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Single-domain Bose condensate magnetometer achieves energy resolution per bandwidth below $\hbar$

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We present a magnetic sensor with energy resolution per bandwidth $E_R < \hbar$. We show how a $^{87}$Rb single-domain spinor Bose–Einstein condensate, detected by nondestructive Faraday rotation probing, achieves single-shot low-frequency magnetic sensitivity of 72(8) fT measuring a volume $V = 1.091(30) \, \mu m^3$ for 3.5 s, and thus, $E_R = 0.075(16) \, \hbar$. We measure experimentally the condensate volume, spin coherence time, and readout noise and use phase space methods, backed by three-dimensional mean-field simulations, to compute the spin noise. Contributions to the spin noise include one-body and three-body losses and shearing of the projection noise distribution, due to competition of ferromagnetic contact interactions and quadratic Zeeman shifts. Nonetheless, the fully coherent nature of the single-domain, ultracold two-body interactions allows the system to escape the coherence vs. density trade-off that imposes an energy resolution limit on traditional spin precession sensors. We predict that other Bose-condensed alkalis, especially the antiferromagnetic $^{23}$Na, can further improve the energy resolution of this method.

Significance

Energy resolution per bandwidth $E_R$ is a cross-technology figure of merit that quantifies the combined spatial, temporal, and field resolution of a magnetic sensor. Today’s best-developed magnetometer technologies, including superconducting quantum interference devices, spin-exchange relaxation-free Rb vapors, and nitrogen-vacancy centers in diamond, are limited by quantum noise to $E_R \gtrsim \hbar$. Meanwhile, important sensing applications, e.g., noninvasive discrimination of individual brain events, would be enabled by $E_R < \hbar$. This situation has motivated proposals for sensors operating by new physical principles. Our result, $E_R = 0.075(16)\hbar$, far beyond the best possible performance of established sensor technologies, confirms the potential of this class of proposed sensors. The result opens horizons for condensed matter, neuroscience, and tests of fundamental physics.

Well-known quantum limits profoundly, but not irretrievably, constrain our knowledge of the physical world. Uncertainty relations forbid precise, simultaneous knowledge of observables such as position and momentum. Parameter estimation limits, e.g., the standard quantum limit and Heisenberg limit, constrain our ability to measure transformations not subject to uncertainty relations, e.g., rotations (1, 2). Both these classes of quantum limits admit trade-offs: uncertainty principles allow an observable to be precisely known if one foregoes knowledge of its conjugate observable, and parameter estimation limits allow better precision in exchange for a greater investment of resources, e.g., particle number.

A qualitatively different sort of quantum limit is found in magnetic field sensing, where well-studied sensor technologies are known to obey a quantum limit on the energy resolution per bandwidth,

$$E_R \equiv \frac{\langle \delta B^2 \rangle VT}{2\mu_0}.$$  \[1\]

Here $\langle \delta B^2 \rangle$ is the mean squared error of the measurement, $V$ is the sensed volume, $T$ is the duration of the measurement, and $\mu_0$ is the vacuum permeability.

A limit on $E_R$ constrains sensitivity when measuring the field in a given space–time region, without reference to any other physical observable, nor to any resource. In contrast to other quantum sensing limits, this allows nothing to be traded for greater precision; it means that details of the field distribution are simply unmeasurable. Known limits on $E_R$, derived from quantum statistical modeling, show that direct current (dc) superconducting quantum interference devices (dc SQUIDs) (3, 5, 6), rubidium vapor magnetometers (7, 8), and immobile spin precession sensors, e.g., nitrogen-vacancy centers in diamond (NVD) (4, 9), are all limited to $E_R \gtrsim \alpha \hbar$, where $\hbar$ is the reduced Planck constant and $\alpha$ is a number of order unity. These limits, though, are imposed by technology-specific mechanisms, not by a universal constraint on all sensor technologies (10).

A variety of exotic sensing techniques, including noble gas spin precession sensors (11–13), levitated ferromagnets (14, 15), and dissipationless superconducting devices (16–18), have been proposed to achieve $E_R < \hbar$ by evading specific relaxation mechanisms (10). If $E_R < \hbar$ can be achieved, it will break an impasse that has held since the early 1980s, when $E_R \approx \hbar$ was reached in dc SQUID sensors (6, 19). In addition to resolving the question of whether $E_R \gtrsim \hbar$ is universal, achieving $E_R < \hbar$ would open horizons in condensed matter physics (20) and neuroscience (21).

For example, to enable single-shot discrimination of brain events, a magnetometer would need $\delta B \sim 1 \, \text{fT}$ sensitivity to $T \sim 10 \, \text{ms}$ events when measuring in $V \sim (3 \, \text{mm})^3$ volumes (22, 23), or $E_R \sim \hbar$.

Here we study an exotic magnetometer technology, the single-domain spinor Bose–Einstein condensate (SDSBE), that freezes out relaxation pathways due to collisions, dipolar...
interactions, and also spin diffusion (24) and domain formation (25, 26), which occur in unconfined condensates. With a ⁸⁷Rb SDSBEC, we find \(E_R = 0.075(16)h\), far beyond what is possible, even in principle, with established technologies (10, 27). Our results demonstrate the possibility of \(E_R \ll h\) sensors and motivate the study of other exotic sensor types.

To understand how the SDSBEC evades the \(h\) limit, it is instructive to first show why other spin precession sensors, which include NVD and alkali vapors, obey such a limit. The principle in instructive to first show why other spin precession sensors, which include NVD and alkali vapors, obey such a limit.

Dipole coupling in NVD (9, 10). This density–coherence trade-off is kinematically forbidden for a sensor operating in the ground hyperfine state (28). Second, because of quantum degeneracy, the elastic two-body interactions (spin-independent and spin-dependent contact interactions) produce a coherent dynamics that does not raise the entropy of the many-body spin state (29). Third, in the single-domain regime, these coherent dynamics cannot reduce the net polarization through domain formation, as happens in extended SBECs (30). As we will show, \(1/T_2\) then contains no contribution \(\propto n\), and we escape the density–coherence trade-off.

To understand the SDSBEC sensitivity, we compute \(\langle \delta \theta^2 \rangle_F\), including quantum statistical effects due to collisional interactions, which can importantly modify the spin distribution from its mean-field behavior (33). We employ the truncated Wigner approximation (TWA) (34, 35), previously applied to study spatial coherence in BECs (36). In the single-mode approximation (SMA), the quantum field describing the condensate factorizes into a spatial distribution \(\phi_N(r)\) and a spinor field operator \(\chi\) describing all atoms in the condensate. \(\chi \equiv (\hat{a}_{1\uparrow}, \hat{a}_{1\downarrow}, \ldots)^T\), where \(\hat{a}_{1\uparrow}\) are bosonic annihilation operators, such that \(N \equiv \chi^\dagger \chi\) is the atomic number operator. \(\phi_N(r)\) is the ground-state solution to the spin-independent part of the Hamiltonian in the Thomas–Fermi approximation and with \(N\) atoms. We normalize \(\phi_N\) such that \(I_2 = 1\), where \(I_2 \equiv \int d^3 r |\phi_N(r)|^4\).

The spinor field \(\chi\) evolves under the SMA Hamiltonian (37)

\[
H_{SMA} = \frac{g}{2} \chi^\dagger \chi \chi \cdot \chi \chi + q \chi^\dagger I_z^2 \chi,
\]

where \(g \equiv g_q I_q \propto N^{-3/5}\) describes the spin-dependent interaction strength and the \(q\) term describes the quadratic Zeeman

Fig. 1. SDSBEC magnetic field sensor. (A) Experimental schematic: crossed, far-off-resonance beams (orange) are used to produce and hold a spinor condensate in a spherical optical dipole trap. A near-resonance probe beam (red) is used to make nondestructive Faraday rotation measurements of the on-axis component of the collective spin \(F\). A reference detector (RD) measures the number of input photons, and quarter- (QWP) and half-wave (HWP) plates are used to set the polarization before a lens (L) focuses the probe onto the atomic cloud. The transmitted light is analyzed for polarization rotation using a second HWP, polarization beamsplitter (PBS), and differential photodetector (DPD). (B) Computed density \(n\) of the prepared SBEC in the x–z plane (dark square in schematic). (C) Evolution of the collective spin statistical distribution during the sensing protocol (not to scale): the atoms are spin polarized parallel to the field direction, with the collective spin \(F\) statistically distributed as shown by red dots, limited by spin projection noise and atom number uncertainty. Spins are then tipped by a radiofrequency pulse to be orthogonal to the field (\(B\)), shown by green dots. During a free-precession time \(T\) the collective spin precesses by an angle \(\theta = \gamma BT\), while also diminishing in magnitude and experiencing shearing of the statistical distribution (green–blue progression). (D) Readout: during the final few precession cycles the spin component \(F_\perp\) is detected by Faraday rotation. Measurements of optical polarization rotation angle \(\varphi\) versus time \(t\) (points) are fit with a free-induction waveform (line) to infer spin rotation angle \(\theta\) at readout time \(T\). (E) Spatial distribution of the polarization defect density \(n - F_\perp\), at \(t = 1\), where \(F_\perp\) is the transverse polarization density, obtained from 3×1D Gross–Pitaevskii equation simulations for the experimental trap conditions and \(q/h = 0.5\) Hz (Left) and \(q/h = 0\) Hz (Right). Scale is as in B. The very small observed spin defect implies a small upper bound to spin noise from ferromagnetism-driven spin segregation and justifies the use of the SMA to compute quantum noise dynamics.
shift, including contributions from the external field and from microwave or optical fields. The combined action of the $g$ and $q$ terms induces a shearing of the condensate’s spin noise distribution from its initial coherently-state distribution. Losses occur at rate $dn'/dt = -\Gamma_1 n - \Gamma_3 n^{3/2}$, where $\Gamma_1$ describes the rate of collisions with background gas and $\Gamma_3$ is proportional to the three-body loss cross section. The evolution of the many-body spin state $\rho$ is described by the master equation $d\rho/dt = [H_{\text{SMA}}, \rho]/(i\hbar) + \mathcal{L}[\rho]$, where $\mathcal{L}[\rho]$ is the Liouvillian

$$\mathcal{L}[\rho] = \sum \kappa_i \left( 2 \hat{O}_i \hat{O}_i^\dagger - \hat{O}_i^\dagger \hat{O}_i - \hat{O}_i \hat{O}_i^\dagger \right),$$

and the jump operators $\hat{O}_i$, with associated rates $\kappa_i$, describe the various loss processes (Mode Shape, Interaction Strengths, and Jump Operators and Quantum Noise Evolution).

Fig. 2 shows the evolution of the noise contribution to $E_N$ over time as computed by TWA. For a given trapping potential and finite $\Gamma_1, q$, and/or $\Gamma_3$, the energy resolution shows a global minimum with $T$. To understand the in-principle limits of this $T$-optimized noise level, we note the following: 1) $\Gamma_1$ can in principle be arbitrarily reduced through improved vacuum conditions, while $q$ can also be made arbitrarily small by compensating the contribution of the external field with microwave or optical dressing, leaving $\Gamma_3$ as the sole factor to introduce spin noise. 2) The noise effects of $\Gamma_3$, which are a strong function of density, can also be made arbitrarily small, by increasing $r_T$ and $N$ to give a large, low-density condensate. 3) The corresponding increase in $V$ is more than offset by the increase in $T_2$, such that $E_N \propto V/T_2$ tends toward zero. 4) At the same time, the SMA and TWA approximations become more accurate in this limit. We conclude that a low-density SBEC in a loose trapping potential can operate well in the single-mode regime, suffer small three-body losses, and achieve $E_N \ll h$.

We now show that an SDBSEC magnetometer can in practice operate with $E_N$ well below $h$. The experimental configuration is illustrated in Fig. 4 and described in detail in Palacios et al. (29). In brief, a pure condensate of $^{87}$Rb atoms in the $F = 1$ manifold with an initial atom number $N_0 = 6.8(5) \times 10^4$ is produced by forced evaporation in a crossed-beam optical dipole trap. The condensate is initially fully polarized along $B$ by evaporation in the presence of a magnetic gradient, tipped by a radiofrequency pulse to be orthogonal to $B$, then allowed to precess for a time $T$ before readout, as depicted in Fig. 1C. A probe light tuned 258 MHz to the red of the $F = 1 \rightarrow F' = 0$ transition of the $D_2$ line is used for nondestructive Faraday rotation measurement of the collective spin of the condensate (Fig. 1D).

Atom number is measured by time-of-flight absorption imaging. From atom-number decay we observe $\Gamma_1 = 8.6(31) \times 10^{-2}$ s$^{-1}$ and $\Gamma_3 = 1.0(6) \times 10^{-5}$ atom$^{-3/2}$s$^{-1}$ one-body and three-body collision rates, respectively. The very small three-body loss rate allows us to approximate atomic losses as exponentially decaying with lifetime $7.1(2)$ s. In this approximation the resulting rate $dn'/dt$ never differs by more than 4% from the numerical solution when both $\Gamma_1$ and $\Gamma_3$ are included. The coherence time is found to be equal to the atomic lifetime in the trap and therefore $T_2 = 7.1(2)$ s.

The curvature of the trapping potential is determined from the measured SBEC oscillation frequencies. We find $\omega_1/2\pi = 67.2(10)$ Hz, $\omega_2/2\pi = 89.0(7)$ Hz, and $\omega_3/2\pi = 97.6(9)$ Hz, where the subscripts index the principal axes of the trap. For our number of atoms $N = 6.8(5) \times 10^4$ these correspond to Thomas–Fermi radii $r_{TF}^{(1,2,3)} = 7.0(1)$ μm, $6.20(9)$ μm, and $6.00(9)$ μm in the Thomas–Fermi approximation (Mode Shape, Interaction Strengths, and Jump Operators). This parabolic geometry defines the volume containing the entire condensate $V = 4\pi r_{TF}^{(1,2,3)}/3 = 1.091(30)$ μm$^3$.

As shown in Fig. 1D, measurements of the spin precession can be taken over several precession cycles with little damage to the polarization, allowing the precession angle to be estimated with readout noise $\langle \delta \theta^2 \rangle_{\text{RO}} = 1.08(24) \times 10^{-4}$ rad$^2$ at the time of optimal readout $T = T_2/2$ (Readout Noise). We note that $\langle \delta \theta^2 \rangle_{\text{RO}}$ could be further reduced through improved probe–atom coupling and/or squeezed light (38, 39).

Combining the above we have volume $V = 1.091(30)$ μm$^3$, readout noise $\langle \delta \theta^2 \rangle_{\text{RO}} = 1.08(24) \times 10^{-4}$ rad$^2$, and spin quantum noise $\langle \delta \theta^2 \rangle_{\text{SN}} = 1.46(100) \times 10^{-3}$ rad$^2$. For an optimum readout time of $T = 3.5$ s, these give a magnetic sensitivity of 72(8) Hz and $E_N = 0.075(16)$ μHz (Duty Cycle). This is a factor of 17 better than any previously reported value (24, 40, 41) and well beyond the level $E_N \approx h$ that constrains the most advanced existing technologies.

In applying the TWA, we assumed the validity of the SMA. To check this, we integrate in time the three-dimensional Gross–Pitaevskii equation (Description of the Condensate) on a graphical processing unit, as described in refs. 42, 43. Spatially resolved polarization column densities are shown in Fig. 1B and E and indicate fractional polarization defects at the $10^{-5}$ level. The defect $N - N_F$ of the condensate as a whole is of order 1 atom. By vector addition, the contribution to the variance of the azimuth spin component $F_\phi$ is then no larger than the projection noise $\langle \delta \theta^2 \rangle_{\text{PN}} = N/2$ and could be far smaller. These mean-field results, together with coherence measurements reported in (29), give a quantitative justification for the use of the SMA.

We extend the analysis to other $F = 1$ alkali species and find that some could perform still better than the $^{87}$Rb system studied here. Two considerations are relevant here. First, we note the conditions for single-mode dynamics: $r_{TF}/\xi < 1$ and $r_{TF}/\lambda < 1$, where $\xi$ is the spin-healing length (37) and $\lambda$ is the threshold wavelength for spin wave amplification (44) (SMA Validity Conditions). In Fig. 3 we show $\max(r_{TF}/\xi, r_{TF}/\lambda)$ versus $V$ and $q$ and note that $^{87}$Rb and $^{23}$Na remain single-domain for smaller volumes and for stronger fields than do $^7$Li and $^{87}$Rb. We note also that the dynamical condition $r_{TF}/\lambda < 1$ favors antiferromagnetic interactions, giving $^{23}$Na a marked advantage by this criterion. The second consideration concerns $\text{Phil.}

Fig. 2. Spin noise contribution to $E_N$ of the SDBSEC sensor, from TWA simulations with measured trap parameters including condensate volume $V$ and one- and three-body decay rates $\Gamma_1$ and $\Gamma_3$, respectively. Blue, orange, green, and red curves show $E_N$ for $q/h = 0.30, 0.12, 0.05$, and 0 Hz, respectively. To separate different effects, we show also conditions $\Gamma_1 = q/h = 0$ (violet) and $\Gamma_1 = \Gamma_3 = q/h = 0$ (brown). Spheres represent the $(F_x, F_y, F_z)$ phase space at time 0 s (Bottom) and at 3 s with $q/h = 0.3$ Hz and 0 Hz (Top Left and Top Right); sphere radius is equal to the number of remaining atoms, and points sample the rotating-frame Wigner distribution. For ease of visualization, dispersion of the Wigner distribution is magnified by a factor of 10.

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the three-body recombination rate (45) \( \Gamma_3 \propto \hbar \omega_0^2 / M \), where \( \omega_0 \) is the s-wavenumber scattering length for the channel of total spin zero. Relative to \(^{87}\text{Rb}\), this rate in \(^7\text{Li}, ^{23}\text{Na}, \) and \(^{41}\text{K} \) is a factor 25, 4, and 2 smaller, respectively, suggesting an advantage for these species when limited by three-body losses.

In conclusion, we have shown that an appropriately confined, quantum degenerate Bose gas, i.e., an SDSBEC, has a qualitative Description of the Condensate. A \( N \) spin-F atoms by the collective spin operator \( F \), i.e., the sum of the vector spin operators for the individual atoms. \( F \) is initialized in a fully polarized state orthogonal to the magnetic field \( B \). The spin angle precesses at a rate \( \delta = \gamma B \), where \( \gamma \) is the gyromagnetic ratio. It is convenient to work with spin components in a frame rotating at the nominal Larmor frequency, such that a small change in angle can be expressed as \( \delta \theta \). The equilibrium magnetic noise is \( \langle \delta B_r^2 \rangle = \langle \delta B_q^2 \rangle / (\gamma^2 T^2) \), by propagation of error. If \( F_r \) experiences Markovian relaxation, then \( F_r \). At time measurement is \( F_T(T) = N \exp (\pi F / (\pi F_2) \), where \( F_2 \) is the transverse relaxation time and \( F_T \) is the initial, fully polarized state has azimuthal spin noise \( \langle \delta S_r \rangle = \langle \delta S_q \rangle = F_r / (2 \hbar) \), i.e., the standard quantum limit, if \( N \) does not decrease during the evolution (as is the case for color center and vapor phase ensembles), this describes a minimum noise for \( F_r \) during the evolution. We thus find \( \langle \delta B_r^2 \rangle \geq \exp (2T / T_2) \geq 2 \gamma^2 T F T N \). Choosing \( T \) to minimize the right-hand side of this inequality, we find \( T = T_2 / 2 \) and thus \( \langle \delta B_r^2 \rangle \geq \exp (1 / (2 T_2 F T N) \), including the sensor volume \( V \), the energy resolution is lower-bounded by \( E_0 \geq \exp (1 / (4 \rho_{\text{eff}}^2 F T \rho N) \), where \( N = V / V \) is the number density.

Writing the relaxation rate as \( 1 / T_2 = A_1 n^2 + A_3 n^3 + \ldots \), where \( A_n \) are non-zero, \( E_0 \approx A_1 n^{-1} + A_3 n^{-3} + \ldots \) is manifestly lower-bounded. First principle calculations for immobilized spin precession sensors (4) and models including measured spin relaxation rates for optimized Rb vapor magnetometers (8) show that these lower bounds are within a factor of 2 of \( E_0 = \hbar / \gamma \).

Description of the Condensate. A \( 1 \) spinor condensate with weak collisional interactions is well described by a three-component field \( \psi_s(t) \) evolving under the Hamiltonian:

\[
H = H_{\text{SDS}} + H_{\text{g}}.
\]

where \( H_{\text{SDS}} \) and \( H_{\text{g}} \) are the spin-independent and spin-dependent parts, respectively. Summing over repeated indices, and omitting position dependence for clarity, these are:

\[
H_{\text{SDS}} = \int d^3r \left[ \frac{\hbar^2}{2M} \frac{\nabla^2}{2M} + U \psi_{\alpha} + \frac{g_1}{2} \psi_{\alpha} \psi_{\beta} \psi_{\gamma} \psi_{\nu} + q \psi_{\alpha} \psi_{\beta} \psi_{\gamma} \psi_{\delta} \right].
\]

Here \( q \), the matrix representing the single-atom spin projection operator onto the axis \( \gamma \). In \( H_{\text{g}} \), the terms are ferromagnetic interaction, linear Zeeman, and quadratic Zeeman energies, respectively, \( p = (x+y)/\hbar \); where \( B = \) the field strength Schwarzschild. 5 The best known spatial energy redistribution is well described by a three-component field strength superposition.

Mode Shape, Interaction Strengths, and Jump Operators. In the Thomas–Fermi approximation (47), a pure condensate in a spherical harmonic potential has the mode function:

\[
|\phi(r)\rangle = \frac{15}{8 \pi r_{\text{SDS}}^2} \left( 1 - \frac{r^2}{r_{\text{SDS}}^2} \right)
\]

for \( r \leq r_{\text{SDS}} \) and zero otherwise, where \( r_{\text{SDS}} \) is the radial coordinate, \( r_{\text{SDS}} = \sqrt{\langle 15g_s N \rangle / (4 \pi M a_0^2) \}^{1/5} \) is the Thomas–Fermi radius, and \( \omega \) is the trap angular frequency. Because \( r_{\text{SDS}} \approx N^{1/5} \), the integrals \( \int_0^{r_{\text{SDS}}} d^3r \) that determine the effective strength of two- and three-body interactions are \( k \propto N^{-1} \) and \( k \propto N^{-2} \), respectively. The rate of three-body collisions can then be written \( \Gamma_{\text{NCF}} = N^{4/5} \), such that atom losses are described by:

\[
\frac{dN}{dt} = \Gamma_{\text{NCF}} N^{4/5}.
\]

We note that in this model, losses are independent of internal state. While this is well established for one-body losses, for three-body losses the state dependence is, to our knowledge, unknown. Two-body losses due to magnetic dipole–dipole scattering and spin-orbit interaction in second order (28) are energetically forbidden in the low-field scenario of interest here.

We use a set of jump operators that reproduces Eq. 9 while also respecting the symmetry of the loss process: one-body losses are described by \( O_{\text{SDS}}^{(1):m} = a_{\alpha} m \), \( m \in \{-1, 0, 1\}, \) where \( a_{\alpha} \) annihilates an atom in internal state \( m \) with strengths \( \rho_{\text{SDS}} = 1/2 \), while three-body losses are described by \( O_{\text{SDS}}^{(3):m} = N^{3/5} a_{\alpha}^2 a_{\beta} a_{\gamma} m, n, \in \{-1, 0, 1\}, \) with \( N \equiv (\delta_{\alpha+1, \beta} + \delta_{\alpha, \beta+1} + \delta_{\alpha+1, \beta+1}) \) and strengths \( \rho_{\text{SDS}}^{(3)} = 51/2 \).
The Fokker-Planck equation describes the evolving probability distribution of a particle undergoing Brownian motion and as such can be described by a stochastic differential equation that is straightforward to integrate numerically. We identify a complex-valued vector \( \mathbf{c} = (c_1, c_2, c_3, c_4) \) with the spinor field \( \chi \) and \( n \)-number functions \( \Omega^{(1)} = \mathbf{c} m, m \in \{-1, 0, 1\} \), \( \Omega^{(2)} = |e^{-i/2s} \mathbf{c}_m \mathbf{c}_n \mathbf{c}_o \mathbf{c}_p, m, n, o, \alpha \in \{-1, 0, 1\} \), with the jump operators \( \mathbf{O}^{(1)} \) and \( \mathbf{O}^{(2)} \), respectively. To account for the uncertainty of the initial state, a collection of starting points are chosen with values \( c_0 = (z+x, z+x, z-x, z-x) \) with \( c_0 \in \mathbb{S}^2 \). The Fokker-Planck equation describes the evolving probability distribution \( d\mathbf{c} = \left( \sum_i \frac{\partial}{\partial c_i} \mathbf{O}^{(1)}_i \right) d\mathbf{c} + \sum_i \sqrt{\mathbf{O}^{(2)}_i} \mathbf{dZ}_i \), respectively. For the simulations shown in Fig. 2, we used 5,000 starting points.

Each initial point evolves by the (Itô) stochastic differential equation

\[
d\mathbf{c}_m = \left( \frac{1}{2} \left( \frac{\partial}{\partial c_n} \mathbf{O}^{(1)}_m \right) \right) d\mathbf{c}_n + \sum_i \sqrt{\mathbf{O}^{(2)}_i} \mathbf{dZ}_i,
\]

where \( d\mathbf{c} = (dx + idy)/\sqrt{2} \) is a complex Wiener increment, in which \( dx \) and \( dy \) are independent Wiener increments, i.e., zero-mean normal deviates with variance \( dt \). Using the jump operators \( \mathbf{O}^{(1)} \) and \( \mathbf{O}^{(2)} \) defined above and adding their noise contributions in quadrature, we find

\[
d\mathbf{c} = \left( \frac{29}{10} \sum_{i} (c_i f_i + c_i f_i - c_i f_i + c_i f_i) \right) + B^{(s)} \cdot d\mathbf{Z},
\]

where \( d\mathbf{Z} = (dx + idy)/\sqrt{2} \) is a complex Wiener increment, in which \( dx \) and \( dy \) are independent Wiener increments, i.e., zero-mean normal deviates with variance \( dt \).

Readout Noise. We experimentally prepare SBECS of \( 87\text{Rb} \) atoms in the \( f = 1 \) \( m = +1 \) ground state under a bias field along direction \( z \) and strength \( B = 29 \mu T \), which induces Larmor precession at angular frequency \( \omega = 2\pi \times 200 \text{ kHz} \). A radiofrequency \( \pi/2 \) pulse is applied to tip the spins to the \( xy \) plane. After a free evolution time \( T \) we detect the spin precession by Faraday rotation, sending 60 pulses, each of 200-ns duration and containing 2 \( \times 10^9 \) photons, to observe rotation angles \( \varphi_T \) at times \( T \), \( 1, \ldots, 60 \). Representative data are shown in Fig. 1D and are well described as a free induction decay signal. We parameterize the signal noise plus as

\[
\varphi = G_i \cos(\omega_i T) f_i(T) + \sin(\omega_i T) F_i(T) e^{-\tau_i/T_{\text{sat}}} + \varphi^{(1)} R_O, \tag{14}
\]

where \( G_i \) is the effective atom-light coupling in radians per spin, \( \tau_i \equiv t - T \) is the time since the start of probing, \( F(T) \) is the collective spin at the start of probing, \( T_{\text{sat}} \) is the spin relaxation rate due to probe scattering, and \( \varphi^{(1)} R_O \) is the readout noise. \( G_i = 2.5(1) \times 10^{-7} \) rad/atom is found by fully polarizing the atoms along \( y \), such that \( F_T = N_i \) and measuring \( \varphi \) by Faraday rotation. \( N_i \) is then measured by absorption imaging. \( T_{\text{sat}} = 29.7 \mu \text{s} \), found by fitting free induction decays as in Fig. 10.

To determine the atomic precession angle from a free induction decay we define the angle estimator \( \delta_i = \arctan(F_i(T), F_i(T)) \) in terms of the parameters \( F_i(T), F_i(T) \) that make the best least-squares fit of Eq. 14 to a given free induction decay \( \varphi \) with the previously determined \( G_i \) and \( T_{\text{sat}} \).

By propagation of errors, and due to the fit function’s linear dependence on \( F_i \) and \( F_i \), the estimator’s mean squared error is

\[
\langle (\delta_i - \delta_i) R_O \rangle = \frac{r^2}{\langle R_O \rangle^2} \exp(-2T/T_{\text{sat}}),
\]

where \( r = \langle \cos \theta - \sin \theta \rangle \) is a projector on the azimuthal direction and \( \langle R_O \rangle \) is the covariance matrix of the contribution made by \( \varphi^{(1)} R_O \) to the fit parameters. To evaluate Eq. 15, we note that \( \langle R_O \rangle \) can be directly measured: we collect 40 traces \( \langle \varphi \rangle \) at time \( T \) with no atoms in the trap. We then fit Eq. 14 using the \( G_i \) and \( T_{\text{sat}} \) obtained previously. The result is

\[
\langle T_{\text{sat}} \rangle = \left( \frac{184 - 2}{222} \right) \pm \left( \frac{38}{30} \right) \times 10^{-7} \text{ rad when } T = 2T_f/2.
\]

SMR Validity Conditions. Two criteria for the validity of the SMA are found in the literature for the scheme of interest, in which a \( f = 1 \) condensate precesses about an orthogonal magnetic field. The first compares the ferromagnetic energy associated with a spatial overlap of the different \( m_i \) states to the kinetic energy associated with a domain wall, to derive the condition \( \gamma_f < \xi_2 < 2\pi \frac{\sqrt{2}}{\gamma_f} n_0 \), where \( n_0 \) is known as the spin-healing length. The second criterion derives from a consideration of dynamical stability (44): in a plane wave scenario, spin wave perturbations to an initially uniform spin precessing at \( \omega = \gamma_f/\hbar \) are nonincreasing for wavelengths smaller than \( \lambda_{\text{min}} = 2\pi \hbar \sqrt{2\gamma_f n_0} \), which is a condition for the SMA to be then \( \nu < \lambda_{\text{min}} \). We note that for ferromagnetic interactions \( g_0 > 0 \), but not for antiferromagnetic ones, this second condition is stricter than the first because \( \lambda_{\text{min}} > \xi_2 \).

Duty Cycle. While the main result of this work is a single-shot sensitivity, i.e., the noise level when measuring a field over a continuous interval \( T \), it is also interesting to consider averaging multiple sequential sensor readings to obtain a time-averaged estimate for the field. In this multishot scenario, the dead time between measurements must be accounted for in the energy resolution per bandwidth. Including the 30-s required to produce the next SBECE sample, we find a multishot sensitivity of 344(39) fT/√Hz and an energy resolution of \( \Delta B^{(1)} V/T \langle 2 \Delta \theta \rangle = 0.48(11) \text{ fT} \), which is also significantly below \( h \) and well below any previously reported value.

Data Availability. Data and data analysis codes are available for download at Zenodo (DOI: 10.5281/zenodo.5751414) (S4).
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19. M. Cromar, P. Carelli, Low-noise tunnel junction dc SQUID’s. 
22. L. J. MacGregor, F. Pulvermüller, M. van Casteren, Y. Shytrov, Ultra-rapid access to words in the brain. 
24. M. Vengalattore et al., High-resolution magnetometry with a spinor Bose-Einstein condensate. 
29. S. Palacios et al., Multi-second magnetic coherence in a single domain spinor Bose-Einstein condensate. 
33. B. Lücke et al., Twin matter waves for interferometry beyond the classical limit. 
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