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Online search for a hyperplane in high-dimensional Euclidean space

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ABSTRACT

We consider the online search problem in which a server starting at the origin of a d-dimensional Euclidean space has to find an arbitrary hyperplane. The best-possible competitive ratio and the length of the shortest curve from which each point on the d-dimensional unit sphere can be seen are within a constant factor of each other. We show that this length is in $\Omega(d) \cap O(d^{3/2})$.

1. Introduction

In d-dimensional Euclidean space, given a set of objects $\mathcal{T} \subseteq \mathcal{P}(\mathbb{R}^d)$, the *online search problem (for an object in $\mathcal{T}$)* asks for a curve (or *search strategy*) $\zeta : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^d$ with $\zeta(0) = 0$ that minimizes the competitive ratio [14]. We say that ζ is *c-competitive* if there exists an $\alpha \in \mathbb{R}_{\geqslant 0}$ such that

$$\text{length}(\zeta|_{[0,t^\star]}) \leqslant c \cdot \text{dist}(0, T) + \alpha$$

for all $T \in \mathcal{T}$, where $t^\star = \inf\{t : \zeta(t) \in T\}$. The infimum over all such c is the *competitive ratio* of ζ. Problems of this type date back to Beck [4] and Bellman [6].

The arguably most basic and well-known such online search problem from this class is the version in which $d = 1$ and $\mathcal{T} = \mathbb{R}$, also known as the cow-path problem, where the best-possible competitive ratio is 9 [5], achieved by visiting the points $(-2)^0, (-2)^1, \ldots$ sequentially. In fact, the class of all 9-competitive strategies has been investigated more closely [1]. Moreover, the competitive ratio can be improved to about 4.591 using randomization [5]. The problem has been generalized to more than two paths starting at the origin [13], searching various types of objects in the plane or lattice [3], and many more scenarios. For a (slightly outdated) survey, see the book by Gal [11].

Another very natural generalization is the case of general d and $\mathcal{T}$ being equal to $\mathcal{H}^{d-1}$, the set of all hyperplanes in $\mathbb{R}^d$. We call this version the *d-Dimensional Hyperplane Search Problem*. When $\mathcal{T}$ is a set of affine subspaces, this is arguably the most interesting case. Note that, if $\mathcal{T}$ contains all affine subspaces of dimension $d' \leq d - 2$, any constant competitive ratio is ruled out, even when d is fixed. Surprisingly, in addition to the case of $d = 1$, results only seem to be known for $d = 2$ and for offline versions [9,2], leaving a gap in the online-algorithms literature. For $d = 2$, it is conjectured that a logarithmic

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spiral that achieves a competitive ratio of about 13.811 is optimal [3,10]. The present paper addresses the aforementioned gap by providing the first asymptotic results for $d \to \infty$.

For the remainder of the paper, we will consider the essentially equivalent but arguably cleaner *Sphere Inspection Problem*: The goal is to find a d-dimensional minimum-length closed curve, γ, that *inspects the unit sphere* in $\mathbb{R}^d$, i.e., γ sees every point p on the unit sphere S^{d-1}. Here, we say that an object $O \subseteq \mathbb{R}^d$ *sees* a point p' on the surface of the unit sphere if there is a point $p \in O$ such that the line segment pp' intersects the unit *ball* exclusively at p'. Note that the curve is not required to start in the origin any more. Such a minimum-length curve exists for any dimension [12]. While this problem is trivial for $d \in \{1, 2\}$, it has only been shown recently that the best-possible length for $d = 3$ is 4π [12]. No results for higher dimensions are known. Such visibility problems have also been considered from an algorithmic point of view, e.g., [7,8].

While the connection between hyperplane search and sphere inspection seems to be folklore (e.g., [12]), we state it formally and provide a short proof for completeness. Let I be an interval, and for $\beta \in \mathbb{R}$, a set $T \subseteq \mathbb{R}^d$, and curve $\gamma : I \to \mathbb{R}^d$, we denote by $\beta \cdot T$ the set $\{\beta \cdot x : x \in T\}$ and by $\beta \cdot \gamma$ the curve $I \to \mathbb{R}^d, t \mapsto \beta \cdot \gamma(t)$.

Proposition 1. *Let $f : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$. The following statements are equivalent:*

- *(i) There exists a length-$O(f(d))$ closed curve in $\mathbb{R}^d$ that inspects S^{d-1}.*
- *(ii) There exists an $O(f(d))$-competitive strategy for the d-Dimensional Hyperplane Search Problem.*

Proof. We first show that (i) implies (ii). Fix $d \in \mathbb{N}$, let γ be the length-ℓ curve for the Sphere Inspection Problem in $\mathbb{R}^d$, and let γ_0 be a point on γ. We show that there exists a (12ℓ)-competitive strategy for the d-Dimensional Hyperplane Search Problem where the additive constant α is equal to 3ℓ. Our strategy works in an infinite number of phases, starting with Phase 0. In each Phase i, the strategy ζ consists of a straight line from the origin to $2^i\gamma_0$, the curve $2^i \cdot \gamma$, and a straight line back from $2^i\gamma_0$ to the origin.

In the following, we show that, in Phase i, all hyperplanes of distance at most 2^i from the origin are visited, and the total distance traversed in Phase i is at most $3 \cdot 2^i \cdot \ell$, which implies the claim. We show two parts separately:

- We show that all claimed hyperplanes are visited by ζ. Consider Phase i and a hyperplane $H \in \mathcal{H}^{d-1}$ at distance at most 2^i from the origin. Let n be a unit normal vector of H. Note that $2^i \cdot \gamma$ sees both the point $2^i n$ and the point $-2^i n$. Let p_1 be a point from which $2^i \cdot \gamma$ sees $2^i n$, and let p_2 be a point from which it sees $-2^i n$. Note that, if $\mathrm{dist}(0, H) = 2^i$, then p_1 or p_2 may be on H. In that case we are done. Otherwise p_1 and p_2 are on different sides of H, and, by the intermediate value theorem, $2^i \cdot \gamma$ intersects H.
- We next bound the distance traversed by ζ in Phase i. Note that it suffices to show that the start point of $2^i \cdot \gamma$, which is equal to its end point, has distance at most $2^i\ell$ from the origin. By scaling, we can restrict to showing $\|\gamma_0\|_2 \leqslant \ell$. Let H be the hyperplane $\langle \gamma_0, x\rangle = 0$. Note that γ must intersect H since otherwise γ does not see $-\gamma_0/\|\gamma_0\|_2$, thus $\mathrm{dist}(\gamma_0, H) \leq \ell$. Since this distance (of γ_0 to H) is precisely $\|\gamma_0\|_2$, it follows that $\|\gamma_0\|_2 \leqslant \ell$.

We now show that (ii) implies (i). Again fix $d \in \mathbb{N}$. Assume that there exists a curve γ starting in the origin and $\alpha \in \mathbb{R}_{\geqslant 0}$ such that, for all hyperplanes $H \in \mathcal{H}^{d-1}$, γ visits H after traversing at most a length of $c \cdot \mathrm{dist}(0, H) + \alpha$. We show that, for any $\varepsilon > 0$, there exists a, not necessarily closed, length-$(c + \varepsilon)$ curve that inspects the sphere. This then implies the existence of a length-$(2c+\varepsilon)$ closed curve that inspects the sphere. We define

$$\zeta := \frac{\varepsilon}{\alpha} \cdot \gamma|_{[0,t^\star]},$$

$$\text{where } t^\star := \sup_{H \in \mathcal{H}^{d-1} : \mathrm{dist}(0,H) \leqslant \alpha/\varepsilon} \min\{t : \gamma(t) \in H\}.$$

Note that the minimum exists because H is closed. Since for each $H \in \mathcal{H}^{d-1}$ with $\mathrm{dist}(0, H) \leqslant \alpha/\varepsilon$, γ visits H after traversing at most a length of $c \cdot \alpha/\varepsilon + \alpha$, the length of $\gamma|_{[0,t^\star]}$ is at most $c \cdot \alpha/\varepsilon + \alpha$, so the length of ζ is at most $c + \varepsilon$.

It remains to be shown that ζ indeed sees every point $p \in S^{d-1}$. Let H be the hyperplane tangent to the unit sphere at p. Further let $t := \min\{t : \gamma(t) \in \alpha/\varepsilon \cdot H\}$. By definition of $t^\star$, we have $t \leqslant t^\star$. Further, by definition of ζ, we have $\zeta(t) \in H$, implying that $\zeta(t)$ sees p. □

In Section 2, we give an auxiliary lemma. In Sections 3 and 4 we will use that lemma and show the following two theorems.

Theorem 1. *Any curve in $\mathbb{R}^d$ that inspects S^{d-1} has length at least $2d$.*

Theorem 2. *There exists a closed curve γ in $\mathbb{R}^d$ of length $(2d)^{3/2}$ that inspects S^{d-1}.*

2. An auxiliary lemma

The following lemma simplifies thinking about the Sphere Inspection Problem.

Lemma 3. *Let $P \subset \mathbb{R}^d$ be the convex hull of some point set V. Then, the following two statements are equivalent.*

- *(i) We have $S^{d-1} \subset P$.*
- *(ii) The set V sees every point $p \in S^{d-1}$.*

Proof. We start by showing that (i) implies (ii). Let H be the hyperplane that is tangent to the unit sphere at p. If H contains a point $v \in V$, then the lemma directly follows since the line segment vp lies completely within H and therefore does not intersect the unit sphere other

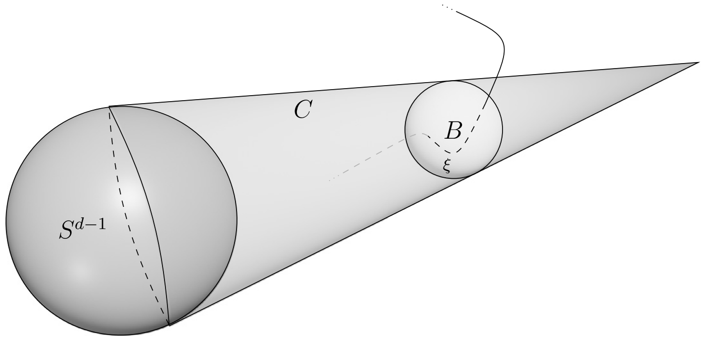

Fig. 1. The apex of the cone C sees all points on the unit sphere that B does.

than at p. So assume that there is no point $v \in V$ that is contained in H. Then, and since P contains one point of the hyperplane ($p \in S^{d-1}$, which is contained in P by (i)), there must exist two points $v_1, v_2 \in V$ which are separated by H. Without loss of generality, let v_1 be the point in the halfspace defined by H whose interior is disjoint from S^{d-1}. Then the line segment $v_1 p$ is completely contained in that halfspace, and since the interior of the halfspace is disjoint from S^{d-1}, v_1 can see p.

Now we show that (ii) implies (i). Towards a contradiction, assume that there exists $p^\star \in S^{d-1}$ with $p^\star \notin P$. Then consider a hyperplane H that separates $p^\star$ from P, and let n be a unit normal vector of H pointing away from P. Consider the hyperplane H' that is tangent to S^{d-1} at n. Clearly, both P and $S^{d-1} \setminus \{n\}$ are contained in one open halfspace defined by H'. Note that no point in this halfspace can see n. Therefore, no $v \in V$ can see n; a contradiction. □

3. Lower bound

The goal of this section is to prove Theorem 1. Towards this, let γ be a curve in $\mathbb{R}^d$ that inspects the unit sphere. We cut γ into a minimum number of contiguous portions of length at most δ for some fixed $\delta < 2$. Let $\xi_1, \ldots, \xi_n$ be the resulting tour portions, where $n = \lceil |\gamma|/\delta \rceil$. Choose a portion ξ, and let x be its midpoint. Clearly ξ is contained in the ball B that has center x and radius $\delta/2$. Further define C to be the cone that is the intersection of all halfspaces that contain both B and S^{d-1} and whose defining hyperplanes are tangent to both B and S^{d-1}. Note that the set of points on the sphere that can be seen by the curve ξ can also be seen from the apex of C, as visualized by Fig. 1. This holds since the radius of B is $\delta/2 < 1$. Note that a single point $p \in \mathbb{R}^d$ can see some subset of an open hemisphere H of the unit sphere. Let $H_1, \ldots, H_n$ denote a set of open hemispheres such that H_i covers the portion of the sphere seen by ξ_i. Since γ inspects the sphere, we have that $(H_i)_{i=1}^n$ covers the sphere.

We need the following lemma.

Lemma 4. *The minimum number of open hemispheres that cover S^{d-1} is $d+1$.*

Proof. To show that $d+1$ points are sufficient, choose a simplex containing the sphere. Now consider the set of open hemispheres whose poles are collinear with the origin and a vertex of that simplex. Indeed, since the simplex contains the sphere, by Lemma 3, the set of vertices of the simplex sees every point $p \in S^{d-1}$. Since each point sees only a subset of the corresponding open hemisphere, the upper bound follows.

For the lower bound, we can use induction on d. Clearly the circle S^1 needs at least three open half-circles to be covered. For S^{d-1}, we have that the boundary of the first hemisphere H_1 is S^{d-2}, and each hemisphere H_i can cover at most an open hemisphere of ∂H_1. So by induction at least $(d-2)+2 = d$ hemispheres are needed to cover ∂H_1, and thus we have that at least $d+1$ hemispheres are needed to cover S^{d-1}. □

By Lemma 4 we have that $n \geqslant d+1$, implying $\lceil \text{length}(\gamma)/\delta \rceil \geqslant d+1$ and therefore $\text{length}(\gamma)/\delta \geqslant d$. With $\delta = 2-\varepsilon$, we have that $\text{length}(\gamma) \geqslant (2-\varepsilon)\cdot d$ for all $\varepsilon > 0$, and thus $\text{length}(\gamma) \geqslant 2d$.

4. Upper bound

In this section we prove Theorem 2. Let C^d be the d-dimensional cross-polytope, i.e., the polytope $\{x \in \mathbb{R}^d : \|x\|_1 \le 1\}$. Define $\bar{C}^d := \sqrt{d} \cdot C^d = \{x \in \mathbb{R}^d : \|x\|_1 \le \sqrt{d}\}$, as shown in Fig. 2. We claim that the vertices of $\bar{C}^d$ inspect the unit sphere.

Lemma 5. *For any point $p \in S^{d-1}$, there exists a vertex v of $\bar{C}^d$ that sees p.*

Proof. We prove that $\bar{C}^d$ contains the unit sphere; then Lemma 3 shows the claim. In other words we prove that for any point $x \in S^{d-1}$, i.e., $\|x\|_2 = 1$, it also holds that $\|x\|_1 \le \sqrt{d}$. Indeed, by the Cauchy-Schwartz inequality we have that $\|x\|_1 \le \sqrt{d} \cdot \|x\|_2$ for any $x \in \mathbb{R}^d$ which in turn can be upper bounded by $\sqrt{d}$ for any $x \in \mathbb{R}^d$ with $\|x\|_2 \le 1$. □

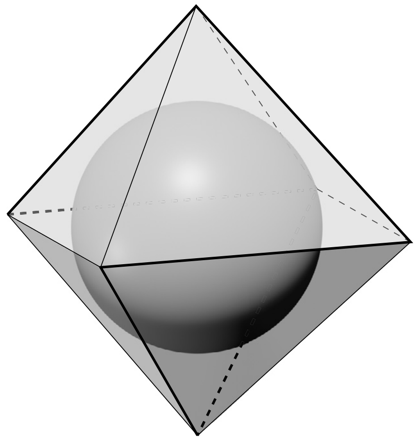

Fig. 2. The cross polytope that has vertices at distance $\sqrt{d}$ from the origin contains the unit sphere, and thus, a tour of its vertices (thick) inspects the unit sphere.

Let $G(\bar{C}^d)$ be the graph[1] of polytope $\bar{C}^d$. Note that $G(\bar{C}^d)$ is the so-called *cocktail party graph* which can be obtained by removing a perfect matching from a complete graph on $2d$ vertices (indeed, in $\bar{C}^d$ any vertex v has an edge to any other vertex except $-v$). We note that Ore's Theorem [15] implies that $G(\bar{C}^d)$ is Hamiltonian, but we provide a short proof for this paper to be self-contained.

Lemma 6. *The graph $G(\bar{C}^d)$ is Hamiltonian.*

Proof. The proof is by induction on d. The statement clearly holds for $\mathbb{R}^2$ where the cross polytope is a square, and $G(\bar{C}^2)$ itself is a Hamiltonian cycle that we denote by c^2. Consider a Hamiltonian cycle c^{d-1} for $G(\bar{C}^{d-1})$. Note that $G(\bar{C}^d)$ can be constructed by $G(\bar{C}^{d-1})$ and adding two nodes, $v^\star$ and $-v^\star$ that are connected to each of the nodes of $G(\bar{C}^{d-1})$. To construct a cycle c^d of $G(\bar{C}^d)$, take any two distinct edges $\{v_1, v_2\}$ and $\{v_3, v_4\}$ contained in c^{d-1} and replace them with the edges $\{v_1, v^\star\}, \{v^\star, v_2\}$ and $\{v_3, -v^\star\}, \{v_3, -v^\star\}$, respectively. The resulting tour is connected, visits all nodes of $G(\bar{C}^d)$ exactly once and the used edges are contained in the edge set of $G(\bar{C}^d)$, thus ensuring the feasibility of c^d. $\square$

By Lemma 5 it directly follows that any closed curve that visits all vertices of $\bar{C}^d$ inspects the unit sphere, and therefore so does the closed curve γ corresponding to the Hamiltonian cycle c^d in the skeleton of $\bar{C}_d$. To complete the proof of Theorem 2, note that each edge of $\bar{C}_d$ has a length of $\sqrt{2d}$ and that γ traverses $2d$ such edges.

[1] The graph $G(P)$ of a polytope P is a graph with a node for each vertex of P and an edge connecting those nodes if P has an edge between the corresponding vertices. The graph of P is also referred to as its 1-skeleton.

5. Conclusion

In this paper, we narrowed down the optimal competitive ratio for the d-Dimensional Hyperplane Search Problem to $\Omega(d) \cap O(d^{3/2})$. The obvious open problem is closing this gap.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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