi_2(4,4) =& -(1 - \hat{\tau}) e^{-2\delta\bar{\tau}} \bar{Q} + e^{-2\delta\bar{\tau}}(-2\bar{R} + \bar{S}_{12} + \bar{S}_{12}^T),\end{aligned}$$

then, objective I (II) of Problem 1 is solved by the distributed controller consisting of input (6) and observer (5) (input (11) and observer (10)). The controller gain matrix is thus designed as $K = YM^{-1}$ ($K_i^1 = YM^{-1}, K_i^2 = U_i - K_i^1 X_i$, (X_i, U_i) is the solution to the output regulation equation (9)).

Proof. For objective I, recalling $\zeta := \tilde{x}_i$, $\varpi := v_i - BK\tilde{\xi}_i(t - \tau_{u_i})$ and $\lim_{t\to\infty} \tilde{\xi}_i(t) = 0$ exponentially, based on (15), if matrix inequalities $\Phi_1 < 0$ (B.2) and (B.3) are feasible, then $\lim_{t\to\infty} \tilde{x}_i(t) = 0$ exponentially if $v_i(t) \equiv 0$; otherwise,

$$\|\tilde{x}_i(t)\|^2 \leq \frac{\gamma}{2\delta\lambda_{\min}(P)} \Delta_i^2, t \to \infty. \tag{18}$$

For objective II, similarly, as $t \to \infty$,

$$\|\tilde{x}_i(t)\|^2 \leq \frac{\gamma \|C_i\|^2}{2\delta\lambda_{\min}(P)} (\Delta_i^2 + \|B_i U_i(x_0 - x_0(t - \tau_{u_i}))\|^2). \tag{19}$$

The problem left is to calculate the controller gain matrix K. Recall that the decay rate δ, which is related to $\bar{\tau}$ in Remark 2, is a system convergence requirement and should be set in advance, meaning δ is known. As K in (8) is unknown, Φ_1 contains two nonlinear terms: $P_2^T BK$ and $P_3^T BK$, since P_2, P_3 are also unknown. From descriptor method, by setting $P_3 = \varepsilon P_2$, nonlinear matrix inequality Φ_1 has then one nonlinear term $P_2^T BK$. Denote $M := P_2^{-1}, \bar{P} := M^T PM, \bar{Q} := M^T QM, \bar{R} := M^T RM, \bar{S} := M^T SM, \bar{S}_{12} := M^T S_{12} M$. Note that from the construction of $\Phi_2(2,2)$, the feasibility of Φ_2 guarantees that M or P_2 is positive definite. Then, inspired from Liu, Fridman, and Xia (2020), multiplying Φ_1 in (B.2) by $\mathrm{diag}(M^T, M^T, M^T, M^T, I_n)$ and $\mathrm{diag}(M, M, M, M, I_n)$ from the left and right side, respectively, and denoting $Y = KM$, Φ_1 in (B.2) is linearized as LMI Φ_2 in (16). Similarly, Φ_1' in (B.3) transforms to Φ_2' in (17). After M and Y are calculated through LMIs (16) and (17), one has $K = YM^{-1}$.

For objective II in heterogeneous MASs, we give each matrix (not scalars) with a subscript i, e.g., replacing A, P as $A_i, P_i, i \in \mathbf{I}_1^N$ and solve all LMIs together to get M_i, Y_i to calculate K_i^1. ■

The application of Lemma 2 is summarized as follows.

Algorithm 1. Controller design for upper bound $\bar{\tau}$

1: **Input:** Delay derivative bound $\hat{\tau}$ and decay rate δ based on system specifications (δ determines how fast (exponentially) the MAS converges).
2: **Initialization:** Set parameters ϵ in (5) and ε.
3: Define $\bar{P}, \bar{Q}, \bar{R}, \bar{S}, \bar{S}_{12}$ and γ, M, Y as variables.
4: Solve LMIs (16) and (17) to get M and Y.
5: **Output:** Compute input parameter $K = YM^{-1}$.

5. Delay size analysis

This section is for objective III of Problem 1, i.e., analyzing how to get an improved upper bound $\bar{\tau}$. As stated in Remark 2, the larger the rate δ, the smaller the upper bound $\bar{\tau}$. Herein, δ is predefined and fixed.

Different delay-dependent LMI conditions (e.g., LMIs (16), (17)) are successfully proposed to prove the stability of single-agent system for different control scenarios; see, e.g., Besançon et al. (2007), Fridman (2014) and Najafi et al. (2013). However, the aforementioned methods do not perform a delay size analysis and the derived conditions are usually restricted to relatively small delays.

Note that LMIs (16) and (17) in Lemma 2 are the only sufficient conditions for MAS stability. The main idea to get an improved upper bound $\bar{\tau}$ is by using the following reasoning. If the upper bound $\bar{\tau}$ is so large that LMIs (16) and (17) are not feasible, then the tracking error would possibly diverge, i.e., $\|\tilde{x}_i(t)\|$ in (18)/(19) would possibly diverge. It may also mean, in this situation, that the value of $\gamma/(2\delta\lambda_{\min}(P))$ would be very large (check the analysis of Fig. 2(d)–(f)). So, it is easily deduced that the objective of minimizing $\gamma/(2\delta\lambda_{\min}(P))$ and keeping $\|\tilde{x}_i(t)\|$ in (18)/(19) bounded simultaneously is preferred. However, the eigenvalue function of unknown LMI variables (e.g., P) is not available, thus some transformation is needed. In addition to the predefined δ, only $\lambda_{\min}(P)$ needs attention. On the contrary to the common setting of $P \succ 0$, design $P \succeq I_n/\mu, \mu > 0$ such that $\lambda_{\min}(P) \geq 1/\mu$. Then, $\gamma/(2\delta\lambda_{\min}(P)) \leq (2\gamma\mu)/(4\delta)$. In this way, based on the law $2\gamma\mu \leq \gamma^2 + \mu^2$, the minimizing objective can change to an optimization constraint, i.e., by asking

$$\frac{\gamma}{2\delta\lambda_{\min}(P)} \leq \frac{2\gamma\mu}{4\delta} \leq \frac{\gamma^2+\mu^2}{4\delta} \leq \chi, \chi > 0$$

with χ predefined, the constraint becomes

$$\gamma^2 + \mu^2 - 4\delta\chi \leq 0 \tag{20}$$

with adding μ as an LMI variable, which can be transformed into an LMI, i.e.,

$$\Phi_3 = \begin{bmatrix} 4\delta\chi & \gamma & \mu \\ \gamma & 1 & 0 \\ \mu & 0 & 1 \end{bmatrix} \succeq 0. \tag{21}$$

One can see (20) can be transformed as $\|[\gamma, \mu]^T\|_2 \leq 2\sqrt{\delta\chi}$, which is a second-order cone constraint. As semi-definite programming contains second-order cone programming, a second-order cone constraint (20) can be written as an LMI (21).

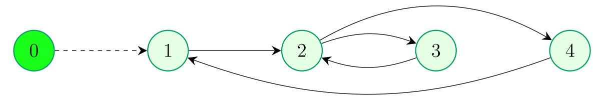

Fig. 1. The digraph $\mathcal{G}$ satisfying Assumption 3.

From $M = P_2^{-1}, \bar{P} = M^T P M$ and $P \succeq I_n/\mu$ we have $\bar{P} - M^T(I_n/\mu)M \succeq 0$. In addition to $\mu I_n \succ 0$, by Schur complement lemma, condition $P \succeq I_n/\mu, \mu > 0$ can be transformed into

$$\Phi_4 = \begin{bmatrix} \bar{P} & M^T \\ * & \mu I_n \end{bmatrix} \succeq 0, \mu > 0. \tag{22}$$

Algorithm 2. Controller design for larger upper bound $\bar{\tau}$.

1: Same steps 1–2 in Algorithm 1 and $\chi > 0$ in (20).
2: Define $\bar{P}, \bar{Q}, \bar{R}, \bar{S}, \bar{S}_{12}$ and γ, M, Y, μ as variables.
3: Solve LMIs (16), (17), (21) and (22) to get M and Y by tuning the value of χ.
4: **Output:** Compute input parameter $K = YM^{-1}$.

Theorem 1. *Based on Lemma 2, additionally $\chi > 0$, by tuning χ if LMIs (16), (17), (21)* and *(22) are feasible, then, for Problem 1, the objectives I, II are solved and the MAS can endure larger delay upper bound $\bar{\tau}$ compared to Lemma 2, i.e., objective III is solved.*

Proof. The proof follows the same lines of the one of Lemma 2. The difference is that now, instead of $P \succ 0$, we add LMIs (21) and (22) here. The intuition explanation is that by adding an objective-function-transformed constraint (i.e., (20)) on the right-hand side of consensus tracking error $\|\tilde{x}_i(t)\|$ (18)/(19), and by tuning the value of χ which provides a freedom to control the bound of $\gamma/(2\delta\lambda_{\min}(P))$ compared to Algorithms 1 which cannot control that bound, the value of $\|\tilde{x}_i(t)\|$ will be more difficult to diverge or become large, i.e., the system could endure larger delay size. ■

Remark 3. Algorithms 1–2 can be applied to single agent system. Unlike Sun and Wang (2009), the dimension of proposed LMIs is not related to agent number N or delay number n, thus will not increase when N or n increases. It means Algorithms 1–2 are also scalable to a large number of agents. Unlike Algorithm 1 in which K is calculated and is fixed for a given delay upper bound $\bar{\tau}$, i.e., the controller is fixed, Algorithm 2 allows for calculating different K for a fixed $\bar{\tau}$ by tuning χ. As aforementioned, χ is tuned for controlling the upper bound of $\gamma/(2\delta\lambda_{\min}(P))$, i.e., tuning χ offers the freedom for Algorithm 2 to design K and then the controller. The χ tuning mechanism is described for different $\bar{\tau}$ in Fig. 2(d)–(f) and for a fixed $\bar{\tau}$ in Fig. 3.

6. Simulations

Heterogeneous MASs are considered here with the graph $\mathcal{G}$ shown in Fig. 1. Set the dynamics of agents 1 and 4 as the platooning dynamics in simulation of Jiang et al. (2021); set agents 2 and 3 respectively as the linearized mobile vehicle and the Caltech wireless tested vehicle in simulation of Jiang, Wen, Peng, Huang, and Rahmani (2019). Set the output matrix as $C_1 = C_4 = [I_2, 0_{2\times1}], C_2 = [I_2, 0_{2\times2}], C_3 = [I_2, 0_{2\times4}]$. Denote $\bar{v} = [\sin(10t), \cos(10t), \sin(20t)]^T$ and set the disturbances as $v_i(t) = 0, t \in [0, 200), i \in \mathbf{I}_1^4$; $v_1(t) = 13\bar{v}, v_2(t) = [2\bar{v}; 0], v_3(t) = [3\bar{v}; 0; 0; 0], v_4(t) = \bar{v}, t \in [200, 400]$. Set $\hat{\tau} = 0.8, \delta = 0.1, \varepsilon = 0.3, \epsilon = 0.3$. Initial conditions are randomly set and $u_i(t) = 0, t \in [-\bar{\tau}, 0], i \in \mathbf{I}_1^4$. Set the communication delays as $\tau_{c_{10}} = 6 + \sin(0.5t), \tau_{c_{21}} = 6 + 2\sin(0.4t), \tau_{c_{23}} = \tau_{c_{32}} =$

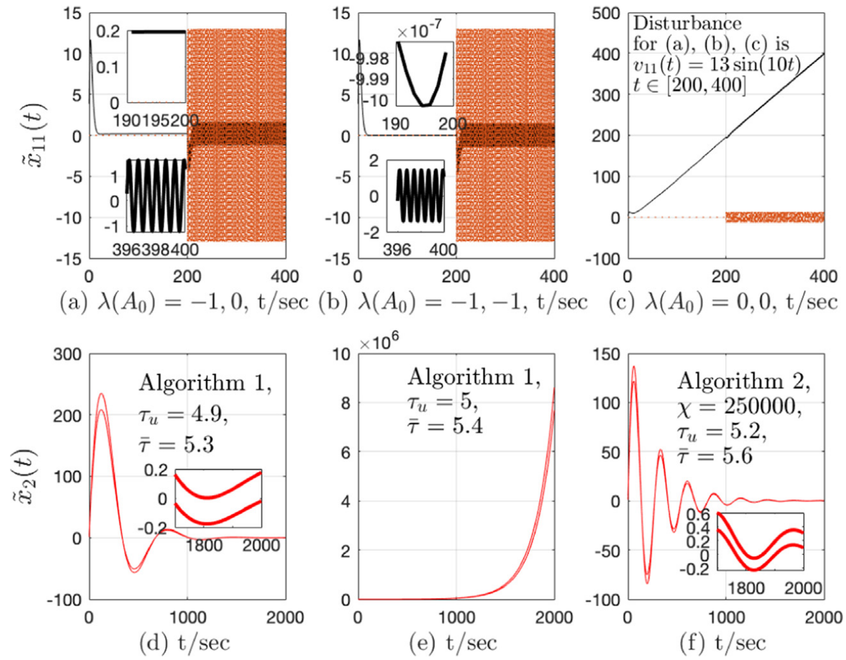

Fig. 2. (a), (b), (c) depict a comparison of the first dimension of output consensus tracking error of agent 1 for different leader dynamics A_0 using Algorithm 1 with $\tau_u = 0.5, \bar{\tau} = 0.9$. (d), (e), (f) depict a comparison of output consensus tracking error of agent 2 related to the input delay bound $\bar{\tau}$ for Algorithms1–2.

$6 + 3\sin(0.5t)$, $\tau_{c_{42}} = 6 + 4\sin(2t)$, $\tau_{c_{14}} = 6 + 5\sin(0.1t)$. Set the input delays as $\tau_{u_1} = \tau_u + 0.1\cos(0.5t)$, $\tau_{u_2} = \tau_u + 0.2\sin(0.5t)$, $\tau_{u_3} = \tau_u + 0.3\cos(0.1t)$, $\tau_{u_4} = \tau_u + 0.4\sin(0.5t)$ with $\tau_u \geq 0.4$ guaranteeing $\tau_{u_i} \geq 0$, $i \in \mathbf{I}_1^4$. It also means $\bar{\tau} = \tau_u + 0.4$ which satisfies $\bar{\tau} \geq \tau_{u_i}$. In the following, by changing the value of τ_u, the upper bound $\bar{\tau}$ can be found and comparison simulations can be provided.

To verify Assumption 6, we set A_0 as marginally stable, asymptotically stable and unstable, respectively as follows:

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The eigenvalues of A_0 are correspondingly shown in Fig. 2(a)–(c). The solutions (X_i, U_i) to the output regulation equation (9) are thus obtained by using [Lemma 4, Cai, Lewis, Hu, and Huang (2017)]. Fig. 2(a)–(c) demonstrate that Assumption 6 is precise and Algorithm 1 is available for MAS consensus tracking control as (i) errors are bounded with effects of external disturbances attenuated during $t \in [200, 400]$; (ii) when $t \in [0, 200)$ without disturbances, $\tilde{x}_{11}(t)$ is bounded (equals 0.2) in (a), zero in (b) and unbounded in (c) which verifies (19).

In the following, to better present the performance comparison, we neglect the disturbance performance, i.e., we set $v_i(t) = 0$. Fig. 2(d)–(f) show that Algorithm 2 can help systems endure larger input delay size. The value setting mechanism of χ in the proposed objective-function-transformed constraint (20) is as follows. In Fig. 2(e), we get $\gamma/(2\delta\lambda_{\min}(P)) = 9.7842 \times 10^3$ from Algorithm 1 for agent 2 which becomes unstable first. So we should choose $\chi > 9.7842 \times 10^3$, e.g., $\chi = 2.5 \times 10^5$ in Fig. 2(f). In fact, it is stable for $\chi = 1 \times 10^5$ with $\bar{\tau} = 5.5$, but not stable with $\bar{\tau} = 5.6$, then we choose $\chi = 2.5 \times 10^5$. The reason that the output tracking error in Fig. 2(e) becomes unbounded is because $\bar{\tau} = 5.4$ is too large for the whole MAS to remain stable under the controller designed using Algorithm 1. This point also proves the superior performance of Algorithm 2 with the freedom of tuning the value of χ to allow for a larger value of $\bar{\tau}$ to be endured by the MAS.

Fig. 3 gives more details about analyzing the influence of constraint (20). When $\chi = 1$ in Fig. 3(a), MAS is unstable, which may be due to the too strong constraint for LMI variables γ and μ in $(\gamma^2 + \mu^2)/(4\delta) \leq 1$. Then, this constraint is relieved in Fig. 3(b)–(f) where one can see MAS becomes stable gradually. Note that further alleviation of the constraint is not very helpful as there is nearly no performance difference between Fig. 3(e) and (f). This finding also verifies (18)/(19).

Remark 4. Fig. 2(e)–(f) and Fig. 3 show when the closed-loop system is on the edge of stable/unstable state with delays, adjusting the constraint of $(\gamma^2 + \mu^2)/(4\delta) \leq \chi$ can improve the system ability of keeping stable, i.e., tuning χ provides a freedom for a system to endure a larger delay size. From Fig. 3 and Fig. 2(f), a basic rule is that for a larger delay upper bound $\bar{\tau}$, a larger χ is required. For a fixed χ, a fast way of finding a good bound for the delay is to use the bisection algorithm.

7. Conclusions and future directions

This paper can address the heterogeneous and time-varying input and communication delays simultaneously by decoupling them when designing observers and using the Lyapunov–Krasovskii functional. A new linear matrix inequality (LMI) is added to existing LMIs to construct an extended LMI approach to help an agent/a system endure larger delays compared to existing LMI solutions. Detailed analysis on how to obtain a larger delay upper bound and better robust control performance are also provided. The proposed controllers and algorithms are without any integral term and thus, can be easily implemented in real applications. It is also easy to apply the proposed theory to large-scale systems because (i) there is no requirement for global information, e.g., the eigenvalues of communication graph; (ii) the LMI dimension will not increase as agent number and delay number increase.

Future work will focus on designing observers that the state matrix can have eigenvalues with positive real parts and that heterogeneous time-varying communication delays can be unknown.

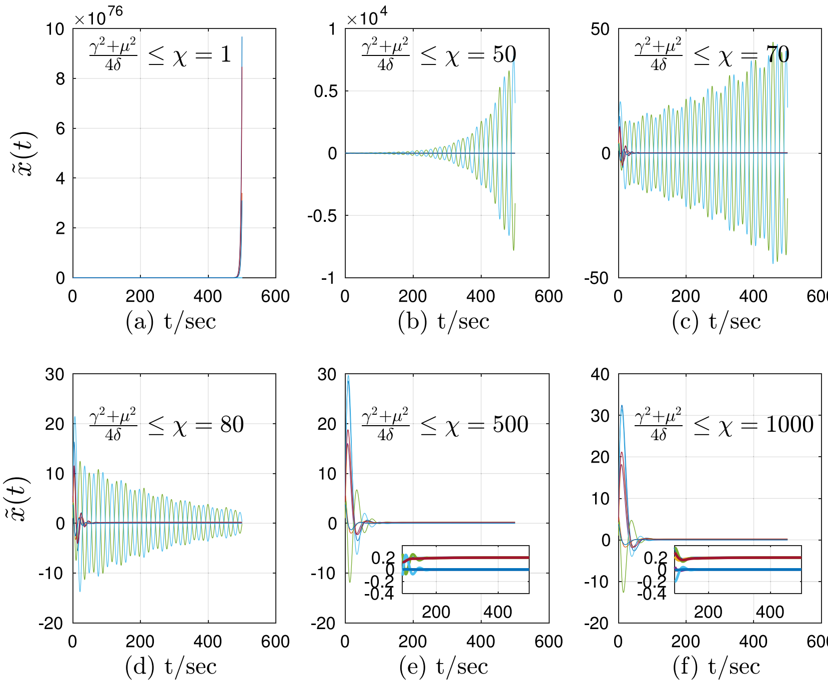

Fig. 3. Influence of the value of χ to MAS stability for Algorithm 2 with a fixed delay upper bound $\bar{\tau} = 2.4$.

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Appendix A. Proof of Lemma 1

Lemma 3 (*Theorem 2, Moreau, 2004*)**.** *Consider the linear system Θ : $\dot{x}(t) = \underline{diag}(A(t))x(t) + (A(t) - \underline{diag}(A(t)))x(t-\tau)$ with $x \in \mathbb{R}^n$, $\tau > 0$ and $\underline{diag}(A(t))$ is the obvious notation for the diagonal matrix obtained from $A(t)$ by setting all off-diagonal entries equal to zero. Assume that the system matrix $A(t)$ is a bounded and piecewise continuous function of time. Assume that, for every time t, the system matrix is Metzler with zero row sums. Under Assumption 3, the equilibrium set of consensus states is uniformly exponentially stable. In particular, all components of any solution of system Θ converge to a common value as $t \to \infty$.*

Lu and Liu (2017) extended Lemma 3 to the case of time-varying delays, i.e., $\tau_{ij}(t)$ in [Lemma 3.1, Lu & Liu, 2017]. Denote $w_0 := e^{-At}x_0$, $w_i := e^{-At}\xi_i$, $i \in \mathbf{I}_1^N$. Then, from (2), (5) and based on $e^{A\tau_{c_{ij}}}x_0(t-\tau_{c_{ij}}) = x_0(t)$, we get

$$\dot{w}_0(t) = -Ae^{-At}x_0(t) + e^{-At}\dot{x}_0(t) = 0, \tag{A.1}$$

$$\dot{w}_i(t) = -\epsilon \sum_{j\in\mathcal{N}, j\neq i} l_{ij}w_j(t-\tau_{c_{ij}}) - \epsilon l_{ii}w_i(t). \tag{A.2}$$

Denote $\bar{w}_j(t) = [w_{0j}(t), w_{1j}(t), \ldots, w_{Nj}(t)]^T$ for each dimension $j, j \in \mathbf{I}_1^n$. Then, we have

$$\dot{\bar{w}}_j(t) = \underline{diag}(-\epsilon\mathcal{L})\bar{w}_j(t) + [-\epsilon\mathcal{L} - \underline{diag}(-\epsilon\mathcal{L})]\bar{w}_j(t-\tau_c)$$

where $\bar{w}_j(t-\tau_c)$ represents the corresponding delay term. Based on the definition of Laplacian matrix $\mathcal{L}$, from Assumption 3, one can see $-\epsilon\mathcal{L}$ is Metzler with zero row sums. Thus, based on Lemma 3 with its extension for time-varying delay case in [Lemma 3.1, Lu & Liu, 2017], we obtain $w_{0j}(t) = w_{1j}(t) = \cdots = w_{Nj}(t), j \in \mathbf{I}_1^n$ exponentially as $t \to \infty$, which also means $w_0(t) = w_1(t) = \cdots = w_N(t), t \to \infty$. Based on definitions of $w_0(t)$ and $w_i(t)$, we arrive at $\lim_{t\to\infty} e^{-At}(\xi_i(t) - x_0(t)) = \lim_{t\to\infty} e^{-At}\tilde{\xi}_i(t) = 0$.

Denote $\tilde{w}_i(t) := e^{-At}\tilde{\xi}_i(t)$, $\tilde{w}(t) = [\tilde{w}_1^T(t), \ldots, \tilde{w}_N^T(t)]^T$. The following follows the proof of Lemma 3.3 in Lu and Liu (2017). Since $\lim_{t\to\infty}\tilde{w}(t) = 0$ exponentially, there exist positive real numbers α_1 and β such that

$$\begin{aligned}\|\tilde{w}(t)\| \le& \alpha_1 e^{-\beta t} \sup_{\theta\in[-\tau,0]} \|\tilde{w}_t(\theta)\| \\ =& \alpha_1 e^{-\beta t} \sup_{\theta\in[-\tau,0]} \|(I_N \otimes e^{-A\theta})\tilde{\xi}(\theta)\| \\ \le& \alpha e^{-\beta t},\end{aligned}$$

where α is some positive real number. Based on Assumption 4 that $\mathrm{Re}(\lambda(A)) \le 0$, there is a polynomial $\Gamma(t)$ such that $\|(I_N \otimes e^{At})\| \le \Gamma(t)$. Then, $\|\tilde{\xi}(t)\| \le \|(I_N \otimes e^{At})\|\|\tilde{w}(t)\| \le \alpha e^{-\beta t}\Gamma(t)$. Therefore, $\lim_{t\to\infty}\tilde{\xi}(t) = 0$ exponentially. ■

Appendix B. Calculation of *W* in (14)

Based on (8) and the LKF in Section 4, we have

$$\begin{aligned}W =& \Xi - (1-\dot{\tau}(t))e^{-2\delta\tau(t)}\zeta^T(t-\tau(t))Q\zeta(t-\tau(t)) \\ &- \bar{\tau}\int_{t-\bar{\tau}}^{t} e^{2\delta(s-t)}\dot{\zeta}^T(s)R\dot{\zeta}(s)ds \\ \le& \Xi - (1-\hat{\tau})e^{-2\delta\bar{\tau}}\zeta^T(t-\tau(t))Q\zeta(t-\tau(t)) \\ &- \bar{\tau}e^{-2\delta\bar{\tau}}\int_{t-\bar{\tau}}^{t} \dot{\zeta}^T(s)R\dot{\zeta}(s)ds \\ &+ 2[\zeta^T P_2^T + \dot{\zeta}^T P_3^T][A\zeta + BK\zeta(t-\tau(t)) + \varpi - \dot{\zeta}] \\ \le& \bar{\zeta}^T\Phi_1\bar{\zeta}\end{aligned} \tag{B.1}$$

with $\Xi = 2\zeta^T P\dot{\zeta} + 2\delta\zeta^T P\zeta - \varpi^T\gamma\varpi + \zeta^T(S+Q)\zeta - e^{-2\delta\bar{\tau}}\zeta^T(t-\bar{\tau})S\zeta(t-\bar{\tau}) + \bar{\tau}^2\dot{\zeta}^T R\dot{\zeta}$ and $\bar{\zeta}(t) = [\zeta^T(t), \dot{\zeta}^T(t), \zeta^T(t-\bar{\tau}), \zeta^T(t-\tau(t)), \varpi^T(t)]^T$,

$$\Phi_1 = \begin{bmatrix} \Phi_1(1,1) & \Phi_1(1,2) & e^{-2\delta\bar{\tau}}S_{12} & \Phi_1(1,4) & P_2^T \\ * & \Phi_1(2,2) & 0 & P_3^T BK & P_3^T \\ * & * & \Phi_1(3,3) & \Phi_1(3,4) & 0 \\ * & * & * & \Phi_1(4,4) & 0 \\ * & * & * & * & -\gamma I_n \end{bmatrix}, \quad \text{(B.2)}$$

$\Phi_1(1,1) = 2\delta P + S + Q - e^{-2\delta\bar{\tau}}R + P_2^T A + A^T P_2$,

$\Phi_1(1,2) = P - P_2^T + A^T P_3$, $\Phi_1(3,4) = e^{-2\delta\bar{\tau}}(R - S_{12}^T)$,

$\Phi_1(1,4) = P_2^T BK + e^{-2\delta\bar{\tau}}(R - S_{12})$,

$\Phi_1(2,2) = \bar{\tau}^2 R - P_3 - P_3^T$, $\Phi_1(3,3) = -e^{-2\delta\bar{\tau}}(S+R)$,

$\Phi_1(4,4) = -(1-\hat{\tau})e^{-2\delta\bar{\tau}}Q + e^{-2\delta\bar{\tau}}(-2R + S_{12} + S_{12}^T)$,

where $\{P_2, P_3, S_{12}\} \in \mathbb{R}^{n\times n}$ will be decided later. The inequality (B.1) comes from $\dot{\tau}(t) \leq \hat{\tau}$ in Assumption 1, the Jensen's inequality and Lemma 3.4 in Fridman (2014) where the matrix S_{12} is introduced to satisfy

$$\Phi_1' = \begin{bmatrix} R & S_{12} \\ * & R \end{bmatrix} \succeq 0. \quad \text{(B.3)}$$

We use the descriptor method in Fridman (2014) where P_2, P_3 are introduced to add the last term in the first inequality of (B.1). That term is identically zero, which comes directly from (8). ■

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