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DISTRIBUTED SOLUTION OF LAPLACIAN EIGENVALUE PROBLEMS*

ANTTI HANNUKAINEN†, JARMO MALINEN†, AND ANTTI OJALAMMI†

Abstract. The purpose of this article is to approximately compute the eigenvalues of the symmetric Dirichlet Laplacian within an interval $(0, \Lambda)$. A novel domain decomposition Ritz method, partition of unity condensed pole interpolation, is proposed. This method can be used in distributed computing environments where communication is expensive, e.g., in clusters running on cloud computing services or networked workstations. The Ritz space is obtained from local subspaces consistent with a decomposition of the domain into subdomains. These local subspaces are constructed independently of each other, using data only related to the corresponding subdomain. Relative eigenvalue error is analyzed. Numerical examples on a cluster of workstations validate the error analysis and the performance of the method.

Key words. eigenvalue problem, domain decomposition, dimension reduction, subspace method

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1. Introduction.

Assume that $\Omega \subset \mathbb{R}^d, d = 2, 3$, is a bounded domain with Lipschitz boundary, and that $\mathcal{V} \subset H_0^1(\Omega)$ is a closed subspace. Consider the following eigenproblem, find $(\lambda_j, u_j) \in \mathbb{R}^+ \times \mathcal{V} \setminus \{0\}$ such that

$$\int_\Omega \nabla u_j \cdot \nabla w \, dx = \lambda_j \int_\Omega u_j w \, dx \qquad \text{and} \qquad \|u_j\|_{L^2(\Omega)} = 1 \tag{1.1}$$

for each $w \in \mathcal{V}$. Here $\mathbb{R}^+ := (0, \infty)$, and the eigenvalues λ_j are numbered in nondecreasing order and repeated by their multiplicities. The purpose of this article is to compute all eigenvalues within the *spectral interval of interest* $(0, \Lambda)$ for $\Lambda \in \mathbb{R}^+$ to a given accuracy in a distributed computing environment. In the following, the *relevant eigenfunctions* are those that are associated with eigenvalues in $(0, \Lambda)$.

Laplacian eigenproblems discretized using the finite element method (FEM) lead to (1.1) with $\mathcal{V}$ being the finite element space of choice. If the dimension of the finite element space is large, e.g., due to the complicated shape of Ω, it may happen that (1.1) cannot be solved using a single workstation. Then two types of distributed solution methods can be applied. First, a parallel eigenvalue iteration (such as shift-and-invert Lanczos) can be used together with a parallel solver for the shifted linear system; see, e.g., [2]. Second, one can use a domain decomposition (DD) method such as AMLS [7], RS-DDS [20], or the CMS variant proposed in [15]; see also [5, 6, 7, 19].

All aforementioned eigensolvers are Ritz methods. That is, instead of (1.1) one solves the problem, find $(\tilde{\lambda}_j, \tilde{u}_j) \in \mathbb{R}^+ \times \widetilde{\mathcal{V}} \setminus \{0\}$ such that for each $w \in \widetilde{\mathcal{V}}$

$$\int_\Omega \nabla \tilde{u}_j \cdot \nabla w \, dx = \tilde{\lambda}_j \int_\Omega \tilde{u}_j w \, dx \qquad \text{and} \qquad \|\tilde{u}_j\|_{L^2(\Omega)} = 1, \tag{1.2}$$

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†Department of Mathematics and Systems Analysis, Aalto University School of Science, Espoo, FI-00076 Aalto, Finland (antti.hannukainen@aalto.fi, jarmo.malinen@aalto.fi, antti.ojalammi@aalto.fi).

where the *method subspace* $\widetilde{\mathcal{V}} \subset \mathcal{V}$ is finite dimensional. The eigenvalues $\tilde{\lambda}_j$ are in a nondecreasing order and repeated according to their multiplicities. In this work, we propose a method subspace $\widetilde{\mathcal{V}}$ of DD type and analyze the relative error between the corresponding eigenvalues of (1.1) and (1.2). We point out that there are also other error sources. For example, there is an error due to applying FEM for discretizing Laplacian eigenproblems and due to solving (1.2) using an algebraic eigensolver, but these error sources are not treated here. For FEM eigenproblem error analysis, see, e.g., [8]. For error analysis of algebraic eigensolvers, see, e.g., [25].

In the context of (1.1), the method subspaces used in CMS, AMLS, and RD-DDS are associated with a decomposition of Ω into *nonoverlapping* subdomains $\{\Omega_j\}$. They are constructed by solving two kinds of eigenproblems: small, inexpensive *local problems* on each Ω_j and *interface problems* related to adjacent subdomains. It is noteworthy that the interface problems are never local, and their solution accrues a significant computational cost in existing DD methods. Nevertheless, the existing DD methods are useful for a wide class of problems especially if a large number of smallest eigenvalues is to be computed [16].

We propose a novel DD eigensolver, *partition of unity condensed pole interpolation* (PU-CPI) for the distributed solution of (1.1). PU-CPI is a Ritz method using a method subspace associated with a *relatively open, finite cover* of Ω, denoted by $\{U^{(p)}\}$, instead of a nonoverlapping decomposition. Since there are no geometric interfaces between the subdomains, solution of nonlocal interface problems is avoided. Consequently, the PU-CPI method subspace is constructed from *local subspaces* on $U^{(p)}$, whose basis is obtained by solving local eigenproblems. A partition of unity on $\{U^{(p)}\}$ is used to bind these local subspaces to a conforming method subspace as in [23]. Similar strategy for solution of direct problems has been presented and analyzed in [3, 11].

Because there are only local problems, PU-CPI does not require any communication between its distributed tasks associated with $\{U^{(p)}\}$. The master and workers communicate to distribute local data at the beginning, and to transfer the finished local results at the end of each task. Thus, PU-CPI can be used even if communication is expensive or nodes are not simultaneously available, e.g., on a cluster running in a cloud computing service or on networked workstations.

We show that the eigenvalue error resulting from PU-CPI depends on how accurately the relevant eigenfunctions (1.1) are approximated by the local subspaces. Thus, the design of the local subspace for $U^{(p)} \subset \Omega$ requires some understanding on the behavior of the these eigenfunctions restricted to $U^{(p)}$. It is well known that (excluding exceptional cases) the restriction of relevant eigenfunctions to any $U^{(p)}$ can be recovered from its trace on $\partial U^{(p)}$. We exploit this property on extended subdomains $\widehat{U}^{(p)}$, $U^{(p)} \subset \widehat{U}^{(p)} \subset \Omega$, and show, intuitively speaking, that the eigenfunction restricted to $U^{(p)}$ only loosely depends on its trace on $\partial\widehat{U}^{(p)}$. Due to this loose dependency, sufficiently good local subspaces, with small dimension, can be defined without referring to boundary values of relevant eigenfunctions on $\partial\widehat{U}^{(p)}$ at all. We expect that such loose dependency is a generic property of elliptic differential operators, making our approach applicable to other problems besides (1.1), e.g., in linear elasticity.

We proceed to review the major steps taken to design $\widetilde{\mathcal{V}}$. In Lemma 3.1 we give a representation formula that relates the restriction of a relevant eigenfunction to U and its trace on $\partial\widehat{U}$. As stated above, this restriction depends on the trace via a boundary-to-interior mapping $Z_U(\cdot)$, which is a nonlinear function from $(0, \Lambda)$ to a space of bounded linear operators. We construct the local subspace for U to approximate the range of this function. Lemma 3.2 shows that the range of Z_U consists of compact operators. We then introduce an *approximate-linearize-compress* strategy in the first main results of this article, Theorems 3.8 and 3.10, to study Z_U. Ultimately, an estimate for the relative eigenvalue error is given in Theorem 4.2.

Constructing the PU-CPI method subspace requires the solution of several local eigenproblems. The practical solution of these problems is always inexact, which is an error source not accounted for in the analysis given. However, numerical experiments given section 6 indicate that the total error is in good agreement with the bound given in Theorem 4.2, and the error associated with inexactly solving eigenproblems seems not to be significant here.

We give a unified analysis valid both for $\mathcal{V} = H_0^1(\Omega)$ or some finite element space as $\mathcal{V}$. Treating the infinite-dimensional setting is necessary for proving the loose dependency of an eigenfunction restricted to $U^{(p)}$ from its trace on $\partial\widehat{U}^{(p)}$ in terms of elliptic regularity theory. In addition, it helps in choosing convenient inner products for the subspaces needed. We envision finite element simulation as a typical application of PU-CPI. Hence, we give a detailed explanation of its application to first-order finite elements in three dimensions.

To demonstrate the potential of PU-CPI, we compute the lowest 200 eigenvalues of (1.1), where $\mathcal{V}$ is a tetrahedral first-order finite element space in $\mathbb{R}^3$. The resulting algebraic eigenvalue problem has approximately 10^7 unknowns. The PU-CPI computation took less than two hours on a cluster of 26 networked workstations; further details are given in section 6.

The article is organized as follows. We begin by reviewing the preliminaries and error analysis of Ritz methods. In section 3, we construct the local subspace for a single subdomain and derive the local error estimate for it. In section 4, we combine the local subspaces to the method subspace $\widetilde{\mathcal{V}}$ and introduce the global error estimates. Section 5 is devoted to standard first-order finite element space. We conclude the article with numerical examples in section 6, followed by a discussion.

2. Background. Let $\Omega', \Omega \subset \mathbb{R}^d$ for $d = 2, 3$ be open bounded sets with Lipschitz boundaries such that $\Omega' \subset \Omega$. The inner products for $H^1(\Omega')$ and $H_0^1(\Omega')$ are

$$(f, g)_{H^1(\Omega')} := (\nabla f, \nabla g)_{L^2(\Omega';\mathbb{R}^d)} + (f, g)_{L^2(\Omega')},$$
$$(f, g)_{H_0^1(\Omega')} := (\nabla f, \nabla g)_{L^2(\Omega';\mathbb{R}^d)}.$$

The corresponding norms are denoted by $\|\cdot\|_{H^1(\Omega')}$ and $\|\cdot\|_{H_0^1(\Omega')}$, respectively.

In the following, we discuss a subspace method for the eigenproblem related to the Laplace operator and its finite element discretization, treated using the formulation in (1.1) with different choices of the space $\mathcal{V}$. The Laplace operator is treated by setting $\mathcal{V} = H_0^1(\Omega)$, and the solution $(\lambda, u) \in \mathbb{R}^+ \times H_0^1(\Omega) \setminus \{0\}$ of (1.1) is required to satisfy

$$-\Delta u = \lambda u \quad \text{in } L^2(\Omega) \quad \text{and} \quad \|u\|_{L^2(\Omega)} = 1. \tag{2.1}$$

The finite element discretization of (2.1) is obtained when $\mathcal{V}$ is chosen as the corresponding finite element space; see section 5.

We work with restrictions of functions from the space $\mathcal{V} \subset H_0^1(\Omega)$ to a subdomain $\Omega' \subset \Omega$. Denote

$$\mathcal{V}(\Omega') := \{\, w|_{\Omega'} \mid w \in \mathcal{V} \,\} \quad \text{and} \quad tr\mathcal{V}(\Omega') := \{\, \gamma_{\partial\Omega'} w \mid w \in \mathcal{V}(\Omega') \,\},$$

where $\gamma_{\partial\Omega'} \in \mathcal{B}(H^1(\Omega'), H^{1/2}(\partial\Omega'))$ is the trace operator on $H^1(\Omega')$.[1] The space of functions with homogeneous boundary values is denoted by

$$\mathcal{V}_0(\Omega') := \{ \, w \in \mathcal{V}(\Omega') \mid \gamma_{\partial\Omega'} w = 0 \, \}.$$

The spaces $\mathcal{V}(\Omega')$, $\mathcal{V}_0(\Omega')$ inherit their inner products and norms from spaces $H^1(\Omega')$, $H^1_0(\Omega')$, respectively. For $tr\mathcal{V}(\Omega')$, we use the norm

$$\|f\|_{tr\mathcal{V}(\Omega')} := \frac{1}{\sqrt{2}} \min_{\substack{w \in \mathcal{V}(\Omega') \\ \gamma_{\partial\Omega'} w = f}} \|w\|_{H^1(\Omega')}. \tag{2.2}$$

We make a standing assumption that all these spaces are complete. This holds, e.g., if $\mathcal{V} = H^1_0(\Omega)$, or it is finite dimensional.

2.1. Subspace methods. If (1.1) is posed on $\Omega' \subset \Omega$ and in any closed subspace $\mathcal{W} \subset H^1_0(\Omega')$ instead of $\mathcal{V} \subset H^1_0(\Omega)$, we denote the set of eigenvalues as $\sigma(\mathcal{W})$.

The relative error between corresponding eigenvalues of (1.1) and (1.2) has been extensively studied (see, e.g., [4, 12, 22], the review article [8], and the references therein). These results are not straightforward, and there exist multiple variants with different assumptions. All such bounds (that the authors are aware of) estimate the error by a product of two terms as in (2.5). The latter one is related to the *eigenfunction approximation error*, i.e., the accuracy of approximation of one, or several, eigenfunctions of (1.1) in the method subspace $\widetilde{\mathcal{V}}$. The first term, denoted by $C(\lambda_j)$ in (2.5), is an expression that may depend on $\mathcal{V}$ and $\widetilde{\mathcal{V}}$ via $\sigma(\mathcal{V})$ and $\sigma(\widetilde{\mathcal{V}})$. Fortunately, this expression can be regarded as a generically unknown constant if the eigenfunction approximation error is sufficiently small.

In this work, we use an estimate adapted from [22, Thm. 3.2] due to its simplicity: the relative eigenvalue error is bounded in terms of the approximability of the corresponding eigenfunction in $\widetilde{\mathcal{V}}$. However, the bound contains a constant that is difficult to estimate a priori. Because the core of our analysis of PU-CPI is to bound the eigenfunction approximation error, our results can be combined with other relative eigenvalue error estimates as well.

The *spectral gap* of $\mathcal{V}$ on $(0, \Lambda)$ is defined as

$$\rho_\Lambda := \min_{\substack{\lambda,\mu \in \sigma(\mathcal{V}) \cap (0,\Lambda) \\ \lambda \neq \mu}} |\mu - \lambda|. \tag{2.3}$$

PROPOSITION 2.1. *Let ρ_Λ be as defined in* (2.3)*, $\widetilde{\mathcal{V}} \subset \mathcal{V}$ a finite-dimensional method subspace, $1 \le j \le \#\{\sigma(\mathcal{V}) \cap (0, \Lambda)\}$, and $(\lambda_j, u_j) \in \sigma(\mathcal{V}) \times \mathcal{V} \setminus \{0\}$ an eigenpair of* (1.1) *corresponding to a simple eigenvalue λ_j. Assume that the Hausdorff distance*

$$dist\left(\sigma(\widetilde{\mathcal{V}}) \cap (0, \Lambda), \sigma(\mathcal{V}) \cap (0, \Lambda)\right) \le \frac{1}{2}\rho_\Lambda. \tag{2.4}$$

Then there exists $\tilde{\lambda} \in \sigma(\widetilde{\mathcal{V}})$ and $C(\lambda_j) \equiv C(\lambda_j; \mathcal{V})$ such that

$$\frac{|\lambda_j - \tilde{\lambda}|}{\lambda_j} \le C(\lambda_j) \min_{v \in \widetilde{\mathcal{V}}} \|u_j - v\|^2_{H^1_0(\Omega)}. \tag{2.5}$$

[1] Here $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$.

Proposition 2.1 is a streamlined version of [22, Thm. 3.2]. The original statement gives an explicit formula for $C(\lambda_j)$ that unfortunately depends on the (a priori unknown) spectra $\sigma(\widetilde{\mathcal{V}})$ and $\sigma(\mathcal{V})$. To guarantee that $C(\lambda_j)$ remains uniformly bounded independently of $\widetilde{\mathcal{V}}$, we have introduced (2.4). If the relevant eigenfunctions are sufficiently well approximated in $\widetilde{\mathcal{V}}$, the Hausdorff distance $dist(\sigma(\widetilde{\mathcal{V}})\cap(0,\Lambda), \sigma(\mathcal{V})\cap(0,\Lambda))$ satisfies (2.4) by [22, Thm. 3.1]. Exactly when this happens in terms of $\widetilde{\mathcal{V}}$, depends on the spectral gap ρ_Λ which is unknown unless the exact spectrum $\sigma(\mathcal{V}) \cap (0, \Lambda)$ is known. Hence, there is no a priori quantitative statement on (2.4). Observe that [22] uses a different normalization of eigenfunctions which affects $C(\lambda_j)$ but is later taken into account in Theorem 4.2.

2.2. The PU-CPI method subspace. Let $\{U^{(p)}\}_{p=1}^M$ for $M \geq 2$ and $U^{(p)} \subset \Omega$ be an open cover of the domain $\Omega \subset \mathbb{R}^d$. In addition, assume that each $U^{(p)}$ has Lipschitz boundary and that there does not exist $p, q \in \{1, \dots, M\}, p \neq q$, satisfying $U^{(p)} \subset U^{(q)}$. We proceed to describe how the PU-CPI method subspace $\widetilde{\mathcal{V}}$ is constructed from the *local method subspaces* $\widetilde{\mathcal{V}}(U^{(p)}) \subset \mathcal{V}(U^{(p)})$.

For $p = 1, \dots, M$, let the *stitching operators* $R^{(p)} \in \mathcal{B}(\mathcal{V}(U^{(p)}), \mathcal{V})$ satisfy

$$(R^{(p)}w^{(p)})|_{\Omega\setminus U^{(p)}} = 0 \quad \text{and} \quad \sum_{p=1}^{M} R^{(p)}(w|_{U^{(p)}}) = w$$

for each $w^{(p)} \in \mathcal{V}(U^{(p)})$ and $w \in \mathcal{V}$. Suitable operators $\{R^{(p)}\}_{p=1}^M$ for $\mathcal{V} = H_0^1(\Omega)$ can be obtained by multiplication with a partition of unity associated with $\{U^{(p)}\}$ as in [23]. The PU-CPI method subspace $\widetilde{\mathcal{V}}$, depending on the local method subspaces $\{\widetilde{\mathcal{V}}(U^{(p)})\}_{p=1}^M$, is defined as

$$\widetilde{\mathcal{V}} := \left\{ w \in \mathcal{V} \mid w = \sum_{p=1}^{M} R^{(p)}w^{(p)} \quad \text{for} \quad w^{(p)} \in \widetilde{\mathcal{V}}(U^{(p)}) \right\}, \tag{2.6}$$

where each $\widetilde{\mathcal{V}}(U^{(p)}) \subset \mathcal{V}(U^{(p)})$ has a low dimension. If $\widetilde{\mathcal{V}}$ satisfies (2.6) and the assumptions of Proposition 2.1, the eigenvalue error depends on approximation properties of the local subspaces. Using a similar technique to [23] gives

$$\min_{v\in\widetilde{\mathcal{V}}} \|u_j - v\|_{H_0^1(\Omega)} \leq \|G\|_{L^\infty(\Omega)} \left(\sum_{p=1}^{M} \mathcal{E}(u_j, U^{(p)}) \right)^{1/2}, \tag{2.7}$$

where function $\mathcal{E}$ is the *local approximation error*,

$$\mathcal{E}(u, U^{(p)}) := \min_{w\in\widetilde{\mathcal{V}}(U^{(p)})} \int_{U^{(p)}} \left| \nabla \left[R^{(p)}(u|_{U^{(p)}} - w) \right] \right|^2 \, dx \tag{2.8}$$

and $G : \Omega \to \{1, \dots, M\}$ is defined as $G(x) := \#\{p \mid x \in U^{(p)}\}$. The aim is to design the local method subspaces $\widetilde{\mathcal{V}}(U^{(p)})$ so that both $\dim(\widetilde{\mathcal{V}}(U^{(p)}))$ and $\mathcal{E}_{j,p} \equiv \mathcal{E}(u_j, U^{(p)})$ are small.

3. Local method subspace. A local method subspace $\widetilde{\mathcal{V}}(U^{(p)}) \subset \mathcal{V}(U^{(p)})$ for a single subdomain $U^{(p)}$ is designed next. For notational convenience, denote $U = U^{(p)}$, $R = R^{(p)}$, and let (λ, u) be some solution to (1.1) satisfying $\lambda < \Lambda$.

3.1. Extended subdomain. Given $r > 0$ and $U \subset \Omega$, let $\widehat{U} \subset \Omega$ be a domain satisfying

$$\{ x \in \Omega \mid dist(x, U) < r \} \subset \widehat{U}. \tag{3.1}$$

Any such $\widehat{U}$ is called an *r-extension* of U, and we make it a standing assumption that both U and $\widehat{U}$ have Lipschitz boundaries. By our assumptions, $U \neq \Omega$, and hence $U \neq \widehat{U}$. In the following, $\widehat{U}$ is fixed unless otherwise stated. The effect of the parameter r is numerically studied in section 6. As shown in the next section, the essential component of the PU-CPI method is the operator-valued function $Z_U : (0, \Lambda) \to \mathcal{B}(tr\mathcal{V}(\widehat{U}), \mathcal{V}(U))$. It will be shown that Z_U is, in fact, analytic and, due to the use of the r-extension and elliptic regularity, also compact operator valued.

3.2. Eigenfunction representation formula. We represent $u|_{\widehat{U}}$ in terms of its boundary trace $\gamma_{\partial\widehat{U}}(u|_{\widehat{U}})$. By (1.1), $u|_{\widehat{U}}$ satisfies

$$\int_{\widehat{U}} \left(\nabla u|_{\widehat{U}} \cdot \nabla w - \lambda u|_{\widehat{U}} w \right) \, dx = 0 \tag{3.2}$$

for each $w \in \mathcal{V}_0(\widehat{U})$. We assume that there exists a right inverse $E \in \mathcal{B}(tr\mathcal{V}(\widehat{U}), \mathcal{V}(\widehat{U}))$ of $\gamma_{\partial\widehat{U}}$ satisfying

$$E : tr\mathcal{V}(\widehat{U}) \to \{ v \in \mathcal{V}(\widehat{U}) \mid v|_U = 0 \}. \tag{3.3}$$

Such an E always exists if $\mathcal{V} = H_0^1(\Omega)$. If the FEM is used for defining $\mathcal{V}$, $\{U^{(p)}\}$ and $\{\widehat{U}^{(p)}\}$ are *constructed* so that E exists; see section 5.

Equation (3.2) is solved by decomposing

$$u|_{\widehat{U}} = u_0 + Eu_B, \quad \text{where} \quad u_0 \in \mathcal{V}_0(\widehat{U}) \quad \text{and} \quad u_B := \gamma_{\partial\widehat{U}} \left(u|_{\widehat{U}} \right). \tag{3.4}$$

It follows from (3.3) that $u|_U = u_0|_U$. Using the decomposition in (3.4), (3.2) gives

$$\int_{\widehat{U}} \left(\nabla u_0 \cdot \nabla w - \lambda u_0 w \right) \, dx = - \int_{\widehat{U}} \left(\nabla w \cdot \nabla E u_B - \lambda w E u_B \right) \, dx \tag{3.5}$$

for each $w \in \mathcal{V}_0(\widehat{U})$, which defines u_0 as a function of λ and u_B. We proceed as in [18] and use an $L^2(\widehat{U})$-orthonormal eigenbasis expansion to solve (3.5). Let $(\mu_k, v_k) \in \mathbb{R}^+ \times \mathcal{V}_0(\widehat{U}) \setminus \{0\}$ be such that

$$\int_{\widehat{U}} \nabla v_k \cdot \nabla w \, dx = \mu_k \int_{\widehat{U}} v_k w \, dx \quad \text{and} \quad \|v_k\|_{L^2(\widehat{U})} = 1 \tag{3.6}$$

for each $w \in \mathcal{V}_0(\widehat{U})$. Assume that $\{\mu_k\}_k \subset \mathbb{R}^+$ are indexed in nondecreasing order and repeated according to their multiplicities. The set $\{v_k\}_k$ is $L^2(\widehat{U})$-orthonormal in $\mathcal{V}_0(\widehat{U})$, hence $\{v_k/\sqrt{\mu_k}\}_k$ is an $H_0^1(\widehat{U})$-orthonormal basis of $\mathcal{V}_0(\widehat{U})$. To solve u_0 from (3.5), expand in $H_0^1(\widehat{U})$

$$u_0 = \sum_{j=1}^{\dim(\mathcal{V}_0(\widehat{U}))} \alpha_j v_j, \quad \text{where each} \quad \alpha_j \in \mathbb{R}. \tag{3.7}$$

Using this expansion with (3.5) and setting $w = v_k$ in (3.6), the orthogonality of the eigenfunctions gives

$$\alpha_k(\mu_k - \lambda) = -\int_{\widehat{U}} (\nabla v_k \cdot \nabla E u_B - \lambda v_k E u_B)\ dx. \tag{3.8}$$

If $\lambda \notin \sigma(\mathcal{V}_0(\widehat{U}))$, u_0 is determined by solving α_k for $k = 1, \ldots, \dim \mathcal{V}_0(\widehat{U})$. To treat any $\lambda \in (0, \Lambda)$, we split the coefficients α_j in (3.7) into two groups. For this purpose, we use the parameter $\tilde{\Lambda}$ defined as

$$\tilde{\Lambda} = \eta\Lambda \quad \text{for} \quad \eta > 1,$$

and the counting function $K : \mathbb{R}^+ \to \mathbb{N}$, given by

$$K(t) := \#\{\ \mu_k \in \sigma(\mathcal{V}_0(\widehat{U}))\ |\ \mu_k \leq t\ \}.$$

In [18], the parameter η is called the *oversampling parameter*. Since $\lambda \in (0, \Lambda)$ the coefficients α_k in (3.7) for $k > K(\tilde{\Lambda})$ are obtained from (3.8). We have now proved the following lemma.

LEMMA 3.1. *Let $\tilde{\Lambda} > \Lambda > 0$. Assume that $(\lambda, u) \in (0, \Lambda) \times \mathcal{V}$ and $U \subset \widehat{U} \subset \Omega$, $\widehat{U} \neq \Omega$, satisfy* (1.1) *and* (3.1), *respectively. Then we have the following orthogonal splitting in $L^2(\widehat{U})$ and in $H_0^1(\widehat{U})$,*

$$u|_U = \sum_{k=1}^{K(\tilde{\Lambda})} \alpha_k\, v_k|_U + (Z(\lambda)u_B)|_U, \tag{3.9}$$

where $u_B = \gamma_{\partial\widehat{U}} u$, $\{\alpha_k\}_{k=1}^{K(\tilde{\Lambda})} \subset \mathbb{R}$, and $Z : (0, \Lambda) \to \mathcal{B}(tr\mathcal{V}(\widehat{U}), \mathcal{V}_0(\widehat{U}))$ is defined as

$$Z(t)w_B := \sum_{k=K(\tilde{\Lambda})+1}^{\dim(\mathcal{V}_0(\widehat{U}))} \frac{v_k}{\mu_k - t} \int_{\widehat{U}} (-\nabla v_k \cdot \nabla E w_B + t v_k E w_B)\ dx. \tag{3.10}$$

The sum converges uniformly for $t \in (0, \Lambda)$ in $H_0^1(\widehat{U})$ and $L^2(\widehat{U})$. Moreover, $Z(t)$ is an analytic function for $t \in (0, \Lambda)$.

There are many ways of showing that $Z(t) \in \mathcal{B}(tr\mathcal{V}(\widehat{U}), \mathcal{V}_0(\widehat{U}))$ for $t \in (0, \Lambda)$; e.g., by using Lemma 3.6. The function Z depends implicitly on $\tilde{\Lambda}, \widehat{U}$, and U in addition to t.

3.3. Evaluation of Z. Following the approach used in [18], we discuss how Z can be evaluated given $K(\tilde{\Lambda})$ the lowest eigenmodes[2] of (3.6). Denote

$$E_{\tilde{\Lambda}} := span\{v_1, \ldots, v_{K(\tilde{\Lambda})}\} \quad \text{for } (v_k, \mu_k) \text{ satisfying (3.6).}$$

Fix $t \in (0, \Lambda)$, $w_B \in tr\mathcal{V}(\widehat{U})$, and solve the auxiliary problem: Find $\hat{z}_0(t) \in \mathcal{V}_0(\widehat{U})$ such that

$$\int_{\widehat{U}} (\nabla \hat{z}_0(t) \cdot \nabla w - t\hat{z}_0(t)w)\ dx = -\int_{\widehat{U}} (\nabla w \cdot \nabla E w_B - t w E w_B)\ dx \tag{3.11}$$

[2]If $\mathcal{V}$ is a finite element space, the value of $K(\tilde{\Lambda})$ can be computed using LDL^T-decomposition and Sylvester's law of inertia. These kinds of decompositions are computed internally in eigensolvers, and we consider evaluating $K(\tilde{\Lambda})$ as an implementation issue.

for each $w \in \mathcal{V}_0(\widehat{U})$. As in section 3.2, each solution admits the orthogonal splitting

$$\hat{z}_0(t) = \sum_{k=1}^{K(\tilde{\Lambda})} \alpha_k v_k + Z(t) w_B \in E_{\tilde{\Lambda}} \oplus E_{\tilde{\Lambda}}^{\perp}$$

even though some α_k's cannot be *uniquely* solved from (3.11) for the exceptional $t \in \sigma(\mathcal{V}_0(\widehat{U}))$. After $\hat{z}_0(t)$ has been solved from (3.11), $Z(t)w_B$ can be evaluated as $Z(t)w_B = P\hat{z}_0(t)$, where $P \in \mathcal{B}(\mathcal{V}_0(\widehat{U}))$ is the $L^2(\widehat{U})$ orthogonal projection onto $E_{\tilde{\Lambda}}^{\perp}$.

3.4. The complementing subspace. Our aim is to design the finite-dimensional subspace $\widetilde{\mathcal{V}}(U)$ such that the local approximation error in (2.8), namely,

$$\min_{v \in \widetilde{\mathcal{V}}(U)} \int_U |\nabla[R(u|_U - v)]|^2 \, dx,$$

can be made arbitrarily small for any $(\lambda, u) \in (0, \Lambda) \times \mathcal{V}$ satisfying (1.1). For $w_B \in tr\mathcal{V}(\widehat{U})$ and $t \in (0, \Lambda)$, denote

$$Z_U(t) w_B = (Z(t)w_B)|_U. \tag{3.12}$$

Obviously by Lemma 3.1 and boundedness of the restriction operator, we have $Z_U : (0, \Lambda) \to \mathcal{B}(tr\mathcal{V}(\widehat{U}), \mathcal{V}(U))$. By Lemma 3.1,

$$u|_U = \sum_{k=1}^{K(\tilde{\Lambda})} \alpha_k \, v_k|_U + Z_U(\lambda) u_B \tag{3.13}$$

for some real-valued α_k's. We construct $\widetilde{\mathcal{V}}(U)$ according to the splitting in (3.13) as

$$\widetilde{\mathcal{V}}(U) = E_{\tilde{\Lambda}}(U) \oplus \mathcal{W}(U), \quad \text{where} \quad E_{\tilde{\Lambda}}(U) := \operatorname{span}\{v_1|_U, \ldots, v_{K(\tilde{\Lambda})}|_U\}, \tag{3.14}$$

and $\oplus$ denotes the orthogonal direct sum in $\mathcal{V}$. The space $\mathcal{W}(U)$ is called the *local complementing subspace*. Let

$$e_U(\mathcal{W}(U)) := \sup_{\substack{t \in (0,\Lambda) \\ w \in \mathcal{V}(\widehat{U})}} \inf_{v \in \mathcal{W}(U)} \frac{\int_U |\nabla[R(Z_U(t)w_B - v)]|^2 \, dx}{\|w\|^2_{H^1(\widehat{U})}}, \tag{3.15}$$

where $w_B = \gamma_{\partial \widehat{U}} w$. As the first term on the right-hand side of (3.13) is included in $\widetilde{\mathcal{V}}(U)$, the local approximation error of u on U has the estimate

$$\mathcal{E}(u, U) \equiv \min_{v \in \widetilde{\mathcal{V}}(U)} \int_U |\nabla[R(u|_U - v)]|^2 \, dx \le e_U(\mathcal{W}(U)) \|u|_{\widehat{U}}\|^2_{H^1(\widehat{U})} \tag{3.16}$$

for each $(\lambda, u) \in (0, \Lambda) \times \mathcal{V}$ satisfying (1.1).

Next, we design the local complementing subspace $\mathcal{W}(U)$ so that the local approximation error in (2.8) can be made arbitrarily small. We begin with the *interpolation step*. Denote the set of $N \ge 1$ Chebyshev nodes on the interval $(0, \Lambda)$ as $\{\xi_i\}_{i=1}^N \subset (0, \Lambda)$. Define the interpolant $\hat{Z} : (0, \Lambda) \to \mathcal{B}(tr\mathcal{V}(\widehat{U}), \mathcal{V}_0(\widehat{U}))$ as

$$\hat{Z}(t) = \sum_{i=1}^N \ell_i(t) Z(\xi_i), \quad \text{where} \quad \ell_i(t) = \prod_{\substack{1 \le j \le N \\ j \ne i}} \frac{t - \xi_j}{\xi_i - \xi_j} \quad \text{for } i = 1, \ldots, N \tag{3.17}$$

are the Lagrange interpolation polynomials. The interpolation error $\widehat{Z} - Z$ is studied in section 3.5. We proceed with a *linearization step*. Define a linear operator[3]

$$B \in \mathcal{B}(tr\mathcal{V}(\widehat{U};\mathbb{R}^N), \mathcal{V}(U)) \quad \text{as} \quad B\boldsymbol{v}_B := \begin{bmatrix} Z_U(\xi_1) & \cdots & Z_U(\xi_N) \end{bmatrix} \boldsymbol{v}_B. \tag{3.18}$$

Here $\hat{Z}_U(t)w_B = (\hat{Z}(t)w_B)|_U$ for all $w_B \in tr\mathcal{V}(\widehat{U})$ and $t \in (0,\Lambda)$. Furthermore,

$$B\boldsymbol{\ell}(t)w_B = B\begin{bmatrix} \ell_1(t)w_B & \cdots & \ell_N(t)w_B \end{bmatrix}^T = \hat{Z}_U(t)w_B$$

and, hence, range$(\hat{Z}_U(t)) \subset$ range(B) for any $t \in (0,\Lambda)$.

We continue with the *finite-rank approximation step*. Given the finite-rank operator $\widehat{B} \in \mathcal{B}(tr\mathcal{V}(\widehat{U};\mathbb{R}^N), \mathcal{V}(U))$, the complementing subspace is fixed as $\mathcal{W}(U) :=$ range$(\widehat{B})$. We will show that $e_U(\mathcal{W}(U))$ in (3.15) is bounded from above by

$$\|R(Z_U(t) - \hat{Z}_U(t))\|_{\mathcal{B}(tr(\mathcal{V}(\widehat{U}),\mathcal{V})} \quad \text{and} \quad \|B - \widehat{B}\|_*,$$

resulting in Theorem 4.2.

If $\mathcal{V} = H_0^1(\Omega)$ each of the operators $Z_U(\xi_i)$, $i = 1, \dots, N$, is compact, which makes finding $\widehat{B}$ feasible.

Lemma 3.2. *Let $U \subset \widehat{U} \subset \Omega \subset \mathbb{R}^d$, $\widehat{U} \neq \Omega$, be as in* (3.1)*, $\mathcal{V} = H_0^1(\Omega)$, and $Z_U : (0,\Lambda) \to \mathcal{B}(tr\mathcal{V}(\widehat{U};\mathbb{R}^N), \mathcal{V}(U))$ be as defined in* (3.12)*. In addition, assume that $\widehat{U}$ is a convex polygonal ($d = 2$) or convex polyhedral domain ($d = 3$). Then $Z_U(t)$ is a compact operator from $tr\mathcal{V}(\widehat{U})$ to $\mathcal{V}(U)$ for all $t \in (0,\Lambda) \setminus \sigma(\mathcal{V}_0(\widehat{U}))$.*

This lemma is proved below.

Representing $u|_U$ in terms of $\gamma_{\partial\widehat{U}}u$ is motivated by Lemma 3.2, keeping in mind that compact operators can be approximated by finite-rank operators in the operator norm. Further, the same holds for B in (3.18) since the number N of Chebyshev nodes is finite. We need the following proposition.

Proposition 3.3. *Let $\mathcal{U}, \mathcal{X}, \mathcal{Y}$ be Banach spaces, $T \in \mathcal{B}(\mathcal{U},\mathcal{Y})$, range$(T) \subset \mathcal{X}$, and $\mathcal{X}$ continuously embedded in $\mathcal{Y}$. Then $T \in \mathcal{B}(\mathcal{U},\mathcal{X})$. In addition, if the embedding $\mathcal{X} \subset \mathcal{Y}$ is compact, then T is a compact operator from $\mathcal{U}$ to $\mathcal{Y}$.*

Proof. Let $u_j \to u$ in $\mathcal{U}$ and $Tu_j \to x$ in $\mathcal{X}$. Since $T : \mathcal{U} \to \mathcal{Y}$ is bounded, $Tu_j \to Tu$ in $\mathcal{Y}$. As $\mathcal{X}$ is continuously embedded in $\mathcal{Y}$, $x = Tu$ holds as an equality in $\mathcal{X}$. We have now shown that $T : \mathcal{U} \to \mathcal{X}$ is a closed linear operator. The first claim follows from the closed graph theorem. The second claim follows since the composition of a compact operator and a bounded operator is compact. ∎

Hence, if $\mathcal{V} = H_0^1(\Omega)$ and $t \in (0,\Lambda)$, the compactness of $Z_U(t) \in \mathcal{B}(tr\mathcal{V}(\widehat{U};\mathbb{R}^N), \mathcal{V}(U))$ follows by showing that $Z_U(t)w_B \in H^2(U)$ for all $t \in (0,\Lambda)$ and $w_B \in tr\mathcal{V}(\widehat{U})$. Due to standing assumptions made on U, $H^2(U)$ is compactly embedded in $H^1(U)$; see, e.g., [1, Thm. 6.3].

Proposition 3.4. *Let the domains $U \subset \hat{U} \subset \Omega \subset \mathbb{R}^d$ be as in* (3.1)*. Let $\mathcal{V} = H_0^1(\Omega)$, and $u \in \mathcal{V}(\widehat{U})$ such that $\Delta u \in L^2(\widehat{U})$. Assume that one of the following holds:*

(i) *$\partial\Omega \cap \partial U = \emptyset$;*

(ii) *$d = 2$, $\widehat{U}$ is a convex polygonal domain, and $\partial\Omega \cap \partial U \neq \emptyset$; or*

(iii) *$d = 3$, $\widehat{U}$ is a convex polyhedral domain, and $\partial\Omega \cap \partial U \neq \emptyset$.*

Then $u|_U \in H^2(U)$.

[3]Here $tr\mathcal{V}(\hat{U};\mathbb{R}^N)$ is defined as $[tr\mathcal{V}(\hat{U})]^N$ and equipped with the natural Hilbert space norm.

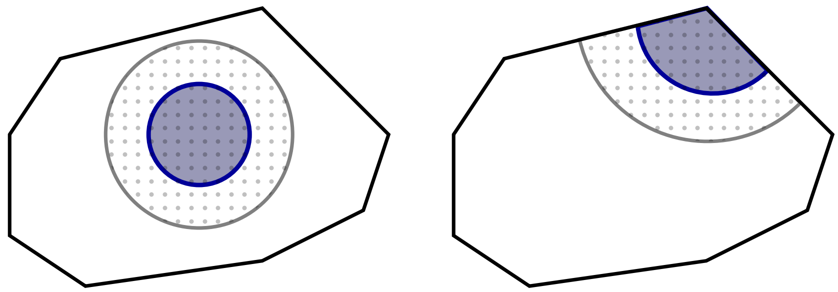

FIG. 1. *Illustration of cases* (i) *and* (ii) *in Proposition* 3.4 *for* $\Omega \subset \mathbb{R}^2$. *The solid black line depicts* $\partial\Omega$, *and the subdomains* U, $\widehat{U}$ *are shown by the blue and the grey dotted areas, respectively. There are two exclusive cases, namely,* (i) $\partial\Omega \cap \partial U = \emptyset$ *(left) and* (ii) $\partial\Omega \cap \partial U \neq \emptyset$ *(right).*

Cases (i) and (ii) are illustrated in Figure 1.

Proof. If (i) holds, the claim follows from the interior regularity estimate; see, e.g., [14, Chap. 6.3]. Assume that (ii) or (iii) holds. Let $\varphi \in C^\infty(\widehat{U})$ be a cutoff function satisfying $\varphi = 1$ in U and $\varphi = 0$ on $\partial\widehat{U} \setminus \partial\Omega$. The function $\varphi u \in H_0^1(\widehat{U})$ satisfies $\Delta(\varphi u) \in L^2(\widehat{U})$ by a straightforward computation. By [17, Chaps. 2.4 and 2.6] and assumptions (ii), (iii), we have $\varphi u \in H^2(\widehat{U})$. The claim follows from $(\varphi u)|_U = u|_U$. ∎

We complete this section by giving a proof of Lemma 3.2.

Proof of Lemma 3.2. Fix $t \in (0,\Lambda) \setminus \sigma(\mathcal{V}_0(\widehat{U}))$ and $w_B \in tr\mathcal{V}(\widehat{U})$. Let $\hat{z} \in \mathcal{V}(\widehat{U})$ be the variational solution of

$$\begin{cases} (\Delta + t)\hat{z} = 0 & \text{in} \quad \widehat{U}, \\ \hat{z} = w_B & \text{on} \quad \partial\widehat{U}, \end{cases}$$

obviously satisfying $\hat{z} \in H^1(\widehat{U})$ and $\Delta\hat{z} \in L^2(\widehat{U})$. Similarly to section 3.2, decompose $\hat{z} = \hat{z}_0 + Ew_B$, where E satisfies (3.3). As $t \notin \sigma(\mathcal{V}_0(\widehat{U}))$,

$$\hat{z}_0 = Z(t)w_B + \sum_{k=1}^{K(\tilde{\Lambda})} \frac{v_k}{\mu_k - t} \int_{\widehat{U}} \left(-\nabla v_k \cdot \nabla Ew_B + tv_k Ew_B\right) \, dx.$$

Further, using $\hat{z} = \hat{z}_0 + Ew_B$ gives

$$Z(t)w_B + Ew_B = \hat{z} - \sum_{k=1}^{K(\tilde{\Lambda})} \frac{v_k}{\mu_k - t} \int_{\widehat{U}} \left(-\nabla v_k \cdot \nabla Ew_B + tv_k Ew_B\right) \, dx.$$

Since the sum on the right-hand side has a finite number of terms, where $\Delta v_k \in L^2(\widehat{U})$, it follows that $\Delta\left(Z(t)w_B + Ew_B\right) \in L^2(\widehat{U})$. Using Proposition 3.4 and the property $(Ew_B)|_U = 0$ gives $\left(Z(t)w_B + Ew_B\right)|_U = Z_U(t)w_B \in H^2(U)$. Since it is already known that $Z_U(t) \in \mathcal{B}(tr\mathcal{V}(\widehat{U}), \mathcal{V}(U))$, Proposition 3.3 completes the proof. ∎

Remark 3.5. The assumption of convexity for the compactness of $Z_U(t)$ can be relaxed using a more technical variant of Proposition 3.4 stated in weighted Sobolev spaces; see, e.g., [24]. In addition, Lemma 3.2 can be extended to cover all values $t \in (0,\Lambda)$.

3.5. Interpolation error. Next, we study how the error terms,

$$e_0 := \left\| \left(\hat{Z}(t) - Z(t) \right) w_B \right\|_{L^2(\widehat{U})} \quad \text{and} \quad e_1 := \left\| \left(\hat{Z}(t) - Z(t) \right) w_B \right\|_{H^1_0(\widehat{U})}, \tag{3.19}$$

depend on N and $\tilde{\Lambda}$. By Lemma 3.1, the function $Z(t)w_B \in \mathcal{V}_0(\widehat{U})$ admits the expansion

$$Z(t)w_B = \sum_{k=K(\tilde{\Lambda})+1}^{\dim(\mathcal{V}_0(\widehat{U}))} \frac{c_{1,k}(w_B) + tc_{0,k}(w_B)}{\mu_k - t} v_k,$$

where the coefficients $c_{0,k} : tr\mathcal{V}(\widehat{U}) \to \mathbb{R}$ and $c_{1,k} : tr\mathcal{V}(\widehat{U}) \to \mathbb{R}$ are defined as

$$c_{0,k}(w_B) := (v_k, Ew_B)_{L^2(\widehat{U})} \quad \text{and} \quad c_{1,k}(w_B) := -(\nabla v_k, \nabla E w_B)_{L^2(\widehat{U};\mathbb{R}^d)} \tag{3.20}$$

for $k = 1, \ldots, \dim(\mathcal{V}_0(\widehat{U}))$. A technical estimate related to these coefficients is given in the following lemma.

Lemma 3.6. *Let $c_{0,k}(w_B)$ and $c_{1,k}(w_B)$ be as in* (3.20)*. Then*

$$\sum_{k=K(\tilde{\Lambda})+1}^{\dim(\mathcal{V}_0(\widehat{U}))} c^2_{0,k}(w_B) \leq \|Ew_B\|^2_{L^2(\widehat{U})} \quad \textit{and} \quad \sum_{k=K(\tilde{\Lambda})+1}^{\dim(\mathcal{V}_0(\widehat{U}))} \frac{c^2_{1,k}(w_B)}{\mu_k} \leq \|\nabla(Ew_B)\|^2_{L^2(\widehat{U};\mathbb{R}^d)}.$$

Proof. Both inequalities follow from almost identical arguments, and we only prove the latter inequality. For any $u \in \mathcal{V}(\widehat{U})$ define $P_1 u \in \mathcal{V}_0(\widehat{U})$ uniquely by the Riesz representation theorem on the Hilbert space $\mathcal{V}_0(\widehat{U})$, requiring

$$(\nabla P_1 u, \nabla v)_{L^2(\widehat{U};\mathbb{R}^d)} = (\nabla u, \nabla v)_{L^2(\widehat{U};\mathbb{R}^d)} \text{ for each } v \in \mathcal{V}_0(\widehat{U}).$$

Then the mapping $u \mapsto P_1 u$ is linear, it satisfies $P_1^2 = P_1$, and $\|\nabla(P_1 E w_B)\|_{L^2(\widehat{U};\mathbb{R}^d)} \leq \|\nabla E w_B\|_{L^2(\widehat{U};\mathbb{R}^d)}$. Since $P_1 u$ is uniquely defined, it also follows that $P_1 u = u$ for all $u \in \mathcal{V}_0(\widehat{U})$. Hence, P_1 is a projection on $\mathcal{V}(\widehat{U})$ with range$(P_1) = \mathcal{V}_0(\widehat{U})$. Since $\{v_k/\sqrt{\mu_k}\}_k$ is an orthonormal basis of $\mathcal{V}_0(\widehat{U})$, we have

$$P_1 E w_B = - \sum_{k=1}^{\dim(\mathcal{V}_0(\widehat{U}))} \frac{v_k}{\sqrt{\mu_k}} \cdot \frac{c_{1,k}(w_B)}{\sqrt{\mu_k}}.$$

The claim follows using Parseval's identity. ∎

Denote

$$f_{m,k}(t) = t^{1-m}(\mu_k - t)^{-1}$$

for $m = 0, 1$ and $k = K(\tilde{\Lambda}) + 1, \ldots, \dim(\mathcal{V}_0(\widehat{U}))$. Recalling (3.17), we have

$$\begin{aligned} \left(Z(t) - \hat{Z}(t) \right) w_B = &\sum_{k=K(\tilde{\Lambda})+1}^{\dim(\mathcal{V}_0(\widehat{U}))} \left(f_{1,k}(t) - \sum_{i=1}^{N} \ell_i(t) f_{1,k}(\xi_i) \right) c_{1,k}(w_B) v_k \\ &+ \sum_{k=K(\tilde{\Lambda})+1}^{\dim(\mathcal{V}_0(\widehat{U}))} \left(f_{0,k}(t) - \sum_{i=1}^{N} \ell_i(t) f_{0,k}(\xi_i) \right) c_{0,k}(w_B) v_k. \end{aligned} \tag{3.21}$$

Observe that the expressions in parentheses in (3.21) are Lagrange interpolation errors with Chebyshev nodes $\{\xi_i\}_{i=1}^N \subset (0,\Lambda)$. The derivatives of $f_{m,k}$ satisfy

$$\frac{1}{N!}\frac{\mathrm{d}^N f_{m,k}}{\mathrm{d}t^N}(t) = \frac{\mu_k^{1-m}}{(\mu_k - t)^{(N+1)}}. \tag{3.22}$$

Hence, we have the estimate for $k = K(\tilde\Lambda)+1,\dots,\dim(\mathcal{V}_0(\widehat U))$,

$$\mu_k^{1+m}\|f_{m,k}(\cdot) - \sum_{i=1}^N \ell_i(\cdot) f_{m,k}(\xi_i)\|^2_{L^\infty(0,\Lambda)} \le \frac{\mu_k^{3-m}\Lambda^{2N}}{4^{2N-1}(\mu_k-\Lambda)^{2(N+1)}} \tag{3.23}$$

for $m = 0, 1$; see, e.g., [13, Chap. 3.3]. We are now in position to give an estimate for the error terms e_0 and e_1.

Lemma 3.7. *Let $t \in (0,\Lambda)$, $w_B \in tr\mathcal{V}(\widehat U)$, and $\hat Z(t)$ be as in* (3.17). *Then the error terms in* (3.19) *satisfy*

$$e_l \le 12\,[4(\eta-1)]^{-N-1}\left(\eta^{l+1}\Lambda^{l-1} + \eta^{l+2}\Lambda^l\right)^{1/2}\|Ew_B\|_{\mathcal{V}(\widehat U)}$$

for $l = 0, 1$ and $\eta := \tilde\Lambda/\Lambda$.

Observe that for $\eta > 5/4$, e_1 and e_0 converge to zero as $N \to \infty$.

Proof. As the estimates for $l = 0,1$ follow from similar arguments, we only consider $l = 1$. By the triangle inequality, Parseval's identity, and (3.21), we have

$$\begin{aligned}\frac{1}{2}e_1^2 \le &\sum_{k=K(\tilde\Lambda)+1}^{\dim(\mathcal{V}_0(\widehat U))} \mu_k^2\left(f_{1,k}(t) - \sum_{i=1}^N \ell_i(t) f_{1,k}(\xi_i)\right)^2 \cdot \frac{c_{1,k}^2(w_B)}{\mu_k}\\ &+ \sum_{k=K(\tilde\Lambda)+1}^{\dim(\mathcal{V}_0(\widehat U))} \mu_k\left(f_{0,k}(t) - \sum_{i=1}^N \ell_i(t) f_{0,k}(\xi_i)\right)^2 \cdot c_{0,k}^2(w_B).\end{aligned} \tag{3.24}$$

We proceed to estimate the right-hand side of (3.23). Since $\mu_k \ge \eta\Lambda = \tilde\Lambda > \Lambda$, we have

$$\frac{\mu_k^{3-m}\Lambda^{2N}}{4^{2N-1}(\mu_k-\Lambda)^{2(N+1)}} \le \frac{\Lambda^{1-m}}{4^{2N-1}\eta^{2N+m-1}}\cdot\left(\frac{\mu_k}{\mu_k-\Lambda}\right)^{2(N+1)}$$

and $\mu_k(\mu_k-\Lambda)^{-1} = (1-\Lambda/\mu_k)^{-1} \le \eta(\eta-1)^{-1}$, recalling $\eta > 1$. Hence,

$$\mu_k^{1+m}\|f_{m,k}(\cdot) - \sum_{i=1}^N \ell_i(\cdot) f_{m,k}(\xi_i)\|^2_{L^\infty(0,\Lambda)} \le \frac{\Lambda^{1-m}\eta^{3-m}}{4^{2N-1}}\cdot\left(\frac{1}{\eta-1}\right)^{2(N+1)}. \tag{3.25}$$

Using Lemma 3.6 and (3.25) together with (3.24) gives

$$e_l^2 \le \frac{2\eta^{l+1}}{4^{2N-1}(\eta-1)^{2N+2}}\left(\Lambda^{l-1}\|\nabla(Ew_B)\|^2_{L^2(\widehat U)} + \eta\Lambda^l\|Ew_B\|^2_{L^2(\widehat U)}\right) \quad \text{for } l = 1.$$

Carrying out similar argumentation leads to the same formula for $l = 0$. Estimating the coefficient completes the proof. ∎

We conclude this subsection by using Lemma 3.7 to obtain an upper bound for the local interpolation error.

THEOREM 3.8. *Let* $Z, \hat{Z} : (0,\Lambda) \to \mathcal{B}(tr\mathcal{V}(\widehat{U};\mathbb{R}^N), \mathcal{V}(\widehat{U}))$ *be as defined in* (3.10) *and* (3.17)*, respectively. In addition, define* $Z_U, \hat{Z}_U : (0,\Lambda) \to \mathcal{B}(tr\mathcal{V}(\widehat{U};\mathbb{R}^N), \mathcal{V}(U))$ *as* $Z_U(t)w = (Z(t)w)|_U$ *and* $\hat{Z}_U(t)w = (Z(t)w)|_U$*, respectively. Then for* $t \in (0,\Lambda)$

$$\|\hat{Z}_U(t) - Z_U(t)\|_{\mathcal{B}(tr\mathcal{V}(\widehat{U}),\mathcal{V})} \le C_E\, e(\eta, N),$$

where $C_E = C_E(\mathcal{V}, U, \widehat{U}) := \|E\|_{\mathcal{B}(tr\mathcal{V}(\widehat{U}),\mathcal{V}(\widehat{U}))}$ *and*

$$e(\eta, N) = 12\eta\,[4(\eta-1)]^{-N-1}\left(2 + \eta\Lambda + \frac{1}{\eta\Lambda}\right)^{1/2}.$$

Recall that $Z_U, \hat{Z}_U$ depend implicitly on $\tilde{\Lambda}, N$. We expect the constant C_E to be inversely proportional to the extension radius r. Note that for $\eta > 5/4$, increasing the number of interpolation points N decreases the error exponentially.

3.6. Low-rank approximation error. Recall the definitions of the operator $B \in \mathcal{B}(tr\mathcal{V}(\hat{U};\mathbb{R}^N), \mathcal{V}(U))$ in (3.18) and $\mathcal{W}(U)$,

$$B\boldsymbol{v}_B := \begin{bmatrix} Z_U(\xi_1) & \cdots & Z_U(\xi_N)\end{bmatrix}\boldsymbol{v}_B \quad \text{and} \quad \mathcal{W}(U) := \operatorname{range}(\widehat{B}),$$

where $\widehat{B} \in \mathcal{B}(tr\mathcal{V}(\widehat{U};\mathbb{R}^N), \mathcal{V}(U))$ is a finite-rank operator. Next, we relate the error term in (3.15) to the operator norm of $B - \widehat{B}$. We define

$$\|\boldsymbol{w}_B\|_{tr\mathcal{V}(\widehat{U};\mathbb{R}^N)} := \left(\sum_{i=1}^{N} \|w_{B,i}\|^2_{tr\mathcal{V}(\widehat{U})}\right)^{1/2} \tag{3.26}$$

and

$$\|w\|_{\mathcal{V}_R(U)} := \left(\int_U |\nabla(Rw)|^2\,dx + \int_U w^2\,dx\right)^{1/2}. \tag{3.27}$$

We proceed with a technical lemma.

LEMMA 3.9. *For any* $t \in (0,\Lambda)$ *and* $w \in \mathcal{V}(\widehat{U})$

$$\|\boldsymbol{\ell}(t)w_B\|_{tr\mathcal{V}(\widehat{U};\mathbb{R}^N)} \le \frac{\Lambda_N}{\sqrt{2}}\|w\|_{H^1(\widehat{U})},$$

where $w_B = \gamma_{\partial\widehat{U}}w$ *and* $\Lambda_N := \max_{t\in[0,\Lambda]}\sum_{i=1}^N |\ell_i(t)|$ *is the Lebesgue constant related to the Chebyshev nodes* $\{\xi_i\}_{i=1}^N \subset (0,\Lambda)$.

For the estimate of the Lebesgue constant see, e.g., [10].

Proof. Using definitions (3.26) and (2.2),

$$\|\boldsymbol{\ell}(t)w_B\|^2_{tr\mathcal{V}(\widehat{U};\mathbb{R}^N)} = \|w_B\|^2_{tr\mathcal{V}(\widehat{U})}\sum_{i=1}^N |\ell_i(\lambda)|^2 \le \frac{1}{2}\|w\|^2_{H^1(\widehat{U})}\sum_{i=1}^N |\ell_i(\lambda)|^2.$$

The proof is completed by observing that $\sum_{i=1}^N |\ell_i(t)|^2 \le \left(\sum_{i=1}^N |\ell_i(t)|\right)^2 \le \Lambda_N^2$. ∎

We are now in position to give an upper bound for the error term $e_U(\mathcal{W}(U))$ in (3.15).

THEOREM 3.10. *Let* $\{\xi_i\}_{i=1}^N$ *be the Chebyshev nodes on* $(0,\Lambda)$, $Z_U(t)$ *be as in* (3.12), *and* e_U *as defined in* (3.15). *Further, let* $R \in \mathcal{B}(\mathcal{V}(U),\mathcal{V})$ *be a stitching operator as defined in section* 2.2, *and* $B \in \mathcal{B}(tr\mathcal{V}(\widehat{U};\mathbb{R}^N),\mathcal{V}(U))$ *be as defined in* (3.18). *For any* $\widehat{B} \in \mathcal{B}(tr\mathcal{V}(\widehat{U};\mathbb{R}^N),\mathcal{V}(U))$,

$$e_U(\mathcal{W}(U))^{1/2} \le \frac{1}{\sqrt{2}}\left[C_E\; e(\eta,N)\|R\|_{\mathcal{B}(\mathcal{V}(U),\mathcal{V})} + \Lambda_N\|B-\widehat{B}\|_*\right],$$

where $\mathcal{W}(U) = \mathrm{range}(\widehat{B})$, *and* $C_E, e(\eta,N)$ *are as defined in Theorem* 3.8. *Here we denote* $\|\cdot\|_* := \|\cdot\|_{\mathcal{B}(tr\mathcal{V}(\widehat{U};\mathbb{R}^N),\mathcal{V}_R(U))}$.

Proof. Let $w \in H^1(\widehat{U})$ and $w_B = \gamma_{\partial\widehat{U}}w$. Observe that

$$\begin{aligned}\inf_{v\in\mathcal{W}(U)}&\left(\int_U |\nabla[R(Z_U(t)w_B - v)]|^2\,dx\right)^{1/2}\\ &\le \frac{1}{\sqrt{2}}\|R\|_{\mathcal{B}(\mathcal{V}(U),\mathcal{V})}\|(Z_U(t)-\hat{Z}_U(t))\|_{\mathcal{B}(tr\mathcal{V}(\widehat{U}),\mathcal{V})}\|w\|_{H^1(\widehat{U})}\\ &\quad+\inf_{v\in\mathcal{W}(U)}\left(\int_U |\nabla[R(\hat{Z}_U(t)w_B - v)]|^2\,dx\right)^{1/2}.\end{aligned}$$

The first term on the right-hand side is bounded by using Theorem 3.8. Choosing $v = \widehat{B}\boldsymbol{\ell}(t)w_B \in \mathcal{W}(U)$ and recalling $\hat{Z}_U(t)w_B = B\boldsymbol{\ell}(t)w_B$ yields

$$\inf_{v\in\mathcal{W}(U)}\left(\int_U |\nabla[R(\hat{Z}_U(t)w_B - v)]|^2\,dx\right)^{1/2} \le \|B-\widehat{B}\|_*\|\boldsymbol{\ell}(t)w_B\|_{tr\mathcal{V}(\widehat{U};\mathbb{R}^N)}.$$

Lemma 3.9 completes the proof. ∎

We have now constructed the local subspace $\widetilde{\mathcal{V}}(U)$ and estimated the local approximation error $\mathcal{E}(u,U)$ for a subdomain $U = U^{(p)} \subset \Omega$ via (3.16). The error estimate for the global reduced problem follows by using the stitching operators.

4. PU-CPI. We proceed to define the local subspaces $\widetilde{\mathcal{V}}(U^{(p)})$ used in the PU-CPI method and to derive a relative eigenvalue error estimate.

We extend the notation of section 3 to the case of several subdomains $\{U^{(p)}\}_{p=1}^M$, and we set $U = U^{(p)}$ for $p \in \{1,\dots,M\}$. Denote the r-extension of $U^{(p)}$ by $\widehat{U}^{(p)}$ as in (3.1). Let $(\mu_k^{(p)}, v_k^{(p)}) \in \mathbb{R}^+ \times \mathcal{V}_0(\widehat{U}^{(p)}) \setminus \{0\}$ satisfy

$$\int_{\widehat{U}^{(p)}} \nabla v_k^{(p)}\cdot\nabla w\,dx = \mu_k^{(p)}\int_{\widehat{U}^{(p)}} v_k^{(p)} w\,dx$$

for each $w \in \mathcal{V}_0(\widehat{U}^{(p)})$ as in (3.6). We further require $\{v_k^{(p)}\}$ to be an $L^2(\widehat{U}^{(p)})$-orthonormal set, that $\mu_k^{(p)}$ are enumerated in nondecreasing order, and $K^{(p)}(\tilde{\Lambda}) := \#\{\, k \in \mathbb{N} \mid \mu_k^{(p)} \le \tilde{\Lambda}\,\}$. Similarly to (3.14), the local subspaces are $\widetilde{\mathcal{V}}(U^{(p)}) = E_{\tilde{\Lambda}}(U^{(p)}) \oplus \mathcal{W}(U^{(p)})$, where

$$E_{\tilde{\Lambda}}(U^{(p)}) = \mathrm{span}\{v_1^{(p)}|_{U^{(p)}},\dots,v_{K^{(p)}}^{(p)}|_{U^{(p)}}\}. \tag{4.1}$$

Define $Z^{(p)} : (0,\Lambda) \to \mathcal{B}(tr\mathcal{V}(\widehat{U}^{(p)}),\mathcal{V}_0(\widehat{U}^{(p)}))$ by replacing μ_k, v_k, and E in (3.12) by $\mu_k^{(p)}$, $v_k^{(p)}$, and the right inverse of the trace operator $E^{(p)} : tr\mathcal{V}(\widehat{U}^{(p)}) \to$

$\{ \ v \in \mathcal{V}(\widehat{U}^{(p)}) \mid v|_{U^{(p)}} = 0 \ \}$. Recall that the existence of $E^{(p)}$ is a structural assumption made on $\mathcal{V}$, $U^{(p)}$, and $\widehat{U}^{(p)}$.

Let $Z_{U^{(p)}} : (0, \Lambda) \to \mathcal{B}(tr\mathcal{V}(\widehat{U}^{(p)}), \mathcal{V}(U^{(p)}))$ be defined as in (3.12) and

$$\begin{aligned} &B^{(p)} \in \mathcal{B}(tr\mathcal{V}(\widehat{U}^{(p)}; \mathbb{R}^N), \mathcal{V}(U^{(p)})) \quad \text{such that} \\ &B^{(p)} = \begin{bmatrix} Z_{U^{(p)}}(\xi_1) & \cdots & Z_{U^{(p)}}(\xi_N) \end{bmatrix}. \end{aligned} \tag{4.2}$$

We choose the complementing subspace as $\mathcal{W}(U^{(p)}) = \mathrm{range}(\widehat{B}^{(p)})$, where $\widehat{B}^{(p)} \in \mathcal{B}(tr\mathcal{V}(\widehat{U}^{(p)}; \mathbb{R}^N), \mathcal{V}(U^{(p)}))$ will later be a low-rank approximation of $B^{(p)}$.

Assumption 4.1. Let $\widetilde{\mathcal{V}} \subset \mathcal{V}$ be a finite-dimensional method subspace, $\Lambda > 0$, $(\lambda_j, u_j) \in (0, \Lambda) \times \mathcal{V}$ be an eigenpair of (1.1) corresponding to a simple eigenvalue λ_j, and $\widetilde{\mathcal{V}}$ satisfy the spectral convergence assumption (2.4). Let $\widetilde{\Lambda} = \eta\Lambda$ for the oversampling parameter $\eta > 1$, and let $\{\xi_i\}_{i=1}^N$ be the Chebyshev interpolation points of $(0, \Lambda)$.

THEOREM 4.2. *Make Assumption* 4.1. *For $p = 1, \ldots, M$, let $E_{\tilde{\Lambda}}(U^{(p)})$ be as defined in* (4.1), *$R^{(p)} \in \mathcal{B}(\mathcal{V}(U^{(p)}), \mathcal{V})$ satisfy the assumptions of section* 2.2, *$B^{(p)}$ be as defined in* (4.2), *and $\widehat{B}^{(p)} \in \mathcal{B}(tr\mathcal{V}(\widehat{U}^{(p)}; \mathbb{R}^N), \mathcal{V}(U^{(p)}))$. Define the PU-CPI method subspace $\widetilde{\mathcal{V}}$ as in* (2.6) *using the local subspaces $\widetilde{\mathcal{V}}(U^{(p)}) = E_{\tilde{\Lambda}}(U^{(p)}) \oplus \mathcal{W}(U^{(p)})$ and the local complementing subspaces $\mathcal{W}(U^{(p)}) = \mathrm{range}(\widehat{B}^{(p)})$.*

Then there exists $\tilde{\lambda} \in \sigma(\widetilde{\mathcal{V}})$ such that

$$\frac{|\lambda_j - \tilde{\lambda}|}{\lambda_j} \leq C_M(\lambda_j) \max_{p=1,\ldots,M} \left[\Lambda_N^2 \|B^{(p)} - \widehat{B}^{(p)}\|_*^2 + C_{E^{(p)}}^2 e(\eta, N)^2 \|R^{(p)}\|^2_{\mathcal{B}(\mathcal{V}(U^{(p)}), \mathcal{V})} \right],$$

where $e(\eta, N)$ and $\|\cdot\|_*$ are as defined in Theorems 3.8 *and* 3.10, *respectively. The constants $C_M(\lambda_j)$ and $C_{E^{(p)}}$ are defined as*

$$C_M(\lambda_j) := C(\lambda_j)(\lambda_j + 1)\|\widehat{G}\|^4_{L^\infty(\Omega)} \quad \text{and} \quad C_{E^{(p)}} := \|E^{(p)}\|_{\mathcal{B}(tr\mathcal{V}(\widehat{U}^{(p)}), \mathcal{V}(\widehat{U}^{(p)}))},$$

where $C(\lambda_j)$ is as defined in Proposition 2.1. *The counting function $\hat{G} : \Omega \to \{1, \ldots, M\}$ is defined as $\widehat{G}(x) := \#\{p \mid x \in \widehat{U}^{(p)}\}$.*

Proof. Proposition 2.1 together with (2.7) and (3.16) give

$$\begin{aligned} \frac{|\lambda_j - \tilde{\lambda}|}{\lambda_j} &\leq C(\lambda_j)\|G\|^2_{L^\infty(\Omega)} \sum_{p=1}^M \mathcal{E}(u_j, U^{(p)}) \\ &\leq C(\lambda_j)\|G\|^2_{L^\infty(\Omega)} \sum_{p=1}^M e_U \left(\mathcal{W}(U^{(p)}) \right) \|u_j|_{\widehat{U}^{(p)}}\|^2_{H^1(\widehat{U}^{(p)})} \end{aligned} \tag{4.3}$$

for the local complementing subspaces $\mathcal{W}(U^{(p)}) = \mathrm{range}(\hat{B}^{(p)})$ constructed in section 3 for $U = U^{(p)}$. Estimating the sum similarly with (2.7) and observing that $\|u_j\|^2_{H^1(\Omega)} = (\lambda_j + 1)\|u_j\|^2_{L^2(\Omega)} = (\lambda_j + 1)$, give

$$\frac{|\lambda_j - \tilde{\lambda}|}{\lambda_j} \leq C(\lambda_j)(\lambda_j + 1)\|G\|^2_{L^\infty(\Omega)}\|\widehat{G}\|^2_{L^\infty(\Omega)} \max_{p=1,\ldots,M} e_U \left(\mathcal{W}(U^{(p)}) \right). \tag{4.4}$$

Since $U^{(p)} \subset \widehat{U}^{(p)}$ we have $\|G\|_{L^\infty(\Omega)} \leq \|\widehat{G}\|_{L^\infty(\Omega)}$. Theorem 3.10 completes the proof. ∎

In a practical application of the PU-CPI method, the foremost challenge is to define the low-rank approximating operators $\widehat{B}^{(p)}$ and to efficiently construct a basis for the local complementing subspaces $\mathcal{W}(U^{(p)}) = \text{range}(\widehat{B}^{(p)})$. In section 5, we use the FEM, i.e., $\mathcal{V} = \mathcal{V}_h$, and singular value decomposition (SVD) for this purpose.

5. Finite element realization of PU-CPI. Define the set function (i.e., *open interior of closure*) intc : $A \mapsto B$ as $A = \text{int}(\overline{B})$ for $B \subset \mathbb{R}^d$. A finite family of sets $\{K_i\}_i \subset \Omega$ is called a triangular or a tetrahedral partition of Ω if $K_i \subset \Omega$ are open simplicial sets satisfying $\Omega = \text{intc}(\cup_i K_i)$ and $K_i \cap K_j = \emptyset$ for $i \neq j$. We make a standing assumption that partitions do not contain hanging nodes.

We consider the finite element (FE) discretization of (2.1) under the following assumptions.

Assumption 5.1.

(i) Let $\{\mathcal{T}_h\}_h$ be a family of quasi-uniform triangular or tetrahedral partitions of Ω with mesh size $h = \max_{K \in \mathcal{T}_h} \text{diam}(K)$ in the sense of [9].

(ii) Let

$$\mathcal{V} = \mathcal{V}_h = \{\, w \in H_0^1(\Omega) \mid w|_K \in P^1(K) \quad \text{for all } K \in \mathcal{T}_h \,\}, \tag{5.1}$$

and $\{\psi_l\}_l$ be the nodal basis functions of $\mathcal{V}_h$.

We call $\boldsymbol{x}$ the *coordinate vector* of $w \in \mathcal{V}_h$ and define the one-to-one correspondence $\boldsymbol{x} \sim w$, where $w = \sum_l x_l \psi_l$. The same convention is used in all subspaces of $\mathcal{V}_h$.

An open cover $\{U^{(p)}\}_{p=1}^M$ is constructed by dividing the vertices of the partition $\mathcal{T}_h$ into nonempty disjoint sets $\{\mathcal{N}_p\}_{p=1}^M$ using, e.g., METIS [21]. The set $U^{(p)}$ is obtained as[4]

$$U^{(p)} = \text{intc}\{\, K \in \mathcal{T}_h \mid K \text{ has at least one vertex index in } \mathcal{N}_p\}. \tag{5.2}$$

The r-extension of a subdomain $U^{(p)}$ is chosen as

$$\widehat{U}^{(p)} = \text{intc}\left\{ K \in \mathcal{T}_h \;\middle|\; dist(K, U^{(p)}) \le r \right\}. \tag{5.3}$$

An example of an open cover and the related r-extensions is given in Figure 2. Note that our definition allows very exotic open covers, not all of which are computationally meaningful.

Let $\{\psi_1^{\widehat{U}}, \ldots, \psi_{\hat{n}}^{\widehat{U}}\}$ and $\{\psi_1^U, \ldots, \psi_n^U\}$ be bases of $\mathcal{V}_h(\widehat{U})$ and $\mathcal{V}_h(U)$ on $\widehat{U} \equiv \widehat{U}^{(p)}$ and $U \equiv U^{(p)}$, respectively:

$$\mathcal{V}_h(\widehat{U}) = \text{span}\{\psi_1^{\widehat{U}}, \ldots, \psi_{\hat{n}}^{\widehat{U}}\}, \quad \mathcal{V}_h(U) = \text{span}\{\psi_1^U, \ldots, \psi_n^U\}. \tag{5.4}$$

We further assume that the basis functions are ordered so that

$$\begin{aligned} tr\mathcal{V}_h(\widehat{U}) &= \text{span}\{\psi_1^{\widehat{U}}|_{\partial\widehat{U}}, \ldots, \psi_{\hat{n}_B}^{\widehat{U}}|_{\partial\widehat{U}}\}, \quad & tr\mathcal{V}_h(U) &= \text{span}\{\psi_1^U|_{\partial U}, \ldots, \psi_{n_B}^U|_{\partial U}\}, \\ \mathcal{V}_{h0}(\widehat{U}) &= \text{span}\{\psi_{\hat{n}_B+1}^{\widehat{U}}, \ldots, \psi_{\hat{n}}^{\widehat{U}}\}, & \mathcal{V}_{h0}(U) &= \text{span}\{\psi_{n_B+1}^U, \ldots, \psi_n^U\}. \end{aligned} \tag{5.5}$$

Denote $n_I = n - n_B$ and $\hat{n}_I = \hat{n} - \hat{n}_B$ and assume that n_I, $\hat{n}_B$, $\hat{n}_I$, and n_B are all nonzero. Because of the ordering in (5.5), it is natural to split the coordinate vectors $\boldsymbol{x} \in \mathbb{R}^{\hat{n}}$ to the boundary and interior coordinates as

$$\boldsymbol{x} := \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_I \end{bmatrix}, \quad \text{where} \quad \boldsymbol{x}_B \in \mathbb{R}^{\hat{n}_B} \quad \text{and} \quad \boldsymbol{x}_I \in \mathbb{R}^{\hat{n}_I}. \tag{5.6}$$

[4]Observe that the sets $\{U^{(p)}\}_p$ consist of simplices in a partition $\mathcal{T}_h$. Thus the diameter of each $U^{(p)}$ is always larger than h, linking the scale h and scales of $U^{(p)}$'s.

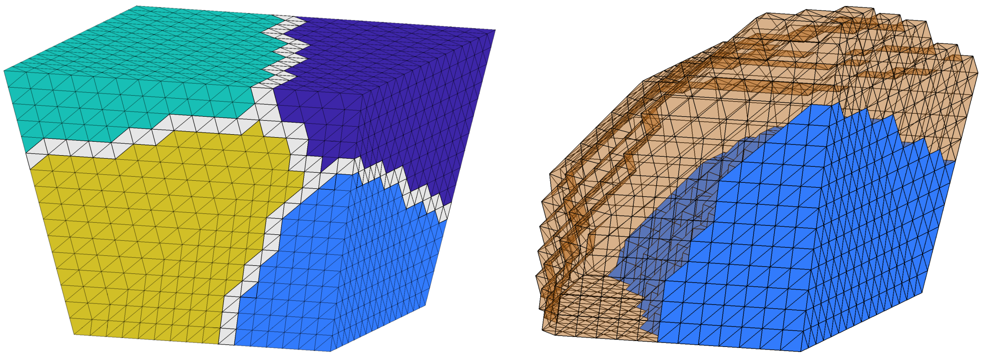

FIG. 2. *A partitioning of a cuboid with four subdomains and a visualization of an extended subdomain on one part. Surface triangles belonging to several* $U^{(p)}$ *and to set* Γ *defined in* (5.4) *are visualized in white.*

This splitting is applied to $\hat{n} \times \hat{n}$-matrices as follows,

$$\mathsf{A} = \begin{bmatrix} \mathsf{A}_{BB} & \mathsf{A}_{BI} \\ \mathsf{A}_{IB} & \mathsf{A}_{II} \end{bmatrix}, \tag{5.7}$$

where $\mathsf{A}_{BB} \in \mathbb{R}^{\hat{n}_B \times \hat{n}_B}$, $\mathsf{A}_{BI} \in \mathbb{R}^{\hat{n}_B \times \hat{n}_I}$, $\mathsf{A}_{IB} \in \mathbb{R}^{\hat{n}_I \times \hat{n}_B}$, and $\mathsf{A}_{II} \in \mathbb{R}^{\hat{n}_I \times \hat{n}_I}$. Let $E_h : tr\mathcal{V}_h(\widehat{U}) \to \mathcal{V}_h(\widehat{U})$ be defined as

$$E_h w_B = \sum_{l=1}^{\hat{n}_B} x_{Bl} \psi_l^{\widehat{U}}, \quad \text{where} \quad \boldsymbol{x}_B = \begin{bmatrix} x_{B1} \\ \vdots \\ x_{B\hat{n}_B} \end{bmatrix} \sim w_B.$$

That is, E_h is a right inverse of the trace operator that satisfies $(E_h w_B)|_U = 0$.

5.1. Evaluation of the trace norm. We discuss evaluation of the norm of $tr\mathcal{V}_h(\widehat{U})$ required to construct $\widehat{B}$ in practice.

LEMMA 5.2. *Let* $\mathcal{V}_h(\widehat{U})$, $tr\mathcal{V}_h(\widehat{U})$ *be as defined in* (5.4) *and assume that* (5.5) *holds. Define* $\mathsf{K} \in \mathbb{R}^{\hat{n} \times \hat{n}}$ *as*

$$\mathsf{K}_{ij} = \int_{\widehat{U}} \left(\nabla \psi_i^{\widehat{U}} \cdot \nabla \psi_j^{\widehat{U}} + \psi_i^{\widehat{U}} \psi_j^{\widehat{U}} \right) dx \quad \text{for } i, j = 1, \dots, \hat{n}$$

and then split K *into* $\mathsf{K}_{BB}, \mathsf{K}_{BI}$, *and* K_{II} *according to* (5.7). *Then for any* $f \in tr\mathcal{V}_h(\widehat{U})$,

$$\|f\|_{tr\mathcal{V}_h(\widehat{U})} = \left(\boldsymbol{x}_B^T \mathsf{S} \boldsymbol{x}_B \right)^{1/2}, \quad \textit{where} \quad \boldsymbol{x}_B \sim f \quad \textit{and} \quad \mathsf{S} = \mathsf{K}_{BB} - \mathsf{K}_{BI} \mathsf{K}_{II}^{-1} \mathsf{K}_{BI}^T. \tag{5.8}$$

Proof. Observe that for $v_1, v_2 \in \mathcal{V}_h(\widehat{U})$ it holds that

$$(v_1, v_2)_{H^1(\widehat{U})} = \boldsymbol{v}_2^T K \boldsymbol{v}_1, \quad \text{where } \boldsymbol{v}_1 \sim v_1,\ \boldsymbol{v}_2 \sim v_2. \tag{5.9}$$

Using the splitting (5.6) and unitary equivalence (5.9) gives

$$\|f\|^2_{trV_h} = \frac{1}{2} \min_{\boldsymbol{y}_I \in \mathbb{R}^{\hat{n}_I}} \begin{bmatrix} \boldsymbol{x}_B & \boldsymbol{y}_I \end{bmatrix} \begin{bmatrix} \mathsf{K}_{BB} & \mathsf{K}_{BI} \\ \mathsf{K}_{BI}^T & \mathsf{K}_{II} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{y}_I \end{bmatrix}, \quad \text{where } \boldsymbol{x}_B \sim f.$$

Direct calculation gives $\boldsymbol{y}_I = -\mathsf{K}_{II}^{-1}\mathsf{K}_{BI}^T\boldsymbol{x}_B$. Hence,

$$\|f\|^2_{tr\mathcal{V}_h} = \boldsymbol{x}_B^T \begin{bmatrix} \mathsf{I} & -\mathsf{K}_{BI}\mathsf{K}_{II}^{-1} \end{bmatrix} \begin{bmatrix} \mathsf{K}_{BB} & \mathsf{K}_{BI} \\ \mathsf{K}_{BI}^T & \mathsf{K}_{II} \end{bmatrix} \begin{bmatrix} \mathsf{I} \\ -\mathsf{K}_{II}^{-1}\mathsf{K}_{BI}^T \end{bmatrix} \boldsymbol{x}_B = \boldsymbol{x}_B^T \mathsf{S} \boldsymbol{x}_B,$$

which completes the proof. ∎

Remark 5.3. The matrix S defined in (5.8) is dense and expensive to construct. To circumvent this, consider the linear system

$$\begin{bmatrix} \mathsf{K}_{BB} & \mathsf{K}_{BI} \\ \mathsf{K}_{BI}^T & \mathsf{K}_{II} \end{bmatrix} \begin{bmatrix} \boldsymbol{y}_B \\ \boldsymbol{y}_I \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_B \\ 0 \end{bmatrix}.$$

By direct calculation $\mathsf{S}\boldsymbol{y}_B = \boldsymbol{x}_B$. Since K is invertible, so is S. Hence,

$$\mathsf{S}^{-1}\boldsymbol{x}_B = \mathsf{F}_B^T\mathsf{K}^{-1}\mathsf{F}_B\boldsymbol{x}_B, \quad \text{where} \quad \mathsf{F}_B \in \mathbb{R}^{n_B\times n},\ (\mathsf{F}_B)_{ij} = \delta_{ij}. \tag{5.10}$$

Using the equation above, the action of S^{-1} can be efficiently computed by solving a linear system with coefficient matrix K, e.g., using the Cholesky factorization. Due to this, our implementation of the PU-CPI method subspace uses S^{-1} instead of S.

5.2. Stitching operators. In section 2.2, the open cover $\{U^{(p)}\}_p$ is related to a family of stitching operators $\{R^{(p)}\}_p$, $R^{(p)} : \mathcal{V}(U^{(p)}) \to \mathcal{V}$. For $\mathcal{V} = \mathcal{V}_h$ we define the stitching operator $R_h : \mathcal{V}_h(U) \to \mathcal{V}_h$ corresponding the subdomain $U = U^{(p)}$ by

$$(R_h)|_{\Omega\setminus U} = 0 \quad \text{and} \quad (R_h w)|_U = \sum_{l=n_B+1}^{n} \psi_l^U x_l. \tag{5.11}$$

Even though[5] $\{\psi_l^U\}_{l=n_B+1}^n$ is a basis of $\mathcal{V}_{h0}(U)$, the embedding $\mathcal{V}_{h0}(U)$ into $\mathcal{V}_h$ by zero extension makes it possible to regard $R_h w$ as an element of $\mathcal{V}_h$. The PU-CPI error estimate in Theorem 4.2 depends on $\|R_h\|_{\mathcal{B}(\mathcal{V}_h(U),\mathcal{V}_h)}$, which we estimate next.

LEMMA 5.4. *Let $U \subset \Omega$ be defined similarly to* (5.2) *and $R_h \in \mathcal{B}(\mathcal{V}_h(U), \mathcal{V}_h)$ as in* (5.11)*. Under Assumption* 5.1 *there exists a constant $C_R = C_R(\{\mathcal{T}_h\}_h)$ such that*

$$\|R_h\|_{\mathcal{B}(\mathcal{V}_h(U),\mathcal{V}_h)} \le C_R h^{-1}.$$

Proof. Recall that $\mathcal{V}_h$ and $\mathcal{V}_h(U)$ inherit their norms from $H_0^1(\Omega)$ and $H^1(U)$, respectively. Let $w \in \mathcal{V}_h(U)$ and $\boldsymbol{x} \sim w$. By the inverse inequality in, e.g., [9, sect. 4.5] there exists a constant $C_{inv} := C_{inv}(\{\mathcal{T}_h\}_h)$, independent of h, such that

$$\|R_h w\|_{H_0^1(\Omega)} = \|\nabla R_h w\|_{L^2(\Omega;\mathbb{R}^d)} \le C_{inv} h^{-1} \|R_h w\|_{L^2(\Omega)}.$$

Observe that $\operatorname{supp}(R_h w) \subset U$ for each $w \in \mathcal{V}_h(U)$. The following norm equivalence is given, e.g., in [9, Lem. 6.2.7]:

$$c_1 h^{d/2}|\boldsymbol{x}| \le \|w\|_{L^2(U)} \le C_1 h^{d/2}|\boldsymbol{x}|, \quad \text{where} \quad |\boldsymbol{x}| = (\boldsymbol{x}^T\boldsymbol{x})^{1/2} \tag{5.12}$$

for any $w \in \mathcal{V}_h(U)$, $\boldsymbol{x} \sim w$, and constants $c_1 = c_1(\{\mathcal{T}_h\}_h), C_1 = C_1(\{\mathcal{T}_h\}_h)$. Using (5.12) and definition (5.11) gives

$$\|R_h w\|_{L^2(\Omega)} \le C_1 h^{d/2} \left(\sum_{l=n_B+1}^{n} x_l^2 \right)^{1/2} \le C_1 h^{d/2}|\boldsymbol{x}| \le C_1 c_1^{-1} \|w\|_{L^2(\Omega)}. \qquad ∎$$

[5] In our implementation of the stitching operator, we select the basis functions $\{\psi_l^U\}_l$ from the set $\{\psi_l\}_l$ to avoid changing bases. Keeping track of the related indexing is challenging and not discussed here nor in the following.

5.3. The local complementing subspace. Recall that the local complementing subspace satisfies $\mathcal{W}_h(U) = \mathrm{range}(\widehat{B}_h)$, where $\widehat{B}_h$ is a low-rank approximation of B_h defined in (3.18) for $\mathcal{V} = \mathcal{V}_h$. In this section, we use SVD to construct this approximation, state the corresponding error estimate, and explain how this can be realized using matrix algebra.

As usual, the finite-dimensional operator $B_h \in \mathcal{B}(tr\mathcal{V}_h(\widehat{U};\mathbb{R}^N), \mathcal{V}_{hR}(U))$ has the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, and there exist rank k, $k < n$, operators B_{hk} satisfying

$$\min_{\mathrm{rank}(T)\leq k} \|B_h - T\|_* = \|B_h - B_{hk}\|_* = \sigma_{k+1}, \tag{5.13}$$

where $\|\cdot\|_* = \|\cdot\|_{\mathcal{B}(tr\mathcal{V}_h(\widehat{U};\mathbb{R}^N),\mathcal{V}_{hR}(U))}$. Now $\mathcal{W}_h(U) = \mathrm{range}(B_{hk})$ for B_{hk} satisfying (5.13) for fixed cutoff index k. Ultimately, a basis for $\mathcal{W}_h(U)$ is provided by the left-singular vectors $\{u_1, \ldots, u_k\}$ corresponding to an SVD of B_h.

Theorem 5.5. *Make Assumption* 4.1 *defining* λ_j *and let* $\mathcal{V} = \mathcal{V}_h$ *satisfy Assumption* 5.1. *Let the stitching operators* $R_h^{(p)}$ *be defined as in* (5.11) *for* $U = U^{(p)}$ *and* $p \in \{1, \ldots, M\}$. *Let the singular values* $\sigma_1^{(p)} \geq \sigma_2^{(p)} \geq \cdots \geq \sigma_{n^{(p)}}^{(p)}$ *and left–singular vectors* $\{u_l^{(p)}\}_{l=1}^{n^{(p)}}$ *be defined as above for* $U = U^{(p)}$. *The local complementing subspaces are defined as*

$$\mathcal{W}_h(U^{(p)}) := \mathrm{span}\{u_1^{(p)}, \ldots, u_{k^{(p)}}^{(p)}\},$$

where $\{k^{(p)}\}_{p=1}^M$ *are local cutoff indices. Define the local subspaces as* $\widetilde{\mathcal{V}}_h(U^{(p)}) := E_{\tilde{\Lambda}}(U^{(p)}) \oplus \mathcal{W}_h(U^{(p)})$, *where* $E_{\tilde{\Lambda}}(U^{(p)})$ *is as in* (4.1), *and the associated (global) PU-CPI method subspace* $\widetilde{\mathcal{V}}_h$ *as in* (2.6). *Then there exists* $\tilde{\lambda} \in \sigma(\widetilde{\mathcal{V}}_h)$ *such that*

$$\frac{|\lambda_j - \tilde{\lambda}|}{\lambda_j} \leq C_M(\lambda_j) \max_{p=1,\ldots,M} \left[\Lambda_N (\sigma_{k^{(p)}+1}^{(p)})^2 + C_R^2 C_{E^{(p)}}^2 h^{-2} e^2(\eta, N) \right],$$

where Λ_N, $C_M(\lambda_j)$, *and* $C_{E^{(p)}}^2$ *are as defined in Theorem* 4.2, C_R *as in Lemma* 5.4, *and* $e(\eta, N)$ *as in Theorem* 3.8.

Theorem 5.5 follows from the above discussion and Theorem 4.2.

We proceed to deal with the details in the practical construction of a basis for $\mathcal{W}_h(U)$. To this end, we represent the linear operators $Z_h(t)$ and B_h as matrices using the bases of $tr\mathcal{V}_h(\widehat{U}), \mathcal{V}_{h0}(\widehat{U})$, and $\mathcal{V}_h(U)$ defined in (5.4)–(5.5). Denote by $\mathsf{A}, \mathsf{M} \in \mathbb{R}^{\hat{n}\times\hat{n}}$ the stiffness and mass matrices of the FE-discretized version of (3.2), respectively. Both of these matrices are split as in (5.7). Following section 3.3, the matrix representation of $Z_h(t)$ is $\mathsf{Z}_h : (0, \Lambda) \to \mathbb{R}^{\hat{n}_I\times\hat{n}_B}$ given by

$$\mathsf{Z}_h(t) := \mathsf{P}_h(\mathsf{A}_{II} - t\mathsf{M}_{II})^\dagger(-\mathsf{A}_{BI}^T + t\mathsf{M}_{BI}^T), \tag{5.14}$$

where $\mathsf{Z}_h(t)$ is real analytic for all $t \in (0, \Lambda)$. Here $\dagger$ is the Moore–Penrose pseudoinverse and $\mathsf{P}_h := \mathsf{I} - \sum_{k=1}^{K(\tilde{\Lambda})} \boldsymbol{v}_k\boldsymbol{v}_k^T\mathsf{M}_{II}$, where $\boldsymbol{v}_k \sim v_k$ for eigenfunctions $v_k \in \mathcal{V}_h(\widehat{U})$ of (3.6).[6] The matrix representation of the operator B_h, defined in (3.18), in the natural basis of the Cartesian product space $tr\mathcal{V}_h(\widehat{U};\mathbb{R}^N)$ is

$$\mathsf{B}_h = \mathsf{F}_U \begin{bmatrix} \mathsf{Z}_h(\xi_1) & \cdots & \mathsf{Z}_h(\xi_N) \end{bmatrix} \in \mathbb{R}^{n\times N\hat{n}_B}, \tag{5.15}$$

[6]This is another way to define Z_h for all $t \in (0, \Lambda)$ compared to section 3.2; also used in [18].

where $n = \dim(\widetilde{\mathcal{V}}_h(U))$ and $\mathsf{F}_U \in \mathbb{R}^{\hat{n}_I \times n}$ is the matrix representation of the restriction operator $F_U : \mathcal{V}_h(\widehat{U}) \to \mathcal{V}_h(U)$ given by $F_U v = v|_U$ in bases (5.4)–(5.5). The norm of the Cartesian product space $tr\mathcal{V}_h(\widehat{U};\mathbb{R}^N)$ in terms of coordinate vectors is given by

$$\|\boldsymbol{v_B}\|_{tr\mathcal{V}_h(\widehat{U};\mathbb{R}^N)} = \|(\mathsf{I}_N \otimes \mathsf{S}^{1/2})\boldsymbol{x}_B\|_2 \quad \text{for} \quad \boldsymbol{x}_B \sim v_B$$

by Lemma 5.2. Here $\mathsf{I}_N \in \mathbb{R}^{N\times N}$ is the identity matrix and $\otimes$ denotes the Kronecker product. Finally, observe that $\|w\|_{\mathcal{V}_{hR}(U)} = \|\mathsf{K}_R^{1/2}\boldsymbol{x}\|_2$ with $\boldsymbol{x} \sim w$ and the symmetric, positive definite matrix $\mathsf{K}_R \in \mathbb{R}^{n\times n}$ defined as

$$(\mathsf{K}_R)_{lm} = \int_U \left(\nabla(R_h\psi_l^U)\cdot\nabla(R_h\psi_m^U) + \psi_l^U\psi_m^U\right)\, dx. \tag{5.16}$$

Matrix representation of operators $\{B_{hk}\}_k$ is obtained by computing the SVD of the $\mathbb{R}^{n\times N\hat{n}}$-matrix

$$\mathsf{C} := \mathsf{K}_R^{1/2}\mathsf{B}_h(\mathsf{I}_N \otimes \mathsf{S}^{-1/2}) = \sum_{l=1}^{n} \sigma_l \boldsymbol{u}_l \boldsymbol{v}_l^T,$$

where $\{\boldsymbol{u}_l\}_{l=1}^n \subset \mathbb{R}^n$ and $\{\boldsymbol{v}_l\}_{l=1}^n \subset \mathbb{R}^{N\hat{n}}$ are the left-singular and the right-singular vectors of C, respectively. Then

$$\mathsf{B}_{hk} = \mathsf{K}_R^{-1/2}\left(\sum_{l=1}^{k}\sigma_l\boldsymbol{u}_l\boldsymbol{v}_l^T\right)(\mathsf{I}_N \otimes \mathsf{S}^{1/2}), \tag{5.17}$$

as can be seen from the definition of the operator norm $\|\cdot\|_*$ by a change of variables.

The basis for $\mathcal{W}_h(U)$ is obtained from the first k left-singular vectors $\{\boldsymbol{u}_l\}_l$ of the matrix C as

$$\mathcal{W}_h(U) = \left\{ \sum_{l=1}^{n} y_l\psi_l^U \in \mathcal{V}_h(U) \;\middle|\; \boldsymbol{y} \in \mathsf{K}_R^{-1/2}\operatorname{span}\{\boldsymbol{u}_1,\ldots,\boldsymbol{u}_k\} \right\}. \tag{5.18}$$

In numerical realization, the vectors $\{\boldsymbol{u}_l\}_l$ are computed by solving the largest k eigenpairs of the $\mathbb{R}^{n\times n}$-matrix[7]

$$\mathsf{C}\mathsf{C}^T = \mathsf{K}_R^{1/2}\left(\sum_{i=1}^{N}\mathsf{F}_U\mathsf{Z}_h(\xi_i)\mathsf{S}^{-1}\mathsf{Z}_h(\xi_i)^T\mathsf{F}_U^T\right)\mathsf{K}_R^{1/2} \tag{5.19}$$

using the Lanczos iteration with the mapping $\boldsymbol{x} \mapsto \mathsf{C}\mathsf{C}^T\boldsymbol{x}$. There are two reasons for using the dual approach. First, the dimension of $\mathsf{C}\mathsf{C}^T$ is independent of N. Second, an explicit construction of S is avoided by utilizing Remark 5.3.

5.4. Assembly of the PU-CPI Ritz eigenproblem. The remaining task is to solve the global Ritz eigenvalue problem (1.2) posed in the PU-PCI method subspace $\widetilde{\mathcal{V}}_h$. Let $\{\varphi_l^{(p)}\}_l$ be a basis of the space $R_h^{(p)}\widetilde{\mathcal{V}}_h(U^{(p)}) \subset \mathcal{V}_h$ and denote $n^{(p)} := \dim(R_h^{(p)}\widetilde{\mathcal{V}}_h(U^{(p)}))$. Then the ordered set

$$\{\varphi_l^{(p)} \mid l = 1,\ldots,n^{(p)},\ p = 1,\ldots,M\ \} = \left\{\phi_k \mid k = 1,\ldots,\sum_{p=1}^{M} n^{(p)}\ \right\} \tag{5.20}$$

[7]In practice, the square roots $K_R^{1/2}$ are replaced by the Cholesky factors of K_R.

is a basis for the PU-CPI method subspace $\widetilde{\mathcal{V}}_h$ defined in (2.6) with dimension $\tilde{n} := \sum_{p=1}^{M} n^{(p)}$. The ordering in (5.20) defines an integer-valued function $\sigma(p, l)$ satisfying

$$\varphi_l^{(p)} = \phi_{\sigma(p,l)} \quad \text{for } l = 1, \ldots, n^{(p)},\ p = 1, \ldots, M.$$

Next, we assemble the matrices A, M in the global eigenproblem find $(\tilde{\lambda}_k, \tilde{\boldsymbol{v}}_k) \in \mathbb{R}^+ \times \mathbb{R}^{\tilde{n}}$ such that

$$\mathsf{A}\tilde{\boldsymbol{v}}_k = \tilde{\lambda}_k \mathsf{M}\tilde{\boldsymbol{v}}_k,$$

where $\mathsf{A}_{lm} = (\nabla\phi_l, \nabla\phi_m)_{L^2(\Omega;\mathbb{R}^d)}$and $\mathsf{M}_{lm} = (\phi_l, \phi_m)_{L^2(\Omega)}$. In our early numerical experiments, a straightforward assembly of A and M proved to be time consuming. Next, we outline a more efficient and numerically more stable strategy.

We only study the entries of A since the entries of M are computed similarly. The entries of A are obtained by computing

$$\mathsf{A}_{\sigma(p,l),\sigma(q,m)} = \int_\Omega \nabla\varphi_l^{(p)} \cdot \nabla\varphi_m^{(q)}\, dx \tag{5.21}$$

for each $l = 1, \ldots, n^{(p)}$, $m = 1, \ldots, n^{(q)}$, and $p, q \in \{1, \ldots, M\}$. If $p \neq q$ in (5.21),

$$\mathsf{A}_{\sigma(p,l),\sigma(q,m)} = \int_\Gamma \nabla\varphi_l^{(p)} \cdot \nabla\varphi_m^{(q)}\, dx, \tag{5.22}$$

where the overlap set $\Gamma \subset \Omega$ is defined as

$$\Gamma = \operatorname{intc}\{\ K \in \mathcal{T}_h \mid K \text{ has vertex indices in at least two sets } \mathcal{N}_p\ \};$$

see Figure 2. The off-diagonal entries in (5.22) can be computed if the functions $\{\varphi_l|_\Gamma\}_{l=1}^{\tilde{n}}$ are known.

If $p = q$ in (5.21),

$$\mathsf{A}_{\sigma(p,l),\sigma(p,m)} = \int_{U^{(p)}} \nabla\varphi_l^{(p)} \cdot \nabla\varphi_m^{(p)}\, dx. \tag{5.23}$$

To store the minimal amount of data, the basis functions $\{\varphi_l^{(p)}\}_{l=1}^{n^{(p)}}$ are solutions of the symmetric eigenvalue problem

$$\int_{U^{(p)}} \nabla\varphi_l^{(p)} \cdot \nabla\varphi_m^{(p)}\, dx = d_l^{(p)} \int_{U^{(p)}} \varphi_l^{(p)}\varphi_m^{(p)}\, dx \quad \text{and} \quad \|\varphi_l^{(p)}\|_{L^2(U^{(p)})} = 1 \tag{5.24}$$

for eigenvalues $d_l^{(p)} \in \mathbb{R}^+$ and for each $l, m = 1, \ldots, n^{(p)}$. Thus, for each p,

$$\mathsf{A}_{\sigma(p,l),\sigma(p,m)} = d_l^{(p)}\delta_{lm} \quad \text{and} \quad \mathsf{M}_{\sigma(p,l),\sigma(p,m)} = \delta_{lm}.$$

To summarize, the matrices A and M can be fully characterized based on the data

$$\{\varphi_l|_\Gamma\}_{l=1}^{\tilde{n}}, \quad \{\nabla\varphi_l|_\Gamma\}_{l=1}^{\tilde{n}}, \quad \text{and} \quad \{d_l^{(p)}\}_{l=1}^{n^{(p)}} \quad \text{for } p = 1, \ldots, M.$$

Restrictions of the basis functions can be stored, e.g., on some inner surface to visualize the eigenfunctions.

Table 1
Input parameters to the PU-CPI algorithm.

Λ	Spectral interval of interest $(0,\Lambda)$
N	Number of interpolation points
η	Oversampling parameter
$\mathcal{T}_h$	Triangular ($d=2$) or tetrahedral ($d=3$) partition of Ω
M	Number of subdomains
r	Extension radius
tol	Cutoff tolerance for singular values

5.5. Overview of the PU-CPI algorithm. The PU-CPI is intended for a distributed computing environment with a single master and multiple workers. The input data for the algorithm are specified in Table 1.

The cutoff tolerance is used to determine the parameters $k^{(p)}$ in Theorem 5.5 so that $\sigma^{(p)}_{k^{(p)}+1} \le tol$. Theorem 5.5 gives the error estimate, for any $\lambda_j \in \sigma(\mathcal{V}_h) \cap (0,\Lambda)$ there exists $\tilde{\lambda} \in \sigma(\widetilde{\mathcal{V}}_h)$ such that

$$\frac{|\lambda_j - \tilde{\lambda}|}{\lambda_j} \le C\left[tol^2 + e^2(\eta, N)\right] \quad \text{for some constant } C.$$

The PU-CPI proceeds in three steps:

1. *Step* 1 *(work division).* METIS is used to partition the vertices of $\mathcal{T}_h$ into M subsets by the master. The submeshes defining $U^{(p)}$ and $\widehat{U}^{(p)}$ are created from these vertex sets as explained in section 5. The submeshes defining $U^{(p)}$ and $\widehat{U}^{(p)}$ for $p = \{1,\dots,M\}$ are submitted to workers.
2. *Step* 2 *(distributed computation).* Each worker receives a submesh and computes a basis for $R_h^{(p)}\widetilde{\mathcal{V}}_h(U^{(p)})$ in the following steps (i)–(v), where all matrices refer to the subdomain $U^{(p)}$.
   - (i) Assemble the stiffness and mass matrices A, M related[8] to $\mathcal{V}_h(\widehat{U}^{(p)})$. Split A, M to interior and boundary parts according to (5.7). Compute the $K(\tilde{\Lambda})$ lowest eigenpairs $(\mu_k, \boldsymbol{v}_k)$ of the pencil $(\mathsf{A}_{II}, \mathsf{M}_{II})$, and form the projection $\mathsf{P}_h = \mathsf{I} - \sum_{k=1}^{K(\tilde{\Lambda})} v_k v_k^T \mathsf{M}_{II}$.
   - (ii) Construct the matrices K_R as in (5.16), F_U as in (5.15), K as in Lemma 5.2, and F_B as in (5.10).
   - (iii) Compute the largest eigenpairs $(\sigma_k^2, \boldsymbol{c}_k)$ of

     $$\mathsf{C}\mathsf{C}^T = \mathsf{L}^T\left(\sum_{i=1}^{N} \mathsf{F}_U \mathsf{Z}_h(\xi_i)\mathsf{S}^{-1}\mathsf{Z}_h(\xi_i)^T\mathsf{F}_U^T\right)\mathsf{L}$$

     using the Lanczos iteration. The action $\boldsymbol{x} \mapsto \mathsf{S}^{-1}\boldsymbol{x}$ is evaluated as explained in Remark 5.3.
   - (iv) An auxiliary basis for $R_h^{(p)}\mathcal{V}_h(\widehat{U}^{(p)})$ is obtained from column vectors of Q,

     $$\mathsf{Q} := \mathsf{R}\left[\mathsf{F}_U\boldsymbol{v}_1, \dots, \mathsf{F}_U\boldsymbol{v}_{K(\widetilde{\Lambda})}, \mathsf{L}^{-T}\boldsymbol{c}_1, \dots, \mathsf{L}^{-T}\boldsymbol{c}_k\right],$$

[8]The homogeneous Dirichlet boundary condition is imposed on $\partial\widehat{U}^{(p)} \cap \partial\Omega$ and this has been communicated to the worker.

where R is the matrix representation of $R_h^{(p)}$ restricted to $\mathcal{V}_{h0}(U^{(p)})$. To satisfy (5.24), we solve the diagonal matrix D and the invertible matrix V from the eigenvalue problem

$$\mathsf{Q}^T\mathsf{A}_0\mathsf{Q}\mathsf{V} = \mathsf{Q}^T\mathsf{M}_0\mathsf{Q}\mathsf{V}\mathsf{D} \quad \text{and} \quad \mathsf{V}^T\mathsf{Q}^T\mathsf{M}_0\mathsf{Q}\mathsf{V} = \mathsf{I},$$

where A_0 and M_0 are the stiffness and mass matrices in $\mathcal{V}_{h0}(U^{(p)})$. The final subspace is obtained from the columns of $\tilde{\mathsf{Q}} = \mathsf{Q}\mathsf{V}$.

(v) Submit diag(D) and $\tilde{\mathsf{Q}}(:, n_\Gamma)$ to the master. Here n_Γ is set of those vertex indices that lie on $\overline{\Gamma}$.

3. *Step* 3 *(solution of the global PU-CPI eigenproblem).* The master solves (1.2) posed in the method subspace $\widetilde{\mathcal{V}}_h$. The required matrices are constructed as outlined in section 5.4 and the resulting problem solved using the Lanczos iteration.

6. Numerical examples. We give numerical examples validating the theoretical results and demonstrating the potential of the PU-CPI variant of section 5. For this purpose, we use a cluster of 26 desktop computers of which 24 had a Xeon E3-1230 CPU, and two were equipped with Xeon W-2133. There was 32 GB of RAM in all but one workstation which had 64 GB. Because solving the smallest eigenvalues of the global Ritz eigenvalue problem (1.2) posed in the PU-PCI method subspace $\widetilde{\mathcal{V}}_h$ using shift-and-invert Lanczos iteration requires lots of memory, the workstation with 64 GB of RAM acted as the master. All data were transferred over NFS, and distributed tasks were launched using GNU parallel [26]. All computations were done using MATLAB R2019a. As the computers were also in other use, the given runtime estimates are conservative.

We study the behavior and convergence of the method using the domain

$$\Omega = F((0,1)^3), \quad \text{where} \quad F : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 + 0.4x_3(2x_1 - 1) \\ x_2 + 0.4x_3(2x_2 - 1) \\ x_3 \end{bmatrix}; \tag{6.1}$$

see Figure 2. As in section 5, problem (1.1) is posed in the space $\mathcal{V} \equiv \mathcal{V}_h$, where $\mathcal{V}_h$ is the FE space of piecewise linear functions over the tetrahedral partition $\mathcal{T}_h$ of domain Ω. The mesh parameter values h are varied by mapping different uniform tetrahedral meshes of $(0,1)^3$ with F. The open cover $\{U^{(p)}\}_p$ of Ω is constructed by splitting the vertex indices of the $\mathcal{T}_h$ into disjoint sets $\{\mathcal{N}_p\}_p$ using METIS as explained in section 5. The subdomains produced in this manner can have significantly different shapes and sizes in a way that cannot be controlled. We observed that choosing the extension radius $r^{(p)}$ proportional to the diameter of the corresponding subdomain $U^{(p)}$ is beneficial for keeping the dimension of the subproblems reasonable. This is done heuristically: define the empirical radius of $U^{(p)}$ by

$$r_c^{(p)} = \frac{1}{2}\left(\max_{m\in\mathcal{N}_p} \boldsymbol{u}^T\boldsymbol{x}_m - \min_{m\in\mathcal{N}_p} \boldsymbol{u}^T\boldsymbol{x}_m\right),$$

where $\boldsymbol{u}$ is the first principal component of the coordinate vector set $\{\boldsymbol{x}_m\}_{m\in\mathcal{N}_p}$. Unless otherwise stated, we choose the extension radius for subdomain $U^{(p)}$ as $r^{(p)} = 0.2r_c^{(p)}$.

Intuitively speaking, we have observed that PU-CPI works best if the subdomains $U^{(p)}$ touch each other as little as possible. So as to domain Ω in (6.1), we observed that METIS produces subdomains that have significant intersections compared to their diameters. This represents the worst-case behavior of PU-CPI.

Throughout this section, we approximate 200 lowest eigenvalues of problem (1.1), and the parameter Λ is chosen accordingly. While experimenting with PU-CPI, it appears that choosing $N = 5$ and $\eta = 2.5$ makes the interpolation error smaller than 10^{-10} for all mesh sizes h used. Hence, these values were kept fixed, and the dependency of the relative eigenvalue error on N and η was not investigated. We focus on the effect of cutoff tolerance of singular values, number of subdomains, problem size, and the extension radius on computational load and accuracy.

6.1. Varying mesh density. The eigenvalue problem (1.1) was solved with different mesh parameters h. Subdomains with about 5000 vertices were used except for the three densest meshes. For these meshes, a smaller number of larger subdomains was required to decrease $\dim(\widetilde{\mathcal{V}}_h)$, so that the eigenvalue problem (1.2) posed in space $\widetilde{\mathcal{V}}_h$ could be solved by the master workstation. Since METIS failed to partition the densest mesh, it was manually divided into cube-shaped subdomains.

The results are shown in Table 2. The maximum relative eigenvalue error was estimated by comparing PU-CPI against the shift-and-invert Lanczos solution of (1.1) using the MATLAB `eigs` function with a tolerance of 10^{-10}. The sparsity of the matrices produced by PU-CPI is shown in Figure 4. A breakdown of time required by each step of PU-CPI is shown in Table 3. The comparable values $\mathrm{t_{CPI}}$ and $\mathrm{t_{FEM}}$ are the wall clock times (in seconds) spent after the mesh structure was constructed. For fair comparison, standard FE solution uses the MATLAB `eigs` function with a tolerance of 10^{-4}. In addition, $\mathrm{t_{CPI}}$ includes file I/O times and network delays, whereas $\mathrm{t_{FEM}}$ includes the time required to assemble the full stiffness and mass matrices.

6.2. Effect of subdomain extension. When using a larger extension radius $r^{(p)}$, the singular values $\sigma_k^{(p)}$ of C in (5.19) are expected to decay faster. This effect is studied using meshes with 54 872 and 195 112 of degrees–of–freedom (DOFs). In both cases, the singular values $\sigma_k^{(p)}$ were computed for a single subdomain with extension radius $r^{(p)} = 0.2r_c^{(p)}$, $0.6r_c^{(p)}$, and $r_c^{(p)}$. The results are shown in Figure 5, and the extended subdomains with different radii are visualized in Figure 3. As expected, the singular values decay much faster for larger $r^{(p)}$. This comes at higher computational cost due to an increase in the extended subdomain DOFs. At the same time, the faster decay of singular values leads to smaller $\dim(\widetilde{\mathcal{V}}_h)$.

6.3. The effect of the cutoff tolerance of singular values. The computations were performed using three different mesh densities and several values of *tol*. The maximum relative eigenvalue error and $\dim(\widetilde{\mathcal{V}}_h)$ are shown in Figure 6. Additionally, relative error for each of the 200 lowest eigenvalues are detailed in Figure 7. These results verify the linear relationship between tol^2 and the relative eigenvalue error predicted in section 5.5. In this example, choosing $tol = 1$ already produces a relative eigenvalue error smaller than 1%.

7. Conclusions. The PU-CPI method for the approximate solution of eigenvalues in $(0, \Lambda)$ of the Dirichlet Laplacian on domain Ω is proposed. PU-CPI is a Ritz method where the method subspace $\widetilde{\mathcal{V}}$ is constructed from the local method subspaces $\{\widetilde{\mathcal{V}}(U^{(p)})\}_p$ for $U^{(p)} \subset \Omega$ as stated in (2.6). Since the local subspaces are independent of each other, PU-CPI can be used in distributed computing environments where communication is costly. Failed distributed tasks can be restarted, making the implementation of PU-CPI very robust.

TABLE 2

Relative fill-in is the ratio of the number of nonzeros in the stiffness matrices from spaces $\widetilde{\mathcal{V}}_h$ and $\mathcal{V}_h$. The last column is the average size of the dimension of the local method subspaces. Relative error is not given if the problem could not be solved using the MATLAB `eigs` *on a single workstation. The number of subdomains in computations, except for those with the three largest* $\dim(\mathcal{V}_h)$, *was chosen so that each subdomain had about* 5000 *vertices; see section* 6.1.

$\dim(\mathcal{V}_h)$	$\dim(\widetilde{\mathcal{V}}_h)$	M	rel. fill-in %	max rel error	$\mathrm{avg_p}\{\dim \widetilde{\mathcal{V}}(U^{(p)})\}$
54 872	2713	13	436.0	4.28×10^{-5}	209
110 592	3777	25	291.9	1.90×10^{-4}	151
195 112	4920	44	171.9	2.99×10^{-4}	112
314 432	6048	69	121.1	3.73×10^{-4}	88
474 552	7686	103	85.6	3.47×10^{-4}	75
681 472	9791	146	76.9	3.81×10^{-4}	67
941 192	12 398	200	68.2	-	62
1 259 712	15 587	267	60.8	-	58
1 643 032	19 735	346	59.9	-	57
2 097 152	24 276	440	56.3	-	55
2 628 072	30 124	549	56.4	-	55
3 241 792	12 820	150	26.7	-	85
5 000 211	19 261	250	24.7	-	77
10 360 232	29 124	308	22.8	-	95

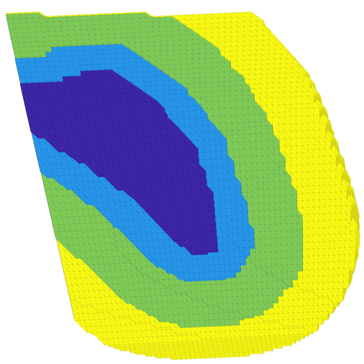

FIG. 3. *Examples of extended subdomains* $\widehat{U}^{(p)}$ *for extension radii* $0.2r_c^{(p)}, 0.6r_c^{(p)}$, *and* $r_c^{(p)}$. *The figure depicts a cross section where* $U^{(p)}$ *is colored in dark blue.*

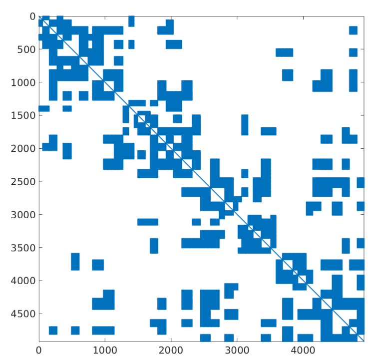

FIG. 4. *Sparsity pattern of the PU-CPI stiffness and mass matrices corresponding to* (1.2) *posed on* $\widetilde{\mathcal{V}}_h$ *with* $\dim(\mathcal{V}_h) = 195\,112$, $\dim(\widetilde{\mathcal{V}}_h) = 492$, *and* $M = 44$.

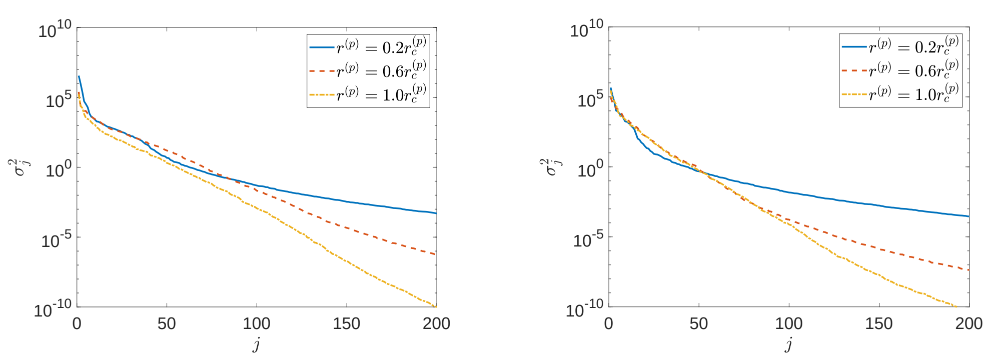

FIG. 5. *Effect of the extension radius on the decay of singular values* $\{\sigma_j\}_{j=1}^{200}$ *for the cube with* 54 872 *(left panel) and* 195 112 *(right panel) DOFs. In these experiments, extended subdomain DOFs range between* 10 610–32 423 *and* 12 644–49 659, *respectively.*

TABLE 3

Second column: Average time over p of computing bases for local method subspaces $\widetilde{\mathcal{V}}_h(U^{(p)})$ by workers. Third column: Time required to partition the mesh by the master. Fourth column: Time required to construct the r-extensions by the master. Remaining columns: Solution time for (1.2) *posed in PU-CPI method subspace with the MATLAB* **eigs** *(t_{red}), total PU-CPI computational time (t_{CPI}), and time required by direct FE-solution of* (1.1) *(t_{FEM}). Cases where the problem could not be solved on a single workstation are marked with –. METIS was not used for the densest mesh, and all times are in seconds.*

$\dim(\mathcal{V}_h)$	avg. t_{sub}	METIS	r-ext.	t_{red}	t_{CPI}	t_{FEM}
54 872	32.3	1.6	6.5	13.3	105.1	53.9
110 592	35.5	3.4	13.6	19.2	130.0	142.2
195 112	48.8	6.2	31.7	25.2	177.2	307.5
314 432	59.1	10.3	48.6	33.8	227.3	661.7
474 552	63.8	16.7	78.3	43.5	303.7	1264.7
681 472	68.0	24.1	123.8	56.1	429.7	1859.7
941 192	71.2	34.7	189.7	79.4	610.8	-
1 259 712	75.5	47.2	263.8	107.0	772.7	-
1 643 032	75.8	65.0	357.6	151.6	1030.5	-
2 097 152	79.9	85.7	511.0	212.5	1397.2	-
2 628 072	79.6	107.3	671.3	302.4	1781.5	-
3 241 792	378.9	114.8	1562.0	165.5	4509.7	-
5 000 211	385.3	327.4	1680.4	306.1	6312.9	-
10 360 232	344.4	-	1365.4	690.3	6525.7	-

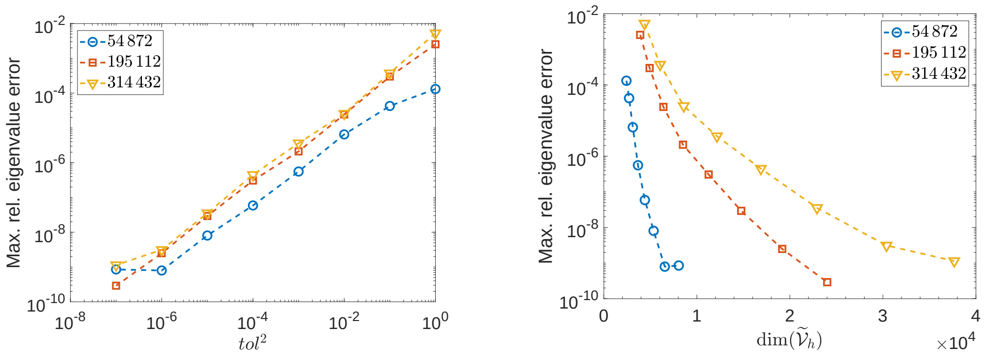

FIG. 6. *Maximum relative eigenvalue error using three mesh densities. Left panel: Given as a function of the cutoff tolerance for singular values* tol. *Right panel: Given as a function of* $\dim(\widetilde{\mathcal{V}}_h)$ *for the same sample points.*

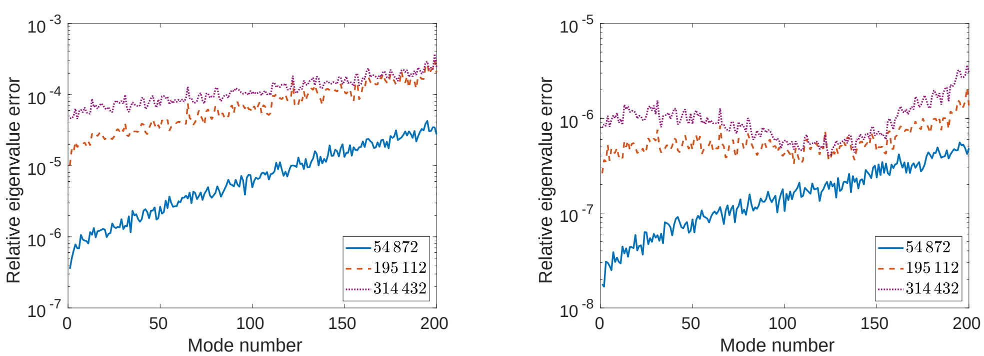

FIG. 7. *Relative eigenvalue errors for the* 200 *lowest modes for three mesh densities. The cutoff tolerance for singular values $tol^2 = 0.1$ (left panel) and $tol^2 = 0.001$ (right panel).*

Let (u, λ) be the solution of (1.1) for $\lambda \in (0, \Lambda)$. According to Proposition 2.1 and (2.7), the local method subspaces should be designed to approximate $u|_{U^{(p)}}$. Local information on $u|_{U^{(p)}}$ is obtained in terms of the operator valued function Z_U in Lemma 3.1. Since Z_U is compact operator valued by Lemma 3.2, its values can be efficiently low-rank approximated.

The local method subspace for the single subdomain $U \equiv U^{(p)}$ is designed to approximate the range of Z_U in the sense of (3.15). This approximation makes use of interpolation, linearization, and low-rank approximation as explained in section 3.4. The local approximation error is estimated in Theorems 3.8 and 3.10. Theorem 4.2 combines these estimates to bound the global relative eigenvalue error.

An example of low-rank approximation is given for the first-order FEM in Theorem 5.5. The key ingredients are Lemma 5.2 and Remark 5.3 that allow numerical treatment of a required boundary trace norm. A basis for each local method subspace is obtained from eigenvectors of the corresponding CC^T in (5.19). The dimension of CC^T is independent of parameters N and η.

Finally, numerical examples validating the theoretical results and demonstrating the potential of PU-CPI are given in section 6. The authors could use inexpensive networked workstations to solve an eigenvalue problem ten times as large as straightforwardly solvable on a single workstation. In contrast to using a supercomputer, such networked workstations are widely available.

The dimension of PU-CPI method subspace $\widetilde{\mathcal{V}}$ is related to the number of singular values of each CC^T larger than given $tol > 0$. Nothing in our theoretical work indicates how the fast singular values decay or estimate the dimension of $\widetilde{\mathcal{V}}$. The numerical results in Figure 5 indicate exponential decay with a rate dependent on the extension radius r, which we believe to be a generic property of similar elliptic problems. All this remains a topic of further research.

Acoustic eigenvalues problem, for example, benefit from treatment of more general boundary conditions. The authors have implemented PU-CPI for mixed homogeneous Dirichlet and Neumann boundary conditions, and the error analysis extends to this case.

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